Banach spaces with many boundedly complete basic sequences failing PCP

Ginés López Pérez

Universidad de Granada, Facultad de Ciencias, Departamento de Análisis Matemático, 18071-Granada, Spain

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To my mother Francisca and my sister Isabel, in memoriam

Abstract
We prove that there exist Banach spaces not containing $\ell_1$, failing the point of continuity property and satisfying that every semi-normalized basic sequence has a boundedly complete basic subsequence. This answers in the negative the problem of Remark 2 in Rosenthal (2007) [12].

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1. Introduction
Recall that a Banach space is said to have the point of continuity property (PCP) provided every non-empty closed and bounded subset admits a point of continuity of the identity map from the weak to norm topologies. It is known that Banach spaces with Radon–Nikodym property, including separable dual spaces, satisfy PCP, but the converse is false (see [2]). The PCP has been characterized for separable Banach spaces in [2] and [5], and this characterization implies that Banach spaces with PCP have many boundedly complete basic sequences, and so many subspaces which are separable dual spaces. As PCP is separably determined [1], that is, a Banach space satisfies PCP if every separable subspace has PCP, it is natural looking for a sequential characterization of PCP. In this sense, it has been proved in [12] that every semi-normalized basic
sequence in a Banach space with PCP has a boundedly complete subsequence. The converse
of the above result is false in general, but it is open for Banach spaces not containing $\ell_1$ (see
Remark 2 in [12]). The goal of this note is to prove in Corollary 2.4 that there exists a family of
Banach spaces failing PCP and not containing $\ell_1$ such that every semi-normalized basic sequence
has a boundedly complete subsequence. Concretely, the space $B_\infty$, the natural predual of the
space $JT_\infty$, constructed in [5] is the desired example (Corollary 2.5).

It seems natural recall now the definition of $JT_\infty$ and $B_\infty$. For this consider the tree $T_\infty = \bigcup_{k=0}^\infty \mathbb{N}^k$. If $t = (n_1, n_2, \ldots, n_k) \in T_\infty$, set $|t| = k$ and for $j \leq k$ set $t | j = (n_1, n_2, \ldots, n_j)$. The
partial order on $T_\infty$ is defined by $s \leq t$ if $|s| \leq |t|$ and $s = t | |s|$. A segment on $T_\infty$ is a totally
ordered subset of $T_\infty$. Finally the space $JT_\infty$ is the completion of vector space of all real-valued,
finitely supported functions on $T_\infty$ with the norm

$$
\|x\| = \sup \left( \sum_{i=1}^{n} \left( \sum_{t \in S_i} x(t) \right)^2 \right)^{1/2}
$$

where the supremum is taken over all families $(S_1, S_2, \ldots, S_n)$ of disjoint segments in $T_\infty$. If
$\{e_t\}_{t \in T_\infty}$ denotes the canonical basis of $JT_\infty$ and $\{e_t^*\}$ is the sequence of biorthogonal functionals
in $JT_\infty^*$, then the space $B_\infty$ is the closed linear span of the sequence $\{e_n\}$.

We begin with some notation and preliminaries. Let $X$ be a Banach space and let $\{e_n\}$ be a basic sequence in $X$. $\{e_n\}$ is said to be semi-normalized if $0 < \inf_n \|e_n\| \leq \sup_n \|e_n\| < +\infty$, $X^*$ denotes the topological dual of $X$ and the closed linear span of $\{e_n\}$ is denoted by $[e_n]$. $\{e_n\}$ is called

(i) **boundedly complete** provided whenever scalars $\{\lambda_i\}$ satisfy $\sup_n \| \sum_{i=1}^{n} \lambda_i e_i \| < +\infty$, then
$\sum_n \lambda_n e_n$ converges;

(ii) **shrinking** if the scalar sequence $\{\|f\|_{e_n, e_{n+1}, \ldots}\|$ converges to zero $\forall f \in X^*$;

(iii) **supershrinking** provided $\{e_n\}$ is shrinking and whenever scalars $\{\lambda_i\}$ satisfy
$\sup_n \| \sum_{i=1}^{n} \lambda_i e_i \| < +\infty$ and $\{\lambda_i\} \to 0$, then $\sum_n \lambda_n e_n$ converges;

(iv) **strongly summing** provided is a weakly Cauchy sequence and whenever scalars $\{\lambda_i\}$ satisfy
$\sup_n \| \sum_{i=1}^{n} \lambda_i e_i \| < +\infty$, then $\sum_n \lambda_n$ converges.

