Synchrotron radiation from massless charge

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ABSTRACT

Classical radiation power from an accelerated massive charge diverges in the zero-mass limit, while some authors suggest that strictly massless charge does not radiate at all. On the other hand, the regularized classical radiation reaction force, though looking odd, is non-zero and finite. To clarify this controversy, we consider radiation problem in massless scalar quantum electrodynamics in the external magnetic field. In this framework, synchrotron radiation is found to be non-zero, finite, and essentially quantum. Its spectral distribution is calculated using Schwinger's proper time technique for *ab initio* massless particle of zero spin. Provided $E^2 \gg eH$, the maximum in the spectrum is shown to be at $\hbar\omega = E/3$, and the average photon energy is $4E/9$. The normalized spectrum is universal, depending neither on E nor on H. Quantum nature of radiation makes classical radiation reaction equation meaningless for massless charge. Classical theory is reliable only as providing the low-frequency part of the true quantum radiation spectrum.

1. Introduction

Recently the problem of radiation from massless charges attracted attention in the framework of classical electrodynamics. It was argued that the line singularity of the Lienard–Wiechert potential for the massless charge forbids using the Green function method to compute radiation [1,2], while, according to [3], the conservation equation for Maxwell energy–momentum tensor implies that the massless charge does not radiate at all [3]. The similar assertion was earlier formulated by Kosyakov [4] basing on conformal invariance of classical electrodynamics with massless charges (see also [5]). On the other hand, a non-zero and finite expression has been derived [6,3] for the regularized radiation reaction force acting upon an accelerated massless charge. Finally, both these alternatives seem to disagree with divergence of the classical formulas for radiation power from an accelerated massive charge [7] in the limit of zero mass. To clarify this confusion, one is led to consider the radiation problem from quantum viewpoint.

In quantum electrodynamics the limit of zero mass also has peculiar features. First, apart from usual infrared divergencies, there are *collinear singularities*, occurring when photon is emitted from massless legs in the Feynman diagrams in the direction of the charge momentum [8]. This is manifestation of degeneracy of states of the charge and the photon moving along the same line. Elimination of collinear divergences is achieved using Kinoshita–

Lee–Nauenberg [9–11] prescription to average over an ensemble of degenerate states (for review and further references see [12, 13]). Note that the above mentioned line singularities of classical retarded potentials look as classical counterparts of collinear singularities of quantum theory. Their regularization was proposed in [3] by imposing condition that the potentials should be defined as distributions, the idea reminiscent of the quantum averaging.

Second, vacuum polarization was shown to induce screening of the massless charges created in high-energy processes [14–16]. However, this screening occurs at very large distances of the order $L \sim E^{-1} \exp(3\pi/\alpha)$ [17], where α is the fine structure constant. It is therefore legitimate to consider processes whose formation length is of the order of E^{-1} (which turns out to be our case). Anyway, though peculiarities of massless QED were often interpreted as manifestation of inconsistencies of the theory or proof of non-existence of massless charges [14,15], the prevailing current opinion is that massless QED is special, but viable theory [18,17,19]. Moreover, massless QED in the *external magnetic field* attracted much attention during two past decades in connection with prediction of the magnetic catalysis of the chiral symmetry breaking [20], and other interesting phenomena relevant to solid state physics and cosmology [21]. These effects are associated with dynamics at low Landau levels in the magnetic field. Here, exploring synchrotron radiation, we will deal with high excitation levels. Note that, once the magnetic field is treated non-perturbatively, the charges are never free, so in this approach there are no collinear singularities in the radiation amplitudes.

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We will show that the theory predicts finite synchrotron radiation from massless charges at the quantum level. We will also investigate whether the zero mass limit of the radiation amplitudes obtained within the massive theory match with genuine massless ones. We find that, unlike the situation in classical theory, the massless limit for synchrotron radiation in quantum theory is smooth in the quasiclassical (high Landau levels) regime.

