JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **227,** 144–165 (1998)
ARTICLE NO. AY986090

# Paracompactness and Full Normality in Ditopological Texture Spaces

Lawrence M. Brown

*Department of Mathematics, Faculty of Science, Hacettepe University, 06532 Beytepe, Ankara, Turkey;*
E-mail: brown@eti.cc.hun.edu.tr

and

Murat Diker

*Department of Science Education, Hacettepe University, 06532 Beytepe, Ankara, Turkey;*
E-mail: mdiker@eti.cc.hun.edu.tr

*Submitted by Ying-Ming Liu*

Received February 10, 1998

In this paper the authors lay the foundation for a theory of dicovers of ditopological texture spaces and use this to define notions of paracompactness and full normality. Some applications to fuzzy topology are also mentioned. 

*Key Words:* texture space; ditopology; dicover; dicover paracompact; dicover co-paracompact; dicover biparacompact; dicover fully normal; fuzzy topology; $\alpha$-paracompact; $\alpha$, $\epsilon$-paracompact

## 1. INTRODUCTION

Texture spaces, initially called fuzzy structures, were introduced in [3, 4]. The motivation was to provide a point set setting for the study of fuzzy sets, and a detailed account of the relation between fuzzy lattices, fuzzy sets, $L$-fuzzy sets [14], generalized fuzzy sets [20], and intuitionistic sets [11] on the one hand and various classes of texture space on the other may be found in [5–7]. The notion of a dichotomous topology (or ditopology for short) on a texture space was also introduced in [3]. A fuzzy topology



0022-247X/98 $25.00

corresponds in a natural way to a ditopology on the corresponding texture space, as we will see below, and the reader is referred to [8] for more details. There is also a close relation between ditopologies and bitopologies, and this has served to motivate notions of compactness [3–5] and connectedness [12].

The aim of this paper is to lay the foundation for a theory of dicovers for ditopological texture spaces, and use this as the basis for a discussion of paracompactness and full normality. Here again we will be motivated by bitopological concepts, particularly the theory of dual covers given in [1]. For the benefit of the reader, and to make the paper reasonably self-contained, we will begin by recalling some definitions from [3].

DEFINITION 1.1. Let $S$ be a set. Then $\mathcal{S} \subseteq \mathcal{P}(S)$ is called a *texturing* of $S$, and $(S, \mathcal{S})$ is called a *texture space*, or simply a *texture*, if

(1) $(\mathcal{S}, \subseteq)$ is a complete lattice containing $S$ and $\emptyset$, which has the property that arbitrary meets coincide with intersections, and finite joins coincide with unions.

(2) $\mathcal{S}$ is completely distributive.

(3) $\mathcal{S}$ separates the points of $S$. That is, given $s_1 \neq s_2$ in $S$, we have $A \in \mathcal{S}$ with $s_1 \in A$, $s_2 \notin A$, or $A \in \mathcal{S}$ with $s_2 \in A$, $s_1 \notin A$.

A surjection $\sigma: \mathcal{S} \to \mathcal{S}$ is called a *complementation* if $\sigma^2(P) = P$ for all $P \in \mathcal{S}$ and $P \subseteq Q$ in $\mathcal{S}$ implies $\sigma(Q) \subseteq \sigma(P)$. A texture with a complementation is said to be *complemented*.

EXAMPLES 1.2. (1) For any set $X$, $(X, \mathcal{P}(X), \pi)$, $\pi(A) = X \setminus A$ is a complemented texture space representing the usual (crisp) set structure of $X$.

(2) Let $L = (0,1]$, $\mathcal{L} = \{(0,r] \mid r \in [0,1]\}$, and $\lambda((0,r]) = (0, 1-r]$, $r \in [0,1]$. Then $(L, \mathcal{L}, \lambda)$ is a complemented texture space. It is the texture corresponding to the classical fuzzy lattice $[0,1]$ (see [6]).

(3) For $I = [0,1]$ define $\mathcal{T} = \{[0,t] \mid t \in [0,1]\} \cup \{[0,t) \mid t \in [0,1]\}$, $\iota([0,t]) = [0, 1-t)$ and $\iota([0,t)) = [0, 1-t]$, $t \in [0,1]$. Again $(I, \mathcal{T}, \iota)$ is a complemented texture space.

As usual, for $s \in S$, we define $P_s \in \mathcal{S}$ by $P_s = \bigcap\{A \mid s \in A \in \mathcal{S}\}$, so that $s \mapsto P_s$ is an embedding of $S$ in $\mathcal{S}$, and the sets $P_s$, $s \in S$, form a base of $\mathcal{S}$ in the sense that for each $A \in \mathcal{S}$ we have $A = \bigvee\{P_s \mid s \in A\}$ (and, in fact, $A = \bigcup\{P_s \mid s \in A\}$). Trivially each $P_s$ is a molecule in $\mathcal{S}$ (i.e., $P_s \neq \emptyset$ and $P_s \subseteq A \cup B$, $A, B \in \mathcal{S} \Rightarrow P_s \subseteq A$ or $P_s \subseteq B$), and $(S, \mathcal{S})$ is called *simple* if all of the molecules have this form. As noted in [6], there is a one-to-one correspondence between complemented simple textures and fuzzy lattices. We will return to this correspondence later in this paper.

Looking at the above examples, we note that in turn, $P_x = \{x\}$, $P_r = (0, r]$, and $P_t = [0, t]$. The first two textures are simple, but the third is not, since the sets $[0, t)$, t $\in (0, 1]$, are also molecules.

As in [8] we will also consider the sets $Q_s \in \mathscr{S}$, $s \in S$, defined by

$$Q_s = \bigvee \{P_t \mid s \notin P_t\}.$$

Referring to the above examples, we have $Q_x = X \setminus \{x\}$, $Q_r = (0, r] = P_r$, and $Q_t = [0, t)$ respectively. The second example shows clearly that we can have $s \in Q_s$, and indeed even $Q_s = S$. Also, in general, the sets $Q_s$ do not have any clear relation with either the set theoretic complement or the complementation on $\mathscr{S}$. They are, however, closely connected with the notion of the core of the sets in $\mathscr{S}$. We recall from [8] that for $A \in \mathscr{S}$ the *core* of $A$ is the set

$$\operatorname{core}(A) = \bigcap \left\{ \bigcup \{A_i \mid i \in I\} \mid A = \bigvee \{A_i \mid A_i \in \mathscr{S}, i \in I\} \right\}.$$

Clearly $\operatorname{core}(A) \subseteq A$, and in general we can have $\operatorname{core}(A) \notin \mathscr{S}$. We will generally denote $\operatorname{core}(A)$ by $A^\flat$. The following facts will be used in the sequel. The reader is referred to [8, Theorem 2.4] for details of the proof, part of which depends heavily on the fact that $(S, \mathscr{S})$ is completely distributive.

LEMMA 1.3. (1) *$s \notin A \Rightarrow A \subseteq Q_s \Rightarrow s \notin A^\flat$ for all $s \in S$, $A \in \mathscr{S}$.*

(2) *$A^\flat = \{s \mid A \not\subseteq Q_s\}$ for all $A \in \mathscr{S}$.*

(3) *For $A_i \in \mathscr{S}$, $i \in I$, we have $(\bigvee_{i \in I} A_i)^\flat = \bigcup_{i \in I} A_i^\flat$.*

(4) *$A$ is the smallest element of $\mathscr{S}$ containing $A^\flat$ for all $A \in \mathscr{S}$.*

(5) *For $A, B \in \mathscr{S}$, if $A \not\subseteq B$, then there exists $s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq B$.*

(6) *$A = \bigcap \{Q_s \mid s \notin A\}$ for all $A \in \mathscr{S}$.*

We now recall the definition of a *dichotomous topology* (or *ditopology* for short) on a texture first given in [3].

DEFINITION 1.4. $(\tau, \kappa)$ is called a *ditopology* on $(S, \mathscr{S})$ if

(1) $\tau \subseteq \mathscr{S}$ satisfies

a. $S, \emptyset \in \tau$

b. $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$

c. $G_i \in \tau$, $i \in I \Rightarrow \bigvee_{i \in I} G_i \in \tau$, and

(2) $\kappa \subseteq \mathscr{S}$ satisfies

a. $S, \emptyset \in \kappa$

b. $F_1, F_2 \in \kappa \Rightarrow F_1 \cup F_2 \in \kappa$

c. $F_i \in \kappa,\ i \in I \Rightarrow \bigcap_{i \in I} F_i \in \kappa.$

The sets of $\tau$ are called *open* and those of $\kappa$ are called *closed*. In general there is no *a priori* relation between $\tau$ and $\kappa$, but if $\sigma$ is a complementation on $(S, \mathscr{S})$ and $\tau,\ \kappa$ are connected by the relation $\kappa = \sigma(\tau)$, then we call $(\tau, \kappa)$ a *complemented ditopology* on $(S, \mathscr{S}, \sigma)$.

Finally, let $Z \subseteq S$. Then the *closure* of $Z$ is the set $[Z] = \bigcap\{F \in \kappa \mid Z \subseteq F\}$, and the *interior* is $]Z[ = \bigvee\{G \in \tau \mid G \subseteq Z\}$.

A complemented ditopology on $(X, \mathscr{P}(X), \pi)$ is precisely a topology. On the other hand, a (not necessarily complemented) ditopology on $(X, \mathscr{P}(X))$ can be associated with a bitopology on $X$. One of the topologies is $\tau$, and the other $\kappa' = \{X \setminus K \mid K \in \kappa\}$. Since $\mathscr{S}$ need not be closed under set-theoretic complementation, this formal relation with bitopologies breaks down in the general case, but even so, many bitopological concepts may be adapted to the ditopological setting by first expressing them in a form involving the open sets from one topology and the closed sets from the other. As a case in point, the following axioms, introduced in [5], reflect Kopperman's subdivision of bitopological joint compactness into compactness and stability properties [17].

