More non-analytic classes of continua

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Abstract

The method of [U. Darji, Topology Appl. 103 (2000) 243–248] is extended to get the coanalytic hardness of many classes of metric continua. For instance: (1) the family of all continua in $I^n$, $n \geq 2$, that admit only arcs (simple closed curves) as chainable (circularly chainable) subcontinua is coanalytic complete; (2) the family of all continua in $I^n$, $n \geq 2$ ($n \geq 3$), which contain no copy of a given nondegenerate chainable (circularly chainable) continuum $Y$ is coanalytic hard; if $Y$ is an arc or a pseudo-arc (a simple closed curve or a pseudo-solenoid), then the family is coanalytic complete; (3) the family of all tree-like continua that contain no hereditarily decomposable subcontinua is coanalytic hard; (4) the family of all $\lambda$-dendroids that contain no arcs is coanalytic complete; (5) the sets of all countable-dimensional continua and of all weakly infinite-dimensional continua in the Hilbert cube are coanalytic hard; strongly countable-dimensional continua form a coanalytic complete family.

Keywords: Coanalytic set; Coanalytic complete; Chainable continuum; Inverse limit; Clump of continua; $\lambda$-dendroid; Tree-like continuum; Pseudo-arc; Pseudo-solenoid; Countable-dimensional; Weakly infinite-dimensional

1. Introduction

The inspiration for this note comes from the Darji paper [5], where it is proved that the set $\mathcal{HD}(I^n)$ of all hereditarily decomposable continua in $I^n$, $n \geq 2$, is a coanalytic complete subset of the hyperspace $C(I^n)$ of all subcontinua of the cube $I^n$ with the
Hausdorff metric. Without any change, the proof in [5] directly yields more corollaries. For example, one can easily observe that the sets of all

- (plane, 1-dimensional) arcwise connected continua,
- (plane) dendroids,
- (plane, countable) fans,
- (plane) continua which are unions of countably many arcs,
- continua which contain no copy of the simplest indecomposable continuum in $I^n$ are coanalytic hard.

Recall that a subset of a Polish space is analytic, if it is a continuous image of a Borel subset of some Polish space. A subset of a Polish space is coanalytic, if its complement is analytic. A subset $B$ of a Polish space $Y$ is said to be coanalytic hard [8] if for any Polish space $X$ and its coanalytic subset $A$ there exists a continuous map $f : X \to Y$ (called a reduction of $A$ to $B$) such that $f^{-1}(B) = A$; if, additionally, $B$ is coanalytic, then $B$ is said to be coanalytic complete. Since $A$ can be taken to be non-analytic (or, equivalently, non-Borel), a coanalytic hard set is not analytic. In practice, one proves the coanalytic hardness of $B$ by reducing to it some known coanalytic complete set $A$. We denote by $K(X)$ the hyperspace of all non-empty compact subsets of $X$ with the Hausdorff metric and by $C(X) \subset K(X)$ the hyperspace of all subcontinua of $X$.

Generally speaking, the method of proving coanalytic hardness in [5] is constructing a continuous reduction $f : K([0, 1]^\omega) \to C(I^n)$ of the Hurewicz coanalytic complete set $K(D)$, where $D = \{\sigma \in [0, 1]^\omega : \exists k \forall m \geq k(\sigma(m) = 0)\}$ [8], to the family $H_D \subset C(I^n)$ ($\sigma(m)$ is the $m$th term of sequence $\sigma$).

In Section 3, we generalize the construction from [5] (by considering inverse limits of arbitrary polyhedra) and discover many families of continua as coanalytic hard or complete. In particular, the set of all continua in $I^2$ ($I^3$) which contain no copy of a given non-locally connected chainable (circularly chainable) continuum is coanalytic hard. The result is supplemented in Section 4—we show that the families of all continua without arcs, all tree-like continua without arcs and all $\lambda$-dendroids without arcs are coanalytic complete; all tree-like continua without hereditarily decomposable subcontinua form a coanalytic hard family. Theorems 4.1 and 4.3 cannot be obtained by the method of Section 3 and require different constructions.

Most families are considered in $I^2$ or $I^3$ but the evaluations hold true for any greater $n$.

2. Preliminaries

All spaces in the paper are metric separable. A continuum is a connected compact space. A continuum $X$ is decomposable if there exist two proper subcontinua $A, B \subset X$ such that $X = A \cup B$; $X$ is indecomposable if it is not decomposable. A continuum is hereditarily decomposable (indecomposable) if each of its subcontinua is decomposable (indecomposable). A continuum $X$ is unicoherent if, for any subcontinua $A, B \subset X$ such that $X = A \cup B$, the intersection $A \cap B$ is connected; $X$ is hereditarily unicoherent if each subcontinuum of $X$ is unicoherent. A hereditarily unicoherent continuum which
is hereditarily decomposable (arcwise connected) is called a \(\lambda\)-dendroid (dendroid). If a dendroid has exactly one ramification point \(p \in X\), then it is a fan with vertex \(p\). A fan is countable if it is a union of countably many arcs emanating from its vertex.