2. Classical theory with quantum cut-off

Classical formula [7] for the total radiation power emitted by a massive charge moving with the energy E along the circle in the magnetic field H,

$$P_{\rm cl} = \frac{2e^4 H^2}{3m^2}\left(\frac{E}{m}\right)^2,\tag{1}$$

diverges in the limit $m \to 0$. To clarify the origin of this divergence let us pass to the spectral-angular distribution. The famous Schott formula

$$dP = \sum_{\nu=0}^{\infty}\frac{e^2\nu^2\omega_H^2}{2\pi}\left[\tan^2\theta\, J_\nu^2(\nu\beta\cos\theta) + \beta^2 J_\nu'^2(\nu\beta\cos\theta)\right]d\Omega,\tag{2}$$

where $E = m/\sqrt{1-\beta^2}$, shows that the frequency of radiation is discrete and consists of the harmonics of the relativistic Larmor frequency:

$$\omega = \nu\omega_H,\quad \omega_H = \frac{eH}{E},\tag{3}$$

and by virtue of the properties of Bessel functions, the effective domain of ν extends to

$$\nu \lesssim \nu_{\rm cr} \sim (1-\beta^2)^{-3/2} = (E/m)^3.\tag{4}$$

The Schott formula itself does not depend on m and admits the limit of the velocity of light $\beta \to 1$. But in the massless theory there will be no frequency cutoff (4), and this is the reason why the total power (1) diverges in this case. Therefore, for the massless charge, the cutoff frequency should necessarily be quantum, $\hbar\omega_{\max} = E$, or $\nu_{\max} = E^2/eH$ (assuming $eH > 0$). This quantum cutoff does not depend on the mass m either and nicely fits with the quantization rule for purely transverse motion [22]: $E^2 = E_n^2 = m^2 + eH(2n+1)$, $n = 0,1,2,\ldots$, provided both initial and final level numbers n, n' are large and $\nu = n - n' \ll n$.

Integrating (2) over angles and passing to continuous frequency distribution $P = \int P(\omega)\,d\omega$, one can show that the low frequency limit of $P(\omega)$ is also mass-independent:

$$P_{\rm low}(\omega) = \frac{e^2\omega_H 3^{1/6}\Gamma(2/3)}{\pi}\left(\frac{\omega}{\omega_H}\right)^{1/3},\tag{5}$$

so we can integrate it up to $\omega_{\max} = E/\hbar$ to get an estimate for the total radiation power:

$$P_{\rm cut} = \frac{e^2\sqrt{3}\Gamma(2/3)}{4\pi\hbar^2}(3e\hbar HE)^{2/3}.\tag{6}$$

This quantity is finite, but contains a factor $\hbar^{-4/3}$, showing that radiation is essentially quantum.

3. Mass operator for $m = 0$

Quantum theory of synchrotron radiation has different formulations. Historically, the first was a direct approach making use of exact solutions of the wave equations in magnetic field [22] (for a more recent review see [23]). Later on Schwinger suggested the "proper time" method to calculate the mass operator of the charge in the constant field $F_{\mu\nu}$ for spins $s = 0, 1/2$; its imaginary part gives the total probability rate of synchrotron radiation [24,25]. Further ramifications of this approach allow to get spectral and spectral-angular distributions of radiation [26] (for alternative constructions of the mass operator in constant electromagnetic field see [27,28]). Here we apply the technique of [24,25] for zero-spin charged particle of strictly zero mass. The massless limit of the corresponding massive theory will be discussed in the next section.

The action term involving the mass operator for the complex scalar field $\phi(x)$ is

$$-\frac{1}{2}\int \phi(x)M(x,x')\phi(x')dxdx',\tag{7}$$

where $M(x,x')$ in Schwinger's operator notation reads:

$$M = ie^2\int\left[(\Pi - k)^\mu\frac{1}{k^2}\frac{1}{(\Pi-k)^2}(\Pi-k)_\mu\right]\frac{dk}{(2\pi)^4} - M_0.\tag{8}$$

Here $\Pi_\mu = -i\partial_\mu - eA_\mu$, A_μ denotes the constant magnetic filed, and M_0 is the subtraction term needed to ensure vanishing of M and its first derivative with respect to Π^2 at $\Pi^2 = 0$. It consists of the first two Taylor expansion terms of M in Π^2 for $F = dA = 0$. The idea of the proper time method is to perform exponentiation of propagators

$$\frac{1}{k^2}\frac{1}{(\Pi-k)^2} = -\int_0^\infty sds\int_0^1 e^{-is\mathcal{H}},$$

$$\mathcal{H} = (k - u\Pi)^2 - u(1-u)\Pi^2,\tag{9}$$

and to replace the k-integration by averaging over the eigenstates of the operator ξ_μ, canonically conjugate to k_μ, $[k_\mu, \xi_\nu] = i\eta_{\mu\nu}$ in the Hilbert space of a fictitious particle:

$$M = ie^2\int_0^\infty sds\int_0^1 du\langle\xi|(\Pi-k)^\mu e^{-is\mathcal{H}}(\Pi-k)_\mu|\xi\rangle - M_0.\tag{10}$$