DEFINITION 1.5. Let $(\tau, \kappa)$ be a ditopology on the texture $(S, \mathscr{S})$. Then $(\tau, \kappa)$ is called

(i) *Compact* if whenever $S = \bigvee_{i \in I} G_i, G_i \in \tau, i \in I$, there is a finite subset $J$ of $I$ with $\bigcup_{j \in J} G_j = S$.

(ii) *Co-compact* if whenever $\bigcap_{i \in I} F_i = \emptyset, F_i \in \kappa, i \in I$, there is a finite subset $J$ of $I$ with $\bigcap_{j \in J} F_j = \emptyset$.

(iii) *Stable* if every $K \in \kappa$ with $K \neq S$ is compact, i.e., whenever $K \subseteq \bigvee_{i \in I} G_i, G_i \in \tau, i \in I$, there is a finite subset $J$ of $I$ with $K \subseteq \bigcup_{j \in J} G_j$.

(iv) *Co-stable* if every $G \in \tau$ with $G \neq \emptyset$ is co-compact, i.e., whenever $\bigcap_{i \in I} F_i \subseteq G, F_i \in \kappa, i \in I$, there is a finite subset $J$ of $I$ with $\bigcap_{j \in J} F_j \subseteq G$.

As noted in [5], for a complemented ditopological texture, compactness and co-compactness coincide, as do stability and co-stability.

## 2. DICOVERS AND PARACOMPACTNESS

The notion of dicover was used in [5] in relation to a concept of compactness for ditopological textures.

DEFINITION 2.1. A subset $\mathscr{C}$ of $\mathscr{S} \times \mathscr{S}$ is called a *difamily* on $(S, \mathscr{S})$. Let $\mathscr{C} = \{(G_\alpha, F_\alpha) \mid \alpha \in A\}$ be a difamily on $(S, \mathscr{S})$. Then $\mathscr{S}$ is called a *dicover* of $(S, \mathscr{S})$ if for all partitions $A_1$, $A_2$ of $A$ (including the trivial partitions) we have

$$\bigcap_{\alpha \in A_1} F_\alpha \subseteq \bigvee_{\alpha \in A_2} G_\alpha .$$

Now let $(\tau, \kappa)$ be a ditopology on $(S, \mathscr{S})$. Then a difamily $\mathscr{C}$ on $(S, \mathscr{S}, \tau, \kappa)$ is called *(co-)open* if $\mathrm{dom}(\mathscr{C}) \subseteq \tau$ $(\mathrm{ran}(\mathscr{C}) \subseteq \tau)$, and *(co-)closed* if $\mathrm{dom}(\mathscr{C}) \subseteq \kappa$ $(\mathrm{ran}(\mathscr{C}) \subseteq \kappa)$.

For a difamily $\mathscr{C}$ we will normally write $G\mathscr{C}F$ in preference to $(G, F) \in \mathscr{C}$.

If $\mathscr{C} = \{(G_\alpha, F_\alpha) \mid \alpha \in A\}$ is an open and co-closed dicover on $(X, \mathscr{P}(X))$, then $\mathscr{D} = \{(G_\alpha, X \setminus F_\alpha) \mid \alpha \in A\}$ is an open *dual cover* on $X$ in the sense of [1]. Dual covers have been studied extensively in the bitopological literature, under a variety of different guises. See, for example, [2, 13, 21]. Our definitions relating to dicovers reflect this relation with dual covers, but it should be noted that for textures in which infinite joins do not necessarily coincide with unions, not all of the properties of dual covers carry over. In particular, for each $x \in X$ we have $\alpha \in A$ with $x \in G_\alpha \cap (X \setminus F_\alpha)$, that is, $x \in G_\alpha$ and $x \notin F_\alpha$. Furthermore, pairs in a dual cover with an empty intersection may be removed without affecting its status as a dual cover, and therefore it may be assumed without loss of generality that all pairs in the cover meet. This corresponds to the condition $G_\alpha \not\subseteq F_\alpha$, $\alpha \in A$, but we now present an example to show that these properties are not satisfied for dicovers in general. First we give the following result, which is important in its own right.

LEMMA 2.2. *Let $(S, \mathscr{S})$ be a texture. Then $\mathscr{P} = \{(P_s, Q_s) \mid s \in S^\flat\}$ is a dicover of $S$.*

*Proof.* Let $S_1^\flat, S_2^\flat$ be a partition of $S^\flat$. We must show that $\bigcap_{s \in S_1^\flat} Q_s \subseteq \bigvee_{s \in S_2^\flat} P_s$. Now $\bigcap_{s \in S_1^\flat} Q_s = \bigcap_{s \in S_1^\flat} \bigvee_{s \notin P_t} P_t$, and since both $(S, \mathscr{S})$ and $(S, \mathscr{P}(S))$ are completely distributive, it is not difficult to show that it will be sufficient to verify

$$\bigcap_{s \in S_1^\flat} \bigcup_{s \notin P_t} P_t \subseteq \bigcup_{s \in S_2^\flat} P_s .$$

Suppose this inclusion does not hold. Then by Lemma 1.3(5) we have $r \in S$ with $P_r \not\subseteq \bigcup_{s \in S_2^\flat} P_s$ and $\bigcap_{s \in S_1^\flat} \bigcup_{s \notin P_t} P_t \not\subseteq Q_r$. This last relation shows $Q_r \neq S$, whence $r \in S^\flat$ by Lemma 1.3(2) applied to $A = S$. Hence $r \in S_1^\flat$ or $r \in S_2^\flat$. However, both possibilities lead to an immediate contradiction, and $\mathscr{P}$ is a dicover as claimed. ∎

COROLLARY. *Given $A \in \mathscr{S}$, $A \neq \emptyset$, there exists $s \in S^\flat$ with $P_s \subseteq A$.*

*Proof.* Suppose that for some $A \in \mathscr{S}$ we have $P_s \not\subseteq A$ $\forall s \in S^\flat$. Then, by Lemma 1.3(1), we have $A \subseteq \bigcap_{s \in S^\flat} Q_s \subseteq \bigvee_{s \in \emptyset} P_s = \emptyset$, whence $A = \emptyset$.

If we apply Lemma 2.2 to the texture of Examples 1.2(2), we see that $\mathscr{P} = \{((0,r],(0,r]) \mid r \in (0,1)\}$ is a dicover $\{(G_\alpha, F_\alpha) \mid \alpha \in A\}$ for which $G_\alpha = F_\alpha$ for all $\alpha \in A$, thus providing a counterexample to the properties mentioned above.

DEFINITION 2.3. Let $(S, \mathscr{S})$ be a texture, $\mathscr{C}$ and $\mathscr{C}'$ difamilies in $(S, \mathscr{S})$. Then $\mathscr{C}$ is said to be a *refinement* of $\mathscr{C}'$, written $\mathscr{C} \prec \mathscr{C}'$, if given $A\mathscr{C}B$ we have $A'\mathscr{C}'B'$ with $A \subseteq A'$ and $B' \subseteq B$.

If $\mathscr{C}$ is a dicover and $\mathscr{C} \prec \mathscr{C}'$, then clearly $\mathscr{C}'$ is a dicover. On the other hand, when we speak of a refinement of a dicover, we shall always mean a dicover refinement unless stated otherwise. Given dicovers $\mathscr{C}$ and $\mathscr{D}$, it is easy to verify that

$$\mathscr{C} \wedge \mathscr{D} = \{(A \cap C, B \cup D) \mid A\mathscr{C}B, C\mathscr{D}D\}$$

is also a dicover. It is the meet of $\mathscr{C}$ and $\mathscr{D}$ with respect to the refinement relation.

We now present some finiteness properties for difamilies.

DEFINITION 2.4. Let $\mathscr{C} = \{(G_i, F_i) \mid i \in I\}$ be a difamily faithfully indexed over $I$. Then $\mathscr{C}$ is said to be

(i) *Finite* (*co-finite*) if $\operatorname{dom}(\mathscr{C})$ (resp., $\operatorname{ran}(\mathscr{C})$) is finite.

(ii) *Locally finite* if for all $s \in S$ there exists $K_s \in \kappa$ with $P_s \not\subseteq K_s$ so that the set $\{i \mid G_i \not\subseteq K_s\}$ is finite.

(iii) *Locally co-finite* if for all $s \in S$ with $Q_s \neq S$ there exists $H_s \in \tau$ with $H_s \not\subseteq Q_s$ so that the set $\{i \mid H_s \not\subseteq F_i\}$ is finite.

(iv) *Point finite* if for each $s \in S$ the set $\{i \mid P_s \subseteq G_i\}$ is finite.

(v) *Point co-finite* if for each $s \in S$ with $Q_s \neq S$ the set $\{i \mid F_i \subseteq Q_s\}$ is finite.

To fully appreciate the symmetry between the definitions of locally finite and locally co-finite, note that there is an implied condition "$P_s \neq \emptyset$" in the definition of locally finite difamily, which remains implicit since it is always satisfied, whereas, as we have seen, it is possible to have $Q_s = S$.

This point is further clarified by the following results. The proofs are trivial and are omitted.

LEMMA 2.5. *The following are equivalent*:

(*a*) $\mathscr{C} = \{(G_i, F_i) \mid i \in I\}$ *is locally finite*.

