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Ten Hadamard Matrices of Order 1852 of Goethals–Seidel Type

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No Hadamard matrices of Goethals–Seidel type of order 1852 appear in the literature. In this note we construct ten such matrices. All of them have the maximal possible excess 79 636. The existence of a Hadamard matrix of order 1852 follows from a recent theorem of Yamada, but this fact has remained unnoticed so far.

A $(1, -1)$-matrix A of order m is a *Hadamard matrix* if $AA^T = mI_m$ (A^T is the transpose of A and I_m is the identity matrix of order m). If such A exists and $m > 2$ then $4 \mid m$ and we shall write $m = 4n$. There exists a Hadamard matrix of order 2. Since the tensor product of Hadamard matrices is again a Hadamard matrix, one is mainly interested in constructing Hadamard matrices of order $m = 4n$ for n odd. According to [7], Hadamard matrices of order $4n$ are known for all odd $n < 500$, except for the following 18 values of n:

$$107, 167, 179, 191, 213, 223, 239, 251, 283,$$
$$311, 347, 359, 419, 443, 463, 479, 487, 491.$$

All these numbers, except 213 which is divisible by 3, are primes congruent to 3 (mod 4).

Two remarks are in order. First, the number 213 should be removed from this list because T-sequences of length 71 have been recently constructed in [5]. (These sequences are also given in [7].) Indeed, this implies that there exists a Baumert–Hall array of order $4 \cdot 71$. Hence one can insert four Williamson type matrices of order 3 into this array to obtain a Hadamard matrix of order $4 \cdot 213$. Second, the number 463 also should not appear in this list. Indeed, $q = 461$ is a prime $\equiv 5 \pmod 8$ and there exists a skew Hadamard matrix of order $(q + 3)/2 = 232 = 8 \cdot 29$. By Theorem 4 of [9] this implies the existence of a Hadamard matrix of order $4(q + 2) = 4 \cdot 463$. It is interesting that this fact was noticed neither by Yamada [9, p. 378] nor Miyamoto [6, p. 107].

In this note we shall construct ten Hadamard matrices of order $4 \cdot 463 = 1852$ which are of a special type, known as the Goethals–Seidel type. Apart from their own instrinsic interest, these matrices have the maximum possible excess, namely $4 \cdot 43 \cdot 463 = 79\,636$. Indeed, this excess coincides with the upper bound due to Kounias and Farmakis [4].

In view of [8, Theorem 8.41, Cor. 8.42] it suffices to construct $4-(n; n_1, n_2, n_3, n_4; \lambda)$ supplementary difference sets modulo n with $n = 463$ and $n + \lambda = \sum n_i$. Recall that four subsets S_1, S_2, S_3, S_4 of $\{1, 2, \ldots, n-1\}$ are said to be $4-(n; n_1, n_2, n_3, n_4; \lambda)$ *supplementary difference sets* (*sds*) modulo n if $|S_k| = n_k$ for $k = 1, 2, 3, 4$ and for each $r \in \{1, 2, \ldots, n-1\}$ we have $\lambda_1(r) + \cdots + \lambda_4(r) = \lambda$, where $\lambda_k(r)$ is the number of solutions of the congruence $i - j \equiv r \pmod n$ with $\{i, j\} \subset S_k$.

Let G be the group of non-zero residue classes modulo the prime 463, and let $H = \langle 251 \rangle = \{1, 21, 33, 34, 118, 163, 169, 178, 182, 190, 196, 200, 230, 251, 286, 308, 318, 412, 441, 449, 450\}$ be its subgroup of order 21. We enumerate the 22 cosets

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α_i, $0 \leq i \leq 21$, of H in G as follows:

$$\alpha_0 = H, \quad \alpha_2 = 2H, \quad \alpha_4 = 4H, \quad \alpha_6 = 5H, \quad \alpha_8 = 7H, \quad \alpha_{10} = 8H,$$
$$\alpha_{12} = 10H, \quad \alpha_{14} = 19H, \quad \alpha_{16} = 25H, \quad \alpha_{18} = 29H, \quad \alpha_{20} = 49H,$$

and $\alpha_{2i+1} = -1 \cdot \alpha_{2i}$ for $0 \leq i \leq 10$.