The quantity $\mathcal{H}$ is then treated as Hamiltonian of this particle and the averaging is performed in the Heisenberg picture passing to s-dependent operators $k(s)$, $\xi(s)$, $\Pi(s)$, which can be found exactly in terms of F. Performing calculations and comparing the results with formulas given in [25] for the massive theory we find that one can start with Eq. (40) of Tsai [25] setting there $m = 0$. The subsequent computations in [25] use approximations of the integrals in terms of the Macdonald functions. These approximations become singular as $m \to 0$, so we develop an alternative integration scheme.

As in [25], our starting formula is valid for the *on shell* mass operator, i.e. for $\phi(x) = \phi(\mathbf{r})e^{-iEt}$ satisfying the wave equation $\Pi^2\phi = 0$, with discrete eigenvalues of the energy (we consider purely transverse motion)

$$E = \sqrt{eH(2n+1)}.\tag{11}$$

It reads:

$$M = \frac{e^2}{4\pi} \int_0^1 du \int_0^\infty \frac{ds}{s} \left[e^{-i\psi} \Delta^{-1/2} \left(E^2 \Phi_1 + 4ieH\Phi_2 + i\Phi_3/s \right) \right.$$
$$\left. - 2i/s \right], \tag{12}$$

with

$$\Phi_1 = 3 - 4u + u^2 - \frac{(1-u)^2}{\Delta}(4\cos 2x - 1) - \frac{u(1-u)}{x\Delta}\sin 2x,$$

$$\Phi_2 = \sin 2x - \frac{2u(1-u)\sin^2 x}{x\Delta}\cos 2x,$$

$$\Phi_3 = 1 + \frac{1-u}{\Delta}(2\cos 2x - 1) + \frac{u\sin 2x}{2x\Delta}(4\cos 2x - 3), \tag{13}$$

where $x = eHsu$, and

$$\Delta = (1-u)^2 + u(1-u)\frac{\sin 2x}{x} + u^2 \left(\frac{\sin x}{x}\right)^2,$$

$$\psi = (2n+1)[\beta - (1-u)x], \quad \tan \beta = \left(\cot x + \frac{u}{x(1-u)}\right)^{-1}.$$

This expression is valid for all Landau levels n.

In what follows we will be interested in the case $n \gg 1$, when the integrals can be evaluated expanding all x-dependent quantities in power series. Indeed, the main contribution to the integral over x comes from the region where the phase $\psi(x, u) \lesssim 1$, in which β for $x \ll 1$ can be approximated as

$$\beta \approx (1-u)x + u(1-u)^2 x^3/3, \tag{14}$$

so that

$$\psi \approx (2n+1)\alpha x^3 = \frac{E^2}{eH}\alpha x^3 = \frac{s^3}{3}(eHE)^2 u^4 (1-u)^2,$$
$$\alpha = u(1-u)^2/3. \tag{15}$$

For large n, apart form the narrow regions around the limiting points of u,

$$u > n^{-1}, \quad 1 - u > n^{-1/2}, \tag{16}$$

which *a posteriori* give negligible contributions, the essential domain of x is

$$x \lesssim n^{-1/3}. \tag{17}$$

Therefore we use (15) in the exponent, expanding the other functions in powers of x[1]:

$$\Delta^{-1} \approx 1 + u(4 - 3x)x^2/3, \tag{18}$$

$$\Phi_1 \approx \left(8 - 32u/3 + 13u^2/3 - u^3 \right)(1-u)x^2, \tag{19}$$

$$\Phi_2 \approx 2(1 - u + u^2)x, \tag{20}$$

$$\Phi_3 \approx 2 - (4 - 10u/3 + u^2)x^2. \tag{21}$$

Using the bound (17) and taking into account different orders of various terms in n, we find that the contribution of Φ_2 will be of the order of $n^{-1/3}$ with respect to the leading term Φ_1, while in Φ_3 one has to keep only the zero order term. Introducing the decay rate via