(b) *There exists a family* $\mathscr{B} = \{B_j \mid j \in J\} \subseteq \mathscr{S} \setminus \{\varnothing\}$ *with the property that for* $A \in \mathscr{S}$ *with* $A \neq \varnothing$, *we have* $j \in J$ *with* $B_j \subseteq A$, *and for each* $j \in J$ *there is* $K_j \in \kappa$ *so that* $B_j \not\subseteq K_j$ *and the set* $\{i \mid G_i \not\subseteq K_j\}$ *is finite*.

LEMMA 2.6. *The following are equivalent*:

(a) $\mathscr{C} = \{(G_i, F_i) \mid i \in I\}$ *is locally co-finite*.

(b) *There exists a family* $\mathscr{B} = \{B_j \mid j \in J\} \subseteq \mathscr{S} \setminus \{S\}$ *with the property that for* $A \in \mathscr{S}$ *with* $A \neq S$ *we have* $j \in J$ *with* $A \subseteq B_j$, *and for each* $j \in J$ *there is* $H_j \in \tau$ *so that* $H_j \not\subseteq B_j$ *and the set* $\{i \mid H_j \not\subseteq F_i\}$ *is finite*.

We are now in a position to prove

THEOREM 2.7. *The difamily* $\mathscr{C} = \{(G_i, F_i) \mid i \in I\}$ *is locally finite if for each* $s \in S$ *with* $Q_s \neq S$ *we have* $K_s \in \kappa$ *with* $P_s \not\subseteq K_s$, *so that the set* $\{i \mid G_i \not\subseteq K_s\}$ *is finite*.

*Proof.* Apply Lemma 2.5 to $\mathscr{B} = \{P_s \mid Q_s \neq S\} = \{P_s \mid s \in S^\flat\}$, which satisfies $A \in \mathscr{S} \setminus \{\varnothing\} \Rightarrow \exists s \in S^\flat$ with $P_s \subseteq A$ by the corollary to Lemma 2.2.

Similar properties hold for point finiteness and point co-finiteness. The details are left to the reader.

The next two theorems give results that will be needed later on.

THEOREM 2.8. *Let* $\mathscr{C}$ *be a locally finite* (*locally co-finite*) *dicover and* $s \in S$ (*resp*., $s \in S^\flat$). *Then there exists* $A \mathscr{C} B$ *with* $s \in A$ *and* $s \notin B$.

*Proof.* Let $\mathscr{C} = \{(A_i, B_i) \mid i \in I\}$ be locally finite and $s \in S$. Choose $K \in \kappa$ with $s \notin K$ and $\{i \in I \mid A_i \not\subseteq K\}$ finite. Consider the partition $I_1 = \{i \in I \mid s \in A_i\}$, $I_2 = \{i \in I \mid s \notin A_i\}$ of $I$. Then $\bigcap_{i \in I_1} B_i \subseteq \bigvee_{i \in I_2} A_i$. The set on the right does not contain $s$, since all but finitely many of the sets $A_i$, $i \in I_2$, are contained in $K$. Hence $s \notin \bigcap_{i \in I_1} B_i$, which gives $i \in I_1$ with $s \notin B_i$, as required.

The proof of the second result is similar.

THEOREM 2.9. *Let* $\mathscr{C} = \{(A_i, B_i) \mid i \in I\}$ *be a difamily*.

(a) *If* $\mathscr{C}$ *is locally finite, then* $\mathrm{dom}(\mathscr{C})$ *is closure preserving*.

(b) *If* $\mathscr{C}$ *is locally co-finite, then* $\mathrm{ran}(\mathscr{C})$ *is interior preserving*.

*Proof.* We prove (a), leaving (b) to the reader. Take $I' \subseteq I$. Trivially $\bigvee_{i \in I'} [A_i] \subseteq [\bigvee_{i \in I'} A_i]$, so suppose the opposite inclusion does not hold. By Lemma 1.3(5) we have $s \in S$ with $[\bigvee_{i \in I'} A_i] \not\subseteq Q_s$ and $P_s \not\subseteq \bigvee_{i \in I'}[A_i]$. Since $\mathscr{C}$ is locally finite, we have $K \in \kappa$ with $P_s \not\subseteq K$, so that $\{i \in I \mid A_i \not\subseteq K\}$ is finite. Let $I_1 = \{i \in I' \mid A_i \not\subseteq K\}$ and $I_2 = I' \setminus I_1$. Then

$$\bigvee_{i \in I'} A_i = \bigvee_{i \in I_2} A_i \cup \bigcup_{i \in I_1} A_i \subseteq K \cup \bigcup_{i \in I_1} [A_i] \in \kappa,$$

so $[\bigvee_{i \in I'} A_i] \subseteq K \cup \bigcup_{i \in I_1} [A_i]$. However, from $P_s \not\subseteq \bigvee_{i \in I'} [A_i]$, $I_1 \subseteq I'$, and $P_s \not\subseteq K$ we may now deduce $[\bigvee_{i \in I'} A_i] \subseteq Q_s$, which is a contradiction. Hence

$$\left[\bigvee_{i \in I'} A_i\right] = \bigvee_{i \in I'} [A_i],$$

as required. ∎

DEFINITION 2.10. A ditopological texture $(S, \mathscr{S}, \tau, \kappa)$ is said to be

(i) *Dicover paracompact* if every open, co-closed dicover of $S$ has an open, co-closed locally finite refinement.

(ii) *Dicover co-paracompact* if every open, co-closed dicover of $S$ has an open, co-closed locally finite refinement.

(iii) *Dicover biparacompact* if it is both dicover paracompact and dicover co-paracompact.

If $(S, \mathscr{S}, \tau, \kappa, \sigma)$ is a complemented ditopological structure and $\mathscr{C} = \{(G_\alpha, F_\alpha) \mid \alpha \in A\}$ is an open, co-closed dicover, then so is $\sigma(\mathscr{C}) = \{(\sigma(F_\alpha), \sigma(G_\alpha)) \mid \alpha \in A\}$, while $\sigma$ maps a family $\mathscr{B}$ satisfying the conditions of Lemma 2.5 into one satisfying the conditions of Lemma 2.6, and conversely. Hence we have proved

THEOREM 2.11. *In a complemented ditopological texture the notions of dicover paracompactness, dicover co-paracompactness, and dicover biparacompactness coincide.*

DEFINITION 2.12. A ditopological texture is *dicover compact* (*dicover co-compact*) if every open, co-closed dicover has a finite (resp., co-finite) subcover. A ditopological fuzzy structure that is both dicover compact and dicover co-compact is *dicover bicompact*.

Clearly a ditopological texture is dicover compact if and only if every element of $\kappa$ is compact, i.e., the space is compact and stable in the sense of Definition 1.5. Likewise, dicover co-compactness is equivalent to co-compactness plus co-stability. The following result follows trivially from the definitions:

THEOREM 2.13. *A dicover compact* (*co-compact, bicompact*) *texture is dicover paracompact* (*resp., co-paracompact, biparacompact*).

EXAMPLE 2.14. Consider the texture $(I, \mathscr{T}, \iota)$ of Examples 1.2(3) and let $\tau = \{[0, r) \mid 0 \leq r \leq 1\} \cup \{I\}$, $\kappa = \{[0, r] \mid 0 \leq r \leq 1\} \cup \{\emptyset\} = \iota(\tau)$. This *lower ditopological unit interval* is dicover bicompact, and hence dicover biparacompact.

## 3. REGULARITY AND NORMALITY CONDITIONS

We begin this section by looking at regularity in relation to dicover (co-)paracompactness. First we recall the following definition from [8].

DEFINITION 3.1. A ditopology $(\tau, \kappa)$ on the texture $(S, \mathscr{S})$ is said to be

(a) $R_1$ if $G \in \tau, G \not\subseteq Q_s, P_t \not\subseteq G \Rightarrow \exists H \in \tau$ with $H \not\subseteq Q_s$, $P_t \not\subseteq [H]$.

(b) Co-$R_1$ if $F \in \kappa, P_s \not\subseteq F, F \not\subseteq Q_t \Rightarrow \exists K \in \kappa$ with $P_s \not\subseteq K$, $]K[ \not\subseteq Q_t$.

(c) *Regular* if $G \in \tau, G \not\subseteq Q_s \Rightarrow \exists H \in \tau$ with $H \not\subseteq Q_s, [H] \subseteq G$.

(d) *Co-regular* if $F \in \kappa, P_s \not\subseteq F \Rightarrow \exists K \in \kappa$ with $P_s \not\subseteq K, F \subseteq ]K[$.

For each property $\mathscr{P}$ in this list, $\mathscr{P}$ and co-$\mathscr{P}$ are equivalent for a complemented ditopology. It should also be noted that in establishing regularity it is sufficient to find $H \in \tau$ satisfying $s \in K, [H] \subseteq G$. Likewise, the requirement $P_s \not\subseteq K$ in the definition of co-regularity may be weakened to $K \subseteq Q_s$. The reader is referred to [8] for details of the proof.

THEOREM 3.2. (i) *A $R_1$ co-paracompact ditopological texture is regular.*

(ii) *A co-$R_1$ paracompact ditopological texture is co-regular.*

*Proof.* We prove (i), leaving the essentially dual proof of (ii) to the reader.