A pointed continuum \((Z, z_0)\) is said to be a clump made of a collection \(\{X_t: t \in T\}\) of continua with vertex \(z_0\) if for every \(t\) there is a topological copy \(X'_t \subset Z\) of \(X_t\), for any \(t_1, t_2 \in T\), \(t_1 \neq t_2\) (cf. [4] for a general concept of a clump).

By a countable clump of circles (arcs) with vertex \(z_0\) we mean a clump made of a countable collection of simple closed curves (arcs) with vertex \(z_0\). Clearly, a countable clump of arcs with vertex \(z_0\) is a countable fan with vertex \(z_0\).

For a sequence \(\sigma\) (finite or infinite) of 0’s and 1’s, \(\sigma|n\) denotes its finite reduction of length \(n\), i.e., \(\sigma|n\) is the sequence of first \(n\) digits of \(\sigma\); put \(\sigma|0 = \emptyset\). If \(\sigma\) is of length \(n\) and \(i \in \{0, 1\}\), then \(\sigma_i\) is the sequence whose last digit is \(i\) and \(\sigma_i|n = \sigma_i\). \(I = I = [0, 1]\) is the unit interval and \(Q = I^\infty = \prod_{i=1}^\infty I_i\) is the Hilbert cube. For a family \(W\) of sets, \(W^*\) denotes their union.

If \(U\) is a finite family of open subsets of a space \(X\), then \(N(U)\) we denote the nerve of \(U\).

We will write \(V \prec U\), if family of sets \(V\) refines \(U\), i.e., if for each \(V \in V\) there exists \(U \in U\) such that \(V \subset U\).

Recall that any compact space \(Y\) can be represented as an inverse limit

\[
Y = \lim_{\leftarrow}(P_n, f_n)
\]

of polyhedra \(P_n\) with surjective continuous bonding maps \(f_n: P_{n+1} \rightarrow P_n\); if \(\dim Y \leq d\), then the polyhedra can also be of dimension \(\leq d\). If \(Y\) and polyhedra \(P_n\) are subsets of \(Q\), then \(P_n\)'s can be regarded as the unions \(N(P_n)^*\) of nerves of \(\frac{1}{n}\)-covers \(P_n\) of \(Y\) consisting of open subsets of \(Q\) such that

- \(P_{n+1} \prec P_n\),
- \(f_n\) is a standard simplicial map induced by the above refinement.

If each \(P_n\) is an arc (a circle, a tree), then \(Y\) is called an arc-like (circle-like, tree-like) or chainable (circularly chainable) continuum; in such the cases the covers \(P_n\) are chains (circular chains, tree-chains). A nondegenerate chainable (circularly chainable) continuum is locally connected if and only if it is an arc (a simple closed curve).

All nondegenerate chainable hereditarily indecomposable continua are homeomorphic and are called pseudo-arcs. All circularly chainable and non-chainable continua which are hereditarily indecomposable are called pseudo-solenoids; all planar pseudo-solenoids are homeomorphic and are called pseudo-circles. A chainable continuum is an arc of pseudo-arcs, if it admits a continuous decomposition, called canonical, into pseudo-arcs with the decomposition space an arc. All arcs of pseudo-arcs are homeomorphic and they contain no arc (for more informations concerning pseudo-arcs, pseudo-solenoids and arcs of pseudo-arcs, see [13]).
A continuous mapping \( f \) from a compact space \( X \) onto \( Y \) is atomic, if for each subcontinuum \( K \) of \( X \) such that \( f(K) \) is nondegenerate, we have \( K = f^{-1}(f(K)) \).

For definitions of countable-dimensional, strongly countable-dimensional, and weakly infinite-dimensional spaces, see [7] or [1].

If \( L \) is an arc (with or without end-points), then \( \partial L \) denotes the set of its end-points and \( \text{int} L = L \setminus \partial L \).

3. Continua which do not contain a given continuum

**Theorem 3.1.** Let \( Y = \lim (P_n, f_n) \) be a nondegenerate continuum, where \( P_n \)'s are polyhedra and \( f_n : P_{n+1} \to P_n \) are surjective continuous bonding maps such that \( Y \) is contained in no clump made of all subcollections of \( \{P_1, P_2, \ldots \} \) with a vertex. If \( \mathcal{B} \subset C(Q) \) contains all such clumps and no member of \( \mathcal{B} \) contains a topological copy of \( Y \), then \( \mathcal{B} \) is coanalytic hard.

**Proof.** We can assume all polyhedra are non-trivial.

Let \( P_0 \) be a finite family of open subsets of \( Q \) such that \( N(P_0)^* \cong P_1 \). Choose a point \( z_0 \in P_0^* \). Let \( P_0 \) and \( P_1 \) be finite families of open subsets of \( Q \) such that, for \( i \in \{0, 1\} \),

- \( N(P_i)^* \cong P_2 \).
- \( P_1 \prec P_0 \).
- the simplicial mapping \( \kappa_i : N(P_i)^* \to N(P_0)^* \) induced by the above refinement is equivalent to \( f_1 \).
- \( \text{mesh} P_i < 1/2 \).
- \( P_0^* \cap P_1^* = \{U\} \) and \( z_0 \in U \) for some \( U \in P_0 \cap P_1 \).