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[1] The initial integral (12), (13) converges at the upper limit $x \to \infty$, while higher terms in Taylor expansion of the integrand will produce divergent quantities, what is typical for asymptotic series. We will keep only the leading terms giving the convergent integrals over x.

$$\Gamma = -\frac{1}{E}\operatorname{Im}M, \tag{22}$$

we obtain

$$\Gamma = \frac{e^2}{4\pi E}\int_0^1 du \int_0^\infty \frac{dx}{x}\left(E^2\Phi_1 \sin\psi + 2\frac{eHu}{x}(1 - \cos\psi)\right), \tag{23}$$

where for ψ one has to use (15), and for Φ_1 — the approximation (19). Taking into account the table integrals

$$\int_0^\infty \sin(z^3)\, z\, dz = \frac{\Gamma(2/3)}{2\sqrt{3}}, \tag{24}$$

$$\int_0^\infty \left[1 - \cos(z^3)\right]\frac{dz}{z^2} = \frac{\sqrt{3}\,\Gamma(2/3)}{2}, \tag{25}$$

we get

$$\Gamma = \frac{e^2 \Gamma(2/3)(3eHE)^{2/3}}{8\pi\sqrt{3}E}\int_0^1 \frac{8 - 32u/3 + 19u^2/3 - 3u^3}{u^{2/3}(1-u)^{1/3}}\,du. \tag{26}$$

The integrand has an integrable singularity at the lower limit. Integrating, we finally obtain the total decay rate

$$\Gamma = \frac{4e^2}{9E}\Gamma(2/3)(3eHE)^{2/3}. \tag{27}$$

With account for radiative decay, the energy levels of a charge in the magnetic field are no longer stationary, but quasi-stationary, provided the level spacing $\Delta E = E_n - E_{n-1} \approx (eH/2n)^{1/2}$ is much larger than $\Gamma/2$. This leads to an upper bound for the Landau level number n:

$$n \ll \frac{81}{16\,[\Gamma(2/3)]^3}\frac{1}{\alpha^3} \approx 0.5 \cdot 10^7, \tag{28}$$

where $\alpha = 1/137$. Thus our calculation is consistent under the following conditions for the energy:

$$eH \ll E^2 \ll 10^7 eH, \tag{29}$$

which leaves enough space of validity, though one has to keep in mind that our formulas are valid in the leading order in $n^{-1/3}$.

4. The spectral power

To get the spectral power of radiation one has to perform Fourier expansion in the mass operator, leading to

$$\operatorname{Im}M = \operatorname{Im}\left(\int d\omega \int_{-\infty}^\infty e^{i\omega\tau} M' \frac{d\tau}{2\pi}\right), \tag{30}$$

where

$$M' = -ie^2 \int_0^\infty s\,ds \int_0^1 du \langle\xi|(\Pi - k)^\mu e^{-is\mathcal{H}} e^{-ik^0\tau}(\Pi - k)_\mu|\xi\rangle$$
$$- M'_0. \tag{31}$$

The spectral power $P(\omega)$ is then introduced via the relation

$$\Gamma = \int P(\omega)\frac{d\omega}{\omega}, \tag{32}$$

so one obtains:

$$P(\omega) = -\frac{\omega}{E} \, \mathrm{Im} \left(\int_{-\infty}^{\infty} e^{i\omega\tau} \, M' \, \frac{d\tau}{2\pi} \right) . \tag{33}$$

With this modification we find:

$$P(\omega) = -\frac{e^2}{4\pi} \frac{\omega}{E} \, \mathrm{Im} \int_0^{\infty} \frac{ds}{s} \int_0^1 du \left[\frac{E^2 x^2}{3} e^{-i\psi} \right.$$
$$\times \left(24 - 56u + 45u^2 - 16u^3 + 3u^4 \right)$$
$$\left. + \left(e^{-i\psi} - 1 \right) \left(\frac{2i}{s} - \frac{i(2-u)}{s} \frac{d}{du} \right) \right] J, \tag{34}$$

where the non-leading terms were omitted, and J denotes the integral over τ:

$$J(\omega, u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\omega - uE)\tau - i\tau^2/4s} d\tau . \tag{35}$$

