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Banach spaces with many boundedly complete basic sequences failing PCP ☆

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To my mother Francisca and my sister Isabel, in memoriam

Abstract

We prove that there exist Banach spaces not containing ℓ_1, failing the point of continuity property and satisfying that every semi-normalized basic sequence has a boundedly complete basic subsequence. This answers in the negative the problem of Remark 2 in Rosenthal (2007) [12].

Keywords: Point of continuity property; Boundedly complete sequences; Supershrinking sequences

1. Introduction

Recall that a Banach space is said to have the point of continuity property (PCP) provided every non-empty closed and bounded subset admits a point of continuity of the identity map from the weak to norm topologies. It is known that Banach spaces with Radon–Nikodym property, including separable dual spaces, satisfy PCP, but the converse is false (see [2]). The PCP has been characterized for separable Banach spaces in [2] and [5], and this characterization implies that Banach spaces with PCP have many boundedly complete basic sequences, and so many subspaces which are separable dual spaces. As PCP is separably determined [1], that is, a Banach space satisfies PCP if every separable subspace has PCP, it is natural looking for a sequential characterization of PCP. In this sense, it has been proved in [12] that every semi-normalized basic

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sequence in a Banach space with PCP has a boundedly complete subsequence. The converse of the above result is false in general, but it is open for Banach spaces not containing ℓ_1 (see Remark 2 in [12]). The goal of this note is to prove in Corollary 2.4 that there exists a family of Banach spaces failing PCP and not containing ℓ_1 such that every semi-normalized basic sequence has a boundedly complete subsequence. Concretely, the space B_∞, the natural predual of the space JT_∞, constructed in [5] is the desired example (Corollary 2.5).

It seems natural recall now the definition of JT_∞ and B_∞. For this consider the tree $T_\infty = \bigcup_{k=0}^{\infty} \mathbb{N}^k$. If $t = (n_1, n_2, \dots, n_k) \in T_\infty$, set $|t| = k$ and for $j \leqslant k$ set $t \mid j = (n_1, n_2, \dots, n_j)$. The partial order on T_∞ is defined by $s \leqslant t$ if $|s| \leqslant |t|$ and $s = t \mid |s|$. A segment on T_∞ is a totally ordered subset of T_∞. Finally the space JT_∞ is the completion of vector space of all real-valued, finitely supported functions on T_∞ with the norm

$$\|x\| = \sup\left(\sum_{i=1}^{n}\left(\sum_{t\in S_i} x(t)\right)^2\right)^{1/2}$$

where the supremum is taken over all families $(S_1, S_2, \dots, S_n)$ of disjoint segments in T_∞. If $\{e_t\}_{t\in T_\infty}$ denotes the canonical basis of JT_∞ and $\{e_t^*\}$ is the sequence of biorthogonal functionals in JT_∞^*, then the space B_∞ is the closed linear span of the sequence $\{e_t^*\}$.

We begin with some notation and preliminaries. Let X be a Banach space and let $\{e_n\}$ be a basic sequence in X. $\{e_n\}$ is said to be semi-normalized if $0 < \inf_n \|e_n\| \leqslant \sup_n \|e_n\| < +\infty$, X^* denotes the topological dual of X and the closed linear span of $\{e_n\}$ is denoted by $[e_n]$. $\{e_n\}$ is called

(i) *boundedly complete* provided whenever scalars $\{\lambda_i\}$ satisfy $\sup_n \|\sum_{i=1}^n \lambda_i e_i\| < +\infty$, then $\sum_n \lambda_n e_n$ converges;
(ii) *shrinking* if the scalar sequence $\{\|f_{|[e_n,e_{n+1},\dots]}\|\}$ converges to zero $\forall f \in X^*$;
(iii) *supershrinking* provided $\{e_n\}$ is shrinking and whenever scalars $\{\lambda_i\}$ satisfy $\sup_n \|\sum_{i=1}^n \lambda_i e_i\| < +\infty$ and $\{\lambda_i\} \to 0$, then $\sum_n \lambda_n e_n$ converges;
(iv) *strongly summing* provided is a weakly Cauchy sequence and whenever scalars $\{\lambda_i\}$ satisfy $\sup_n \|\sum_{i=1}^n \lambda_i e_i\| < +\infty$, then $\sum_n \lambda_n$ converges.