Inductively, suppose a finite family \( P_\tau \) of open subsets of \( Q \) has been defined for an arbitrary sequence \( \tau \) of 0’s and 1’s of length \( n \) such that \( P_\tau \) refines \( P_\emptyset \) and \( z_0 \in P_\emptyset^* \). If \( \tau \) contains \( k \) digits 1 (0 \( \leq k \leq n \)), then one can find families \( P_{\tau_0} \) and \( P_{\tau_1} \) of open subsets of \( Q \) such that, for \( i \in \{0, 1\} \),

1. \( N(P_{\tau_0})^* \cong N(P_{\tau})^* \),
2. \( N(P_{\tau_1})^* \cong P_{k+1} \),
3. \( P_{\tau_1} \prec P_{\tau} \),
4. the simplicial map \( \kappa_{\tau_0} : N(P_{\tau_0})^* \to N(P_{\tau})^* \) induced by the refinement \( P_{\tau_0} \prec P_{\tau} \) is a homeomorphism,
5. the simplicial map \( \kappa_{\tau_1} : N(P_{\tau_1})^* \to N(P_{\tau})^* \) induced by the refinement \( P_{\tau_1} \prec P_{\tau} \) is equivalent to the bonding map \( f_{k+1} : P_{k+1} \to P_k \),
6. \( \text{mesh} P_{\tau_i} < \frac{1}{2^i} \),
7. \( P_{\tau_0}^* \cap P_{\tau_1}^* = V \) and \( z_0 \in V \) for some \( V \in P_{\tau_0} \cap P_{\tau_1} \).
For $\sigma \in \{0, 1\}^\omega$, define a continuum
\[ M_\sigma = \bigcap_{n=0}^{\infty} P^n_{\sigma|^n}. \]

Observe that if $\sigma(m) = 1$ for finitely many $m$’s, e.g., for $m \in \{m_1, m_2, \ldots, m_k\}$, where $m_1 < m_2 < \cdots < m_k$, then
\[ M_\sigma \overset{\text{top}}{=} P_{k+1}. \quad (3.1) \]

Otherwise,
\[ M_\sigma \overset{\text{top}}{=} Y. \quad (3.2) \]

By the construction, the mapping $\phi : [0, 1]^{\omega} \to C(Q)$, $\phi(\sigma) = M_\sigma$ is continuous, hence the mapping
\[ \Phi : K([0, 1]^{\omega}) \to C(Q), \quad \Phi(A) = \bigcup_{\sigma \in A} M_\sigma \]

is continuous. It follows from (3.1), (3.2) and 7 that $A \subset D$ if and only if $\Phi(A)$ is homeomorphic to a clump made of a subcollection of $\{P_1, P_2, \ldots\}$ with vertex $z_0$. It means that $A \subset D$ if and only if $\Phi(A) \in B$, so $\Phi$ is a continuous reduction of the Hurewicz set $K(D)$ to $B$. \hfill \Box

**Remark 3.2.** If there is an upper bound for the dimensions of all polyhedra $P_n$, then the construction in the above proof can be made in a cube of sufficiently high finite dimension instead of the Hilbert cube.

**Corollary 3.3.** The following subsets of $C(Q)$ are coanalytic hard:

- all continua in $Q$ which contain no copy of $Q$,
- all countable-dimensional continua in $Q$,
- all weakly infinite-dimensional continua in $Q$.

The family of all strongly countable-dimensional continua in $Q$ is coanalytic complete.

**Proof.** In Theorem 3.1, take $P_n = I^n$ and let $f_n : I^{n+1} \to I^n$ be defined by $f_n(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$. Then $Y = Q$ and the hypotheses of Theorem 3.1 are satisfied in each case, since no continuum in either class contains $Q$ (see [7] and [1, p. 534]).

The family $S$ of all strongly countable-dimensional continua in $Q$ is, moreover, coanalytic. Indeed, if $\mathcal{F} \subset K(Q)$ denotes the set of all finite-dimensional compacta in $Q$, then $\mathcal{F}$ is a hereditary family (i.e., each closed subset of $F \in \mathcal{F}$ belongs to $\mathcal{F}$) and $\mathcal{F}$ is $G_\delta$ (since the subfamily $\mathcal{F}_n \subset \mathcal{F}$ of all compacta of dimension at most $n$ is known to be $G_\delta$ and $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$). Therefore, it follows from [8, Proposition (35.37)] that the family
\[ \mathcal{F}_\sigma = \{ F \in K(Q) : F \text{ is the union of countably many members of } \mathcal{F} \} \]

is coanalytic and so is $S = C(Q) \cap \mathcal{F}_\sigma$. \hfill \Box
3.1. Continua which do not contain a given non-locally connected chainable or circularly chainable continuum

Corollary 3.4. Let $Y$ be a non-locally connected chainable (circularly chainable) continuum. If a family $B \subset C(I^2)$ ($B \subset C(I^3)$) contains all countable fans in $I^2$ (all countable clumps of circles in $I^3$ with a vertex) and no member of $B$ contains a topological copy of $Y$, then $B$ is coanalytic hard. If $Y$ is a non-locally connected circularly chainable planable continuum, then we can assume $B \subset C(I^2)$.

