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Journal of Combinatorial Theory, Series A 103 (2003) 281–289

Journal of Combinatorial Theory Series A

http://www.elsevier.com/locate/jcta

Affine semipartial geometries and projections of quadrics

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Received 14 October 2002

Abstract

Debroey and Thas introduced semipartial geometries and determined the full embeddings of semipartial geometries in $\mathrm{AG}(n,q)$ for $n=2$ and 3. For $n>3$ there is no such classification. A model of a semipartial geometry fully embedded in $\mathrm{AG}(4,q)$, q even, due to Hirschfeld and Thas, is the $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ constructed by projecting the quadric $Q^-(5,q)$ from a point of $\mathrm{PG}(5,q)\backslash Q^-(5,q)$. In this paper this semipartial geometry is characterized amongst the $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ (of which there is an infinite family of non-classical examples due to Brown) by its full embedding in $\mathrm{AG}(4,q)$.

Keywords: Semipartial geometry; Affine embedding; Quadric; Generalized quadrangle

1. Introduction

A *semipartial geometry* [9] with parameters s,t,α,μ, also denoted by $\mathrm{spg}(s,t,\alpha,\mu)$, is a partial linear space $\mathcal{S}=(\mathcal{P},\mathcal{B},\mathrm{I})$ of order (s,t), such that for each anti-flag (x,L), the incidence number $\alpha(x,L)$, being the number of points on L collinear with x, equals 0 or a constant α $(\alpha>0)$ and such that for any two points which are not collinear, there are μ $(\mu>0)$ points collinear with both (μ-condition).

A semipartial geometry with $\alpha=1$ is called a *partial quadrangle*. It was introduced by Cameron [5] as a generalization of a generalized quadrangle. Semipartial geometries generalize at the same time the partial quadrangles and the partial

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0097-3165/03/$ - see front matter
doi:10.1016/S0097-3165(03)00093-1

geometries, which are partial linear spaces of order (s,t), such that for each anti-flag (x,L), the incidence number $\alpha(x,L)=\alpha$ (the μ-condition is automatically satisfied). Partial geometries with $\alpha=1$ are the well-known generalized quadrangles. See for instance [13] for more information on generalized quadrangles and [6,7] for more information on partial and semipartial geometries. A semipartial which is not a partial geometry, nor a partial quadrangle will be called *proper*.

The point graph Γ of a semipartial geometry is strongly regular. For a point x of $\mathcal{S}$ we will denote by $\Gamma(x)$ the set of points of $\mathcal{S}$ different from x and collinear to x.

2. Semipartial geometries and generalized quadrangles

In [2] Brown gives the following general construction method for $\mathrm{spg}(q-1,q^2,2,2q(q-1))$. Let $\mathcal{S}$ be a generalized quadrangle of order (q,q^2) containing a subquadrangle $\mathcal{S}'$ of order q. If x is a point of $\mathcal{S}\backslash\mathcal{S}'$, then each line of $\mathcal{S}$ incident with x is incident with a unique point of $\mathcal{S}'$ and the set $\mathcal{O}_x$ of such points is an ovoid of $\mathcal{S}'$. (An ovoid of a generalized quadrangle is a set of points such that each line of the generalized quadrangle is incident with a unique point of the set.) The ovoid $\mathcal{O}_x$ is said to be *subtended* by x. A *rosette* of ovoids of $\mathcal{S}'$ is a set of q ovoids meeting pairwise in an exactly one fixed point of $\mathcal{S}'$. If L is a line of $\mathcal{S}\backslash\mathcal{S}'$, then the ovoids of $\mathcal{S}'$ subtended by the points of $\mathcal{S}\backslash\mathcal{S}'$ incident with L form a rosette of $\mathcal{S}'$.