To prove regularity, take $s \in S$ and $G \in \tau$ with $G \not\subseteq Q_s$. By the $R_1$ axiom, for all $t \in S$ with $t \notin G$, we have $H_t \in \tau$ with $H_t \not\subseteq Q_s, t \notin [H_t]$. Using Lemma 1.3(5) we may establish $\bigcap_{t \notin G} [H_t] \subseteq G$, whence

$$\mathscr{C} = \{(S, [H_t]) \mid t \notin G\} \cup \{(G, \emptyset)\}$$

is an open, co-closed dicover of $(S, \mathscr{S})$. Let $\mathscr{D} = \{(G_\alpha, F_\alpha) \mid \alpha \in A\}$ be an open, co-closed locally co-finite refinement of $\mathscr{C}$. Since $G \not\subseteq Q_s$ implies $Q_s \neq S$, we have $H \in \tau$ with $H \not\subseteq Q_s$, so that the set $\{\alpha | H \not\subseteq F_\alpha\}$ is finite. Let $A_1^1 = \{\alpha | H \subseteq F_\alpha\}$, $A_1^2 = \{\alpha | H \not\subseteq F_\alpha$ and $G_\alpha \not\subseteq G\}$, and $A_2 = A \setminus (A_1^1 \cup A_1^2)$. Since $\mathscr{D}$ is a dicover we have

$$\bigcap_{\alpha \in A_1^1} F_\alpha \cup \bigcap_{\alpha \in A_1^2} F_\alpha \subseteq \bigvee_{\alpha \in A_2} G_\alpha \subseteq G.$$

Clearly $s \in H \subseteq \bigcap_{\alpha \in A_1^1} F_\alpha \in \kappa$. For $\alpha \in A_1^2$ we have $G_\alpha \not\subseteq G$, whence since $\mathscr{D} \prec \mathscr{C}$, we have $t_\alpha \in S$ with $[H_{t_\alpha}] \subseteq F_\alpha$. From $H_{t_\alpha} \not\subseteq Q_s$ we deduce $s \in \bigcap_{\alpha \in A_1^2} H_{t_\alpha} \subseteq \bigcap_{\alpha \in A_1^2} F_\alpha \in \kappa$. Since $A_1^2$ is finite, $M = H \cap \bigcap_{\alpha \in A_1^2} H_{t_\alpha} \in \tau$. Clearly $s \in M$ and $[M] \subseteq G$. By the note following Definition 3.1, this shows that $(\mathscr{S}, \tau, \kappa)$ is regular. ∎

We now turn to normality.

DEFINITION 3.3. The ditopology $(\tau, \kappa)$ on $(S, \mathscr{S})$ is called

(1) *Normal* if given $G \in \tau$ and $F \in \kappa$ with $F \subseteq G$ there exists $H \in \tau$ with $F \subseteq H \subseteq [H] \subseteq G$.

(2) *Dicover normal* if given an open, co-closed difamily $\mathscr{D}$ and $G \in \tau$, $F \in \kappa$ so that $\mathscr{D} \cup \{(G, F)\}$ is a dicover, there exists $H \in \tau$ with $[H] \subseteq G$ so that $\mathscr{D} \cup \{(H, F)\}$ is a dicover.

(3) *Dicover co-normal* if given an open, co-closed difamily $\mathscr{D}$ and $G \in \tau$, $F \in \kappa$ so that $\mathscr{D} \cup \{(G, F)\}$ is a dicover, there exists $K \in \kappa$ with $F \subseteq ]K[$ so that $\mathscr{D} \cup \{(G, K)\}$ is a dicover.

(4) *Dicover binormal* if it is dicover normal and dicover co-normal.

Normality reduces to (one expression for) the usual normality axiom in the topological case and to the pairwise normality of J. C. Kelly [16] in the bitopological case, and corresponds to the normality property usually considered in fuzzy topology [19]. The dicover normality conditions are generalizations of a stronger normality axiom (binormality) for bitopological spaces discussed in [1]. However, the treatment in [1] is based on the joint topology, a concept that has no clear counterpart in ditopological textures, and the notion of dicover has been used here in its place. Let us first note some elementary results concerning these concepts.

THEOREM 3.4. (1) *A dicover normal ditopological texture is normal.*

(2) *A dicover co-normal ditopological texture is normal.*

(3) *The following are equivalent for a ditopological texture* $(\mathscr{S}, \tau, \kappa)$ *on* $S$:

a. $(\mathscr{S}, \tau, \kappa)$ *is dicover binormal.*

b. *Given an open, co-closed difamily* $\mathscr{D}$ *and* $G \in \tau$, $F \in \kappa$, *so that* $\mathscr{D} \cup \{(G, F)\}$ *is a dicover, there exists* $H \in \tau$ *and with* $[H] \subseteq G$ *and* $K \in \kappa$ *with* $F \subseteq ]K[$, *so that* $\mathscr{D} \cup \{(H, K)\}$ *is a dicover.*

(4) *In a complemented ditopological texture the notions of dicover normal, dicover co-normal, and dicover binormal coincide.*

*Proof.* (1) Take $G \in \tau$ and $F \in \kappa$ with $F \subseteq G$. If we let $\mathscr{D} = \{(S, F)\}$, then it is trivial to verify that $\mathscr{D} \cup \{(G, \emptyset)\}$ is a dicover. Hence we have $H \in \tau$ with $[H] \subseteq G$, so that $\mathscr{D} \cup \{(H, \emptyset)\}$ is a dicover. But then $F \subseteq H$, which proves normality.

(2) Similar to (1).

(3) Straightforward.

(4) Suppose that $(\mathscr{S}, \tau, \kappa, \sigma)$ is dicover normal. Let $\mathscr{D}$ be an open, co-closed difamily and $G \in \tau$, $F \in \kappa$ be such that $\mathscr{D} \cup \{(G, F)\}$ is a dicover. Then $\sigma(\mathscr{D})$ is also an open, co-closed difamily, $\sigma(\mathscr{D}) \cup \{(\sigma(F), \sigma(G))\}$ is a dicover, $\sigma(F) \in \tau$ and $\sigma(G) \in \kappa$. Hence we have $H \in \tau$ with $[H] \subseteq \sigma(F)$, so that $\sigma(\mathscr{D}) \cup \{(H, \sigma(G))\}$ is a dicover. Now letting $K = \sigma(H)$, we deduce that $\mathscr{D} \cup \{(G, K)\}$ is a dicover, $K \in \kappa$ and $F \subseteq ]K[$. Hence $(\mathscr{S}, \tau, \kappa, \sigma)$ is dicover co-normal. The converse is similar. ∎

We may now state

THEOREM 3.5. (1) *A regular, dicover paracompact ditopological texture is dicover normal.*

(2) *A co-regular, dicover co-paracompact ditopological texture is dicover co-normal.*

*Proof.* Since these results are essentially dual to one another, we content ourselves with proving (1).

Let $\mathscr{D} = \{(G_\alpha, F_\alpha) \mid \alpha \in A\}$ be an open, co-closed difamily and let $G \in \tau$, $F \in \kappa$ be such that $\mathscr{C} = \mathscr{D} \cup \{(G, F)\}$ is a dicover. We wish to find $H \in \tau$ with $[H] \subseteq G$ so that $\mathscr{D} \cup \{(H, F)\}$ is a dicover. Clearly we may assume without loss of generality that $\mathscr{D}$ is not a dicover, for otherwise we may simply take $H = \emptyset$.

For $s \in S$ with $G \not\subseteq Q_s$ we have $H_s \in \tau$ with $[H_s] \subseteq G$ by regularity. Using a by now familiar argument, we may verify $G = \bigvee_{G \not\subseteq Q_s} H_s$, whence $\mathscr{E} = \mathscr{D} \cup \{(H_s, F) \mid G \not\subseteq Q_s\}$ is a dicover of $(S, \mathscr{S})$, which is clearly open and co-closed. Let $\mathscr{F} = \{(M_\beta, N_\beta) \mid \beta \in B\}$ be an open, co-closed locally finite refinement of $\mathscr{E}$, and define

$$B' = \left\{ \beta \in B \mid \alpha \in A \Rightarrow M_\beta \not\subseteq G_\alpha \text{ or } F_\alpha \not\subseteq N_\beta \right\}.$$

Note that $B' \neq \emptyset$, for we are assuming $\mathscr{D}$ is not a dicover, whence $\mathscr{F}$ cannot refine $\mathscr{D}$. Define $H = \bigvee\{M_\beta \mid \beta \in B'\} \in \tau$. We wish to show $H$ has the required properties. Suppose first that $[H] \not\subseteq G$ and take $s \in S$ with $P_s \not\subseteq G$ and $[H] \not\subseteq Q_s$. Since $\mathscr{F}$ is locally finite, we may choose $K \in \kappa$ with $s \notin K$ so that $\{\beta \mid M_\beta \not\subseteq K\}$ is finite. Let $\{\beta \mid \beta \in B',\ M_\beta \not\subseteq K\} = \{\beta_1, \beta_2, \dots, \beta_n\}$, whence

$$H \subseteq K \cup M_{\beta_1} \cup \cdots \cup M_{\beta_n}.$$

Since $\mathscr{F} \prec \mathscr{E}$, for $i = 1, 2, \dots, n$ we may choose $s_i \in S$ with $G \not\subseteq Q_{s_i}$, $M_{\beta_i} \subseteq H_{s_i}$, and $F \subseteq N_{\beta_i}$. Hence

$$[H] \subseteq K \cup \left[H_{s_1}\right] \cup \cdots \cup \left[H_{s_n}\right].$$

However, $K \subseteq Q_s$ since $s \notin K$, and $[H_{s_i}] \subseteq G \subseteq Q_s$ for each $i$ since $s \notin G$, which gives the contradiction $[H] \subseteq Q_s$. This establishes that $[H] \subseteq G$.