A boundedly complete basic sequence spans a dual space and a shrinking basic sequence $\{e_n\}$
spans a subspace whose dual has a basis $\{f_n\}$, called the sequence of associated functionals to
$\{e_n\}$. A boundedly complete and shrinking basic sequence spans a reflexive subspace and a basic
sequence in a reflexive space is both boundedly complete and shrinking [7].

The supershrinking basic sequences appear in [8] and [9], where it is proved that a Banach
space $X$ with a supershrinking basis not containing $c_0$ is somewhat order one quasireflexive.
Then $X$ has many boundedly complete basic sequences. The space $B_\infty$ has a supershrinking
basis (see [8] and Theorem IV.2 in [5]), does not contain $c_0$ and fails PCP [5], so $B_\infty$ is a good
candidate to be the desired example. Other examples with a supershrinking basis are $c_0$ and $B$, the
natural predual of James tree space $JT$ [5]. It is worth to mention that, by a separation argument,
a semi-normalized basis of a Banach space $X$ is supershrinking if and only if

$$
\left\{ x^{**} \in X^{**}; \lim_n x^{**}(f_n) = 0 \right\} = X
$$

where $\{f_n\}$ is the associated functional sequence.
The strongly summing basic sequences appear in [11], where it is proved the remarkable $c_0$-theorem, which assures that every weak Cauchy non-trivial sequence in a Banach space not containing $c_0$, has a strongly summing basic subsequence. A weak Cauchy sequence in a Banach space is said to be non-trivial if does not converge weakly. Finally, we recall that if $\{e_n\}$ is a strongly summing sequence, then $\{v_n\}$ is a basic sequence, where $\{v_n\}$ is the difference sequence of $\{e_n\}$, that is, $v_1 = e_1$ and $v_n = e_n - e_{n-1}$ for $n > 1$ [11].

The next lemma shows a very easy connection between supershrinking, strongly summing and boundedly complete basic sequences.

**Lemma 1.1.** Let $\{e_n\}$ be a semi-normalized strongly summing basic sequence with difference sequence $\{v_n\}$. If $\{v_n\}$ is supershrinking, then $\{e_n\}$ is boundedly complete. In fact, $\{e_n\}$ is order one quasireflexive, that is, $\{e_n\}$ has codimension 1 in $[e_n]^\ast\ast$.

**Proof.** Let $\{\lambda_n\}$ be scalars so that $\sup \lambda_n \sum_{i=1}^n \lambda_i e_i < \infty$. We have to prove that $\sum \lambda_n e_n$ converges in order to obtain that $\{e_n\}$ is boundedly complete. Define $\mu_n = \sum_{i=n}^{\infty} \lambda_i$ $\forall n$. Then $\{\mu_n\}$ converges to zero and

$$\sum_{i=1}^n \mu_i v_i = \sum_{i=1}^{n-1} \lambda_i e_i + \mu_n e_n \quad \forall n \in \mathbb{N}.$$ (1.2)

So, $\sup \mu_n \sum_{i=1}^n \mu_i v_i < +\infty$ and then $\sum \mu_n v_n$ converges, by hypothesis. Finally, $\sum \lambda_n e_n$ converges by (1.2), since $\{\mu_n\} \to 0$.