A boundedly complete basic sequence spans a dual space and a shrinking basic sequence $\{e_n\}$ spans a subspace whose dual has a basis $\{f_n\}$, called the sequence of associated functionals to $\{e_n\}$. A boundedly complete and shrinking basic sequence spans a reflexive subspace and a basic sequence in a reflexive space is both boundedly complete and shrinking [7].

The supershrinking basic sequences appear in [8] and [9], where it is proved that a Banach space X with a supershrinking basis not containing c_0 is somewhat order one quasireflexive. Then X has many boundedly complete basic sequences. The space B_∞ has a supershrinking basis (see [8] and Theorem IV.2 in [5]), does not contain c_0 and fails PCP [5], so B_∞ is a good candidate to be the desired example. Other examples with a supershrinking basis are c_0 and B, the natural predual of James tree space JT [5]. It is worth to mention that, by a separation argument, a semi-normalized basis of a Banach space X is supershrinking if and only if

$$\left\{x^{**} \in X^{**}\colon \lim_n x^{**}(f_n) = 0\right\} = X \tag{1.1}$$

where $\{f_n\}$ is the associated functional sequence.

The strongly summing basic sequences appear in [11], where it is proved the remarkable c_0-theorem, which assures that every weak Cauchy non-trivial sequence in a Banach space not containing c_0, has a strongly summing basic subsequence. A weak Cauchy sequence in a Banach space is said to be non-trivial if does not converge weakly. Finally, we recall that if $\{e_n\}$ is a strongly summing sequence, then $\{v_n\}$ is a basic sequence, where $\{v_n\}$ is the difference sequence of $\{e_n\}$, that is, $v_1 = e_1$ and $v_n = e_n - e_{n-1}$ for $n > 1$ [11].

The next lemma shows a very easy connection between supershrinking, strongly summing and boundedly complete basic sequences.

Lemma 1.1. *Let $\{e_n\}$ be a semi-normalized strongly summing basic sequence with difference sequence $\{v_n\}$. If $\{v_n\}$ is supershrinking, then $\{e_n\}$ is boundedly complete. In fact, $[e_n]$ is order one quasireflexive, that is, $[e_n]$ has codimension 1 in $[e_n]^{**}$.*

Proof. Let $\{\lambda_n\}$ be scalars so that $\sup_n \|\sum_{i=1}^n \lambda_i e_i\| < +\infty$. We have to prove that $\sum_n \lambda_n e_n$ converges in order to obtain that $\{e_n\}$ is boundedly complete. As $\{e_n\}$ is strongly summing, hence $\sum_n \lambda_n$ converges. Define $\mu_n = \sum_{i=n}^{+\infty} \lambda_i$ $\forall n$. Then $\{\mu_n\}$ converges to zero and

$$\sum_{i=1}^{n} \mu_i v_i = \sum_{i=1}^{n-1} \lambda_i e_i + \mu_n e_n \quad \forall n \in \mathbb{N}. \tag{1.2}$$

So, $\sup_n \|\sum_{i=1}^n \mu_i v_i\| < +\infty$ and then $\sum_n \mu_n v_n$ converges, by hypothesis. Finally, $\sum_n \lambda_n e_n$ converges by (1.2), since $\{\mu_n\} \to 0$.

