A thermodynamics-based cohesive model for interface debonding and friction

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A constitutive model for interface debonding is proposed which is able to account for mixed-mode coupled debonding and plasticity, as well as further coupling between debonding and friction including post-delamination friction. The work is an extension of a previous model which focuses on the coupling between mixed-mode delamination and plasticity. By distinguishing the interface into two parts, a cracked one where friction can occur and an integral one where further damage takes place, the coupling between frictional dissipation and energy loss through damage is seamlessly achieved. A simple framework for coupled dissipative processes is utilised to derive a single yield function which accurately captures the evolution of interface strength with increasing damage, for both tensile and compressive regimes. The new material model is implemented as a user-defined interface element in the commercial package ABAQUS and is used to predict delamination under compressive loads in several test cases.

1. Introduction

One of the most conspicuous mechanisms of material damage is the formation of well defined regions where inelastic deformation localises and subsequently propagates. When the size of these regions is small enough (compared to the structural scale), it can be computationally advantageous to idealise the localised damage zone as an interface; this way, the full continuum constitutive law is replaced by a formulation relating the interfacial tractions to the separation between the two surfaces.

Examples are laminated composite materials where the excess resin accumulated between consecutive plies usually fosters the initiation of delamination; in the Arctic sea ice, the fracture lines or leads separating blocks of ice that are hundreds of kilometres in size; in concrete, the width of the damage zone is usually of the order of only a few aggregates in size. In all these cases, the computational cost of the numerical model can be reduced by several orders of magnitude by using an interface constitutive formulation.

The problem with this simplification is that traditional interface decohesive models (Hillerborg et al., 1976; Xu and Needleman, 1994; Scheider, 2001; Allix and Corigliano, 1996; Camanho et al., 2003; Davies et al., 2006) tend to focus on the loss of stiffness (modelled with a damage parameter) and are not capable of correctly capturing the frictional dissipation and inelastic deformation that accompany the damage process. Drastic improvements in predicting the permanent set observed experimentally for some materials, especially polymeric and cementitious materials can be obtained by coupling damage and plasticity. While most of these models are based on plasticity theory with the loss of stiffness considered secondary to the plastic-like behaviour (Tvergaard and Hutchinson, 1993; Su et al., 2004; Matzenmiller et al., 2010; Kolluri, 2011), it is our opinion that the reverse (loss of stiffness as the primary failure process) is more faithful to the physics of interface damage. An energy-based formulation has hence been proposed by the authors (Guiamatsia and Nguyen, 2012) to capture the coupling between damage and friction, in a simple and elegant way rooting from the partition of the total dissipation into plastic/frictional and damage ones. However, like many other damage–plasticity models, such a model does not have the capability to explain the apparent increase in strength and toughness of the material when the loading of the damaging area consists of the combination of friction and transverse compression.

Such increase is well documented in a large body of experiments found in the literature. For instance, Wisnom and Jones (1996) pointed out and carried out experiments to investigate the role of friction in increasing mode II fracture energy of the interface of laminated composite; de Teresa et al. (2004), within the context of composite delamination, tested cylindrical specimens made of laminated composite materials under combined shear and compression loading and observed a clear increase of shear strength with increasing pressure; Lugovy et al. (2005)
measured increased values of fracture toughness for Si₃N₄ laminates under the same loading conditions; finally, Hallett and Li (2005) performed numerical experiments of the impact of cross-ply carbon fibre laminated composites and concluded that, to obtain a realistic prediction, both the fracture toughness and the strength of delaminating interfaces under compression had to be artificially increased in the numerical model.

The easiest way to handle the frictional interaction between delaminating interfaces is to adopt the very popular two-step approach consisting of delamination first, then frictional contact. This is the approach taken by Tvergaard (1990) whose model features an unphysical unloading–reloading in the constitutive response if the material is loaded to and past the failure point in compressive shear. This is somewhat similar to what can be obtained with standard tools in most commercial finite element packages: by using element deletion for example, post-damage frictional contact of the newly created surfaces can be activated; but the technique is unable to capture the way in which friction actually affects the damage evolution itself.

A variety of solutions have been proposed by researchers over the years, two popular ones being (a) the enhancement of Tvergaard model proposed by Chaboche et al. (1997), where the shear load smoothly decreases to the Mohr–Coulomb limit when under compression, and (b) the (phenomenological) adjustment of the yield function and/or the fracture toughness to include a normal tractions term (Matzenmiller et al., 2010; Li et al., 2008; Hou et al., 2001; Christensen and De Teresa, 2004). A comprehensive review can be found in Raous (2011).

In 2006, Alfano and Sacco (2006) proposed the representation of the interface as a two-phase material consisting of a damaged part and an integral part, hence defining the damage parameter as the proportion of fully damaged interface. Using subscripts c and i for ‘cracked’ and ‘integral’, A for the interface area, the damage variable D is the area ratio as follows:

\[
A = A_c + A_i, \quad \text{with} \quad D = \frac{A_c}{A} \tag{1}
\]

Using the classical mixture theory, the constitutive law of this ‘composite’ material is obtained through appropriate compatibility relations and mixed stress:

\[
\sigma = D\sigma_c + (1 - D)\sigma_i \tag{2}
\]

It hence becomes straightforward to derive a constitutive model that seamlessly introduces the effect of compressive friction¹ by using a traditional frictional contact model for the damaged part of the material \(\sigma_c\). In their model, Alfano and Sacco (2006) use a non-associative plasticity model with the Mohr–Coulomb cone as the dissipative function. In our model, a fully coupled constitutive formulation is entirely derived from standard thermodynamic principles. The approach is similar to that used in our previous coupled damage/plasticity interface model, but this time, the expression of the free energy is expanded to include the behaviour of the two-phase material described above.

The model is unifying in the sense that it is capable of accounting for damage and plasticity in mixed-mode loading conditions, as well as frictional effects on both strength and toughness under transverse compression. In Fig. 1, we graphically present various scenarios that the new model is capable of capturing.

The new model is simple, requiring the calibration of only two parameters in addition to the previous ‘tension-only’ mixed-mode delamination model (Guiamatsia and Nguyen, 2012): (a) a compressive (elastic) stiffness, \(K_c\), being the slope of the shear traction/shear displacement plot measured for the interface under compression, and (b) the Mohr–Coulomb coefficient of friction denoted \(\mu\). It is noted that in the previous model being extended here (Guiamatsia and Nguyen, 2012), the term “friction” and “plasticity” were used interchangeably to denote the partition of the energy dissipation resulting in the non-reversible deformation of the interface. In the present paper, there is a distinction between that dissipative process linked to residual deformation of the interface and the sliding friction at microcracks which occurs only when the interface is subjected to combined compression and shear. Therefore, “plasticity” is used to refer to the former process, while “friction” is reserved for only the second.