Now, we conclude that $\{e_n\}$ is order one quasireflexive. For this, put $e^*_n = v^*_n - v^*_n$, where $\{v^*_n\}$ is the associated functional sequence to $\{v_n\}$. Then $\{e^*_n\}$ is the associated functional sequence to $\{e_n\}$. Observe that $[e^*_n]^\ast = [v^*_n]^\ast$, hence $\{e^*_n\}$ has codimension 1 in $[e_n]^\ast$, since $x^\ast(y^\ast) = 0$ for every $n$ and $x^\ast(y^\ast) = 1$, where $x^\ast y = \lim x^\ast e_n$ for every $x \in [e_n]^\ast$ exists because $\{e_n\}$ is weakly Cauchy. In fact, $[e^*_n]^\ast = [e^*_n]^\ast$ $\oplus [v^*_n]^\ast$. But $[e^*_n]^\ast$ is canonically isomorphic to $[e_n]^\ast$, since $\{e_n\}$ is a boundedly complete sequence. Then $\{e_n\}$ has codimension 1 in $[e_n]^\ast\ast$. $\square$

**2. Main results**

Corollaries 2.4 and 2.5 announced in the introduction will be deduced from the following more general result.

**Theorem 2.1.** Let $X$ be a Banach space with a semi-normalized supershrinking basis, not containing $c_0$. Then every non-trivial weak Cauchy sequence has a boundedly complete basic subsequence.

Before prove this theorem, we need the following stability property of supershrinking basic block sequences.

**Lemma 2.2.** Let $X$ be a Banach space with a semi-normalized supershrinking basis $\{e_n\}$. If $v_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} \lambda_k e_k$ is a basic block of $\{e_n\}$ with $\{\lambda_k\}$ bounded, then $\{v_n\}$ is a supershrinking basic sequence.
Proof. Let \( \{f_n\}, \{g_n\} \) be the sequences of associated functionals to \( \{e_n\} \) and \( \{v_n\} \), respectively. If we do \( Y = [v_n] \) we claim that \( f_k|_Y = \lambda_k g_n \) whenever \( \sigma(n-1) + 1 \leq k \leq \sigma(n) \). Indeed, for each \( n, k \) such that \( \sigma(n-1) + 1 \leq k \leq \sigma(n) \) and \( \lambda_k \neq 0 \) one has that \( \frac{f_k}{\lambda_k} |_{(v_n)} = 1 \) and \( \frac{f_k}{\lambda_k} |_{(v_n)} = 0 \) for every \( m \neq n \). Furthermore, if \( \lambda_k = 0 \) for such \( n, k \) then \( f_k|_Y = 0 \) (\( f|_Y \) denotes \( f \) restricted to \( Y \) for every \( f \in X^* \)). Then, from the uniqueness of the sequence of associated functionals to the basic sequence \( \{v_n\} \), the claim is proved.

In order to show that \( \{v_n\} \) is a supershrinking basic sequence we check the equality (1.1).

Pick \( y^{**} \in Y^{**} \) with \( \lim_n y^{**}(g_n) = 0 \) then \( \lim_n y^{**}(f_n|_Y) = 0 \) by the claim, since \( \{\lambda_n\} \) is bounded. Now \( y^{**} \in Y^{**} \subset X^{**} \) and \( y^{**}(f_n) = y^{**}(f_n|_Y) \) for every \( n \in \mathbb{N} \). So, \( y^{**} \in X \cap Y^{**} = Y \), since \( \{e_n\} \) is supershrinking, and then \( \{v_n\} \) is also supershrinking. \( \square \)

Now, we show that Banach spaces with a supershrinking basis without copies of \( c_0 \) contain many reflexive subspaces.

**Proposition 2.3.** Let \( X \) be a Banach space with a semi-normalized supershrinking basis \( \{e_n\} \) without isomorphic subspaces to \( c_0 \). Then every subsequence of \( \{e_n\} \) has a further subsequence whose closed linear span is a reflexive subspace.

**Proof.** It is clear that it is enough to prove that \( \{e_n\} \) has a subsequence whose closed linear span is a reflexive subspace.

For this, we apply the Elton Theorem [3] to obtain \( \{e_{\sigma(n)}\} \) a basic subsequence of \( \{e_n\} \) such that

\[
\lim_k \left\| \sum_{i=1}^{k} a_i e_{\sigma(i)} \right\| = +\infty \quad \forall \{a_i\} \notin c_0.
\]

We put \( Y = [e_{\sigma(n)}] \). To see that \( Y \) is reflexive it suffices to prove that \( \{e_{\sigma(n)}\} \) is a boundedly complete basic sequence in \( Y \), since \( \{e_{\sigma(n)}\} \) is a shrinking basic sequence.