If for a subtended ovoid $\mathcal{O}_x$ there is a point y of $\mathcal{S}\backslash\mathcal{S}'$, $y\neq x$, such that $\mathcal{O}_y=\mathcal{O}_x$, then $\mathcal{O}_x$ is said to be *doubly subtended*. If each ovoid of $\mathcal{S}'$ subtended by a point of $\mathcal{S}\backslash\mathcal{S}'$ is doubly subtended, then $\mathcal{S}'$ is said to be *doubly subtended* in $\mathcal{S}$. If $\mathcal{S}'$ is doubly subtended in $\mathcal{S}$, then the incidence structure with point set the subtended ovoids of $\mathcal{S}'$; line set the rosettes of subtended ovoids of $\mathcal{S}'$; and incidence containment is an $\mathrm{spg}(q-1,q^2,2,2q(q-1))$.

The generalized quadrangle $Q(4,q)$ is doubly subtended in $Q^-(5,q)$ and hence by Brown's construction yields a semipartial geometry which is better known as the Metz model of $\mathrm{TQ}(4,q)$ (we use the notation as introduced in [6]). For q odd and $\sigma\in\mathrm{Aut}(\mathrm{GF}(q))$ the generalized quadrangle $Q(4,q)$ is also doubly subtended in the Kantor translation generalized quadrangle associated with σ [12]. Two such generalized quadrangles associated with field automorphisms σ_1 and σ_2, respectively, are isomorphic if and only if $\sigma_1=\sigma_2$ or $\sigma_1=\sigma_2^{-1}$, and similarly for the $\mathrm{spg}(q-1,q^2,2,2q(q-1))$. In the case where σ is the identity the Kantor construction yields $Q^-(5,q)$ and the associated $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ is the Metz model of $\mathrm{TQ}(4,q)$.

An *embedding* of a partial linear space in $\mathrm{AG}(n,q)$ is a representation of the geometry with point set a subset of the point set of $\mathrm{AG}(n,q)$; line set a subset of the line set of $\mathrm{AG}(n,q)$; and incidence inherited from $\mathrm{AG}(n,q)$. The geometry is *fully* embedded if the embedding has the additional property that for every line L of $\mathrm{AG}(n,q)$ that is also a line of the geometry, each point of $\mathrm{AG}(n,q)$ that is incident with L is a point of the geometry. It is also required that $\mathrm{AG}(n,q)$ is generated by the point set of the geometry. In the same way one can define a full embedding of a partial linear space in $\mathrm{PG}(n,q)$.

Let $\mathcal{S}$ be a generalized quadrangle fully embedded in a projective space $\mathrm{PG}(n,q)$, hence $\mathcal{S}$ is classical and $n=3,4$ or 5 [4]. Let p be a point of $\mathrm{PG}(n,q)$ and let Π be a hyperplane of $\mathrm{PG}(n,q)$ not containing p. Let $\mathcal{P}_1$ be the projection of the point set of $\mathcal{S}$ from p onto Π and let $\mathcal{P}_2$ be the set of points of Π on a tangent through p at $\mathcal{S}$. Consider the incidence structure $\mathcal{S}_p=(\mathcal{P}_p,\mathcal{L}_p,\mathrm{I}_p)$ with $\mathcal{P}_p=\mathcal{P}_1\backslash\mathcal{P}_2$, $\mathcal{L}_p$ the set of lines of Π with q points in $\mathcal{P}_p$ and incidence I_p inherited from the projective space. If $\mathcal{S}=Q^-(5,q)$ (fully embedded in $\mathrm{PG}(5,q)$) or $\mathcal{S}=H(4,q^2)$ (fully embedded in $\mathrm{PG}(4,q^2)$) the incidence structure $\mathcal{S}_p$ is a semipartial geometry.

Assume $\mathcal{S}=Q^-(5,q)$ is fully embedded in $\mathrm{PG}(5,q)$ and p is not on the quadric $Q^-(5,q)$, then Hirschfeld and Thas [11] proved that projection yields an $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ that is isomorphic to the semipartial geometry $\mathrm{TQ}(4,q)$. For the other examples we refer to [7]. If q is even, the Hirschfeld–Thas model of $\mathrm{TQ}(4,q)$ yields a semipartial geometry which is fully embedded in $\mathrm{AG}(4,q)$.