Now, we conclude that $[e_n]$ is order one quasireflexive. For this, put $e_n^* = v_n^* - v_{n+1}^*$, where $\{v_n^*\}$ is the associated functional sequence to $\{v_n\}$. Then $\{e_n^*\}$ is the associated functional sequence to $\{e_n\}$. Observe that $[e_n]^* = [v_n^*]$, since $\{v_n\}$ is shrinking. Hence, $[e_n^*]$ has codimension 1 in $[e_n]^*$, since $x^{**}(e_n^*) = 0$ for every n and $x^{**}(v_1^*) = 1$, where $x^{**}(x^*) = \lim_n x^*(e_n)$ for every $x^* \in [e_n]^*$ exists because $\{e_n\}$ is weakly Cauchy. In fact, $[e_n]^* = [e_n^*] \oplus [v_1^*]$. But $[e_n^*]^*$ is canonically isomorphic to $[e_n]$, since $\{e_n\}$ is a boundedly complete sequence. Then $[e_n]$ has codimension 1 in $[e_n]^{**}$. $\square$

2. Main results

Corollaries 2.4 and 2.5 announced in the introduction will be deduced from the following more general result.

Theorem 2.1. *Let X be a Banach space with a semi-normalized supershrinking basis, not containing c_0. Then every non-trivial weak Cauchy sequence has a boundedly complete basic subsequence.*

Before prove this theorem, we need the following stability property of supershrinking basic block sequences.

Lemma 2.2. *Let X be a Banach space with a semi-normalized supershrinking basis $\{e_n\}$. If $v_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} \lambda_k e_k$ is a basic block of $\{e_n\}$ with $\{\lambda_n\}$ bounded, then $\{v_n\}$ is a supershrinking basic sequence.*

Proof. Let $\{f_n\}$, $\{g_n\}$ be the sequences of associated functionals to $\{e_n\}$ and $\{v_n\}$, respectively. If we do $Y = [v_n]$ we claim that $f_k|_Y = \lambda_k g_n$ whenever $\sigma(n-1)+1 \leqslant k \leqslant \sigma(n)$. Indeed, for each n, k such that $\sigma(n-1)+1 \leqslant k \leqslant \sigma(n)$ and $\lambda_k \neq 0$ one has that $\frac{f_k|_Y(v_n)}{\lambda_k} = 1$ and $\frac{f_k|_Y(v_m)}{\lambda_k} = 0$ for every $m \neq n$. Furthermore, if $\lambda_k = 0$ for such n, k then $f_k|_Y = 0$ ($f|_Y$ denotes f restricted to Y for every $f \in X^*$). Then, from the uniqueness of the sequence of associated functionals to the basic sequence $\{v_n\}$, the claim is proved.

In order to show that $\{v_n\}$ is a supershrinking basic sequence we check the equality (1.1).

Pick $y^{**} \in Y^{**}$ with $\lim_n y^{**}(g_n) = 0$ then $\lim_n y^{**}(f_n|_Y) = 0$ by the claim, since $\{\lambda_n\}$ is bounded. Now $y^{**} \in Y^{**} \subset X^{**}$ and $y^{**}(f_n) = y^{**}(f_n|_Y)$ for every $n \in \mathbb{N}$. So, $y^{**} \in X \cap Y^{**} = Y$, since $\{e_n\}$ is supershrinking, and then $\{v_n\}$ is also supershrinking. $\square$

Now, we show that Banach spaces with a supershrinking basis without copies of c_0 contain many reflexive subspaces.

Proposition 2.3. *Let X be a Banach space with a semi-normalized supershrinking basis $\{e_n\}$ without isomorphic subspaces to c_0. Then every subsequence of $\{e_n\}$ has a further subsequence whose closed linear span is a reflexive subspace.*

Proof. It is clear that it is enough to prove that $\{e_n\}$ has a subsequence whose closed linear span is a reflexive subspace.

For this, we apply the Elton Theorem [3] to obtain $\{e_{\sigma(n)}\}$ a basic subsequence of $\{e_n\}$ such that

$$\lim_k \left\| \sum_{i=1}^{k} a_i e_{\sigma(i)} \right\| = +\infty \quad \forall \{a_i\} \notin c_0.$$

We put $Y = [e_{\sigma(n)}]$. To see that Y is reflexive it suffices to prove that $\{e_{\sigma(n)}\}$ is a boundedly complete basic sequence in Y, since $\{e_{\sigma(n)}\}$ is a shrinking basic sequence.