Proof. This follows from Theorem 3.1. We have $Y = \lim\leftarrow (P_n, f_n)$, where $f_n$'s are continuous surjections and, for each $n$, $P_n$ is an arc (a circle, respectively). Now, if $Y$ is chainable or circularly chainable and planable, the construction in the proof of Theorem 3.1 can be done in $I^2$ (in $I^3$ for arbitrary non-locally connected circularly chainable $Y$) (see Remark 3.2). □

In the sequel, $A(I^n)$ and $S(I^n)$ denote, respectively, the subspaces of $C(I^n)$ of all arcs and of all simple closed curves, $n \geq 2$. Recall that both the families are $F_\sigma\delta$ (see, e.g., [10, 11]). The families of all circularly chainable continua, $C(I^n)$ of all chainable continua and $T(I^n)$ of all tree-like continua in $I^n$, respectively. It is known that $P(I^n)$ is a $G_\delta$-subset of $C(I^n)$ [12, p. 225]. The set $P(I^n)$ is the intersection of a $G_\delta$ and an $F_\sigma$ sets in $C(I^n)$, since the classes of hereditarily indecomposable subcontinua and circularly chainable continua are $G_\delta$-subsets of $C(I^n)$ (see [12, p. 206]).

Corollary 3.5. The class of all continua in $I^n$, $n \geq 2$, that contain no pseudo-solenoid is coanalytic complete.

Proof. If $B$ denotes the class, then, clearly,

$$ B = \{ C \in C(I^n) : \forall D \in C(I^n) (D \subset C \implies D \not\in PS(I^n)) \} $$

is coanalytic. It is also coanalytic hard, by Corollary 3.4 (take $Y$ to be a pseudo-circle). □

Proposition 3.6. The set $\mathcal{HU}(X)$ of all hereditarily unicoherent continua in any compact space $X$ is an absolute true $G_\delta$ set. If $X$ contains $I^2$, then $\mathcal{HU}(X)$ is a true $G_\delta$-set and $\mathcal{HU}(I^2) = \mathcal{HU}(X) \cap C(I^2)$ is a dense $G_\delta$-set in $C(I^2)$.

Proof. Let $\{U_1, U_2, \ldots \}$ be an open base of $X$. Denote by $F$ the set of all finite subsets of $\mathbb{N}$ and let

$$ T = \{ (\alpha, \beta) \in F^2 : \overline{\bigcup_{i \in \alpha} U_i} \cap \overline{\bigcup_{j \in \beta} U_j} = \emptyset \}. $$

First, observe that the set

$$ W = \{ (K, L) \in C(X)^2 : K \cap L \text{ is not connected} \} $$

is $F_\sigma$ in $C(X)^2$. (3.3)
Indeed, for any \((\alpha, \beta) \in T\), the set
\[
W(\alpha, \beta) = \left\{ (K, L) \in C(X)^2: K \cap L \subseteq \bigcup_{i \in \alpha, j \in \beta} (U_i \cup U_j), \quad K \cap L \nsubseteq \bigcup_{i \in \alpha} U_i \neq \emptyset \neq K \cap L \nsubseteq \bigcup_{j \in \beta} U_j \right\}
\]
is the intersection of an open and a closed subsets of \(C(X)^2\), hence it is \(F_\sigma\) and so
\[
W = \bigcup_{(\alpha, \beta) \in T} W(\alpha, \beta)
\]
is \(F_\sigma\).

Since, by (3.3), the set
\[
\{ (K, L) \in C(X)^2: K \cap L \in C(X) \} = C(X)^2 \setminus W
\]
is a \(G_\delta\)-set, too.

Suppose now that \(I^2 \subset X\) and observe that family \(\mathcal{H}(I^2)\) is a \(G_\delta\)-set and dense in \(C(I^2)\), because it contains all arcs in \(I^2\). The complement \(Z\) of \(\mathcal{H}(I^2)\) in \(C(I^2)\) also is dense in \(C(I^2)\), since it contains all simple closed curves in \(I^2\).

If \(\mathcal{H}(X)\) were an \(F_\sigma\)-set, then \(\mathcal{H}(I^2)\) would be a \(F_\sigma\)-set, too. Hence, \(Z\) would be a \(G_\delta\) and dense subset of \(C(I^2)\) disjoint with another dense \(G_\delta\)-set \(\mathcal{H}(I^2)\), which contradicts the Baire category theorem.

\(\blacksquare\)

**Corollary 3.7.** The set \(\Lambda(I^2)\) of all \(\lambda\)-dendroids in \(I^2\), is coanalytic complete.

**Proof.** It follows from the definition of a \(\lambda\)-dendroid that
\[
\Lambda(I^2) = \mathcal{H}(I^2) \cap \mathcal{H}(I^2),
\]
so \(\Lambda(I^2)\) is coanalytic, by [5] and Proposition 3.6.

Since \(\Lambda(I^2)\) contains all countable plane fans and no indecomposable chainable continuum \(Y\), it follows from Corollary 3.4 (or from the construction in [5]) that \(\Lambda(I^2)\) is coanalytic hard.