Let \( \{\lambda_n\} \subset \mathbb{R} \) such that \( \sup_n \left\| \sum_{k=1}^{n} \lambda_k e_{\sigma(k)} \right\| < +\infty \). Then \( \{\lambda_n\} \in c_0 \) and \( \sum_n \lambda_n e_{\sigma(n)} \) converges, since \( \{e_{\sigma(n)}\} \) is supershrinking, that is, \( Y \) is reflexive. \( \square \)

**Proof of Theorem 2.1.** Let \( \{f_n\} \) be the functional sequence associated to \( \{e_n\} \) and assume, without loss of generality that \( \{e_n\} \) is monotone, that is, \( \|Q_n\| \leq 1 \forall n \in \mathbb{N} \), where \( Q_n = \sum_{k=1}^{n} f_k \) is the sequence of the projections of the basis \( \{e_n\} \). Put \( M = \sup_n \|e_n\| \) and let \( \{x_n\} \) be a non-trivial weak Cauchy in \( X \). By the \( c_0 \)-theorem, we can assume that there is a strongly summing basic subsequence of \( \{x_n\} \), so we in fact assume that \( \{x_n\} \) itself is a non-trivial weak Cauchy strongly summing basic sequence.

We claim that there exist integers \( 0 < \sigma(1) < \sigma(2) < \cdots, 0 = m_0 < 1 = m_1 < m_2 < \cdots \) and \( \{v_n\} \) a basic sequence such that

\[
\begin{align*}
\text{(i)} & \quad \left| f_h(x_{\sigma(n)}) - f_h(x_k) \right| < \frac{1}{2^{n+3}m_n M} \quad \forall k \geq \sigma(n), \ h \leq m_n, \ n \in \mathbb{N}, \\
\text{(ii)} & \quad v_n \in [e_k: m_{n-1} + 1 \leq k \leq m_{n+1}] \quad \forall n \in \mathbb{N}, \\
\text{(iii)} & \quad \|v_n - z_n\| < 1/2^{n+1} \quad \forall n \in \mathbb{N},
\end{align*}
\]
where \( \{z_n\} \) is the difference sequence of \( \{x_{\sigma(n)}\} \), that is, \( z_1 = x_{\sigma(1)}, z_n = x_{\sigma(n)} - x_{\sigma(n-1)} \) for all \( n > 1 \).

As \( \{x_n\} \) is weakly Cauchy, there is \( \sigma(1) \in \mathbb{N} \) such that
\[
\left| f_1(x_{\sigma(1)}) - f_1(x_k) \right| < 1/2^4 M \quad \forall k \geq \sigma(1).
\] (2.2)

Choose \( m_2 > m_1 \) such that \( \| \sum_{n=m_2+1}^{+\infty} f_n(x_{\sigma(1)})e_n \| < 1/2^2 \) and put \( v_1 = \sum_{n=1}^{m_2} f_n(x_{\sigma(1)})e_n \).

Then \( \| z_1 - v_1 \| = \| \sum_{n=m_2+1}^{+\infty} f_n(x_{\sigma(1)})e_n \| < 1/2^2 \).

Pick now \( \sigma(2) > \sigma(1) \) such that
\[
\left| f_h(x_{\sigma(2)}) - f_h(x_k) \right| < \frac{1}{2^5 m_2 M} \quad \forall k \geq \sigma(2), \ h \leq m_2.
\] (2.3)

Choose \( m_3 > m_2 \) such that \( \| \sum_{n=m_3+1}^{+\infty} (f_n(x_{\sigma(2)}) - f_n(x_{\sigma(1)}))e_n \| < 1/2^4 \).