Let $\{\lambda_n\} \subset \mathbb{R}$ such that $\sup_n \|\sum_{k=1}^n \lambda_k e_{\sigma(k)}\| < +\infty$. Then $\{\lambda_n\} \in c_0$ and $\sum_n \lambda_n e_{\sigma(n)}$ converges, since $\{e_{\sigma(n)}\}$ is supershrinking, that is, Y is reflexive. $\square$

Proof of Theorem 2.1. Let $\{f_n\}$ be the functional sequence associated to $\{e_n\}$ and assume, without loss of generality that $\{e_n\}$ is monotone, that is, $\|Q_n\| \leqslant 1$ $\forall n \in \mathbb{N}$, where $\{Q_n = \sum_{k=1}^n f_k\}$ is the sequence of the projections of the basis $\{e_n\}$. Put $M = \sup_n \|e_n\|$ and let $\{x_n\}$ be a non-trivial weak Cauchy in X. By the c_0-theorem, we can assume that there is a strongly summing basic subsequence of $\{x_n\}$, so we in fact assume that $\{x_n\}$ itself is a non-trivial weak Cauchy strongly summing basic sequence.

We claim that there exist integers $0 < \sigma(1) < \sigma(2) < \cdots$, $0 = m_0 < 1 = m_1 < m_2 < \cdots$ and $\{v_n\}$ a basic sequence such that

$$\text{(i)} \quad \left| f_h(x_{\sigma(n)}) - f_h(x_k) \right| < \frac{1}{2^{n+3} m_n M} \quad \forall k \geqslant \sigma(n),\ h \leqslant m_n,\ n \in \mathbb{N}, \tag{2.1}$$

$$\text{(ii)} \quad v_n \in [e_k \colon m_{n-1}+1 \leqslant k \leqslant m_{n+1}] \quad \forall n \in \mathbb{N},$$

$$\text{(iii)} \quad \|v_n - z_n\| < 1/2^{n+1} \quad \forall n \in \mathbb{N},$$

where $\{z_n\}$ is the difference sequence of $\{x_{\sigma(n)}\}$, that is, $z_1 = x_{\sigma(1)}$, $z_n = x_{\sigma(n)} - x_{\sigma(n-1)}$ for all $n > 1$.

As $\{x_n\}$ is weakly Cauchy, there is $\sigma(1) \in \mathbb{N}$ such that

$$\left|f_1(x_{\sigma(1)}) - f_1(x_k)\right| < 1/2^4 M \quad \forall k \geqslant \sigma(1). \tag{2.2}$$

Choose $m_2 > m_1$ such that $\| \sum_{n=m_2+1}^{+\infty} f_n(x_{\sigma(1)})e_n \| < 1/2^2$ and put $v_1 = \sum_{n=1}^{m_2} f_n(x_{\sigma(1)})e_n$. Then $\|z_1 - v_1\| = \| \sum_{n=m_2+1}^{+\infty} f_n(x_{\sigma(1)})e_n \| < 1/2^2$.

Pick now $\sigma(2) > \sigma(1)$ such that

$$\left|f_h(x_{\sigma(2)}) - f_h(x_k)\right| < \frac{1}{2^5 m_2 M} \quad \forall k \geqslant \sigma(2),\ h \leqslant m_2. \tag{2.3}$$

Chose $m_3 > m_2$ such that $\| \sum_{n=m_3+1}^{+\infty} (f_n(x_{\sigma(2)}) - f_n(x_{\sigma(1)}))e_n \| < 1/2^4$.

Put now $v_2 = \sum_{n=m_1+1}^{m_3} (f_n(x_{\sigma(2)}) - f_n(x_{\sigma(1)}))e_n$. Then $\|z_2 - v_2\| \leqslant \|(f_1(x_{\sigma(2)}) - f_1(x_{\sigma(1)}))e_1\| + \| \sum_{n=m_3+1}^{+\infty} (f_n(x_{\sigma(2)}) - f_n(x_{\sigma(1)}))e_n \| < 1/2^4 + 1/2^4 = 1/2^3$, by (2.2) and (2.3).