\(\blacksquare\)

**Corollary 3.8.** The following subsets of \(C(I^2)\) are coanalytic complete:

1. of all continua that admit only arcs (simple closed curves) as chainable (circularly chainable) subcontinua;
2. of all tree-like continua that admit only arcs as chainable subcontinua;
3. of all hereditarily unicoherent continua that admit only arcs as chainable subcontinua;
4. of all continua which contain no pseudo-arc;
5. of all tree-like continua which contain no pseudo-arc.

**Proof.** It follows from Corollary 3.4 that each of the above families \(\mathcal{B}\) is coanalytic hard.

To prove that it is coanalytic, use the fact that \(\mathcal{A}(I^2) = (S(I^2))\) is a true \(F_\sigma\)-set and each
of the following families is a $G_δ$-set in $C(I^2)$: of all circularly chainable continua, $C(I^2)$, $T(I^2)$, $Hd(I^2)$ (Proposition 3.6) and $P(I^2)$. Hence, the set

$$B = \{ C \in T(I^2) : \forall D \in C(I^2) ((D \in C(I^2) \text{ and } D \subset C) \Rightarrow D \in A(I^2)) \}$$

is coanalytic. All other cases can be evaluated similarly. \(\square\)

3.2. Continua without simple closed curves

**Proposition 3.9.** The family of all continua in $I^2$ which contain no simple closed curve is coanalytic complete.

**Proof.** The family $B$ of all continua in $I^2$ that contain no simple closed curves is coanalytic hard by [8, pp. 256–257]. Since the family $S(I^2)$ of all simple closed curves in $I^2$ is $F_\sigma\delta$ and

$$B = \{ C \in C(I^2) : \forall D \in C(I^2) (D \subset C \Rightarrow D \notin S(I^2)) \},$$

the set $B$ is coanalytic. \(\square\)

Corollary 3.4 and Proposition 3.9 imply the following fact.

**Corollary 3.10.** Given an arbitrary nondegenerate circularly chainable continuum $Y$, the class of all continua in $I^3$ which contain no topological copy of $Y$ is coanalytic hard.

In order to get an analogous statement for continua that contain no copy of an arbitrary chainable continuum, we need to have a counterpart of Proposition 3.9 for continua containing no arcs. This requires more complicated constructions which are described in the next section.

4. Continua without arcs

**Theorem 4.1.**

1. The family of all tree-like continua (all continua) in $I^2$ which contain no arcs is coanalytic complete.
2. The family of all tree-like continua (all continua) in $I^2$ that contain no hereditarily decomposable subcontinua is coanalytic hard.

**Proof.** Let $F \in \{ C(I^2), T(I^2) \}$. Then

$$B = \{ C \in C(I^2) : C \in F \text{ and } \forall D \in C(I^2) (D \subset C \Rightarrow D \notin A(I^2)) \}$$

is coanalytic.

To prove $B$ is coanalytic hard, we need the following proposition.
Proposition 4.2. Let $C$ be the standard ternary Cantor set. Suppose $E$ is a countable subset of $C$. There exists a compact space $X$ in the Euclidean plane $\mathbb{R}^2$ and a continuous map $q : \mathbb{R}^2 \to \mathbb{R}^2$ such that

- $q(X) = C \times I$,
- $q^{-1}(c, 0)$ is an arc for $c \in C \setminus E$,
- $q^{-1}(c, 0)$ is an arc of pseudo-arcs for $c \in E$,
- $q^{-1}(x)$ is a singleton for all $x \in \mathbb{R}^2 \setminus C \times \{0\}$, and
- if $p : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first coordinate space, then the map $pq \restriction X : X \to C$ is open.

Proof. Let $E = \{e_1, e_2, \ldots\}$. Let $W$ be an arc of pseudo-arcs in $\mathbb{R}^2$. Consider the canonical decomposition of $W$ into pseudo-arcs $P_t \subset W$, $t \in T$, and the decomposition $D$ of $\mathbb{R}^2$ consisting of all pseudo-arcs $P_t$, $t \in T$, and singletons. It follows from the Moore decomposition theorem that the decomposition space $\mathbb{R}^2/D$ is homeomorphic to $\mathbb{R}^2$. In other words, there exists a continuous surjection $q_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $q_1(W)$ is an arc $A_1$ and for each $a \in A_1$ there is $t \in T$ such that $q_1^{-1}(a) = P_t$, whereas all other point-inverses of $q_1$ are singletons. We can also assume that $A_1 = \{e_1\} \times I$. Suppose continuous surjections $q_1, q_2, \ldots, q_n : \mathbb{R}^2 \to \mathbb{R}^2$ have been defined such that, for $1 < i \leq n$,

- $q_i(W)$ is an arc $A_i = (q_1 \cdots q_{i-1})^{-1}(\{e_i\} \times I)$,
- for each $a \in A_i$ there is $t \in T$ such that $q_i^{-1}(a) = P_t$ and all other point-inverses of $q_i$ are singletons.

Then again, using the Moore decomposition theorem, we define a continuous surjection $q_{n+1} : \mathbb{R}^2 \to \mathbb{R}^2$ such that

- $q_{n+1}(W)$ is the arc $A_{n+1} = (q_1 \cdots q_n)^{-1}(\{e_{n+1}\} \times I)$,
- for each $a \in A_{n+1}$ there is $t \in T$ such that $q_{n+1}(a) = P_t$ and all other point-inverses of $q_{n+1}$ are singletons.