The presentation begins with the thermo-mechanical description of the model and derivation of the yield function and evolution of internal variables. This is followed by the description of the stress–return algorithm, as implemented in a user-defined interface element with the commercial package ABAQUS/Explicit (2010). Finally the model is applied to the analysis of a laminated composite plate under low-velocity impact and the modelling of a fibre pull out test.

2. Thermo-mechanical formulation

The following expression of the Helmholtz energy potential, \(\Psi\), is considered for the two-phase integral/cracked material:

\[
\Psi = \frac{1}{2} \left(1 - D\right)K_n \left(u_n - u_n^0\right)^2 + \frac{1}{2} D \left[1 - H\left(u_n - u_n^0\right)\right] K_n \left(u_n - u_n^0\right)^2
\]

\[
+ \frac{1}{2} \left(1 - D\right)K_i \left(u_i - u_i^0\right)^2 + \frac{1}{2} D \left[1 - H\left(u_i - u_i^0\right)\right] K_i \left(u_i - u_i^0\right)^2 \tag{3}
\]

For each loading direction (n or s), there are two terms corresponding to the sum of the cracked (D) and integral (1 – D) contributions. Here, \(D\) is a scalar variable representing the interface damage state; \(u\) is the vector of interfacial separation, with normal and shear components, respectively represented by subscripts n and s; \(K_n\) is the elastic stiffness corresponding to the normal or transverse direction; \(K_n\) and \(K_i\) are elastic shear stiffness corresponding, respectively, to the tensile and compressive loading regimes; finally, \(H(\cdot)\) is the Heaviside function, taking the value of unity if the argument (•) is positive, and zero otherwise. The superscripts p and f indicate plasticity and friction, respectively. In the above expression of the energy potential, \(\Psi\), it is implicitly assumed that the normal stiffness \(K_n\), once completely lost in tension, is fully recovered upon compression. However it is not the case with the shear stiffness \(K_s\); only a fraction can be recovered, e.g. \(K_s' < K_s\), depending on the roughness of the cracked surface. Further details on \(K_i'\) will be elaborated later.

It is easy to visualise the compatibility relation between the interfacial displacement jump at the integral part \(u_i\) and that at the cracked part, \(u_c\) (also cf. Alfano and Sacco (2006)).

\[
u = u_i = u_c = \begin{cases} u_n = u_n^0 = u_i^0 \\ u_s = u_s^0 = u_i^0 \end{cases} \tag{4}
\]

Since the distinction has already been established between the inelastic deformation of the integral part being referred to as ‘plastic’ and that of the cracked part as ‘frictional’, the indices i and c can be dropped altogether, yielding:

\[
\Psi = \frac{1}{2} \left(1 - D\right)K_n \left(u_n - u_n^0\right)^2 + \frac{1}{2} D \left[1 - H\left(u_n - u_n^0\right)\right] K_n \left(u_n - u_n^0\right)^2
\]

\[
+ \frac{1}{2} \left(1 - D\right)K_i \left(u_i - u_i^0\right)^2 + \frac{1}{2} D \left[1 - H\left(u_i - u_i^0\right)\right] K_i \left(u_i - u_i^0\right)^2 \tag{5}
\]

In the above expression, the classical additive decomposition of the total jump into elastic and plastic components has been used, with

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¹ The frictional dissipation between two fully damaged surfaces in compression is distinguished from that taking place at an integral interface, and which is modelled through as plastic deformation.
\( \mathbf{u}^f = \mathbf{u} - \mathbf{u}^p \) being the elastic displacement jump. There are four internal variables: the plastic normal jump \( \mathbf{u}_n^p \), the plastic shear jump \( \mathbf{u}_s^p \), the frictional shear jump \( \mathbf{u}_s^f \), and the damage variable \( D \). The corresponding generalised forces, interface tractions \( t_n, t_s, t_{sc} \) and damage dissipative force \( \chi \) are obtained as derivatives of the free energy potential (5) with respect to the associated internal variables:

\[
\begin{align*}
\frac{\partial \Psi}{\partial u_n^p} = t_n = \frac{\partial \Psi}{\partial u_s^p} = t_s = \frac{\partial \Psi}{\partial u_{sc}^p} = t_{sc} = \frac{\partial \Psi}{\partial D} = \chi
\end{align*}
\]

(6)

Expanding,

\[
\begin{align*}
\frac{\partial \Psi}{\partial u_n^p} &= (1 - D)K_n(u_n - u_n^p) + DK_nH(-[u_n - u_n^p])(u_n - u_n^p) + \frac{1}{2}D\delta(u_n - u_n^p)[K_n(u_n - u_n^p)^2 + K_s^e(u_s - u_s^f)^2] \quad (7)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \Psi}{\partial u_s^p} &= (1 - D)K_s(u_s - u_s^p) \quad (8)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \Psi}{\partial u_{sc}^p} &= DK_s^eH(-[u_n - u_n^p])(u_n - u_n^p) \quad (9)
\end{align*}
\]

The last term in the expression of the normal traction involves the Dirac delta, which is infinite when the elastic normal displacement jump vanishes, i.e. \( u_n^p = u_n - u_n^p = 0 \). Physically, as the loading regime goes from tension to compression, the normal ‘elastic’ separation will tend to zero, as should the term \( u_n - u_n^p \). These terms must indeed vanish and remain zero throughout the tensile loading regime. The result, only at \( u_n^p = u_n - u_n^p = 0 \), is hence infinity times zero which is indeterminate, but for practical purpose, it will be assumed that the contribution can be neglected. Consequently, the tractions can be expanded in the reduced form:

\[
\begin{align*}
t_n &= \begin{cases} 
(1 - D)K_n(u_n - u_n^p), & u_n^p > 0 \\
K_n(u_n - u_n^p), & u_n^p < 0
\end{cases} \quad (10)
\end{align*}
\]

\[
\begin{align*}
t_s &= (1 - D)K_s(u_s - u_s^p), \quad u_s^p > 0 \\
t_n + t_{sc} = (1 - D)K_s(u_s - u_s^p) + DK_s^e(u_s - u_s^f), \quad u_s^p \leq 0 \quad (11)
\end{align*}
\]