Put now \( v_2 = \sum_{n=m_3+1}^{m_2} (f_n(x_{\sigma(2)}) - f_n(x_{\sigma(1)}))e_n \). Then \( \| z_2 - v_2 \| \leq \| (f_1(x_{\sigma(2)}) - f_1(x_{\sigma(1)}))e_1 \| + \| \sum_{n=m_3+1}^{+\infty} (f_n(x_{\sigma(2)}) - f_n(x_{\sigma(1)}))e_n \| < 1/2^4 + 1/2^4 = 1/2^3 \), by (2.2) and (2.3).

Assume, inductively, that \( m_2 < m_3 < \cdots < m_{n+1} \), \( \sigma(2) < \sigma(3) < \cdots < \sigma(n) \), \( v_1, v_2, \ldots, v_n \) have been constructed such that
\[
\left| f_h(x_{\sigma(n)}) - f_h(x_k) \right| < \frac{1}{2^{n+3} m_n M} \quad \forall k \geq \sigma(n), \ h \leq m_n.
\] (2.4)

Pick now \( m_{n+2} > m_{n+1} \) such that
\[
\left\| \sum_{n=m_{n+2}+1}^{+\infty} (f_n(x_{\sigma(n+1)}) - f_n(x_{\sigma(n)}))e_n \right\| < 1/2^{n+3}.
\] (2.5)

Put \( v_{n+1} = \sum_{i=m_{n+1}}^{m_{n+2}} (f_i(x_{\sigma(n+1)}) - f_i(x_{\sigma(n)}))e_i \). Then \( \| z_{n+1} - v_{n+1} \| \leq \| \sum_{i=1}^{m_n} (f_i(x_{\sigma(n+1)}) - f_i(x_{\sigma(n)}))e_i \| + \| \sum_{i=m_{n+1}+1}^{+\infty} (f_i(x_{\sigma(n+1)}) - f_i(x_{\sigma(n)}))e_i \| < 1/2^{n+3} + 1/2^{n+3} = 1/2^{n+2} \), by (2.4) and (2.5).

Now, choose \( \sigma(n+1) > \sigma(n) \) such that
\[
\left| f_h(x_{\sigma(n+1)}) - f_h(x_k) \right| < \frac{1}{2^{n+4} m_{n+1} M} \quad \forall k \geq \sigma(n+1), \ h \leq m_{n+1}.
\] (2.6)

Then the induction is complete and the claim is proved.

From the claim, it is clear that \( \{v_n\} \) is a basic sequence equivalent to \( \{z_n\} \), the difference sequence of \( \{x_{\sigma(n)}\} \), since \( \sum_{n=1}^{+\infty} \| z_n - v_n \| < 1/2 \) (see Proposition 1.a.9 in [7]). Also, we obtain from (ii) of the claim that \( [v_n, v_{n+1}, \ldots] \subset [e_{m_{n-1}+1}, e_{m_{n-1}+2}, \ldots] \forall n \in \mathbb{N} \). Then \( \{v_n\} \) is a shrinking basic sequence, since \( \{e_n\} \) is shrinking.

Now, let us see that \( \{v_n\} \) is a supershrinking basic sequence. For this, we choose \( \{\lambda_n\} \) a scalar sequence such that \( \sum_{n=1}^{+\infty} \lambda_k v_k < +\infty \) and we have to prove that \( \sum_{n=1}^{+\infty} \lambda_n v_n \) converges, whenever \( \{\lambda_n\} \to 0 \).

From the proof of the claim \( v_1 = \sum_{n=1}^{m_2} f_n(x_{\sigma(1)})e_n \), and for every \( n > 1 \), \( v_n = \sum_{k=m_{n-1}+1}^{m_{n+1}} (f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)}))e_k \).
Put \( \mu_i = \lambda_1 f_i(x_{\sigma(1)}) \) for \( 1 \leq i \leq m_1 \), \( \mu_i = \lambda_1 f_i(x_{\sigma(1)}) + \lambda_2 (f_i(x_{\sigma(2)}) - f_i(x_{\sigma(1)})) \) for \( m_1 + 1 \leq i \leq m_2 \) and \( \mu_i = \lambda_{k-1} (f_i(x_{\sigma(k-1)}) - f_i(x_{\sigma(k-2)})) + \lambda_k (f_i(x_{\sigma(k)}) - f_i(x_{\sigma(k-1)})) \) for \( m_{k-1} + 1 \leq i \leq m_k \) and \( k > 2 \).