In [8] Debroey and Thas classified the proper semipartial geometries that may be fully embedded in $\mathrm{AG}(n,q)$ for $n=2$ and 3, as well as the possible models for the embeddings in these cases. For $n>3$ there is no such classification. There are two examples known, one being the Hirschfeld–Thas model of $\mathrm{TQ}(4,q)$, q even.

We will prove the following main theorem.

Main Theorem. *Let $\mathcal{S}$ be a semipartial geometry* $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ *fully embedded in* $\mathrm{AG}(4,q)$. *Then $q=2^h$ and $\mathcal{S}$ is the Hirschfeld–Thas model of* $\mathrm{TQ}(4,q)$.

3. The spg$(q-1,q^2,2,2q(q-1))$ embedded in AG$(4,q)$

In this section, let $\mathcal{S}$ be an $\mathrm{spg}(q-1,q^2,2,2q(q-1))$ fully embedded in $\mathrm{AG}(4,q), q\neq 2$.

Let Π_∞ denote the hyperplane at infinity of $\mathrm{AG}(4,q)$. The line set of $\mathcal{S}$ is a subset of the line set of $\mathrm{AG}(4,q)$, which in turn is a subset of the line set of $\mathrm{PG}(4,q)$, the projective completion of $\mathrm{AG}(4,q)$. Thus a line of $\mathcal{S}$ will be said to intersect Π_∞ in the point of Π_∞ incident with the line in $\mathrm{PG}(4,q)$. The same symbol will be used to refer to such a line in the three different contexts.

For a point x of $\mathcal{S}$, let θ_x denote the set of q^2+1 points in Π_∞ determined by the intersection of Π_∞ with the lines of $\mathcal{S}$ through x. Since $\alpha=2$ any line N of Π_∞ intersects θ_x in at most three points. A line of Π_∞ intersecting θ_x in 0,1,2 or 3 points will be referred to as an external line, tangent, secant or 3-secant, respectively.

Let (x,L) be an antiflag of $\mathcal{S}$, $M=\langle x,L\rangle\cap\Pi_\infty$ and $p=L\cap\Pi_\infty$. If $\alpha(x,L)=0$, then M is either a tangent of θ_x at p or an external line of θ_x, while for $\alpha(x,L)=2$, we obtain that either $p\notin\theta_x$ and M intersects θ_x in two points, or $p\in\theta_x$ and M intersects θ_x in three points.

Lemma 1. *Let x be a point of the semipartial geometry $\mathcal{S}$ and let M be a projective line of Π_∞ intersecting θ_x in three points p_1, p_2, p_3. Then all of the points of $\langle M, x\rangle \backslash M$ are points of $\mathcal{S}$ and the $3q$ affine lines in $\langle M, x\rangle$ through p_1, p_2 or p_3 are exactly the lines of $\mathcal{S}$ contained in the plane $\langle M, x\rangle$. Furthermore, q is a power of 3.*

Proof. Let y be a point of $\langle x, p_1\rangle \backslash \{x, p_1\}$. Since $\alpha(y, \langle x, p_2\rangle) = 2$ we obtain a line $\langle y, z\rangle$ of $\mathcal{S}$ with $z \in \langle x, p_2\rangle \backslash \{x, p_2\}$ which also intersects $\langle x, p_3\rangle$ in the point u. If $u \neq p_3$, then $\alpha(x, \langle y, z\rangle) > 2$, a contradiction, and so $u = p_3$. Similarly since $\alpha(y, \langle x, p_3\rangle) = 2$ it follows that the line $\langle y, p_2\rangle$ is a line of $\mathcal{S}$. Since this is true for any $y \in \langle x, p_1\rangle \backslash \{x, p_1\}$ we have that each affine line through p_2 or p_3 is a line of $\mathcal{S}$. Clearly by similar arguments we also have that each affine line through p_1 is a line of $\mathcal{S}$. If N is any line of $\langle M, x\rangle$ not incident with p_1, p_2 or p_3, then N cannot be a line of $\mathcal{S}$ since for any point y of $\langle M, x\rangle \backslash M$ not on N we would have $\alpha(y, N) > 2$.