The damage energy is:

\[
\begin{align*}
\chi = -\frac{\partial \Psi}{\partial D} = \begin{cases} 
\frac{1}{2}K_n(u_n - u_n^p)^2 + \frac{1}{2}(1 - H(u_n - u_n^p))K_n(u_n - u_n^p)^2 \\
- \frac{1}{2}K_s^e(u_s - u_s^f)^2 - \frac{1}{2}(1 - H(u_s - u_s^f))K_s^e(u_s - u_s^f)^2 \end{cases}
\end{align*}
\]

\[
= \chi_n + \chi_s - \chi_{sc} \quad (12)
\]

Therefore, three terms contribute to the energy driving the damage process, including a negative contribution from the damaged interface in friction. This is consistent with physical observations, as the (compressive) friction should have the effect of slowing down the damage evolution. However, that term must remain smaller or equal to the sum of the first two terms, in such a way that the damage energy is always positive, so as to fulfill the irreversibility condition. For this reason, \( K_s^e \) should be much smaller than \( K_n \) (\( K_s^e < K_n \)). This also makes general physical sense as the shear stiffness of a cracked interface upon compression must be much smaller than that of the intact material.\(^2\) Note that damage stops evolving once the critical value \( D = 1 \) corresponding to full loss of cohesion being reached. Thermodynamically, the damage energy as per Eq. (12) ceases to exist at that moment.

Similar to the approach taken in Guiamatsia and Nguyen (2012), a strong coupling between damage, plasticity and friction is assumed and formulated through the specification of damage potentials as homogenous functions of the rates of internal variables, after the concept described in Einav et al. (2007). The dissipation rate is assumed of the following quadratic form, using the indicial notation corresponding to the associated internal variable:

\[
\Phi = \sqrt{\phi_n^2 + \phi_s^2 + \phi_{sc}^2 + \phi_f^2} \quad (16)
\]

where different individual contributions \( \phi_n, \phi_s, \phi_{sc} \) and \( \phi_f \) are described in Table 1. We note the appearance of the additional term \( -\phi_n \) in these expressions to account for the increase of both strength and toughness due to compression.

The rate of frictional dissipation,

\[
\phi_f = [\mu(-t_s) + X][\partial u_s^f] \quad (17)
\]

includes the Mohr–coulomb shear strength, \( \mu (-t_s) \), because we are seeking a yield in the form of the Mohr–Coulomb model for the cracked interface. There is also the traction-like term, \( X \), which is utilised to impose a condition that the yield function should be

\(^2\) The realistic value for \( K_s^e \) is discussed again later in the manuscript.
continuous for $t_n = 0$, at the transition of loading regimes, as described in Appendix A.

The other dissipation rates (damage, plastic normal, plastic shear) feature the parameter $r_D = G_{ij}/G_{ij}^c$; this is the partition of the energy dissipation that is allocated to the creation of new surfaces (pure fracture), $G_c$ and $G_{ij}$, respectively, the total interface toughness and the total dissipation from pure fracture; $F(D)$ is a monotonic function controlling the damage dissipation that verifies:

$$\int_{D=0}^{D=1} F(D) \delta D = G_c$$  \hspace{1cm} (22)

For a given ratio of mode mixity $k = \frac{K_{II}}{K_{I}}$ defined as the ratio between energy for shear and the total damage energy (see Eqs. (12)–(15)), the (mixed) mode dissipative process controlled by function $F(D)$ is a combination of functions $F_D(D)$ and $F_I(D)$ in pure modes I and II, respectively (see Guiamatsia and Nguyen, 2012 for details on the mixed mode formulation and behaviour). In addition, for the effects of matrix plasticity, a simple averaging of pure mode I ($r_{DII}$) and pure mode II ($r_{DIII}$) parameters is also assumed:

$$r_D = kr_{DII} + (1 - k)r_{DIII}$$ \hspace{1cm} (23)

$$F(D) = \beta[|F(I)(D)| + (1 - k)|F_I(D)|]$$

where

$$F_{n,s}(D) \approx \left( \frac{2N^2}{3k_{n,s}} \right)^2 \left( \frac{1 + \frac{(N)^2}{2k_{n,s}K_{II}G_c(1 - N^2)}}{1 - D + \frac{(N)^2}{2k_{n,s}K_{II}G_c(1 - N^2)}} \right)^2$$ \hspace{1cm} (24)

Here, classical notation is used for the interface fracture toughness in normal ($G_{II}$) and shear ($G_{III}$) modes, and the strength in normal (N) and shear (S) modes. It can be readily verified that the integral of the proposed damage function is equal to the fracture toughness,

$$\int_{D=0}^{D=1} F_{n,s}(D) \delta D = G_{II}$$ \hspace{1cm} (25)

Therefore, the mode-mixity parameter $\beta$ can be tuned with experiments performed with the ratio $k$ held constant, like Reeder and Crews (1990) mixed-mode bending experiments, by using:

$$\beta(k) = \frac{G_c(k,D)}{K_{II}G_c + (1 - k)|G_{III}|}$$ \hspace{1cm} (26)

In the model implementation, either a power-law interaction (Whitcomb, 1984) or the BK criterion for composite failure (Beneggagh and Kenane, 1996) are used to determine $G_c(k,D)$, but any other function can be utilised for the mixed-mode toughness, provided that it corresponds to constant mode-mixity experiments. Further details on the mixed mode interaction can be found in our earlier paper (Guiamatsia and Nguyen, 2012).  