We have constructed ten non-equivalent $4-(463; 210, 231, 231, 231; 440)$ sds's S_1, S_2, S_3, S_4. (Two sds's are said to be equivalent if one can be obtained from the other by permuting and/or shifting the four sets S_k, and/or by multiplying each S_k by some fixed element of G.) In all ten cases the sets S_k have the form

$$S_k = \bigcup \alpha_i, \qquad i \in J_k, \quad k = 1, 2, 3, 4,$$

where $J_k \subset \{0, 1, \ldots, 21\}$. Hence, instead of listing the sets S_k, we shall list the index sets J_k. The ten solutions are as follows:

(1) $J_1 = \{9, 10, 12, 14, 15, 17, 18, 19, 20, 21\}$,
$J_2 = \{0, 1, 4, 5, 9, 10, 11, 14, 16, 19, 21\}$,
$J_3 = \{0, 1, 4, 6, 9, 10, 12, 14, 15, 17, 20\}$,
$J_4 = \{1, 3, 4, 6, 7, 8, 10, 13, 19, 20, 21\}$;

(2) $J_1 = \{1, 4, 6, 7, 8, 10, 13, 14, 15, 21\}$,
$J_2 = \{0, 1, 2, 3, 4, 7, 11, 12, 13, 19, 21\}$,
$J_3 = \{0, 1, 4, 5, 6, 7, 13, 15, 19, 20, 21\}$,
$J_4 = \{2, 7, 8, 9, 11, 12, 13, 14, 17, 19, 20\}$;

(3) $J_1 = \{4, 7, 8, 9, 10, 14, 16, 17, 19, 21\}$,
$J_2 = \{0, 4, 6, 7, 12, 13, 14, 17, 18, 20, 21\}$,
$J_3 = \{1, 3, 4, 8, 10, 11, 13, 16, 19, 20, 21\}$,
$J_4 = \{1, 4, 5, 6, 7, 9, 14, 15, 16, 17, 20\}$;

(4) $J_1 = \{4, 7, 8, 9, 10, 14, 16, 17, 19, 21\}$,
$J_2 = \{0, 4, 5, 6, 7, 8, 14, 15, 16, 17, 21\}$,
$J_3 = \{0, 4, 6, 7, 12, 13, 14, 17, 18, 20, 21\}$,
$J_4 = \{1, 3, 4, 8, 10, 11, 13, 16, 19, 20, 21\}$;

(5) $J_1 = \{2, 3, 4, 8, 9, 10, 12, 15, 17, 20\}$,
$J_2 = \{0, 1, 2, 8, 10, 11, 12, 13, 17, 19, 21\}$,
$J_3 = \{0, 3, 5, 6, 8, 9, 13, 15, 19, 20, 21\}$,
$J_4 = \{0, 3, 7, 8, 10, 11, 13, 16, 19, 20, 21\}$;

(6) $J_1 = \{3, 6, 7, 10, 12, 13, 15, 17, 18, 19\}$,
$J_2 = \{0, 1, 2, 3, 7, 13, 14, 17, 18, 20, 21\}$,
$J_3 = \{0, 2, 4, 5, 8, 10, 12, 17, 18, 19, 21\}$,
$J_4 = \{1, 3, 4, 6, 7, 10, 15, 17, 19, 20, 21\}$;

(7) $J_1 = \{8, 11, 13, 14, 15, 16, 18, 19, 20, 21\}$,
$J_2 = \{0, 1, 4, 5, 9, 10, 11, 14, 16, 19, 21\}$,
$J_3 = \{0, 1, 4, 6, 9, 10, 12, 14, 15, 17, 20\}$,
$J_4 = \{1, 3, 4, 6, 7, 8, 10, 13, 19, 20, 21\}$;