Now, consider the inverse sequence

$$\mathbb{R}^2 \leftarrow q_1 \leftarrow q_2 \leftarrow \cdots$$

whose limit $L$ is homeomorphic to $\mathbb{R}^2$ because the bonding maps are cell-like, so they are near-homeomorphisms [6, p. 189] and we can apply the Brown approximation theorem [3] (to spheres instead of planes). Let

$$X_1 = C \times I \leftarrow X_2 = q_1^{-1}(X_1) \leftarrow X_3 = (q_2)^{-1}(X_2) \leftarrow \cdots$$

be the inverse sequence with bonding maps $q_1 \restriction X_2$, $q_2 \restriction X_3$, $\ldots$. Put $X = \lim(X_n, q_n \restriction X_{n+1})$.

The projection $q : L \to \mathbb{R}^2$ onto the first coordinate space satisfies the conclusion of Proposition 4.2. □
Proceeding with the proof of Theorem 4.1, let $F_1, F_2, \ldots$ be all the components of $I \setminus C$. Choose mutually disjoint pseudo-arcs $P_1, P_2, \ldots$ in $\mathbb{R}^2$ such that $P_n \cap (C \times I) = \emptyset$ and $\operatorname{diam} P_n \leq 0$, for each $n \in \mathbb{N}$, and $\operatorname{diam} P_n = \operatorname{diam} F_n$. Then $M = C \times [0] \cup \bigcup_{n=1}^{\infty} P_n$ is a continuum in $\mathbb{R}^2$ which contains no hereditarily decomposable subcontinuum.

Let us identify the Cantor set $\{0, 1\}^\omega$ with $C$. Put $E = D$ in Proposition 4.2. Then $q^{-1}(M)$ is a tree-like continuum which contains no hereditarily decomposable subcontinuum. Without loss of generality we can assume that $\mathcal{I}^{-1}(M) \subset I^2$. It follows from the construction of $X$ that the map

$$\phi : C \to C(I^2), \quad \phi(c) = q^{-1}\{(c) \times I \cup M\}$$

is continuous. Hence, the map

$$\Phi : K(C) \to C(I^2), \quad \Phi(A) = \bigcup_{c \in A} \phi(c)$$

is a continuous reduction of $K(D)$ to $B$ as well as to the family of all tree-like continua (all continua) in $I^2$ which contain no hereditarily decomposable subcontinua.

**Theorem 4.3.** The family of all $\lambda$-dendroids (all hereditarily decomposable continua) in $I^2$ which contain no arcs is coanalytic complete.

**Proof.** Let $\mathcal{F} \in \{A(I^2), \mathcal{H}(I^2)\}$. The family

$$\mathcal{B} = \mathcal{F} \cap \{C \in C(I^2) : \forall D \subset C \implies D \notin A(I^2)\}$$

is coanalytic, since $\mathcal{F}$ is coanalytic (Corollary 3.7, Proposition 3.6) and $A(I^2)$ is Borel.

In order to prove that $\mathcal{B}$ is coanalytic hard we will use the following proposition by Maćkowiak [14, p. 8]:

**Proposition 4.4.** If $X$ is a continuum, $M$ is a compactum, $A$ is a 0-dimensional compact subset of $X$ and $A$ is a decomposition space of $M$ into components, then there exist a continuum $\hat{X}$ containing $M$ and an atomic surjection $r : \hat{X} \to X$ such that $r \upharpoonright \hat{X} \setminus M$ is a homeomorphism onto $X \setminus A$.

First, we describe a construction of a Janiszewski hereditarily decomposable chainable continuum $J$ without arcs.

Let $S_0 = I$ and a dense subset $D_0 = \{d^0_1, d^0_2, \ldots\} \subset \text{int } I$ be given. Substitute in Proposition 4.4: $X = S_0$, $A = \{d^0_1\}$, $M = M_0 = I$ and get a chainable continuum $S_1 = \mathcal{S}_0$ and an atomic surjection $r_0 : S_1 \to S_0$ with $r_0 \upharpoonright S_1 \setminus M$ a homeomorphism onto $S_0 \setminus A$. Put $D_1 = r_0^{-1}(D_0 \setminus \{d^0_1\}) = \{d^1_1, d^1_2, \ldots\}$ and let $D_1 = \{d^1_1, d^1_2, \ldots\}$ be a dense subset of $\text{int } M_0$. Inductively, suppose chainable continua $S_k$, disjoint arcs $M_{\{i_1, \ldots, i_k\}}$, atomic surjections $r_{k-1} : S_k \to S_{k-1}$, and subsets $D_{\{i_1, \ldots, i_k\}} = \{d_{i_1, \ldots, i_k}^1, d_{i_1, \ldots, i_k}^2, \ldots\}$ are defined for $1 \leq k \leq n$, where $i_1, \ldots, i_k \in \{0, 1\}$. Then, in Proposition 4.4, substitute

$$X = S_n,$$

$$A = \bigcup_{i_1, \ldots, i_n \in \{0, 1\}} \{d_{i_1, \ldots, i_n}^1\},$$
\[
M = \bigcup_{i_1, \ldots, i_n \in [0,1]} M_{(i_1, \ldots, i_n)}
\]