Assume, inductively, that $m_2 < m_3 < \cdots < m_{n+1}$, $\sigma(2) < \sigma(3) < \cdots < \sigma(n)$, $v_1, v_2, \ldots, v_n$ have been constructed such that

$$\left|f_h(x_{\sigma(n)}) - f_h(x_k)\right| < \frac{1}{2^{n+3} m_n M} \quad \forall k \geqslant \sigma(n),\ h \leqslant m_n. \tag{2.4}$$

Pick now $m_{n+2} > m_{n+1}$ such that

$$\left\| \sum_{n=m_{n+2}+1}^{+\infty} \big(f_n(x_{\sigma(n+1)}) - f_n(x_{\sigma(n)})\big)e_n \right\| < 1/2^{n+3}. \tag{2.5}$$

Put $v_{n+1} = \sum_{i=m_n+1}^{m_{n+2}} (f_i(x_{\sigma(n+1)}) - f_i(x_{\sigma(n)}))e_i$. Then $\|z_{n+1} - v_{n+1}\| \leqslant \| \sum_{i=1}^{m_n} (f_i(x_{\sigma(n+1)}) - f_i(x_{\sigma(n)}))e_i \| + \| \sum_{i=m_{n+2}+1}^{+\infty} (f_i(x_{\sigma(n+1)}) - f_i(x_{\sigma(n)}))e_i \| < 1/2^{n+3} + 1/2^{n+3} = 1/2^{n+2}$, by (2.4) and (2.5).

Now, choose $\sigma(n+1) > \sigma(n)$ such that

$$\left|f_h(x_{\sigma(n+1)}) - f_h(x_k)\right| < \frac{1}{2^{n+4} m_{n+1} M} \quad \forall k \geqslant \sigma(n+1),\ h \leqslant m_{n+1}. \tag{2.6}$$

Then the induction is complete and the claim is proved.

From the claim, it is clear that $\{v_n\}$ is a basic sequence equivalent to $\{z_n\}$, the difference sequence of $\{x_{\sigma(n)}\}$, since $\sum_{n=1}^{+\infty} \|z_n - v_n\| < 1/2$ (see Proposition 1.a.9 in [7]). Also, we obtain from (ii) of the claim that $[v_n, v_{n+1}, \ldots] \subset [e_{m_{n-1}+1}, e_{m_{n-1}+2}, \ldots]$ $\forall n \in \mathbb{N}$. Then $\{v_n\}$ is a shrinking basic sequence, since $\{e_n\}$ is shrinking.

Now, let us see that $\{v_n\}$ is a supershrinking basic sequence. For this, we chose $\{\lambda_n\}$ a scalar sequence such that $\sup_n \| \sum_{k=1}^{n} \lambda_k v_k \| < +\infty$ and we have to prove that $\sum_n \lambda_n v_n$ converges, whenever $\{\lambda_n\} \to 0$.

From the proof of the claim $v_1 = \sum_{n=1}^{m_2} f_n(x_{\sigma(1)})e_n$, and for every $n > 1$, $v_n = \sum_{k=m_{n-1}+1}^{m_{n+1}} (f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)}))e_k$.

Put $\mu_i = \lambda_1 f_i(x_{\sigma(1)})$ for $1 \leqslant i \leqslant m_1$, $\mu_i = \lambda_1 f_i(x_{\sigma(1)}) + \lambda_2(f_i(x_{\sigma(2)}) - f_i(x_{\sigma(1)}))$ for $m_1 + 1 \leqslant i \leqslant m_2$ and $\mu_i = \lambda_{k-1}(f_i(x_{\sigma(k-1)}) - f_i(x_{\sigma(k-2)})) + \lambda_k(f_i(x_{\sigma(k)}) - f_i(x_{\sigma(k-1)}))$ for $m_{k-1} + 1 \leqslant i \leqslant m_k$ and $k > 2$.