Finally, to show that $\mathscr{D} \cup \{(H, F)\}$ is a dicover it is sufficient, since we know that $\mathscr{D} \cup \{(G, F\})$ is a dicover, to prove that

$$\bigcap_{\alpha \in A_1} F_\alpha \subseteq H \cup \bigvee_{\alpha \in A_2} G_\alpha$$

for all partitions $A_1, A_2$ of $A$. Let $B_1 = \{\beta \mid \exists \alpha \in A_1 \text{ with } M_\beta \subseteq G_\alpha \text{ and } F_\alpha \subseteq N_\beta\}$, $B_2 = B \setminus (B_1 \cup B')$. Since $B_1 \cap B' = \emptyset$, we see that $B_1$, $B_2$, and $B'$ form a partition of $B$. For $\beta \in B_1$ choose $\alpha_\beta \in A_1$ with $M_\beta \subseteq G_{\alpha_\beta}$ and $F_{\alpha_\beta} \subseteq N_\beta$, and let $A'_1 = \{\alpha_\beta \mid \beta \in B_1\}$. Likewise, recalling that $\mathscr{F} \prec \mathscr{E}$, we see that for $\beta \in B_2$ we may choose $\alpha_\beta \in A_2$ in a similar way, and we let $A'_2 = \{\alpha_\beta \mid \beta \in B_2\}$. Then

$$\bigcap_{\alpha \in A'_1} F_\alpha \subseteq \bigcap_{\beta \in B_1} N_\beta \subseteq \bigvee_{\beta \in B' \cup B_2} M_\beta = H \cup \bigvee_{\beta \in B_2} M_\beta \subseteq H \cup \bigvee_{\alpha \in A'_2} G_\alpha,$$

whence

$$\bigcap_{\alpha \in A_1} F_\alpha \subseteq \bigcap_{\alpha \in A'_1} F_\alpha \subseteq H \cup \bigvee_{\alpha \in A'_2} G_\alpha \subseteq H \cup \bigvee_{\alpha \in A_2} G_\alpha,$$

which is the required result. ∎

Combining Theorems 3.4 and 3.5 gives us

THEOREM 3.6. (1) *A regular dicover paracompact ditopological texture is normal.*

(2) *A co-regular dicover co-paracompact ditopological texture is normal.*

Likewise, Theorems 3.2 and 3.5 give us

THEOREM 3.7. *A $R_1$ (co-$R_1$) dicover biparacompact ditopological texture is dicover normal (resp., co-normal).*

COROLLARY 1. *A $R_1$ and co-$R_1$ dicover paracompact ditopological texture is dicover binormal.*

COROLLARY 2. *A $R_1$ or co-$R_1$ dicover biparacompact ditopological texture is normal.*

## 4. A CONCEPT OF FULL NORMALITY

We begin by describing the notion of (co-)shrinkability for dicovers. As in the case of classical topology, this will play an important role in our treatment of full normality.

DEFINITION 4.1. Let $\mathscr{D} = \{(M_\alpha, N_\alpha) \mid \alpha \in A\}$ be a dicover of $(S, \mathscr{S})$, faithfully indexed over $A$.

(1) $\mathscr{D}$ is called *shrinkable* if for each $\alpha \in A$ we have $G_\alpha \in \tau$ with $[G_\alpha] \subseteq M_\alpha$ so that $\mathscr{E} = \{(G_\alpha, N_\alpha) \mid \alpha \in A\}$ is a dicover of $(S, \mathscr{S})$. In this case $\mathscr{E}$ is called a *shrinking* of $\mathscr{D}$.

(2) $\mathscr{D}$ is called *co-shrinkable* if for each $\alpha \in A$ we have $F_\alpha \in \kappa$ with $N_\alpha \subseteq ]F_\alpha[$ so that $\mathscr{F} = \{(M_\alpha, F_\alpha) \mid \alpha \in A\}$ is a dicover of $(S, \mathscr{S})$. In this case $\mathscr{F}$ is called a *co-shrinking* of $\mathscr{D}$.

THEOREM 4.2. (i) *Every point finite open, co-closed dicover of a dicover normal ditopological texture is shrinkable.*

(ii) *Every point co-finite open, co-closed dicover of a dicover co-normal ditopological texture is co-shrinkable.*

*Proof.* We concentrate on (i), leaving the proof of (ii) to the reader. Let $\mathscr{C} = \{(G_\alpha, F_\alpha) \mid \alpha \in A\}$ be a point finite open, co-closed dicover. Consider the set $\mathscr{H}$ of functions $h\colon D(h) \subseteq A \to \tau$, where $[h(\alpha)] \subseteq G_\alpha$ $\forall \alpha \in D(h)$ and

$$\mathscr{C}_h = \{(G_\alpha, F_\alpha) \mid \alpha \in A \setminus D(h)\} \cup \{(h(\alpha), F_\alpha) \mid \alpha \in D(h)\}$$

is a dicover.

(a) $\mathscr{H} \neq \emptyset$. Indeed, take any $\mu \in A$. Then letting $\mathscr{D} = \{(G_\alpha, F_\alpha) \mid \alpha \in A \setminus \{\mu\}\}$, we see that $\mathscr{D} \cup \{(G_\mu, F_\mu)\}$ is a dicover, $G_\mu \in \tau$, and $F_\mu \in \kappa$, so since $(\mathscr{S}, \tau, \kappa)$ is dicover normal, we have $H \in \tau$ satisfying $[H] \subseteq G_\mu$ and such that $\mathscr{D} \cup \{(H, F_\mu)\}$ is a dicover. If we define $D(h) = \{\mu\}$ and $h(\mu) = H$, we see that $h \in \mathscr{H}$.

(b) If for $h, k \in \mathscr{H}$ we define $h \leq k \Leftrightarrow D(h) \subseteq D(k)$ and $h(\alpha) = k(\alpha)\ \forall \alpha \in D(h)$, then $(\mathscr{H}, \leq)$ is inductive. To see this, let $\mathscr{H}' \subseteq \mathscr{H}$ be a chain and define $D(h)$ as the set $\bigcup\{D(h') \mid h' \in \mathscr{H}'\}$. For each $\alpha \in D(h)$ there exists $h' \in \mathscr{H}'$ with $\alpha \in D(h')$, and the value of $h'(\alpha)$ is independent of the particular $h'$ so chosen, since $\mathscr{H}'$ is a chain. Hence we may define a function $h$: $D(h) \to \tau$ by $h(\alpha) = h'(\alpha)$. We prove that $h \in \mathscr{H}$, whence $h$ will be an upper bound of $\mathscr{H}'$ in $\mathscr{H}$. That $[h(\alpha)] \subseteq G_\alpha\ \forall \alpha \in D(h)$ is trivial, so we prove that $\mathscr{C}_h$ is a dicover. Suppose not; then for some partition $A_1$, $A_2$ of $A$ we have

$$\bigcap_{\alpha \in A_1} F_\alpha \not\subseteq \bigvee_{\alpha \in A_2 \cap D(h)} h(\alpha) \cup \bigvee_{\alpha \in A_2 \setminus D(h)} G_\alpha.$$

By Lemma 1.3(5) we have $s \in S$ with

$$P_s \not\subseteq \bigvee_{\alpha \in A_2 \cap D(h)} h(\alpha) \cup \bigvee_{\alpha \in A_2 \setminus D(h)} G_\alpha \tag{1}$$

$$\bigcap_{\alpha \in A_1} F_\alpha \not\subseteq Q_s. \tag{2}$$

Since $\mathscr{C}$ is point finite the set $\{\alpha \mid s \in G_\alpha\}$ is finite, and hence so, too, is the subset $\{\alpha \mid \alpha \in A_2 \cap D(h)$ and $s \in G_\alpha\}$. This subset is nonempty, for otherwise we should have $\bigvee_{\alpha \in A_2 \cap D(h)} G_\alpha \subseteq Q_s$, and noting that (1) implies $\bigvee_{\alpha \in A_2 \setminus D(h)} G_\alpha \subseteq Q_s$, we could deduce $\bigcap_{\alpha \in A_1} F_\alpha \subseteq Q_s$ from the fact that $\mathscr{C}$ is a dicover, so contradicting (2). Denote it by $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$, and for each $i$, $1 \leq i \leq n$, choose $h_i \in \mathscr{H}'$ with $\alpha_i \in D(h_i)$. If we let $h'$ denote the largest member of the set $\{h_1, h_2, \dots, h_n\}$ with respect to the ordering $\leq$ on $\mathscr{H}$, then $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq D(h') \subseteq D(h)$. Since $\mathscr{C}_{h'}$ is a dicover, we have

$$\begin{aligned}\bigcap_{\alpha \in A_1} F_\alpha &\subseteq \bigvee_{\alpha \in A_2 \cap D(h')} h'(\alpha) \cup \bigvee_{\alpha \in A_2 \setminus D(h')} G_\alpha \\ &= \bigvee_{\alpha \in A_2 \cap D(h')} h(\alpha) \cup \bigvee_{\alpha \in A_2 \cap (D(h) \setminus D(h'))} G_\alpha \cup \bigvee_{\alpha \in A_2 \setminus D(h)} G_\alpha.\end{aligned}$$

Each of the three sets on the right is a subset of $Q_s$. Indeed this follows immediately from (1) for the third set, and for the first set if we note that $D(h') \subseteq D(h)$. Finally, $\alpha \notin D(h') \Rightarrow \alpha \notin \{\alpha_1, \alpha_2, \dots, \alpha_n\} = \{\alpha \mid \alpha \in A_2 \cap D(h)$ and $s \in G_\alpha\}$, whence $\alpha \in A_2 \cap (D(h) \setminus D(h')) \Rightarrow s \notin G_\alpha \Rightarrow G_\alpha \subseteq Q_s$, which proves the result for the second set. However, we now have $\bigcap_{\alpha \in A_1} F_\alpha \subseteq Q_s$, which again contradicts (2). Hence, $\mathscr{C}_h$ is a dicover, and $h \in \mathscr{H}$ as required. Thus, $(\mathscr{H}, \leq)$ is inductive.