and get \( S_{n+1} = \tilde{S}_n \) with an atomic surjection \( r_n : S_{n+1} \to S_n \) such that \( r_n \mid S_n+1 \setminus M \) is a homeomorphism onto \( S_n \setminus A \). Put

\[
D_{(i_1, \ldots, i_n)} = r_n^{-1}(D_{(i_1, \ldots, i_n)} \setminus \{d^1_{(i_1, \ldots, i_n)}, d^2_{(i_1, \ldots, i_n)}\}) = \{d^1_{(i_1, \ldots, i_n)}, d^2_{(i_1, \ldots, i_n)}\}
\]

and choose a dense subset

\[
D_{(i_1, \ldots, i_n)} = \{d^1_{(i_1, \ldots, i_n)}, d^2_{(i_1, \ldots, i_n)}\}
\]

of \( \text{int} M_{(i_1, \ldots, i_n)} \).

The Janiszewski continuum is now defined as the inverse limit \( J = \lim_{\leftarrow} (S_n, r_n) \). It is well-known that \( J \) is a non-decomposable continuum without arcs. If \( r : J \to S_0 = I \) denotes the projection, then it follows from the construction that \( J \) is irreducible between points \( r^{-1}(0) \) and \( r^{-1}(1) \).

We are going to construct an auxiliary chainable continuum \( H \) as the inverse limit \( \lim \{X_n, r_n'\} \) of chainable continua \( X_0, X_1, \ldots \) in the following way. As in the proof of Proposition 4.2, let \( C \) be the standard ternary Cantor set and let \( F_1, F_2, \ldots \) be the sequence of all components of \( I \setminus C \) such that \( F_1 \) is the open segment \((\frac{1}{3}, \frac{2}{3}) \), and \( F_2, F_3, \ldots \) are two open segments of length \( \frac{1}{2^n} \), \( F_4, \ldots, F_7 \) are of length \( \frac{1}{2^n} \), etc. Consider a chainable continuum \( X_0 \) which is the union of \( C \times I \) and all segments of the form \( F_j \times \{ j \} \), where \( j = 1 \) if \( F_j \) is of length \( \frac{1}{2^n} \) for \( m \) even, and \( j = 0 \) for an odd \( m \) (see [12, p. 191]).

Suppose \( E = \{e_0, e_1, \ldots \} \subset C \). We now mimic the construction of \( J \) on each fiber \( \{e_n\} \times I \) of the Cantor bundle \( C \times I \subset X_0 \). In Proposition 4.4, take \( X = X_0 \), \( M = M_0 \), \( A = \{\{e_0, d^1_{(0)}\}\} \); then there is a chainable continuum \( X_1 = \tilde{X} \) and an atomic surjection \( r_0' : X_1 \to X_0 \) with \( r_0' \mid X_1 \setminus M \) a homeomorphism onto \( X_0 \setminus A \). Put \( D_0 = (r_0')^{-1}(D_0 \setminus \{d^1_{(0)}\}) = \{d^1_{(0)}, d^2_{(0)}\} \) and let \( D_1 = \{d^1_{(0)}, d^2_{(0)}\} \) be a dense subset of \( \text{int} M_0 \).

Inductively, assume chainable continua \( X_k \), arcs \( M_{(i_1, \ldots, i_k)} \), atomic surjections \( r_{k+1}' : X_k \to X_{k-1} \) are defined for \( 1 \leq k \leq n \), where \( i_1, \ldots, i_k \in [0,1] \). In Proposition 4.4, substitute \( X = X_n \),

\[
A = \bigcup_{k=0}^n \left( \{e_k\} \times \bigcup_{i_1, \ldots, i_{k-1} \in [0,1]} \{d^1_{(i_1, \ldots, i_{k-1})}\} \right),
\]

\[
M = \bigcup_{k=0}^n \left( \{e_k\} \times \bigcup_{i_1, \ldots, i_{k-1} \in [0,1]} M_{(i_1, \ldots, i_{k-1})} \right)
\]

(assume \( i_0 = \emptyset \)) and take \( X_{n+1} = \tilde{X}_n \), with an atomic surjection \( r_{n+1}' : X_{n+1} \to X_n \) such that \( r_{n+1}' \mid X_n+1 \setminus M \) is a homeomorphism onto \( X_n \setminus A \).