Now let the affine lines of $\langle M, x\rangle$ through p_1 be labelled $L_1, \ldots, L_q$. For any L_i there are $q^3(q-1)/2$ antiflags (y, L_i) of $\mathcal{S}$ with incidence number 2, and hence $q^2(q^2-1)/2 - q^3(q-1)/2 - q = (q^3 - q^2)/2 - q$ antiflags (z, L_i) with incidence number 0. Counting the number of points z of $\mathcal{S}$ such that $z \in \langle M, x\rangle$ or $\alpha(z, L_i) = 0$ for some L_i we have at most $q^3(q-1)/2$ points, fewer than the total number of points of $\mathcal{S}$. Consequently there exists a point x' of $\mathcal{S}$ such that $x' \notin \langle M, x\rangle$ and $\alpha(x', L_i) = 2$ for all L_i. Hence there are $2q$ points of $\langle M, x\rangle$ collinear with x' in $\mathcal{S}$. Let this set of points be Ω. Since $|\Omega| = 2q$ it follows that each affine line through p_2 or p_3 is incident with 2 points of Ω. If N is any line of $\langle M, x\rangle$ not incident with p_1, p_2 or p_3, then $\langle N, x'\rangle$ contains at most 3 lines of $\mathcal{S}$ on x' and so N contains at most 3 points of Ω. So now consider any $y \in \Omega$. Then lines $\langle y, p_1\rangle$, $\langle y, p_2\rangle$ and $\langle y, p_3\rangle$ cover 4 points of Ω while the remaining $q-2$ lines of $\langle M, x\rangle$ on y must cover the remaining $2q-4$ points of Ω with at most 2 points of $\Omega \backslash \{y\}$ on a line. Consequently, each such line contains exactly 3 points of Ω. It follows that each line of $\langle M, x\rangle$ not incident with p_1, p_2 or p_3 is incident with 0 or 3 points of Ω. Now let p be any point of $M \backslash \{p_1, p_2, p_3\}$. By considering the lines of $\langle M, x\rangle$ on p we see that Ω may be partitioned into sets of size 3 and so $3 | q$. $\square$

Lemma 2. *Let x and y be two collinear points of $\mathcal{S}$, then a line M of Π_∞ incident with $p = \langle x, y\rangle \cap \Pi_\infty$ is either a tangent of both θ_x and θ_y, a secant of both θ_x and θ_y with $M \cap \theta_x \cap \theta_y = \{p\}$, or a 3-secant of both θ_x and θ_y with $|M \cap \theta_x \cap \theta_y| = 3$.*

Proof. Let M be a line of Π_∞ incident with $p = \langle x, y\rangle \cap \Pi_\infty$. Since $\alpha = 2$, both $|M \cap \theta_x|$ and $|M \cap \theta_y|$ are at most 3. If $M \cap \theta_y = \{p\}$ and $|M \cap \theta_x| > 1$, then this contradicts $\alpha = 2$. Hence if $M \cap \theta_y = \{p\}$, then it is also the case that $M \cap \theta_x = \{p\}$, that is, M is a tangent of both θ_x and θ_y.

Assume that $|M \cap \theta_x| = 3$, then by Lemma 1 every point of the affine plane $\langle x, M\rangle$ is incident with three lines of $\mathcal{S}$ and belonging to that plane; more particular this holds for the point y and so $|M \cap \theta_y| = 3$.

Hence the only possibility which is left is $|M \cap \theta_x| = |M \cap \theta_y| = 2$ with $M \cap \theta_x \cap \theta_y = \{p\}$. $\square$

If $|M \cap \theta_x \backslash \theta_y| = |M \cap \theta_y \backslash \theta_x| = 1$, then M is said to be of type (A) with respect to x and y. If $|M \cap \theta_x \cap \theta_y| = 3$, and hence $|M \cap \theta_x \backslash \theta_y| = |M \cap \theta_y \backslash \theta_x| = 0$, then M is said to be of type (B) with respect to x and y.