By Zorn's lemma, $\mathscr{H}$ contains a maximal element $h$. We prove that $D(h) = A$. Suppose this is not so and take $\mu \in A \setminus D(h)$. For $\mathscr{D} = \{(h(\alpha), F_\alpha) \mid \alpha \in D(h)\} \cup \{(G_\alpha, F_\alpha) \mid \alpha \in A \setminus (D(h) \cup \{\mu\})\}$, we see that $\mathscr{D} \cup \{(G_\alpha, F_\alpha)\}$ is a dicover, $G_\alpha \in \tau$, and $F_\alpha \in \kappa$, so since $(\tau, \kappa)$ is dicover normal, we have $H \in \tau$ with $[H] \subseteq G_\mu$, so that $\mathscr{D} \cup \{(H, F_\alpha)\}$ is a dicover. If we define $h'$ by

$$h'(\alpha) = \begin{cases} h(\alpha) & \text{if } \alpha \in D(h) \\ H & \text{if } \alpha = \mu, \end{cases}$$

then clearly $h' \in \mathscr{H}$ and $h < h'$, since $D(h) \subset D(h') = D(h) \cup \{\mu\}$, which contradicts the maximality of $h$. Hence $D(h) = A$, and it follows that

$$\mathscr{E} = \{(h(\alpha), F_\alpha) \mid \alpha \in A\}$$

is a shrinking of $\mathscr{C}$. ∎

Now we define the notion of star and co-star of a set in $\mathscr{S}$ with respect to a difamily.

DEFINITION 4.3. Let $\mathscr{C} = \{(A_i, B_i) \mid i \in I\}$ be a difamily, $C \in \mathscr{S}$. Then the *star* of $C$ with respect to $\mathscr{C}$ is the set

$$\mathrm{St}(\mathscr{C}, C) = \bigvee \{A_i \mid i \in I, C \not\subseteq B_i\} \in \mathscr{S},$$

while the *co-star* is the set

$$\mathrm{CSt}(\mathscr{C}, C) = \bigcap \{B_i \mid i \in I, A_i \not\subseteq C\} \in \mathscr{S}.$$

LEMMA 4.4. *If $\mathscr{C}$ be a dicover and $C \in \mathscr{S}$, then $C \subseteq \mathrm{St}(\mathscr{C}, C)$ and $\mathrm{CSt}(\mathscr{C}, C) \subseteq C$.*

*Proof.* Let $\mathscr{C} = \{(A_i, B_i) \mid i \in I\}$, $I_2 = \{i \in I \mid C \not\subseteq B_i\}$, and $I_1 = I \setminus I_2$. Then, since $\mathscr{C}$ is a dicover,

$$C \subseteq \bigcap_{i \in I_1} B_i \subseteq \bigvee_{i \in I_2} A_i = \mathrm{St}(\mathscr{C}, C).$$

The second result is proved likewise. ∎

DEFINITION 4.5. Let $\mathscr{C}$ and $\mathscr{D}$ be dicovers.

(a) We say that $\mathscr{C}$ is a *delta refinement* of $\mathscr{D}$, and write $\mathscr{C} \prec (\Delta)\mathscr{D}$, if

$$\{(\mathrm{St}(\mathscr{C}, P_s), \mathrm{CSt}(\mathscr{C}, Q_s)) \mid s \in S^\flat\} \prec \mathscr{D}.$$

(b) We say that $\mathscr{C}$ is a *star refinement* of $\mathscr{D}$, and write $\mathscr{C} \prec (*)\mathscr{D}$, if

$$\{(\mathrm{St}(\mathscr{C}, A), \mathrm{CSt}(\mathscr{C}, B)) \mid A\mathscr{C}B\} \prec \mathscr{D}.$$

EXAMPLE 4.6. Consider the dicover $\mathscr{P} = \{(P_s, Q_s) \mid s \in S^\flat\}$. The reader may easily verify that $\mathrm{St}(\mathscr{P}, P_s) = P_s$ and $\mathrm{CSt}(\mathscr{P}, Q_s) = Q_s$ for all $s \in S^\flat$. Hence $\mathscr{P} \prec (\Delta)\mathscr{P}$ and $\mathscr{P} \prec (*)\mathscr{P}$.

LEMMA 4.7. *Let $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{E}$ be dicovers of $(S, \mathscr{S})$.*

(1) $\mathscr{C} \prec (*)\mathscr{D} \Rightarrow \mathscr{C} \prec \mathscr{D}$.

(2) *If $\mathscr{P} \prec \mathscr{C}$ then $\mathscr{C} \prec (*)\mathscr{D} \Rightarrow \mathscr{C} \prec (\Delta)\mathscr{D}$.*

(3) *If $\mathscr{C}$ has the property $A\mathscr{C}B \Rightarrow A \not\subseteq B$ then*

  (i) $\mathscr{C} \prec (\Delta)\mathscr{D} \Rightarrow \mathscr{C} \prec \mathscr{D}$.

  (ii) $\mathscr{C} \prec (\Delta)\mathscr{D} \prec (\Delta)\mathscr{E} \Rightarrow \mathscr{C} \prec (*)\mathscr{E}$.

*Proof.* (1) Immediate from Lemma 4.4.

(2) Given $s \in S^\flat$ there exists $A\mathscr{C}B$ with $P_s \subseteq A$ and $B \subseteq Q_s$, whence $\mathrm{St}(\mathscr{C}, P_s) \subseteq \mathrm{St}(\mathscr{C}, A)$ and $\mathrm{CSt}(\mathscr{C}, B) \subseteq \mathrm{CSt}(\mathscr{C}, Q_s)$.

(3)(i) Given $A\mathscr{C}B$ we have $s \in S^\flat$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq B$, whence $A \subseteq \mathrm{St}(\mathscr{C}, P_s)$ and $\mathrm{CSt}(\mathscr{C}, Q_s) \subseteq B$.

(3)(ii) With $A$, $B$ and $s$ as in (i) we have

  (a) $\mathrm{St}(\mathscr{C}, A) \subseteq \mathrm{St}(\mathscr{D}, P_s)$, and

  (b) $\mathrm{CSt}(\mathscr{D}, Q_s) \subseteq \mathrm{CSt}(\mathscr{C}, B)$.

We prove (a), leaving (b) to the reader. For $A'\mathscr{C}B'$ with $A \not\subseteq B'$ choose $t \in S^\flat$ with $A \not\subseteq Q_t$ and $P_t \not\subseteq B'$. Since $\mathscr{C} \prec (\Delta)\mathscr{D}$ we have $C\mathscr{D}D$ with $\mathrm{St}(\mathscr{C}, P_t) \subseteq C$ and $D \subseteq \mathrm{CSt}(\mathscr{C}, Q_t)$. Since $P_t \not\subseteq B'$ we have $A' \subseteq \mathrm{St}(\mathscr{C}, P_t) \subseteq C$, and from $A \not\subseteq Q_t$ we obtain $D \subseteq \mathrm{CSt}(\mathscr{C}, Q_t) \subseteq B$. But now $P_s \not\subseteq B$ implies $P_s \not\subseteq D$, whence $A' \subseteq C \subseteq \mathrm{St}(\mathscr{D}, P_s)$, from which we may deduce (a). ∎

DEFINITION 4.8. A ditopological texture is called *dicover fully normal* if every open, co-closed dicover has an open, co-closed star refinement.

We are now ready to present our main theorem of this section.

THEOREM 4.9. *A $R_1$, co-$R_1$ dicover biparacompact ditopological texture is dicover fully normal.*

*Proof.* Let $\mathscr{C}$ be an open, co-closed dicover of $(\mathscr{S}, \tau, \kappa)$. We have a locally co-finite open, co-closed dicover $\mathscr{E} = \{(H_\beta, K_\beta) \mid \beta \in B\}$ with $\mathscr{D} \prec \mathscr{C}$, since $(\mathscr{S}, \tau, \kappa)$ is co-paracompact. By Theorem 3.7, $(\mathscr{S}, \tau, \kappa)$ is dicover co-normal. Hence, since $\mathscr{E}$ is point co-finite, by Theorem 4.2(ii) it is co-shrinkable. Hence for each $\beta \in B$ we have $Z_\beta \in \kappa$ with $K_\beta \subseteq ]Z_\beta[$, so that $\mathscr{E}^* = \{(H_\beta, Z_\beta) \mid \beta \in B\}$ is a dicover. Using dicover paracompactness we have a locally finite open, co-closed dicover $\mathscr{D} = \{(G_\alpha, F_\alpha) \mid \alpha \in A\}$ with $\mathscr{D} \prec \mathscr{E}^*$. Again by Theorem 3.7, $(\mathscr{S}, \tau, \kappa)$ is dicover normal, so the point finite dicover $\mathscr{D}$ is shrinkable by Theorem 4.2(i). Hence for each $\alpha \in A$ we have $T_\alpha \in \tau$ with $[T_\alpha] \subseteq G_\alpha$, so that $\mathscr{D}^* = \{(T_\alpha, F_\alpha) \mid \alpha \in A\}$ is a dicover.