### 3. Model behaviour

#### 3.1. Strength

With all the dissipative terms being homogeneous functions (in terms of the rates of change of the internal variables) of order 1, the resulting loading function (or yield curve) is obtained, in generalised stress space, as the quadratic expression, Einav et al. (2007):

$$y^r = \left( \frac{Y}{N^2} \right)^2 + \left( \frac{t_n}{N^2} \right)^2 + \left( \frac{t_{II}}{N^2} \right)^2 + \left( \frac{t_{III}}{N^2} \right)^2 - 1 \leq 0$$ \hspace{1cm} (27)

From the thermodynamic formulation described in Einav et al. (2007), $y^r$ is obtained from the Legendre transformation of the dissipation potential (16) and plays the role of the plastic potential in classical plasticity. It gives the evolution rules for the internal variables. Expanding and simplifying the above using functions defined in Table 1 gives the yield condition, denoted here as $y$, in stress space as:

$$y = \frac{Y}{F(D)} + \frac{2\sqrt{2}K_{II}(1 - D)^2}{[(1 - D)\sqrt{2F(D)}K_{II} + \mu(-t_n)]^2} + \frac{t_{II}}{(-\mu(-t_n) + X)^2} - 1 \leq 0$$ \hspace{1cm} (28)

This can also be written separately for tensile (+) and compressive (−) loading as follows, expanding the damage forces in terms of damage and respective tractions:

$$y = \begin{cases} \left( + \right): & \left( \frac{t_{II}}{(-\mu(-t_n) + X)^2} - 1 \right) = 0 \\ \left( - \right): & \left( \frac{t_{II}}{(-\mu(-t_n) + X)^2} - 1 \right) = 0 \end{cases}$$ \hspace{1cm} (29)

Continuity is then imposed for $t_n = 0$ (cf. Appendix A) to obtain the following expression for the term $X$ that, interestingly, vanishes for a damage value of either 0 or 1:

$$X = \frac{2F(D)D(1 - D)^2K_{II}K_{II}^c}{2(1 - D)K_{II} + DK_{II}^c}$$ \hspace{1cm} (30)

Fig. 2 shows isodamage yield curves for an interface with properties as specified on the graph, obtained by imposing and keeping constant the normal traction $t_n$ and loading in shear until the yield condition is met. It can be verified that the curves are indeed continuous at the tension/compression transition. It is also noted that the curves in the compressive regime are straight lines, and are parallel to one another, suggesting a variation of the yield shear stress with pressure that is more or less linear:

$$t_{II}^{\text{yield}} \approx t_{II}^{\text{yield}} + \mu(-t_n)$$ \hspace{1cm} (31)
The rate of energy dissipation is obtained through multiplying the generalised forces by the rate of change of the associated internal variables.

\[
\delta \Phi = \delta \Phi_0 + \delta \Phi'_D + \delta \Phi'_n + \delta \Phi'_s = \chi \delta D + \tau_n \delta u'_n + \tau_d \delta u'_d + \tau_d \delta u'_l
\]

(32)

Because of the strong coupling used here, the consequence of which is a single loading function, the increments of each internal variable are related to a single plastic-type multiplier \( \delta \lambda \) through the following "flow rules" obtained from the loading function, Eq. (27):

\[
\delta D = \delta \lambda \frac{\partial \Phi^*}{\partial \lambda} = 2 \delta \lambda \frac{X_n}{X} = 2 \delta \lambda \frac{r_0}{X} \left[ \frac{X_n}{F(D)} + \frac{2 \int K_s \mu(1-D)^2}{(1-D) \sqrt{2K_s F(D) + \mu(-t_n)}} \right] \]

(33)

\[
\delta u'_n = \delta \lambda \frac{\partial \Phi^*}{\partial \lambda} = 2 \delta \lambda \frac{\tau_n}{\partial \lambda u'_n} = 2 \delta \lambda \frac{r_0}{F(D) t_n}
\]

(34)

\[
\delta u'_d = \delta \lambda \frac{\partial \Phi^*}{\partial \lambda} = 2 \delta \lambda \frac{\tau_d}{\partial \lambda u'_d} = 2 \delta \lambda \frac{r_0}{t_d} \left[ \frac{2 \int K_s \mu(1-D)^2}{(1-D) \sqrt{2K_s F(D) + \mu(-t_n)}} \right]
\]

(35)

\[
\delta u'_l = \delta \lambda \frac{\partial \Phi^*}{\partial \lambda} = 2 \delta \lambda \frac{\tau_d}{\partial \lambda u'_l} = 2 \delta \lambda \frac{t_d}{\left[ \mu(-t_n) + X_n \right]^2}
\]

(36)

In the tensile loading case, the expressions simplify, showing that increments of all other internal variables are proportional to that of damage, yielding an analytical expression of the total energy dissipation in function of the interface properties. In compression, however, the expressions can only be further reduced as far as the following:

\[
\delta D = 2 \delta \lambda \frac{r_0}{X_n} \left[ \frac{2 \int K_s \mu(1-D)^2}{(1-D) \sqrt{2K_s F(D) + \mu(-t_n)}} \right]
\]

\[
= 2 \delta \lambda \frac{r_0}{X_n} \left[ \frac{t_c}{\left[ \mu(-t_n) + X_n \right]^2} \right]
\]

(37)

\[
\delta u'_n = 0
\]

(38)

\[
\delta u'_d = \frac{1}{r_d} \frac{X_n}{t_d} \delta D
\]

(39)

\[
\delta u'_l = \frac{\delta D}{r_d} \frac{t_c}{\mu(-t_n) + X_n^2 - t_c^2}
\]

(40)

The yield equation in compression is used in Eq. (37). From the above, the rate of total energy loss becomes:

\[
\delta \Phi = \chi \frac{\delta D}{r_D} \left[ \frac{[\mu(-t_n) + X_n^2 - t_c^2]}{\mu(-t_n) + X_n^2 - t_c^2} \right]
\]

(41)

In reference to our previous model without friction (Guiamatsia and Nguyen, 2012), the calculated rate of energy dissipation was simply:

\[
\delta \Phi = \chi \frac{\delta D}{r_D} \left[ \frac{[\mu(-t_n) + X_n^2 - t_c^2]}{\mu(-t_n) + X_n^2 - t_c^2} \right]
\]

(42)

Therefore, the new expression clearly shows that the rate of dissipation predicted by this model is higher, with the ratio \( \frac{[\mu(-t_n) + X_n^2 - t_c^2]}{\mu(-t_n) + X_n^2 - t_c^2} \) being larger than 1. It is, however, not possible to simplify the expression further, meaning that the rate of dissipation is also dependent, in this case, on the loading path. Considering the same loading paths utilised to obtain the yield curves of Fig. 2, the total energy dissipation is numerically integrated, assuming total interface damage for \( D = 0.9999 \). For the purpose of comparison, we refer to the work of Li et al. (2008), who studied the effect of compressive delamination with traction–separation laws that were enhanced with the transverse pressure. Two of their proposed models (A and B) are shown in Table 2, along with the corresponding formula for the damage dissipation. The total energy loss, i.e. the energy dissipated to bring the compressed interface to complete failure, \( D = 1 \), is calculated numerically for several values of transverse pressure and compared with models A and B in Fig. 3. The sensitivities of the predicted total dissipation with respect to (1) the coefficient of friction and (2) the relative stiffness of the fully damaged partition \( a = K'/K \) are investigated; in this simulation \( r_0 = 1 \).