Lemma 3. *Let x be a point of $\mathscr{S}$, then θ_x is an ovoid of Π_∞ and q is even.*

Proof. Let y be a point of $\mathscr{S}$ collinear with x and let $p = \langle x, y \rangle \cap \Pi_\infty$. By the proof of Lemma 2 a line of Π_∞ containing p is either tangent to both θ_x and θ_y, or is either of type (A) or of type (B) with respect to x and y. To prove that θ_x is an ovoid, we have to show that there are no lines of type (B) with respect to x and y.

Let L be the line $\langle x, y \rangle$ of $\mathscr{S}$. For any line N of $\mathscr{S}$ intersecting L (in a point of $\mathscr{S}$) $\langle L, N \rangle \cap \Pi_\infty$ is a line of type (A) or (B) (with respect to x and y). Let this line be M and suppose that M is of type (A) such that $\theta_x \cap M = \{p, p_1\}$ and $\theta_y \cap M = \{p, p_2\}$. Let $z = \langle p_2, y \rangle \cap \langle p_1, x \rangle$ and suppose that z is incident with a third line of $\mathscr{S}$ in $\langle M, L \rangle$. Since z is collinear with x and y and $\alpha = 2$ this third line must be $\langle z, p \rangle$. As $\alpha(y, \langle z, p \rangle) = 2$ there is a third line of $\mathscr{S}$ on y in $\langle M, L \rangle$, contradicting the fact that M is a 2-secant of θ_y. Consequently z is incident with exactly two lines of $\mathscr{S}$ in $\langle M, L \rangle$. Similar arguments show that each point of $\langle x, p_1 \rangle \backslash \{p_1\}$ is incident with exactly two lines of $\mathscr{S}$ in $\langle M, L \rangle$. It follows that there are exactly $q+1$ lines of $\mathscr{S}$ in $\langle M, L \rangle$. Also $\alpha = 2$ implies that no two of these lines meet on M. Hence the lines of $\mathscr{S}$ in $\langle M, L \rangle$ form a dual oval with nucleus M, from which it follows that q is even.

By Lemma 1 if M is a line of type (B), then q is a power of 3. Thus we have two distinct cases for the lines of Π_∞ through p that are not tangent to both x and y: either they are all of type (A) or all of type (B). In the latter case $3|q$ and the lines through p partition $\theta_x \backslash \{p\}$ into sets of size 2 which implies that $2|q$, a contradiction. So we must be in the former case and q is even.

Now suppose that x is an arbitrary point of $\mathscr{S}$ and p a point of θ_x. If $y \in \langle x, p \rangle \backslash \{x, p\}$, then by applying the above argument it follows that every line of Π_∞ on p is either a tangent or a secant of θ_x. Hence there are no 3-secants of θ_x and θ_x is an ovoid. $\square$

Corollary 4. *Let x and y be two collinear points of $\mathscr{S}$, then $|\theta_x \cap \theta_y| = 1$.*

Proof. Every line of Π_∞ incident with $p = \langle x, y \rangle \cap \Pi_\infty$ is either a tangent to both θ_x and θ_y or is of type (A) with respect to x and y. $\square$

Lemma 5. *Let x and y be two non-collinear points of $\mathscr{S}$ and $p = \langle x, y \rangle \cap \Pi_\infty$. Let M be any line of Π_∞ incident with p. Then one of the following is the case*:

(i) *M is secant to both θ_x and θ_y and $M \cap \theta_x \cap \theta_y = \emptyset$*;
(ii) *M is tangent to both θ_x and θ_y at a point of $\theta_x \cap \theta_y$; or*
(iii) *M is external to both θ_x and θ_y.*

Furthermore $\theta_x \cap \theta_y$ is an oval with nucleus p.

Proof. Suppose that $r \in \theta_x \cap \theta_y$. We show that $\langle r, p \rangle$ is tangent to both θ_x and θ_y. Suppose that $\langle r, p \rangle$ is a secant line of at least one of the ovoids, say θ_x. Hence $(\theta_x \cap \langle r, p \rangle) \backslash \{r\} = \{u\}$ for some point u. Let $z = \langle u, x \rangle \cap \langle r, y \rangle$. Then x and z are collinear in $\mathscr{S}$ while $|\theta_x \cap \theta_z| \geqslant 2$, contradicting Corollary 4. Hence $\langle p, r \rangle$ is a tangent line of both ovoids.