As $\{\lambda_n\} \to 0$, $\{e_n\}$ is a semi-normalized basis of X and $\{x_n\}$ is bounded, we deduce that $\{\mu_n\} \to 0$. Furthermore, we have the following equality for all $n \in \mathbb{N}$:

$$\sum_{k=1}^{n} \lambda_k v_k = \sum_{k=1}^{m_n} \mu_k e_k + \sum_{k=m_n+1}^{m_{n+1}} \lambda_n \bigl(f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)})\bigr) e_k. \tag{2.7}$$

Hence, whenever $m_n + 1 \leqslant p < m_{n+1}$, $n > 1$ we have

$$\begin{aligned} \sum_{k=1}^{p} \mu_k e_k = {} & \sum_{k=1}^{n} \lambda_k v_k + \sum_{k=m_n+1}^{p} \lambda_{n+1} \bigl(f_k(x_{\sigma(n+1)}) - f_k(x_{\sigma(n)})\bigr) e_k \\ & - \sum_{k=p+1}^{m_{n+1}} \lambda_n \bigl(f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)})\bigr) e_k. \end{aligned} \tag{2.8}$$

Now, as $\{x_n\}$ and $\{Q_n\}$ are bounded and $\{\lambda_n\} \to 0$, we obtain that

$$\begin{aligned} & \lim_n \sum_{k=p+1}^{m_{n+1}} \lambda_n \bigl(f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)})\bigr) e_k \\ & \quad = \lim_n \sum_{k=m_n+1}^{p} \lambda_{n+1} \bigl(f_k(x_{\sigma(n+1)}) - f_k(x_{\sigma(n)})\bigr) e_k = 0, \end{aligned} \tag{2.9}$$

since for every $m_n + 1 \leqslant p < m_{n+1}$, $n \in \mathbb{N}$, $n > 1$ we have:

$$\begin{aligned} \sum_{k=p+1}^{m_{n+1}} \lambda_n \bigl(f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)})\bigr) e_k &= \lambda_n (Q_{m_{n+1}} - Q_p)(x_{\sigma(n)} - x_{\sigma(n-1)}), \\ \sum_{k=m_n+1}^{p} \lambda_{n+1} \bigl(f_k(x_{\sigma(n+1)}) - f_k(x_{\sigma(n)})\bigr) e_k &= \lambda_{n+1} (Q_p - Q_{m_n})(x_{\sigma(n+1)} - x_{\sigma(n)}). \end{aligned} \tag{2.10}$$

From (2.8) and (2.10), it can be deduced that $\sup_p \| \sum_{n=1}^{p} \mu_n e_n \| < +\infty$ and so, $\sum_n \mu_n e_n$ converges, since $\{\mu_n\} \to 0$ and $\{e_n\}$ is supershrinking. Then $\sum_n \lambda_n v_n$ converges by (2.8) and (2.9) and we have proved that $\{v_n\}$ is a supershrinking basic sequence equivalent to the difference sequence of $\{x_{\sigma(n)}\}$. Finally, $\{x_{\sigma(n)}\}$ is boundedly complete by Lemma 1.1, since it is strongly summing. In fact, $[x_{\sigma(n)}]$ is order one quasireflexive, by Lemma 1.1. □

Corollary 2.4. *Let X be a Banach space with a semi-normalized supershrinking basis not containing c_0. Then every semi-normalized basic sequence in X has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace of X.*

Proof. Let $\{x_n\}$ be a semi-normalized basic sequence in X. As X does not contain isomorphic subspaces to ℓ_1, we can assume that $\{x_n\}$ itself is weakly Cauchy, by the ℓ_1-theorem [10]. If $\{x_n\}$ is not weakly convergent, then $\{x_n\}$ is a semi-normalized non-trivial weak Cauchy sequence and $\{x_n\}$ has a boundedly complete subsequence spanning an order one quasireflexive subspace, by Theorem 2.1, and we are done.