As seen in Fig. 3(a), the total dissipation does not appear to depend on the stiffness ratio \( a \), although the partitioning between damage dissipation and frictional dissipation does, as per Fig. 3(b). This is physically reasonable, as smaller \( a \), e.g. lower \( K'/K \) leads to lower \( t_c \) (see Eq. (9)) and hence delayed frictional sliding for the same shear displacement under same normal stress. In such cases the elastic strain energy \( \frac{\partial K}{\partial \gamma} \) will also decrease correspondingly, resulting in higher damage energy driving the debonding process (see Eq. (12)) and hence increasing damage dissipation. Therefore it can be said that the ratio \( a \) controls the energy partition between the damage/plasticity dissipation on one side and the frictional dissipation on the other, analogous to \( r_0 \) which controls the partition between pure damage and plasticity. Therefore, the only parameter that controls the total energy dissipated is the coefficient of friction \( \mu \). In other words, if \( \mu \) can be properly measured, the model can predict the effects of compression on the increase of both strength and toughness of the interface without the need of any unphysical tuning parameters. Practically, this friction parameter \( \mu \) can normally be calibrated experimentally based on yield locii in the compressive section of the stress space such as that obtained in the experiments of De Teresa et al. (2004).
both strength and toughness is left to the model prediction, without having to impose any phenomenological rule on the strength and fracture properties of the material model.

### 3.3. One element tests

The constitutive model was implemented in ABAQUS/Explicit (2010) as a user-defined interface element with the stress-return algorithm provided in Appendix B. This is an 8-node directional element (4-node in two dimensions), with upper and lower faces clearly specified through node numbering. Nodal integration scheme was used, to be consistent with ABAQUS generic elements for the purpose of comparison in the numerical examples. Single interface element tests with a fully constrained lower face and displacement loading on the upper face (cf. Fig. 4) were used for the validation of the constitutive model implementation.

#### 3.3.1. Cyclic loading test 1

The interface used here had the following properties: $K_n = K_s = 0.5 \times 10^{14}$ N/m$^2$; $G_c = 281$ N m/m$^2$; $G_{tc} = 800$ N m/m$^2$; $S = N = 5 \times 10^7$ Pa; $r_D = r_D = 1.0$; $K_f = 0.5 \times 10^{12}$ N/m$^3$; $\mu = 0.3$. Fig. 4 also illustrates the loading profile consisting of initial compression ($u_2$) that was kept at a constant value $u_2^{\text{min}}$ while it is being loaded/unloaded/reloaded in shear ($u_1$). Three different levels of compression were applied, namely $u_2^{\text{min}} = -3 \times 10^{-7}$ m, $-2 \times 10^{-7}$ m and 0 m, corresponding to normal tractions $t_n = -0.27$ kN, $-0.18$ kN and 0 kN; the responses are reported in the traction–separation plots of Fig. 5(a).

Although the partition of damage dissipation is one in this case, i.e. there is no plastic deformation $u_1^p$ linked to debonding, the curve will not return to the origin upon unloading, because the frictional deformation $u_f$ is also inelastic, as shown. Under compression, the shear behaviour of the interface varies smoothly from ‘debonding’- dominated to friction – dominated. Under increasing shear loads, the interface shear traction progressively reduces and converges towards the critical shear traction.

### Table 2

<table>
<thead>
<tr>
<th>Model</th>
<th>Shape of the compression-enhanced traction separation law</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$G_c(1 - \mu \phi)$</td>
</tr>
<tr>
<td>B</td>
<td>$G_c(1 - 2\mu \phi + \mu(\psi)^2)$</td>
</tr>
</tbody>
</table>

Fig. 3. Variation of the energy dissipation with transverse compression; (a) total dissipation, (b) damage dissipation for $\mu = 0.4$.

Fig. 4. Single element with load (left) and loading history (right).
\( \tau_a = \mu(-\tau_s) \), as expected. What was less expected is the asymptotic character of that convergence, observed in Fig. 5(a); in fact, whilst under compression, the constitutive model predicts that complete damage \( (D = 1) \) will actually never happen and this is linked to the assumed form of the dissipative functions. However, since the main features of compressive damage are captured by the model, a practical solution to the ‘asymptotic’ damage evolution is setting a threshold value of the damage variable, for which the debonding is assumed to be complete. In the results presented here, this threshold value is 0.9999.

Fig. 5 also shows that the energy dissipated (area under curve) increases during the progression of coupled debonding/friction failure process, as expected, with the applied transverse pressure. The energy dissipation was also integrated numerically and values of damage dissipation and total dissipation are reported in Table 3. Note that, for this table and Fig. 6, only one stage of shear loading/unloading was applied at a maximum amplitude of separation of \( 20 \times 10^{-5} \) m instead of the maximum \( 5 \times 10^{-5} \) m applied to obtain the plot in Fig. 4; this was to allow the threshold damage value to be reached for all simulations. A further illustration of the time profile for the case \( \tau_a = -0.27 \) kN is compared to the frictionless case, \( \tau_a = 0 \) kN, in Fig. 6.

Fig. 6 clearly shows that damage is the main source of energy loss at the interface during the initial stages of failure (little gap between damage and total curves), but as the damaged area evolves, frictional effects become increasingly important and dominate by the end of the failure process. It is also interesting to note that damage may continue to progress when the shear loading is inverted; this indeed happens for the compressive case shown in Fig. 6, and the damage variable (and dissipation) also increases, albeit, insignificantly in this occasion. This response seems sensible as one would expect further damage as a result of continued friction at the incompletely delaminated interface; this is achieved here thanks to the coupling between damage and friction.

The effect of varying the compressive stiffness \( (a = K_s'/K_s) \) and partition of damage dissipation \( (r_D = G_{fr}/G_s) \) was examined in more detail, for the case \( \tau_a = -0.18 \) kN, and reported in Fig. 7(a) and (b), respectively.