Now we show that M is secant to θ_x if and only if it is secant to θ_y. Therefore we first assume that M intersects θ_x in the point v and θ_y in the point w, with $v \neq w$. Then $\langle v, x \rangle$ intersects $\langle w, y \rangle$, and so $\alpha(y, \langle x, v \rangle) = 2$. This implies that M intersects θ_y in the distinct points w and w', and moreover $w, w' \notin \theta_x \cap \theta_y$. Similarly, since $\alpha(x, \langle y, w \rangle) = 2$, it follows that M intersects θ_x in the distinct points v and v', with $v, v' \notin \theta_x \cap \theta_y$. In other words, M intersects both ovoids in two points outside their intersection. Since $|\Gamma(x) \cap \Gamma(y) \cap \langle x, y, M \rangle| = 4$ and $|\Gamma(x) \cap \Gamma(y)| = \mu = 2q(q-1)$ it follows that there are exactly $q(q-1)/2$ lines incident with p that are secant to both θ_x and θ_y. Since this is the number of secants of an ovoid incident with a point not on the ovoid this means that the set of lines of Π_∞ incident with p and secant to θ_x is also the set of lines incident with p and secant to θ_y.

By Lemma 3, q is even and consequently the $q+1$ tangents of θ_x incident with p are contained in a plane π_x on p and similarly the $q+1$ tangents of θ_y incident with p are contained in a plane π_y. There are two cases to consider: $\pi_x = \pi_y$ and $\pi_x \cap \pi_y$ is a line incident with p. First suppose that $\pi_x = \pi_y$. It follows that the tangents of θ_x incident with p are precisely the tangents of θ_y incident with p with a common point of tangency. Consequently $\theta_x \cap \theta_y$ is an oval of π_x with nucleus p. So in this case $|\theta_x \cap \theta_y| = q+1$. Now suppose that $\pi_x \cap \pi_y$ is a line L incident with p. The line L is a tangent of both θ_x and θ_y at a point $o \in \theta_x \cap \theta_y$. If $M \neq L$ then by arguments above M must be external to θ_y. From this it follows that π_x is the tangent plane of θ_y at o and similarly π_y is the tangent plane of θ_x at o. Since $\langle p, o \rangle$ is the only line of Π_∞ incident with p that is tangent to both θ_x and θ_y it follows that $\theta_x \cap \theta_y = \{o\}$, and so $|\theta_x \cap \theta_y| = 1$.

It is now shown that the case $|\theta_x \cap \theta_y| = 1$ cannot occur. Suppose that $|\theta_x \cap \theta_y| = 1$. Let M be secant of both θ_x and θ_y. It follows by arguments above that if $\theta_x \cap M = \{v, v'\}$ and $\theta_y \cap M = \{w, w'\}$, then $\{v, v', w, w'\}$ are four distinct points. Let $\{x = x_1, x_2, \ldots, x_q\}$ be the set of q points of $\mathscr{S}$ incident with the line $L = \langle x, v \rangle$. By Corollary 4, $\theta_{x_i} \cap \theta_{x_j} = \{v\}$ for $i, j \in \{1, \ldots, q\}$, $i \neq j$, and by a consequence of Lemma 2, the ovoids $\theta_{x_1}, \ldots, \theta_{x_q}$ have a common tangent plane at v, π_v say. It follows that the ovoids $\theta_{x_1}, \ldots, \theta_{x_q}$ partition the points of $\Pi_\infty \backslash \pi_v$ into q sets of size q^2. Without loss of generality assume that y is collinear with the points x_2 and x_3 of L, so by Corollary 4 $|\theta_y \cap \theta_{x_2}| = |\theta_y \cap \theta_{x_3}| = 1$. By above arguments it follows that for $i = 4, \ldots, q$, $|\theta_y \cap \theta_{x_i}| = 1$ or $q+1$.