Take $s \in S^\flat$. Since $\mathscr{D}^*$ is locally finite, by Theorem 2.8 we have $\alpha_s \in A$, so that $s \in T_{\alpha_s}$ and $s \notin F_{\alpha_s}$. Furthermore, $\mathscr{D}^* \prec \mathscr{D} \prec \mathscr{E}^* \prec \mathscr{E} \prec \mathscr{C}$, so we have $\beta_s \in B$ and $A_s \mathscr{C} B_s$, satisfying

$$P_s \subseteq T_{\alpha_s} \subseteq \left[T_{\alpha_s}\right] \subseteq G_{\alpha_s} \subseteq H_{\beta_s} \subseteq A_s \quad \text{and}$$
$$B_s \subseteq K_{\beta_s} \subseteq \left]Z_{\beta_s}\right[ \subseteq Z_{\beta_s} \subseteq F_{\alpha_s} \subseteq Q_s.$$

For $t \in S^\flat$ define

$$M_t = \bigcap \{G_\alpha \mid P_t \subseteq G_\alpha\} \quad \text{and} \quad N_t = \bigvee \{[T_\alpha] \mid P_t \not\subseteq [T_\alpha]\}.$$

(a) The set $\{\alpha \mid P_t \subseteq G_\alpha\}$ is finite since $\mathscr{D}$ is point finite. Thus $M_t \in \tau$, and clearly $P_t \subseteq M_t$.

(b) Since $\mathscr{D}$ is locally finite, so is $\mathscr{D}^*$. By Theorem 2.9, dom$(\mathscr{D}^*)$ is closure preserving, whence $N_t \in \kappa$. In addition we clearly have $N_t \subseteq Q_t$.

By (a) and (b), $\mathscr{G} = \{(M_t, N_t) \mid t \in S^\flat\}$ is an open, co-closed difamily, and $\mathscr{P} \prec \mathscr{G}$, whence by Lemma 2.2 it is a dicover. We show that

$$\mathrm{St}(\mathscr{G}, P_s) \subseteq A_s.$$

Take $t \in S^\flat$ with $P_s \not\subseteq N_t$. Then $P_t \not\subseteq [T_\alpha] \Rightarrow P_s \not\subseteq [T_\alpha]$. But $P_s \subseteq T_{\alpha_s} \subseteq [T_{\alpha_s}]$ so $P_t \subseteq [T_{\alpha_s}] \subseteq G_{\alpha_s}$, which gives $M_t \subseteq G_{\alpha_s} \subseteq A_s$. Hence $\mathrm{St}(\mathscr{G}, P_s) = \bigvee\{M_t \mid P_s \not\subseteq N_t\} \subseteq A_s$, as required.

Now for $t \in S^\flat$ define

$$U_t = \bigcap \left\{]Z_\beta[ \mid ]Z_\beta[ \not\subseteq Q_t\right\} \quad \text{and} \quad V_t = \bigvee \left\{K_\beta \mid K_\beta \subseteq Q_t\right\}.$$

Using the local co-finiteness of $\mathscr{E}$, we deduce $U_t \in \tau$ and $V_t \in \kappa$. Defining $\mathscr{H} = \{(U_t, V_t) \mid t \in S^\flat\}$, we have $\mathscr{P} \prec \mathscr{H}$, so $\mathscr{H}$ is an open, co-closed dicover, and it is easy to verify that

$$B_s \subseteq \mathrm{CSt}(\mathscr{H}, Q_s).$$

If now we let $\mathscr{K} = \mathscr{G} \wedge \mathscr{H}$, then $\mathscr{K}$ is an open, co-closed dicover and $\mathscr{K} \prec \mathscr{C}$. Repeating the argument above with $\mathscr{K}$ in place of $\mathscr{C}$ gives an open, co-closed dicover $\mathscr{R}$ with $\mathscr{R} \prec (\Delta)\mathscr{K}$. Finally let $\mathscr{N}$ be a locally finite open, co-closed refinement of $\mathscr{R}$. For each $s \in S^\flat$ we have $C_s \mathscr{N} D_s$ with

$s \in C_s$ and $s \notin D_s$ by Theorem 2.8. Let $\mathscr{M} = \{(C_s, D_s) \mid s \in S^\flat\}$. Since $\mathscr{P} \prec \mathscr{M}$ we see $\mathscr{M}$ is an open, co-closed dicover, $\mathscr{M}$ satisfies $C \mathscr{M} D \Rightarrow C \not\subseteq D$, and $\mathscr{M} \prec (\Delta)\mathscr{K} \prec (\Delta)\mathscr{C}$. By Lemma 4.7(3)(ii) we have $\mathscr{M} \prec (*)\mathscr{C}$, whence $(\mathscr{S}, \tau, \kappa)$ is dicover fully normal, as required. ∎

In the case of a topology, i.e., a complemented ditopology on $(X, \mathscr{P}(X), \pi)$, the converse is also true (Coincidence theorem of Stone [22]), but it is not true in general, as the following examples show.

EXAMPLES 4.10. (1) Consider the texture space $(L, \mathscr{L}, \lambda)$ of Examples 1.2(2) with the (necessarily complemented) *discrete ditopology* $\tau = \kappa = \mathscr{L}$, and define the open, co-closed difamily $\mathscr{C}$ by

$$\mathscr{C} = \{((0, r - \epsilon], (0, r]) \mid 0 < r \le 1, \epsilon > 0\},$$

where for $r - \epsilon \le 0$ we are setting $(0, r - \epsilon] = \emptyset$. The reader may verify that $\mathscr{C}$ is in fact a dicover of $(L, \mathscr{L})$. Moreover, $\mathscr{C}$ refines every dicover of $(L, \mathscr{L})$ and $\mathscr{C} \prec (*)\mathscr{C}$. In view of Lemma 4.7(1) it is immediate that $(\tau, \kappa)$ is dicover fully normal. As for any discrete ditopology, $(\tau, \kappa)$ is $R_1$ and co-$R_1$. Finally, that $(\tau, \kappa)$ is not dicover paracompact or dicover co-paracompact may be seen directly, or by noting that $\mathscr{C}$ is an open, co-closed dicover, whose only open, co-closed refinement is itself, which does not satisfy the conclusion of Theorem 2.8.

(2) The lower ditopological unit interval of Example 2.13 is dicover paracompact and trivially $R_1$ and co-$R_1$, so by Theorem 4.9 it is dicover fully normal. Let us consider instead the *lower ditopological real line*, which is the texturing $\mathscr{R}$ of $\mathbb{R}$ defined by $\mathscr{R} = \{(-\infty, r] \mid r \in \mathbb{R}\} \cup \{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, with the complementation $\rho((-\infty, r]) = (-\infty, -r)$, $\rho((-\infty, r)) = (-\infty, -r]$, $\rho(\emptyset) = \mathbb{R}$, $\rho(\mathbb{R}) = \emptyset$, and complemented ditopology $\tau = \{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, $\kappa = \{(-\infty, r] \mid r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. It is not difficult to see that $(\tau, \kappa)$ is again $R_1$ and co-$R_1$, but not dicover paracompact or dicover co-paracompact. To see that it is dicover fully normal, let $\mathscr{C} = \{(G_\alpha, F_\alpha) \mid \alpha \in A\}$ be an open, co-closed dicover. Since join and union coincide, given $r \in \mathbb{R}$, we have $\alpha \in A$ with $r \in G_\alpha$, $r \notin F_\alpha$, so we may choose $\delta_r$, $0 < \delta_r < 1$, with $(-\infty, r + 4\delta_r) \subseteq G_\alpha$ and $F_\alpha \subseteq (-\infty, r - 4\delta_r)$. By Lemma 4.7(3ii) it will be sufficient to show that

$$\mathscr{D} = \{((-\infty, r + \delta_r), (-\infty, r - \delta_r]) \mid r \in \mathbb{R}\} \prec (\Delta)\mathscr{C}.$$

However, if we let $\delta = \sup\{\delta_s \mid P_r \not\subseteq (-\infty, s - \delta_s)$ and $(-\infty, s + \delta_s) \not\subseteq Q_s\} = \sup\{\delta_s \mid s - \delta_s < r < s + \delta_s\}$ and choose $t \in \mathbb{R}$ so that $t - \delta_t < r < t + \delta_t$ and $\delta_t > 2\delta/3$, then it is easy to verify that

$$\mathrm{St}(\mathscr{D}, P_r) \subseteq (-\infty, t + 4\delta_t) \quad \text{and} \quad (-\infty, t - 4\delta_t) \subseteq \mathrm{CSt}(\mathscr{D}, Q_r),$$

which gives the required result.

## 5. RELATIONS WITH FUZZY TOPOLOGY

As pointed out in the Introduction, textures may be used to represent lattices of fuzzy sets of various kinds [5–7]. Indeed, if $\mathbb{L}$ is any fuzzy lattice, i.e., a completely distributive lattice with order reversing involution $'$, $L$ is the set of molecules (join irreducible elements) of $\mathbb{L}$, and we set $\varphi(a) = \{m \in L \mid m \le a\}$, $a \in \mathbb{L}$, and $\mathscr{L} = \{\varphi(a) \mid a \in \mathbb{L}\}$, then

THEOREM 5.1 [6]. *$(L, \mathscr{L})$ is a simple texture space with complement $\lambda(\varphi(a)) = \varphi(a')$, $a \in \mathbb{L}$, and $\varphi\colon \mathbb{L} \to \mathscr{L}$ is a complete lattice isomorphism that preserves complementation.*

*Conversely, every complemented simple texture space may be obtained in this way from a suitable fuzzy lattice.*

Now let $T$ be a topology on $\mathbb{L}$, i.e., a subset of $\mathbb{L}$ containing the largest and smallest element of $\mathbb{L}$, and closed under finite meets and arbitrary joins. Then $\varphi(T)$ is the set of open sets and $\varphi(T') = \lambda(\varphi(T))$, where $T' = \{a' \mid a \in T\}$, the set of closed sets for a complemented ditopology on $(L, \mathscr{L}, \lambda)$. Moreover, every complemented ditopology on $(L, \mathscr{L}, \lambda)$ may be obtained in this way [8]. This correspondence permits the notions considered here to be expressed quite easily in the language of topological fuzzy lattices, and the details are left to the interested reader. Insofar as this description involves the use of molecules in $\mathbb{L}$, it falls naturally within the scope of, for example, Wang's theory of topological molecular lattices [23]. However, Wang and others working in this area elect to work almost exclusively with the closed set structure, in contrast to our approach, which involves both the open and closed sets. In fact, from a topological point of view, our bitopologically based ditopological view often leads to concepts quite closely related to those considered by mathematicians who favor a "point free" approach to topological fuzzy lattices. Indeed in [8] it is shown that the regularity axioms of Definition 3.1, as well as the normality axiom, correspond precisely to the corresponding point free concepts in [15]. Moreover, the notion of dicover is point free, and many of the concepts associated with it are also either intrinsically so, or can be given such a characterization. See, for example, Lemma 2.5 and Lemma 2.6 in relation to local finiteness and local co-finiteness. Thus, while we feel that the concept of "point" is basic, and for this reason have placed the notion of texture firmly in a point set framework, this convergence between pointed and point free concepts, which parallels that for standard topology, is nonetheless highly significant.