As expected, increasing the plasticity (by reducing \( r_p \)) causes an increase of the permanent deformation. However, increasing the compressive stiffness, via increasing ratio \( a \), has a much more significant effect on increasing the permanent deformation (in this case it is \( u_s' \)), while the overall stress-separation curves seem invariable (as seen in Fig. 7 for values of a sufficiently small to complete the analysis). This is consistent with the result presented in Section 3.2 where the frictional dissipation increases for a higher value of the ratio \( a \). Note must be made, however, that the parameter \( a \) must not be too high as per the basic model assumption \( K_s' < K_s \). As a rule of thumb, \( K_s' \) should be at least two orders of magnitude smaller than \( K_s \) and in Fig. 7(a), the analyses corresponding to \( a > 0.1 \) were not completed due to numerical difficulties.

The reason for a small \( K_s' \) is, however, not merely numerical, but is underpinned by the physical significance of our fundamental assumption, being the additive decomposition of the interface as damaged and undamaged parts. It is straightforward to visualise that the apparent shear stiffness \( K_s' \) at a fully debonded interface is, in fact, dependent on the applied pressure. If that pressure is relatively small, the force needed to impose a relative sliding displacement between the surfaces will also be relatively small, depending on the surface roughness; hence a small \( K_s' \). On the other hand, applying a very large pressure between the two surfaces crush surface asperities and result, in the limit, to remerging of the two materials, meaning an effective shear stiffness of the interface that is of the same order of magnitude as the parent material, i.e. \( K_s' \sim K_s \) (in Alfano and Sacco (2006), \( K_s' = K_s \)). This latter scenario is, in our view, in conflict with the premise that damaged and undamaged zones at the interface can be distinguished, as illustrated in Fig. 8. Since the effects of loading on the magnitude of \( K_s' \) are not taken into account in this paper, a simple rule proposed above is adopted to simplify the implementation. Since theoretically \( K_s \) is very high, we found that the current condition on the magnitude of \( K_s' \) is not restrictive, at least for the examples in this paper.

### Table 3: Energy dissipated up to interface debonding, for various levels of pressure \( (r_D = 1) \).

<table>
<thead>
<tr>
<th>Interface pressure ( \tau_a (kN) )</th>
<th>Damage dissipation ( \phi = \int f_D dD (N m) )</th>
<th>Total dissipation ( \phi = \int f_D dD + \int f_{fr} du_s' (N m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>799.9</td>
<td>799.9</td>
</tr>
<tr>
<td>-0.18</td>
<td>2574.1</td>
<td>4941.5</td>
</tr>
<tr>
<td>-0.27</td>
<td>3511.7</td>
<td>5591.1</td>
</tr>
</tbody>
</table>

Fig. 6. Total and damage partition of the energy dissipated for the loading scenario.
is coupled with damage during the delamination of the interface, e.g. due to plastic deformation of the resin or frictional sliding of micro cracks during delamination (Guiamatsia and Nguyen, 2012). Hence the coupling coefficient is initially taken to be $r_D = 1$, such that, accordingly, no plasticity is present in the interface debonding process. With these settings, the response to the cyclic loading is shown in Fig. 9(a) for two values of the friction coefficient $\mu$. As expected, there is not much residual deformation as the compressive stiffness is very small. If however, plasticity during delamination is coupled into the model, in addition to friction from delaminated part of the interface, the prediction is that produced in Fig. 9(b), which shows significantly more residual deformation. The prediction of our model in these two cases is straightforward and very similar to that of a damage/plasticity model, with the addition of the increased yield point. In both cases, it is clear that in our model, the damage component (loss of stiffness) is always present and accompany the other dissipative processes, whereas the loading/unloading curves in Alfano and Sacco’s prediction are more or less parallel to one another and to the elastic loading curve, suggesting that all energy dissipation is linked to the inelastic frictional deformation.

4. Simulation of impact driven delamination

The new interface element is now utilised to predict the extent of delamination in laminated composite plates subjected to impact loading. The experiment considered is that of Aymerich et al. (2008) to which the reader is referred to for more details on the experiment that was performed for several energy levels. For the sake of completeness, key material properties and model parameters are presented in Table 4.

The case chosen for the present simulation was the 5.1 J impact, which corresponds to an initial velocity of 2108.5 mm/s of the 2.3 kg impactor. The finite element model, shown in Fig. 10, considered only a quarter of the plate, taking advantage of the problem symmetry; only the two 0/90 interfaces are modelled with interface elements, as experimental observations showed the confinement of delamination to these locations. These interfaces are designated here by ‘upper’ and ‘lower’ according to their distance from the impactor. The upper interface is expected to be subjected to compression while the lower one is loaded in tension.

![Fig. 7. Shear traction against shear separation.](image7)

![Fig. 8. Interface decomposition. Top: under low compressive stress, the interface can be partitioned into pristine $(1-D)$ and fully damaged $(D)$. Bottom: high compressive stress cause crushing of asperities to the extent that no separate fully damaged partition can be isolated.](image8)

![Fig. 9. Shear traction against shear separation of a single interface. (a) Effect of the friction coefficient, (b) effect of the damage/plasticity coupling.](image9)
from the bending of adjacent CFRP layers. A common practise when using standard commercial tools to model this sort of impact problem is to enhance the interface under compression (the upper one) by specifying higher values of shear strength and toughness. Such enhancement is purely artificial and the higher values utilised are more or less arbitrarily chosen with the sole purpose of fitting the experimental results available, hence the models are devoid of any predictive capabilities.

Here, the following scenarios were examined:

(a) generic interface element utilised as is at both interfaces;
(b) generic interface element with higher shear strength (100 MPa vs. 80 MPa) and toughness (2.5 N mm/mm² vs. 0.97 N mm/mm²) at the upper interface, elements with normal properties at the lower interface;
(c) interface elements with the current constitutive model, without friction activated \( (K_f = 0 \text{ MPa/mm}, \mu = 0) \), used at both interfaces;
(d) interface elements with the current constitutive model, with friction activated, used at both interfaces \( (K_f' = 40 \text{ MPa/mm}, \mu = 0) \).

The predicted delaminated areas on the bottom and upper interfaces are compared with the experimental findings (e) in Fig. 11. When the upper interface is made artificially stronger with the ABAQUS generic cohesive element (b), the prediction of the delaminated area at the lower interface is also slightly smaller than the area predicted without such a fix. The new interface model without friction activated (c) yields more or less the same delaminated area as ABAQUS generic element, which is too large. By using the new model with friction activated (d), a smaller delaminated area is also predicted at both interfaces. If the coefficient of friction is chosen to be \( \mu = 0.8 \), then an excellent match with the experimental findings is obtained, as seen in Fig. 11. It is noted that the partition of damage to total dissipation \( r_D \) is set to 1 since there is no significant plasticity reported in coupon testing also reported in Aymerich et al. (2008).