Suppose that $|\pi_v \cap \theta_y| = 1$, then since $v \notin \theta_y$ the ovoids $\theta_{x_1}, \ldots, \theta_{x_q}$ partition the q^2 points of $\theta_y \backslash (\pi_v \cap \theta_y)$ into q sets with size either 1 or $q+1$. This requires $q-1$ sets of size $q+1$ and 1 set of size 1. However $|\theta_{x_i} \cap \theta_y| = 1$ for $i = 1, 2$ and 3, a contradiction. Now suppose that $|\pi_v \cap \theta_y| = q+1$, then since $v \notin \theta_y$ the ovoids $\theta_{x_1}, \ldots, \theta_{x_q}$ partition the $q^2 - q$ points of $\theta_y \backslash (\pi_v \cap \theta_y)$ into q sets with size either 1 or $q+1$. This requires $q-2$ sets of size $q+1$ and 2 sets of size 1, again a contradiction.

It follows that $|\theta_x \cap \theta_y|$ cannot be 1 and so $\pi_x = \pi_y$ and $\theta_x \cap \theta_y$ is an oval of π_x with nucleus p. $\square$

Theorem 6. *Let* $\mathscr{S}$ *be a semipartial geometry* $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ *fully embedded in* $\mathrm{AG}(4,q)$. *Then* $q = 2^h$, $\mathscr{S}$ *is isomorphic to* $\mathrm{TQ}(4,q)$ *and is fully embedded as the Hirschfeld–Thas model.*

Proof. Let $\mathscr{S}$ be a semipartial geometry $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ fully embedded in $\mathrm{AG}(4,q)$. If $q=2$, then $\mathscr{S}$ coincides with its point graph which is the unique complete graph on six vertices and the result follows. Hence we may assume that $q>2$.

Let $\mathscr{K} = \Pi_\infty \cup \mathscr{P}$, where $\mathscr{P}$ is the point set of $\mathscr{S}$. The intersections of $\mathscr{K}$ with a plane of $\mathrm{PG}(4,q)$ are now considered which will allow the use of a result of Hirschfeld and Thas in [10] in order to prove the theorem. So let π be a plane of $\mathrm{PG}(4,q)$. If $\pi \subset \Pi_\infty$, then $\pi \subset \mathscr{K}$; so suppose that $\pi \not\subset \Pi_\infty$ and that $\pi \cap \Pi_\infty$ is the line M.

Suppose that π contains a point x of $\mathscr{S}$. Then M may either be a secant, tangent or external line of θ_x.

Suppose that M is a secant line of θ_x. This is the case if and only if there exists an antiflag (x, L) of $\mathscr{S}$ contained in π such that $\alpha(x, L) = 2$. By the proof of Lemma 3 the lines of $\mathscr{S}$ in π form a dual oval $\mathscr{D}$ with nucleus M and these are all the lines of $\mathscr{S}$ in π. Let z be any point of $\mathscr{S} \cap \pi$ and not collinear in $\mathscr{S}$ with x. Then by Lemma 5, M is a secant of θ_z; hence z is incident with exactly two lines of the dual oval $\mathscr{D}$. It follows that $\pi \cap \mathscr{K}$ is a dual hyperoval; or equivalently the complement of a maximal arc of type $(0, q/2)$.

Next suppose that $M \cap \theta_x = \{p\}$. Hence M is a tangent of θ_x at p and all points of $\pi \cap \mathscr{S}$ are not collinear in $\mathscr{S}$ with x. If y is such a point of $\mathscr{S}$ on π, then by Lemma 5 M is a tangent of θ_y at p and so $\langle p, y \rangle$ is a line of $\mathscr{S}$. It follows that lines of $\mathscr{S}$ in π are incident with p and that all points of $\mathscr{S}$ on π are incident with such a line. Let z be a point of $M \backslash \{p\}$, and let N be a secant of θ_x incident with z. By the above the plane $\langle N, x \rangle$ meets $\mathscr{K}$ in a dual hyperoval and since $z \notin \theta_x$ it follows that the line $\langle z, x \rangle$ is not a line of $\mathscr{S}$. Hence $\langle z, x \rangle$ is incident with exactly $q/2$ points of $\mathscr{S}$ and so π meets the line set of $\mathscr{S}$ in exactly $q/2$ lines each intersecting M in p. So M is a tangent of θ_x if and only if π meets $\mathscr{K}$ in the point set of $q/2+1$ concurrent lines.