As \( \{\lambda_n\} \to 0 \), \( \{e_n\} \) is a semi-normalized basis of \( X \) and \( \{x_n\} \) is bounded, we deduce that \( \{\mu_n\} \to 0 \). Furthermore, we have the following equality for all \( n \in \mathbb{N} \):

\[
\sum_{k=1}^{n} \lambda_k e_k = \sum_{k=1}^{m_n} \mu_k e_k + \sum_{k=m_n+1}^{m_{n+1}} \lambda_n (f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)})) e_k. \tag{2.7}
\]

Hence, whenever \( m_n + 1 \leq p < m_{n+1}, n > 1 \) we have

\[
\sum_{k=1}^{p} \mu_k e_k = \sum_{k=1}^{n} \lambda_k e_k + \sum_{k=m_n+1}^{m_{n+1}} \lambda_n (f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)})) e_k - \sum_{k=p+1}^{m_{n+1}} \lambda_n (f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)})) e_k. \tag{2.8}
\]

Now, as \( \{x_n\} \) and \( \{Q_n\} \) are bounded and \( \{\lambda_n\} \to 0 \), we obtain that

\[
\lim_n \sum_{k=p+1}^{m_{n+1}} \lambda_n (f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)})) e_k = \lim_n \sum_{k=m_n+1}^{p} \lambda_{n+1} (f_k(x_{\sigma(n+1)}) - f_k(x_{\sigma(n)})) e_k = 0, \tag{2.9}
\]

since for every \( m_n + 1 \leq p < m_{n+1}, n \in \mathbb{N}, n > 1 \) we have:

\[
\sum_{k=p+1}^{m_{n+1}} \lambda_n (f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)})) e_k = \lambda_n (Q_{m_n+1} - Q_p) (x_{\sigma(n)} - x_{\sigma(n-1)}),
\]

\[
\sum_{k=m_n+1}^{p} \lambda_{n+1} (f_k(x_{\sigma(n+1)}) - f_k(x_{\sigma(n)})) e_k = \lambda_{n+1} (Q_p - Q_{m_n}) (x_{\sigma(n+1)} - x_{\sigma(n)}). \tag{2.10}
\]

From (2.8) and (2.10), it can be deduced that \( \sup_p \| \sum_{n=1}^{p} \mu_n e_n \| < +\infty \) and so, \( \sum_n \mu_n e_n \) converges, since \( \{\mu_n\} \to 0 \) and \( \{e_n\} \) is supershrinking. Then \( \sum_n \lambda_n v_n \) converges by (2.8) and (2.9) and we have proved that \( \{v_n\} \) is a supershrinking basic sequence equivalent to the difference sequence of \( \{x_{\sigma(n)}\} \). Finally, \( \{x_{\sigma(n)}\} \) is boundedly complete by Lemma 1.1, since it is strongly summing. In fact, \( \{x_{\sigma(n)}\} \) is order one quasireflexive, by Lemma 1.1. \( \square \)

**Corollary 2.4.** Let \( X \) be a Banach space with a semi-normalized supershrinking basis not containing \( c_0 \). Then every semi-normalized basic sequence in \( X \) has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace of \( X \).
Proof. Let \( \{x_n\} \) be a semi-normalized basic sequence in \( X \). As \( X \) does not contain isomorphic subspaces to \( \ell_1 \), we can assume that \( \{x_n\} \) itself is weakly Cauchy, by the \( \ell_1 \)-theorem [10]. If \( \{x_n\} \) is not weakly convergent, then \( \{x_n\} \) is a semi-normalized non-trivial weak Cauchy sequence and \( \{x_n\} \) has a boundedly complete subsequence spanning an order one quasireflexive subspace, by Theorem 2.1, and we are done.