It is worth mentioning, however, that the parameters needed for the frictional part of the model were not validated for the specific material submitted to the impact test; \( \mu = 0.5 \) is roughly within the range of measurements by Schon (2000), but it was necessary here to use \( \mu = 0.8 \) in order to obtain a good match with experimental results. For a truly predictive model, it is desirable that the coefficient of friction be calibrated independently, for instance, through combined compression–shear loading as suggested in Section 2, although it may also vary in function of

<table>
<thead>
<tr>
<th>Table 4</th>
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<tbody>
<tr>
<td>Test parameters and material properties, from Aymerich et al. (2008).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Interface Thickness</th>
<th>( K_R = 8000 \text{ MPa/0.02 mm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_t = 4000 \text{ MPa/0.02 mm} )</td>
<td></td>
</tr>
<tr>
<td>( \mu = 1.61 \times 10^{-3} \text{ kg/mm}^3 )</td>
<td></td>
</tr>
<tr>
<td>( G_a = 0.52 \text{ N mm/mm}^2 )</td>
<td></td>
</tr>
<tr>
<td>( G_m = 0.97 \text{ N mm/mm}^2 )</td>
<td></td>
</tr>
<tr>
<td>Power law failure, ( n = 1.0 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CFRP layer Ply thickness: 0.125 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{E}<em>{11} = 93.7 \text{ GPa}; \tilde{E}</em>{22} = \tilde{E}_{33} = 7.45 \text{ GPa} )</td>
</tr>
<tr>
<td>( G_{12} = G_{13} = 3.97 \text{ GPa} )</td>
</tr>
<tr>
<td>( \gamma_{12} = \gamma_{13} = \gamma_{23} = 0.261 )</td>
</tr>
<tr>
<td>( \mu = 1.61 \times 10^{-3} \text{ kg/mm}^3 )</td>
</tr>
</tbody>
</table>

Parameters for Hashin yield (ABAQUS, 2010):

| \( Y_{1t} = 2400 \text{ MPa}; Y_{1c} = 2000 \text{ MPa} \) |
| \( Y_{2t} = 100 \text{ MPa}; Y_{2c} = 300 \text{ MPa} \) |
| \( Y_{12} = 300 \text{ MPa}; Y_{23} = 300 \text{ MPa} \) |

Progressive damage parameters (ABAQUS, 2010):

| \( G_{1t} = 40 \text{ N/mm}; G_{1c} = 40 \text{ N/mm} \) |
| \( G_{2t} = 2 \text{ N/mm}; G_{2c} = 3.5 \text{ N/mm} \) |
| \( E_{11} = 93.7 \text{ GPa}; E_{22} = E_{33} = 7.45 \text{ GPa} \) |

Other test parameters

<table>
<thead>
<tr>
<th>CFRP [0(\overline{3},90\overline{3})_s. Plate dimensions: 45 mm \times 67.5 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impactor: ( M = 2.3 \text{ kg}, \text{Diameter} = 12.5 \text{ mm (spherical)} )</td>
</tr>
</tbody>
</table>

Fig. 10. ABAQUS quarter finite element model for the impact test.

Fig. 11. Numerical predictions vs. experiment: delamination at the lower (plots and damage map) and the upper (damage map only) interface.

---

\[ \text{Table 4} \]

Test parameters and material properties, from Aymerich et al. (2008).
the transverse compressive traction, which varies during loading. In addition, we emphasise that the aim of this demonstration is to achieve a better numerical modelling of the friction enhancement to delamination. We do not aim for a high-fidelity modelling of damage propagation in composite laminates, in which the interaction between matrix cracking and delamination (Choi and Chang, 1992; de Moura and Goncalves, 2004) must be taken into account.

5. Simulation of a fibre push out experiment

The experiment of pushing out a polyester fibre embedded in an epoxy resin was performed by Bechel and Sottos (1998) and studied by other researchers (e.g. Hutchinson and Jensen, 1990; Lin et al., 2001). Fig. 12 illustrates the experimental setup and the finite element model used in this analysis, and the geometric and material parameters are as follows:

- \( r_f = 0.95 \text{ mm} \)
- \( r_s = 1.025 \text{ mm} \)
- \( r_m = 4.3 \text{ mm} \)
- \( H = 5.36 \text{ mm} \)
- \( E_f = 2500 \text{ MPa} \)
- \( E_m = 4000 \text{ MPa} \)
- \( G_{xy} = G_{xy}^{sk} = 0.11 \text{ N/mm} \)
- \( \eta = 1.0 \)
- \( \kappa = 0.1 \text{ MPa} \)
- \( \kappa' = 2000/0.001 \text{ MPa/mm} \)
- \( r_D = 1.0 \)

To simulate the reported initial compressive matrix strain of \(-0.0022\), an initial temperature field of \(-2.2 \text{ °C}\) was imposed onto the matrix which is assigned a coefficient of thermal expansion of \(\alpha = 10^{-5}\text{°C}^{-1}\), resulting in a compressive strain along most of the fibre/matrix interface. Then the displacement \(\Delta P\) is applied directly onto the fibre nodes in contact with the punch. A relatively fine mesh was utilised around the interfacial region in both the fibre and the matrix, resulting in a total of 9509 nodes and 9104 elements.

The quasi-static analysis was performed with the current model with and without friction activated, as well as with the ABAQUS/Explicit solver. Fig. 13 shows the final deformations obtained with both the ABAQUS generic interface element and the user-defined interface element implementing the current constitutive model.

The standard (generic) decohesive model is compared with the new unified constitutive model based on the predicted peak load. Fig. 14 shows the various predictions for the total reaction force recorded at the punch against its downward displacement.