Finally suppose that M is an external line of θ_x. Let y be any point of M and let L be a secant of θ_x incident with y. The line $\langle x, y \rangle$ is incident with $q/2$ points of $\mathscr{S}$. Hence each line of π incident with x is incident with $q/2$ points of $\mathscr{S}$. If z is any other point of $\mathscr{S}$ in π, then since x and z are not collinear and M is an external line of θ_x it follows by Lemma 5 that M is also an external line of θ_z. Hence π meets $\mathscr{K}$ in a maximal arc of type $(0, q/2)$, and M is an external line to this maximal arc.

By the above discussion a plane section of $\mathscr{K}$ is one of the following sets: (i) a single line; (ii) the entire plane; (iii) a maximal arc of type $(0, q/2)$, plus an external line; (iv) a dual hyperoval, or equivalently, the complement of a maximal arc of type $(0, q/2)$; or (v) $q/2+1$ concurrent lines.

From this list it follows that with respect to the intersection with lines $\mathcal{K}$ is a set of points of type $(1, q/2+1, q+1)$.

Actually, it is possible to show that no planes of type (i) occur, but we do not need this. The set $\mathcal{K}$ does contain plane sections of type (iv), and for $q=4$, $\mathcal{K}$ has no plane section that is either a unital or a subplane. Hence by [10, Theorem 6] the set $\mathcal{K}$ is the projection of a non-singular quadric of $\mathrm{PG}(5,q)$ onto $\mathrm{PG}(4,q)$. Any plane contained in $\mathcal{K}$ is also contained in Π_∞ which can only be the case if $\mathcal{K}$ is the projection of an elliptic quadric $Q^-(5,q)$ onto $\mathrm{PG}(4,q)$. □

We can rephrase as follows our result for an $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ constructed from a doubly subtended subquadrangle of order q of a generalized quadrangle of order (q, q^2).

Corollary 7. *Let $\mathcal{G}$ be a generalized quadrangle of order (q, q^2), $\mathcal{G}'$ a doubly subtended subquadrangle of $\mathcal{G}$ of order q, and $\mathcal{S}$ the* $\mathrm{spg}(q-1, q^2, 2, 2q(q-1))$ *constructed from $\mathcal{G}$ and $\mathcal{G}'$. If $\mathcal{S}$ may be fully embedded in* $\mathrm{AG}(4,q)$, *then* $\mathcal{S} = \mathrm{TQ}(4,q)$, $\mathcal{G} = Q^-(5,q)$, $\mathcal{G}' = Q(4,q)$ *and* $q = 2^h$.

Proof. By Theorem 6 $\mathcal{S} \cong \mathrm{TQ}(4,q)$ and $q=2^h$. Since $\mathcal{S}$ (in the model of Metz) may be constructed from the doubly subtended subquadrangle $Q(4,q)$ of $Q^-(5,q)$, it follows from [2, Theorem 3.3] that $\mathcal{G}' = Q(4,q)$ and $\mathcal{S}$ is the model of Metz in $Q(4,q)$. Since $Q(4,q)$ is doubly subtended in $\mathcal{G}$ with all subtended ovoids being elliptic quadrics on $Q(4,q)$, it follows that $\mathcal{G} = Q^-(5,q)$ [1,3]. □

Acknowledgments

During the writing of this paper, the first author was a Postdoctoral Research Fellow supported by a research grant of Ghent University, No. GOA 12050300, while the third author was a Research Fellow supported by the Flemish Institute for the Promotion of Scientific and Technological Research in Industry (IWT), Grant No. IWT/SB/971002.

The authors thank one of the referees for his comments which shortened the first version of this paper.

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