For large E, this integral is formed in the region $uE\tau \lesssim 1$, where the quadratic factor in the exponent has the order

$$\frac{\tau^2}{4s} \sim \frac{eH}{4uE^2 x} = \frac{1}{4(2n+1)ux} . \tag{36}$$

For $n \gg 1$, the effective domain in the x-integral is $x \lesssim n^{-1/3}$, thus the whole factor has the order $n^{-2/3}$. Omitting this term in the exponent, we obtain the delta-function

$$J(\omega, u) \approx \delta(\omega - uE) , \tag{37}$$

which is then used to integrate over u. The derivative term is evaluated by parts. Differentiating the phase ψ, one has to take into account that the independent variables are (s, u), so one has to use for ψ the last expression in (15) giving

$$\partial_u \psi = \frac{2E^2 x^3}{3eH} (2 - 3u)(1 - u) . \tag{38}$$

Then, denoting $v = \omega/E$, we obtain

$$P(\omega) = \frac{e^2 v}{4\pi E} \int_0^{\infty} \left(E^2 (8 - v^2)(1 - v)^2 x \sin\psi \right.$$
$$\left. + \frac{eHv}{x^2} (1 - \cos\psi) \right) dx . \tag{39}$$

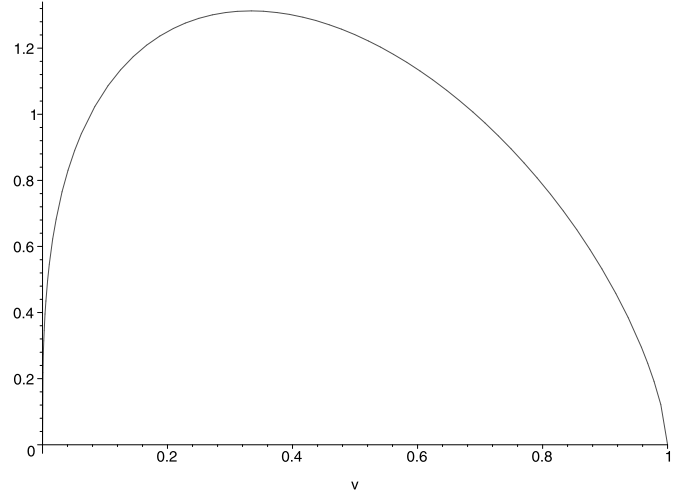The integral over x is evaluated as before, and finally we get

$$P(\omega) = \frac{2e^2 \, \Gamma(2/3)}{27\hbar E} \, (3e\hbar HE)^{2/3} \, \mathcal{P}(\hbar\omega/E) , \tag{40}$$

where the Planck constant is restored, and the normalized spectral function is introduced

$$\mathcal{P}(v) = \frac{27}{2\pi\sqrt{3}} \, v^{1/3} (1 - v)^{2/3} , \qquad \int_0^1 \mathcal{P}(v) \, dv = 1 . \tag{41}$$

This spectrum, shown in Fig. 1, is perfectly smooth function, whose maximum lies at

$$\hbar\omega_{\max} = \frac{1}{3} E . \tag{42}$$

Fig. 1. The universal normalized spectral distribution $\mathcal{P}(v)$ (40) of synchrotron radiation from massless scalar charge with the energy E satisfying $E^2 \gg eH$ versus $v = \hbar\omega/E$. The curve has maximum at $v = 1/3$, the average photon energy $\hbar\omega = 4E/9$.

The average photon energy is

$$\langle \hbar\omega \rangle = E \int_0^1 v \mathcal{P}(v) \, dv = \frac{4}{9} E . \tag{43}$$

The total energy loss per unit time,

$$P = \int_0^{E/\hbar} P(\omega) d\omega = \frac{2e^2 \, \Gamma(2/3)}{27\hbar^2} \, (3e\hbar HE)^{2/3} , \tag{44}$$

differs from the estimate (6) only by a numerical coefficient.

Synchrotron radiation of the massless charge is therefore purely quantum, its intensity being divergent when $\hbar \to 0$. It consists of energetic quanta of the order of the charge energy. The spectral distribution of radiation in the leading order in inverse quantum number $1/n$ is given by a universal formula which does not depend on the magnetic field at all. So even in a weak magnetic field the massless charge converts its energy into quanta of the same order of energy.