The continuum \( H \) is now defined as \( H = \lim(X_n, r_n') \). One can easily see that \( H \) is an hereditarily decomposable chainable continuum. Since chainable continua are planar [2], we can assume that \( H \subset I^2 \). Put \( Z = (r')^{-1}(C \times I) \), where \( r' : H \to X_0 \) is the projection. Observe that, for \( c \in C \), the preimage \( (r')^{-1}([c] \times I) \) is a Janiszewski continuum if \( c \in E \), otherwise it is an arc. One can view the set \( Z \) as a Cantor bundle of arcs and countably many
Janiszewski continua. Notice that the decomposition of $Z$ into components is continuous; in other words, the mapping

$$\phi: C \to K(I^2), \quad \phi(c) = (r')^{-1}((c) \times I)$$

is continuous. The set $C' = (r')^{-1}(C \times \{0\})$ is a Cantor set in $I^2$, so there is an arc $\alpha \subset I^2$ containing $C'$ with end-points in $C'$ [12, p. 539]. Let $G_1, G_2, \ldots$ be all the components of $\alpha \setminus C'$. For each $n \in \mathbb{N}$, replace $G_n$ with a Janiszewski continuum $J_n \subset I^2$ such that $J_n \cap Z = \partial G_n$, $\text{diam} G_n$ and $J_1, J_2, \ldots$ are mutually disjoint. Then $B = C' \cup \bigcup_{n=1}^{\infty} J_n$ is a chainable hereditarily decomposable continuum in $I^2$ which contains no arc.

The continuum $H' = B \cup Z$ is a $\lambda$-dendroid in $I^2$. If $E = D$, then the mapping

$$\Phi: K(C) \to C(I^2), \quad \Phi(K) = B \cup \bigcup_{c \in K} \phi(c) \subset H'$$

is a continuous reduction of the Hurewicz set $K(D)$ to $B$. \hfill \Box

The following fact is a consequence of Corollary 3.4 and Theorem 4.1.

**Corollary 4.5.** Given an arbitrary nondegenerate chainable continuum $Y$, the class of all continua in $I^2$ which contain no topological copy of $Y$ is coanalytic hard.

### 5. Remarks on non-existence of models

A model for a class $C$ of continua is a continuum $M$ such that each member of $C$ is a continuous image of $M$ (usually, in continuum theory, we do not require that $M \in C$).

As an interesting application of the coanalytic completeness of $\mathcal{HD}(I^n)$, it is shown in [5] that all arcwise connected continua in $I^n$ have no arcwise connected model. It should, however, be noted that this fact has been known in a much more general form since long. Krasinkiewicz and Minc [9] proved that there is no hereditarily decomposable model for planar countable fans. Russo [16] showed that there is no model for arcwise connected continua and planar $\lambda$-dendroids. The question of whether there is a model for the class of all planar arcwise connected continua remains open. A good reference for these and other results, containing alternative proofs, is [15].

The argument given in [5] can easily be extended to get another proof of the Krasinkiewicz and Minc theorem as follows.

**Theorem 5.1** [9]. The classes of countable fans and of countable clumps of circles in $I^2$ with vertices have no hereditarily decomposable models.

**Proof.** We refer to Corollary 3.4 and the proof of Theorem 3.1. Let $B$ be the class of all countable fans (countable clumps of circles with vertices) in $I^2$ and suppose an hereditarily decomposable continuum $M$ is a model for $B$. Assume in Corollary 3.4 that the continuum $Y$ is indecomposable and planar. Let $\Phi: K([0, 1]^\omega) \to C(I^2)$ be the continuous reduction of $K(D)$ to $B$ described in the proof of Theorem 3.1 and related to the context
of Corollary 3.4. Put $R = \Phi(K\{0,1\}^\omega)$. The set $M$ of all continuous images of $M$ in $I^2$ is an analytic subset of $C(I^2)$.

We are going to show that $B \cap R = M \cap R$. (5.1)

Suppose that $g(M) \in R \setminus B$ for a continuous map $g : M \to I^2$. Then $g(M)$ is a clump of arcs (simple closed curves) and of at least one copy $Y'$ of $Y$, with vertex $z_0$. Since a continuous image of an hereditarily decomposable continuum cannot be indecomposable [12, p. 208], $Y' \neq g(M)$. Let $S$ be a composant of $Y'$ which does not contain $z_0$. Choose a point $p \in M$ such that $g(p) \in S$. Denote by $M_0$ the component of $g^{-1}(Y')$ containing $p$.

We will show that $g(M_0) = Y'$. Suppose not. Then $g(M_0) \subseteq S$. Let $U$ be an open subset of $g(M)$ such that $z_0 \notin U$ and $g(M_0) \subseteq U$. It follows from the continuity of $g$ that there is an open subset of $M$ such that $g(\text{cl} V) \subseteq U$ and $M_0 \subseteq V$. Then there is a continuum $C$ such that $M_0 \subseteq C \subseteq \text{cl} V$ (see [12, p. 173]). Hence $C \subseteq g^{-1}(Y')$, which is a contradiction.

Since $g(M_0) = Y'$ is nondegenerate indecomposable, $M_0$ contains a nondegenerate indecomposable subcontinuum by [12, p. 208], which is impossible.

The inclusion $B \cap R \subseteq M \cap R$ is clear.

Now, since $R$ is compact, the set $M \cap R$ is analytic and, by (5.1), we have that $B \cap R$ is analytic. On the other hand, map $\Phi$ reduces the Hurewicz set $K(D)$ to $B \cap R$, so $B \cap R$ is coanalytic hard and cannot be analytic, a contradiction.

Acknowledgements

The author thanks J.J. Charatonik for his remarks concerning a construction in the proof of Theorem 4.3, J.R. Prajs for suggesting a proof of Proposition 4.2, S. Solecki for indicating that strongly countable-dimensional continua form a coanalytic family and the referee for all remarks that improved the final version of the paper.

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