(8)
$$J_1 = \{2, 6, 7, 11, 12, 13, 14, 16, 18, 19\},$$
$$J_2 = \{0, 1, 2, 3, 7, 13, 14, 17, 18, 20, 21\},$$
$$J_3 = \{0, 2, 4, 5, 8, 10, 12, 17, 18, 19, 21\},$$
$$J_4 = \{1, 3, 4, 6, 7, 10, 15, 17, 19, 20, 21\};$$

(9)
$$J_1 = \{0, 1, 4, 9, 10, 12, 13, 18, 19, 20\},$$
$$J_2 = \{0, 2, 4, 6, 7, 11, 12, 15, 18, 20, 21\},$$
$$J_3 = \{0, 3, 5, 10, 11, 12, 16, 17, 18, 20, 21\},$$
$$J_4 = \{0, 4, 5, 6, 7, 8, 10, 12, 16, 17, 21\};$$

(10)
$$J_1 = \{0, 1, 4, 9, 10, 12, 13, 18, 19, 20\},$$
$$J_2 = \{0, 3, 5, 10, 11, 12, 16, 17, 18, 20, 21\},$$
$$J_3 = \{0, 4, 5, 6, 7, 8, 10, 12, 16, 17, 21\},$$
$$J_4 = \{1, 3, 5, 6, 7, 10, 13, 14, 19, 20, 21\}.$$

The above ten sds's are pairwise non-equivalent because they have different intersection patterns. For instance, in the first solution we have

$$|J_2 \cap J_3| = 6, \qquad |J_2 \cap J_4| = |J_3 \cap J_4| = 5,$$

while in the second solution we have

$$|J_2 \cap J_3| = 7, \qquad |J_2 \cap J_4| = 6, \qquad |J_3 \cap J_4| = 4.$$

(Shifting cannot be used since it destroys the property of the S_i's being unions of cosets α_j.)

Needless to say, these sds's were found by a computer search. The computation was carried out partly on a Sun Sparc-station 2 and partly on a MIPS machine. The main idea used in the computer search was to try to construct the required sds's from the cosets of a suitable subgroup of non-zero residue classes of integers mod $n = 463$. The same method (with some modifications when n is not a prime) was used successfully by the author recently to construct skew Hadamard matrices of Goethals–Seidel type for 19 orders for which no skew Hadamard matrices were known previously (see [1–3]).

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References

1. D. Ž. Ðoković, Skew Hadamard matrices of order $4 \cdot 37$ and $4 \cdot 43$, *J. Combin. Theory, Ser. A* (to appear).
2. D. Ž. Ðoković, Construction of some new Hadamard matrices, *Bull. Austral. Math. Soc.*, **45** (1992), 327–332.
3. D. Ž. Ðoković, Ten new orders for Hadamard matrices of skew type (submitted).
4. S. Kounias and N. Farmakis, On the excess of Hadamard matrices, *Discr. Math.*, **68** (1988), 59–69.
5. C. Koukouvinos, S. Kounias, J. Seberry, C. H. Yang and J. Yang, On sequences with zero autocorrelation (preprint).
6. M. Miyamoto, A construction for Hadamard matrices, *J. Combin. Theory, Ser. A*, **57** (1991), 86–108.

7. J. Seberry and M. Yamada, Hadamard matrices, sequences and block designs (preprint), to appear in: *Contemporary Design Theory: A Collection of Surveys*, J. Dinitz and D. Stinson (eds), New York, John Wiley, 1992.
8. J. Seberry Wallis, Hadamard matrices, in: *Combinatorics: Room Squares, Sum Free Sets, Hadamard Matrices*, W. D. Wallis, A. P. Street and J. Seberry Wallis, Lecture Notes in Mathematics, vol. 292, part IV, Springer-Verlag, Berlin, 1972.
9. M. Yamada, Some new series of Hadamard matrices, *J. Austral. Math. Soc., Ser. A*, **46** (1989), 371–383.

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