5. Massless limit of the massive theory

Transition to the massless limit in the quantum theory of synchrotron radiation of massive charge is subtle, since the results of the latter depend on two dimensionless parameters (in units $\hbar = c = 1$):

$$\eta = \frac{H}{H_0} = \frac{eH}{m^2} , \qquad \chi = \frac{H}{H_0} \frac{E}{m} = \frac{eHE}{m^3} , \tag{45}$$

which both diverge as $m \to 0$. This is why we started by considering radiation in the massless theory *ab initio*. Now we are going to discuss the massless limit in some results of the massive theory which are known analytically.

Simple closed formulas for the spectral power and the total intensity of synchrotron radiation from scalar charge exist for $\eta \ll 1$ and arbitrary χ [29,30]. This case corresponds to transitions between Landau levels $n \gg 1$ and $n' \gg 1$ giving the dominant contribution in this case. Sub-expansions in terms of η were explored in [31–33]. It was shown that for large χ the leading term remains dominant provided $\chi \gg \eta$ even if η itself is not very small. Note,

that the two parameters (45) have different orders in $1/m$, so that the ratio $\eta/\chi = m/E$ tends to zero in the massless limit. Therefore, one can hope to get sensible results using formulas obtained in the case $\eta \ll 1$.

The total probability and the total power of radiation in the massive $\eta \ll 1$ case are known to be mass-independent in the so-called ultra-quantum limit $\chi \gg 1$ [33].[2] Closer look reveals that they are given *precisely* by our formulas (27) and (44) obtained in the strictly massless theory. Thus we have proven that even in the case $\eta \to \infty$ the leading term in χ of the massive theory remains untouched if $\chi/\eta \to \infty$ as well. Moreover, the *spectral distribution* of synchrotron radiation in the massive quantum theory obtained for $\eta \ll 1$ also tends to our universal power spectrum (41). Indeed, the spectral distribution found in [29,30,25] for $\eta \ll 1$ and arbitrary χ in our notation reads:

$$P_{\mathrm{mass}}(\omega) = \frac{e^2 m^2 \omega}{\sqrt{3}E^2} \int_y^\infty K_{5/3}(\xi)d\xi, \quad y = \frac{2\omega}{3(E-\omega)\chi}, \quad (46)$$

where $K_{5/3}$ is the Macdonald function. In the limit $m \to 0$ one has $\chi \to \infty$ and $y \to 0$, and, using l'Hopital rule

$$\int_y^\infty K_{5/3}(\xi)d\xi \sim 2^{2/3}\Gamma(2/3)\, y^{-2/3}, \quad (47)$$

we see that the spectral power (46) reduces to (40), (41) indeed. Thus our calculations in the preceding section prove that the standard quasiclassical approximation derived in the massive theory for $\eta \ll 1$ and any χ remains valid in the ultra-quantum limit $\chi \gg 1$ not only for finite η, but also for $\eta \to \infty$, provided $\chi/\eta \to \infty$ as well.

Similarly, using the results of [30], one can perform transition to zero mass in the amplitudes for emission of photons with different polarizations. One finds that the linear polarization in the plane orthogonal to magnetic field P_σ prevails above the orthogonal component P_π:

$$P_\sigma = 3P_\pi. \quad (48)$$

6. Magnetically induced m^2

In scalar massless QED it is natural to consider magnetic generation of the *square* of mass m^2, since just this quantity enters the Klein–Gordon equation. In the definition (7) the mass operator has dimensionality of m^2, so its real part gives correction δm^2. This quantity is finite for $m = 0$. Had we extracted (as in [28,32]) the linear quantity δm instead, we would get an infinite result: since $\delta m^2 = 2m\,\delta m$, the linear correction $\delta m = \delta m^2/(2m)$ diverges as $m \to 0$. Thus δm is meaningless in the massless case.

Keeping only the leading terms in the real part of (12) we find:

$$\delta m^2 = \mathrm{Re}M = \frac{e^2}{4\pi} \int_0^1 du \int_0^\infty \frac{dx}{x}\left(\cos\psi\,\Phi_1 + 2\sin\psi\,\frac{eHu}{x}\right). \quad (49)$$

Using the integrals

$$\int_0^\infty \cos(z^3)\,z\,dz = \frac{1}{6}\Gamma(2/3), \quad (50)$$

$$\int_0^\infty \sin(z^3)\frac{dz}{z^2} = \frac{1}{2}\Gamma(2/3), \quad (51)$$

we obtain:

$$\begin{aligned}\delta m^2 &= \frac{e^2}{4\pi} \int_0^1 \frac{8 - 32u/3 + 19u^2/3 - 3u^3}{u^{2/3}(1-u)^{1/3}}du \\ &= \frac{4e^2\,\Gamma(2/3)}{9\sqrt{3}}(3eHE)^{2/3}.\end{aligned} \quad (52)$$

This expression, non-perturbative in H, is valid for any H, provided the condition (29) holds.