If \( \{x_n\} \) is weakly convergent, then \( \{x_n\} \) converges weakly to zero, because \( \{x_n\} \) is a basic sequence. Now, it is straightforward construct a subsequence of \( \{x_n\} \) equivalent to a basic block of the basis. So, we can assume that \( \{x_n\} \) is a semi-normalized basic sequence equivalent to a basic block of the basis. Following the proof of Proposition 1.a.11 in [7], it is easy to construct this basic block satisfying the hypothesis of Lemma 2.2. Then \( \{x_n\} \) is a supershrinking basic subsequence and, by Proposition 2.3, \( \{x_n\} \) has a boundedly complete subsequence spanning a reflexive subspace, so we are done. \( \square \)

As we announced in the introduction, it is enough to apply Corollary 2.4 to obtain the following

**Corollary 2.5.** \( B_\infty \) fails PCP, does not contain isomorphic subspaces to \( \ell_1 \) and every semi-normalized basic sequence in \( B_\infty \) has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace.

**Proof.** The fact that \( B_\infty \) has a semi-normalized supershrinking basis is a consequence of Theorem IV.2 in [5]. So \( B_\infty \) has separable dual and does not contain subspaces isomorphic to \( \ell_1 \). Now, \( B_\infty \) fails PCP and does not contain subspaces isomorphic to \( c_0 \) [5]. Finally, by Corollary 2.4, every semi-normalized basic sequence in \( B_\infty \) has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace of \( B_\infty \). \( \square \)

Let \( B \) be the natural predual of James tree space \( JT \). It is known that \( B \) satisfies PCP, and also \( B \) has a semi-normalized supershrinking basis. (See [6] and [9].) As \( B \) does not contain isomorphic subspaces to \( c_0 \), [6], we can apply Corollary 2.4, as in Corollary 2.5, to obtain the following

**Corollary 2.6.** Every semi-normalized basic sequence in \( B \) has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace of \( B \).

**Remark 2.7.** (i) It has been proved in [4] that a Banach space \( X \) with separable dual satisfies PCP if, and only if, every weakly null tree in the unit sphere of \( X \) has a boundedly complete branch. Also, it is shown in [4] that this characterization of PCP is not true for sequences, by proving that every weakly null sequence in the unit sphere of \( B_\infty \) has a boundedly complete subsequence, while \( B_\infty \) fails PCP. Hence Corollary 2.5 improves this result, since every weakly null sequence in the unit sphere of a Banach space has a semi-normalized basic subsequence.

(ii) From Corollary 2.4 one might think that the good sequential property in order to imply PCP for Banach spaces with separable dual is that every semi-normalized basic sequence has a subsequence spanning a reflexive subspace. And this is true, but this property implies reflexivity. Indeed, assume that \( X \) is a Banach space satisfying that every semi-normalized basic sequence has a subsequence spanning a reflexive subspace. Take a bounded sequence \( \{x_n\} \) in \( X \) and prove that \( \{x_n\} \) has a weakly convergent subsequence. As \( X \) does not contain subspaces isomorphic
to \( \ell_1 \), then \( \{x_n\} \) has a weak Cauchy subsequence \( \{y_n\} \), by the \( \ell_1 \)-theorem. If \( \{y_n\} \) is not semi-normalized, then \( \{y_n\} \) and so \( \{x_n\} \) has a subsequence weakly convergent to zero and we are done. Hence, assume that \( \{y_n\} \) is a semi-normalized weak Cauchy sequence in \( X \). If \( \{y_n\} \) is not weakly convergent, then, by the \( c_0 \)-theorem, for example, \( \{y_n\} \) has a semi-normalized basic subsequence, since \( X \) does not contain isomorphic subspaces to \( c_0 \). By hypothesis, \( \{y_n\} \) has a subsequence spanning a reflexive subspace and hence, this subsequence is weakly convergent to zero, so \( \{x_n\} \) has a weakly convergent subsequence and we are done.

(iii) It is known that \( B_{\infty} \) satisfies the convex point of continuity property CPCP [5], a weaker property than PCP. So it is natural to ask whether a Banach space (maybe not containing \( \ell_1 \)) satisfies CPCP, whenever every semi-normalized basic sequence has a boundedly complete subsequence.

References