It can be noted that the prediction of the new model with no friction activated is in good agreement the prediction of the ABAQUS generic interface element; this is a good verification test. For these two analyses, the delamination is driven only by the relative shear displacement of the fibre respective to the matrix, without consideration of friction. The introduction of friction through the coefficient \(\mu\) results in an increase of the predicted peak load beyond the point of first decohesion (approximately 210 N). In this instance, it is found that taking the value of \(\mu = 0.1\) for the coefficient of friction yields the best match to the experimental finding, with the peak load of about 400 N predicted to occur at a punch displacement of 0.15 mm. The coefficient of friction was again tuned to obtain the best prediction, similar to the impact-driven delamination test, but should really be determined from independent experiments that were not available for the specific material system used here. In addition, as can be seen in Fig. 14, the effects of friction on both the strength and toughness of the interface cannot be captured using ABAQUS generic elements with post failure friction. Our proposed model therefore provides a unified and consistent approach to treat both pre failure (increase of strength and toughness in compression) and post failure frictional effects.

Fig. 12. Fibre push out test: experimental setup (left) and axisymmetric finite element model (right).

Fig. 13. Deformed finite element model; left: ABAQUS generic element, right: User-defined element with new model.

Fig. 14. Various predictions for the total reaction force recorded at the punch against its downward displacement.
6. Conclusion

This paper presented an interface constitutive model coupling damage, plasticity and friction within a consistent thermodynamic framework. Two main ingredients constitute the basis of the model: (1) an expression for the free energy that unifies the three sources of dissipation and couples the evolution of friction to that of damage by separating a unit interface area into distinct damaged area undergoing friction and integral area undergoing damage; and (2) assumed functions of the damage dissipation potentials that accounts for the effect of compression on the evolution of all internal variables. These ingredients were introduced into the existing framework developed by Einav et al. (2007), leading to a set of constitutive equations that are able to capture the observed behaviour of interfaces under a wide spectrum of loading scenarios, in particular, the increased yield point and total dissipation observed when the interface is loaded in combined compression and shear.

Three key parameters control the response of the constitutive model:

- The friction coefficient, \( \mu \), which directly determines the compressed interface shear strength. This total shear strength is equal to the sum of the shear strength under zero compression and the equivalent critical shear strength of the Mohr–Coulomb cone (\(-\mu t_\sigma\)).

- The compressive shear stiffness \( K_s \) of the debonded part of the interface, which controls the amount of inelastic frictional deformation and hence directly controls the partition of energy dissipation by friction.

- The partition of fracture dissipation which is the ratio \( r_D \) of pure damage (loss of stiffness) dissipation to the cumulative dissipation due to damage and plasticity under tensile loading only. In this work, it was assumed that \( r_D = 1 \) as the emphasis was placed on illustrating the ability of the interface model to capture the response under compressive loading.

The constitutive model was implemented as a user-defined element in a commercial finite element package and utilised for the prediction of a number of test cases with readily available experimental data. It was shown that toggling on the frictional dissipation capability resulted in predictions that were more faithful to experimental results and expected trends. The key advantages of this model are (a) the thermodynamic framework used in its development, from which all the constitutive relations are consistently derived with a single yield curve and (b) the meaningfulness of the model parameters which have a clear physical significance and can be straightforwardly calibrated from relatively simple experiments.

In the examples presented, the coefficient of friction was simply tuned until a good match with experiments was achieved. This is because the experimental results used were taken from the existing literature and no coefficient of friction was provided for the specific material systems. As the range for the coefficient of friction can be rather wide and strongly depend on the interfacing materials, an experimental programme needs to be developed for further validation and we are aiming to address this aspect in future work.

We also acknowledge the strong assumption made in the formulation of this model, which is the lumping of the interface roughness into a single stiffness parameter \( K_s \). While the demonstrations in this study show the usefulness and practicality of this assumption, further development to properly take into account the effects of (cracked) surface roughness on both compressive shear stiffness and strength, e.g. (Serpieri and Alfano, 2011) is also planned for the next steps.

Acknowledgments

The authors gratefully acknowledge the support of funding from the Australian Research Council discovery grant DP1093485.

Appendix A. Derivation of \( X(D) \)

Enforcing the continuity of the yield curve between tensile (+) and compressive (−) side of the stress space gives:

\[
y = \frac{2X^{\alpha}_D K_s(1-D)^2}{[1-D]\sqrt{2F(D)K_s^2}} + \left(\frac{t_{ec}}{X}\right)^2 - 1 = \frac{2X^{\alpha}_D K_s(1-D)^2}{[1-D]\sqrt{2F(D)K_s^2}} \Rightarrow X^{\alpha}_D F(D) + \left(\frac{t_{ec}}{X}\right)^2 = X^{\alpha}_D F(D)
\]

In tension, the maximum shear traction is found for zero normal traction:

\[
X^{\alpha}_D F(D) = \frac{(t_{ec})^2}{2(1-D)^2K_s F(D)} = 1 \Rightarrow t_{ec} = t_{ec} = (1-D)\sqrt{2K_s F(D)}
\]

That maximum must be identical to that in the compressive regime at zero normal traction, i.e. \( t^{+}_c = t^{-}_c = t_{ec} + t_{ec} \).

Rewriting the yield function in compression:

\[
X^{\alpha}_D F(D) + \left(\frac{t_{ec}}{X}\right)^2 = 1 \Rightarrow t_{ec} = \frac{(t_{ec})^2}{t_{ec} + t_{ec}} = \frac{(t_{ec})^2}{t_{ec} + t_{ec}} = \frac{t_{ec}}{X} = \sqrt{\frac{1 - (t_{ec})^2}{(t_{ec} + t_{ec})^2}}
\]

Using \( t_{ec} = \frac{\alpha}{K_s} \) and \( t_{ec} + t_{ec} = (1-D)\sqrt{2K_s F(D)} \) gives
\[ X = \frac{t_{sc}}{\sqrt{(t_{sc})^2 (2t_{n} + t_{R})}} = \frac{t_{sc} + t_{R}}{\sqrt{2t_{n} + t_{R}}} \]
\[ = \frac{(1 - D) \sqrt{2K_F(D)}}{(2(1 - D) \sqrt{K_F(D) + 1}} = \frac{(1 - D) \sqrt{2K_F(D)}}{\sqrt{2(1 - D) \sqrt{K_F(D) + 1}} + 1} \]
\[ = \frac{2P(D)D(1 - D)^2 K_s K'_{s}}{2(1 - D)K_s + DK'_{s}} \]

where the individual components of increments of plastic and friction shear separation are, with \( j = 1, 2 \):

\[ \delta u_{j}^{pl} = \delta u_{j}^{pl} \quad \delta u_{j}^{f} = \delta u_{j}^{f} \]

Step 2 is repeated iteratively until the value of the yield is within a certain accuracy tolerance, \(|y| < \text{tol}|\).

### References