7. Stochastic nature of radiation reaction force

Recently radiation reaction problem for massless charge attracted some attention. In [6] the closed formula for classical reaction force was derived including three divergent terms. Comparing these results with those known in the massive case, one notices several strange features. First, there is no intrinsic parameters which could absorb divergencies, like mass in the massive theory, while the number of divergent terms is increased from one to three. Second, there are divergent terms of non-lagrangian nature which cannot be incorporated into the action. Third, the finite part of the reaction force contains quite a high derivative (fifth), and no examples are known to compare it with the radiation power. Finally, the reaction force cannot be obtained from the finite Dirac–Lorentz force known in the massive case since the latter diverges in the limit of zero mass.

The formal result of [6] was confirmed in [3] using different regularization, but the conclusion of [3] was that the reaction force has no physical meaning, since massless charge does not radiate at all. As we have shown here, the massless charge does radiate and the radiation power is finite. However, it is essentially quantum and the average energy of the photons is of the order of the particle energy. The reaction force is therefore stochastic, and consequently, the classical Lorenz–Dirac type equation derived in [6] is meaningless indeed.

Moreover, although the stochastic nature of the radiation recoil is already enough to preclude any classical radiation reaction equation, in the theory of synchrotron radiation of massive charges there also exists a much stronger restriction on the validity of such an equation, known as the $E_{1/5}$ bound [22]. The reason is that the quantum states of the charge in the magnetic field for the transverse motion depend on *two* quantum numbers, the second one being responsible for location of the center of the orbit [22]. Quantum fluctuations due to excitation of the orbit center start at much lower energies than quantum recoil comes into play. Namely, for $\eta \ll 1$,

$$E_{\mathrm{recoil}} \sim m\frac{H_0}{H}, \qquad E_{\mathrm{fluct}} \sim E_{1/5} = m\left(\frac{H_0}{H}\right)^{1/4}, \quad (53)$$

so $E_{\mathrm{fluct}} \ll E_{\mathrm{recoil}}$. Classical description of radiation reaction is valid for $E \ll E_{1/5}$. Now, for massless particles $E_{1/5} = 0$, so classical radiation reaction is twice meaningless.

8. Discussion

We have shown that quantum scalar electrodynamics in the external magnetic field unambiguously predicts finite synchrotron radiation from massless charges in the form of emission of hard photons with energy of the order of the particle energy. We have

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[2] I am indebted to A.V. Borisov for drawing my attention to this fact.

found this first using the strictly massless theory, and then considering the zero-mass limit of the massive theory. Both give the same results for the total probability, the total radiation power and the spectral distribution of radiation, provided the condition (29) holds. Classical theory is still reliable, but only as describing the low frequency region of the true quantum radiation spectrum. This view is different from previous proposals by Kosyakov [4] and Lechner [3]. Althoug this is not directly applicable to quantum electrodynamics in the external field, it is also worth mentioning that the renormalization group flow of massless QED has stable infrared point corresponding to vanishing charge [34].

The normalized spectrum of photons emitted by the massless charge in the magnetic field does not depend on any parameter. Thus, even in a very small magnetic field like that of the Earth, massless charges of the energy satisfying $E^2 \gg eH$ will emit photons with energies of the same order. Also, since we have shown that the limit to zero mass in the massive theory is smooth, our universal spectrum (41) has not only academic interest, but also applies to massive particles with energies $E/m \gg \eta^{-1}$ for any η.

Another interesting prediction following from our calculations is magnetic generation of the square of mass in the linear order in α. This was obtained taking the real part of the mass operator, and also can be derived via dispersion relation from the synchrotron radiation rate.

We have considered here only the quasiclassical case $n \gg 1$. Note that in Schwinger approach summation over the finite quantum numbers n' is performed automatically, so in the leading in $1/n^{1/3}$ approximation our results are not restricted by the condition $\nu = n - n' \ll n$. For low initial Landau levels n the problem must be treated numerically. Partially, the results may be extracted from the existing numerical calculations performed in the massive theory (see [33] and the references therein).

Acknowledgements

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