If $\{x_n\}$ is weakly convergent, then $\{x_n\}$ converges weakly to zero, because $\{x_n\}$ is a basic sequence. Now, it is straightforward construct a subsequence of $\{x_n\}$ equivalent to a basic block of the basis. So, we can assume that $\{x_n\}$ is a semi-normalized basic sequence equivalent to a basic block of the basis. Following the proof of Proposition 1.a.11 in [7], it is easy to construct this basic block satisfying the hypothesis of Lemma 2.2. Then $\{x_n\}$ is a supershrinking basic subsequence and, by Proposition 2.3, $\{x_n\}$ has a boundedly complete subsequence spanning a reflexive subspace, so we are done. □

As we announced in the introduction, it is enough to apply Corollary 2.4 to obtain the following

Corollary 2.5. *B_∞ fails PCP, does not contain isomorphic subspaces to ℓ_1 and every semi-normalized basic sequence in B_∞ has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace.*

Proof. The fact that B_∞ has a semi-normalized supershrinking basis is a consequence of Theorem IV.2 in [5]. So B_∞ has separable dual and does not contain subspaces isomorphic to ℓ_1. Now, B_∞ fails PCP and does not contain subspaces isomorphic to c_0 [5]. Finally, by Corollary 2.4, every semi-normalized basic sequence in B_∞ has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace of B_∞. □

Let B be the natural predual of James tree space JT. It is known that B satisfies PCP, and also B has a semi-normalized supershrinking basis. (See [6] and [9].) As B does not contain isomorphic subspaces to c_0, [6], we can apply Corollary 2.4, as in Corollary 2.5, to obtain the following

Corollary 2.6. *Every semi-normalized basic sequence in B has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace of B.*

Remark 2.7. (i) It has been proved in [4] that a Banach space X with separable dual satisfies PCP if, and only if, every weakly null tree in the unit sphere of X has a boundedly complete branch. Also, it is shown in [4] that this characterization of PCP is not true for sequences, by proving that every weakly null sequence in the unit sphere of B_∞ has a boundedly complete subsequence, while B_∞ fails PCP. Hence Corollary 2.5 improves this result, since every weakly null sequence in the unit sphere of a Banach space has a semi-normalized basic subsequence.

(ii) From Corollary 2.4 one might think that the good sequential property in order to imply PCP for Banach spaces with separable dual is that every semi-normalized basic sequence has a subsequence spanning a reflexive subspace. And this is true, but this property implies reflexivity. Indeed, assume that X is a Banach space satisfying that every semi-normalized basic sequence has a subsequence spanning a reflexive subspace. Take a bounded sequence $\{x_n\}$ in X and prove that $\{x_n\}$ has a weakly convergent subsequence. As X does not contain subspaces isomorphic to ℓ_1, then $\{x_n\}$ has a weak Cauchy subsequence $\{y_n\}$, by the ℓ_1-theorem. If $\{y_n\}$ is not semi-normalized, then $\{y_n\}$ and so $\{x_n\}$ has a subsequence weakly convergent to zero and we are done. Hence, assume that $\{y_n\}$ is a semi-normalized weak Cauchy sequence in X. If $\{y_n\}$ is not weakly convergent, then, by the c_0-theorem, for example, $\{y_n\}$ has a semi-normalized basic subsequence, since X does not contain isomorphic subspaces to c_0. By hypothesis, $\{y_n\}$ has a subsequence spanning a reflexive subspace and hence, this subsequence is weakly convergent to zero, so $\{x_n\}$ has a weakly convergent subsequence and we are done.

(iii) It is known that B_∞ satisfies the convex point of continuity property CPCP [5], a weaker property than PCP. So it is natural to ask weather a Banach space (maybe not containing ℓ_1) satisfies CPCP, whenever every semi-normalized basic sequence has a boundedly complete subsequence.

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