We end by considering very briefly the special case of $\mathbb{L}$-fuzzy sets, which include classical fuzzy sets as the special case $\mathbb{L} = \mathbb{I} = [0, 1]$. If $X$ is a nonempty set, then the molecules of the fuzzy lattice $\mathbb{W} = \mathbb{L}^X$ are just the "fuzzy points" $x_m$ for $x \in X$ and $m \in L$. These are in one-to-one correspondence with the points of $W = X \times L$, and $\varphi$ now takes the form $\varphi(f) = \{(x, m) \mid m \le f(x)\}$, $f \in \mathbb{W}$. Setting $\mathscr{W} = \varphi(\mathbb{W})$ and $\omega(\varphi(f)) = \varphi(f')$, $f'(x) = f(x)'$, gives us the complemented texture space corresponding to $\mathbb{L}^X$. In fact, $(W, \mathscr{W}, \omega)$ is the complemented product of the crisp set structure $(X, \mathscr{P}(X), \pi)$ of $X$ and the complemented texture $(L, \mathscr{L}, \lambda)$ corresponding to $\mathbb{L}$. For the definition of the complemented product of textures, and the proof of this result, the reader is referred to [6]. A fuzzy topology $T$ on $X$ in the sense of Chang [10], i.e., a topology on $\mathbb{W}$, corresponds as before to a complemented ditopology, this time on $(W, \mathscr{W}, \omega)$. The structure of $\mathbb{W} = \mathbb{L}^X$ is quite special, and this is reflected in the various forms of paracompactness discussed in the literature. As expected, there is little relation between these notions and the general concept of paracompactness considered in this paper, but we will present two forms of fuzzy paracompactness, based on ideas presented in [9, 19], which are implied by the dicover paracompactness of the corresponding ditopology on $(W, \mathscr{W})$. For simplicity we restrict our attention to the case $\mathbb{L} = \mathbb{I}$. Let $T$ be a fuzzy topology on $X$. It will be natural for us to call a subset $C$ of $\mathbb{W} = \mathbb{I}^X$ locally finite if the difamily $\mathscr{C} = \{(\varphi(f), \emptyset) \mid f \in C\}$ is locally finite in $(W, \mathscr{W})$. This leads to the following.

DEFINITION 5.2. $C \subseteq \mathbb{W}$ is *locally finite* for the fuzzy topology $T$ if given $x \in X$ and $0 < \epsilon \le 1$ there exists $g \in T$ with $g(x) > 1 - \epsilon$, so that the set $\{f \in C \mid f \not\le 1 - g\}$ is finite.

This notion of local finiteness is weaker than that used in [9, 19], which requires the existence of $g \in T$ with $g(x) = 1$.

DEFINITION 5.3. Let $T$ be a fuzzy topology on $X$.

(1) $(X, T)$ is *α-paracompact* if whenever $B \subseteq T$ and $\alpha \in (0, 1]$ satisfies $\bigvee B \ge \overline{\alpha}$, there exists a locally finite refinement $C \subseteq T$ of $B$ satisfying $\bigvee C \ge \overline{\alpha}$.

(2) $(X, T)$ is *α, ϵ-paracompact* if whenever $B \subseteq T$ and $\alpha \in (0, 1]$ satisfies $\bigvee B \ge \overline{\alpha}$, there exists for each $\epsilon$, $0 < \epsilon \le 1$, a locally finite refinement $C \subseteq T$ of $B$ satisfying $\bigvee C \ge \overline{\alpha - \epsilon}$.

Here local finiteness is taken in the sense of Definition 5.2, and $\overline{\alpha}$ denotes the closure in $(X, T)$ of the constant function on $X$ with value $\alpha$. In the case of a stratified fuzzy topology $T$ these functions are closed, and then $\overline{\alpha}$ may be replaced by $\alpha$ in Definition 5.3. We may now give

THEOREM 5.4. *Let $T$ be a fuzzy topology on $X$, and suppose that the corresponding ditopology on $(W, \mathscr{W})$ is dicover paracompact. Then $(X, T)$ is α-paracompact.*

*Proof.* Take $B \subseteq T$ and $\alpha \in (0,1]$ with $\bigvee B \geq \overline{\alpha}$. If we let

$$\mathscr{B} = \{(S, \varphi(\overline{\alpha}))\} \cup \{(\varphi(f), \emptyset) \mid f \in B\},$$

then it is easy to verify that $\mathscr{B}$ is an open, co-closed dicover of $(W, \mathscr{W})$, whence it has a locally finite open, co-closed refinement $\mathscr{C}$. Let

$$C = \{h \in T \mid \varphi(h) \in \operatorname{dom} \mathscr{C} \text{ and } \exists f \in B \text{ with } \varphi(h) \subseteq \varphi(f)\}.$$

Since $\mathscr{C}$ is a dicover and refines $\mathscr{B}$, it is easy to verify that $\bigvee B \geq \overline{\alpha}$. Moreover, for $x \in X$ and $0 < \epsilon \leq 1$ we have $(x, \epsilon) \in W = X \times (0,1]$, whence for some $g \in T$ we have $(x, \epsilon) \notin \varphi(1 - g)$, and $\{h \in C \mid \varphi(h) \not\subseteq \varphi(1 - g)\}$ finite. Hence $g(x) > 1 - \epsilon$ and $\{h \in C \mid h \not\leq 1 - g\}$ is finite, i.e., $\mathscr{C}$ is locally finite in the sense of Definition 5.2. Finally $C$ is clearly an open refinement of $B$, so $(X, T)$ is $\alpha$-paracompact.

Since every $\alpha$-paracompact fuzzy topology on $X$ is also clearly $\alpha$, $\epsilon$-paracompact, we have the following

COROLLARY. *Let $T$ be a fuzzy topology on $X$, and suppose that the corresponding ditopology on $(W, \mathscr{W})$ is dicover paracompact. Then $(X, T)$ is $\alpha$, $\epsilon$-paracompact.*

Naturally, since the ditopology corresponding to a fuzzy topology is complemented by Theorem 2.11, dicover paracompactness may be replaced by dicover co-paracompactness in the statement of Theorem 5.4 and its corollary. A trivial modification of the proof of [9, Theorem 4.1] may be made to show that $\alpha, \epsilon$-paracompactness is actually a good extension of paracompactness in the sense of Lowen [18]. The reader is referred to [8] for a discussion of goodness of extensions from a ditopological perspective.

## REFERENCES


1. L. M. Brown, "Dual Covering Theory, Confluence Structures and the Lattice of Bicontinuous Functions," Ph.D. thesis, Glasgow University, 1981.
2. L. M. Brown, Semi-sequentially normal bitopological spaces, *Topology Appl.* **44** (1992), 57–62.
3. L. M. Brown, Ditopological fuzzy structures, I, *Fuzzy Syst. A.I. Magazine* **3**(1) (1993).
4. L. M. Brown, Ditopological fuzzy structures, II, *Fuzzy Syst. A.I. Magazine* **3**(2) (1993).
5. L. M. Brown and M. Diker, Ditopological texture spaces and intuitionistic sets, *Fuzzy Sets Syst.* **98** (1998), 217–224.
6. L. M. Brown and R. Ertürk, Fuzzy sets as texture spaces, I. Representation theorems, *Fuzzy Sets Syst.*, to appear.
7. L. M. Brown and R. Ertürk, Fuzzy sets as textures spaces. II. Subtextures and quotient textures, *Fuzzy Sets Syst.*, to appear.

8. L. M. Brown and R. Ertürk, Ditopological texture spaces and fuzzy topology, in preparation.
9. A. Bülbül and M. W. Warner, On the goodness of some types of fuzzy paracompactness, *Fuzzy Sets Syst*. **55** (1993), 187−191.
10. C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl*. **24** (1968), 182−190.
11. D. Çoker, A note on intuitionistic sets and intuitionistic points, *Turkish J. Math*. **20-3** (1996), 343−351.
12. M. Diker, Connectedness in ditopological texture spaces, *Fuzzy Sets Syst*., to appear.
13. T. E. Gantner and R. C. Steinlage, Characterization of quasi-uniformities, *J. London Math. Soc*. **11**(5) (1972), 48−52.
14. J. A. Goguen, *L*-fuzzy sets, *J. Math. Anal. Appl*. **18** (1967), 145−174.
15. B. Hutton and I. Reilly, Separation axioms in fuzzy topological spaces, *Fuzzy Sets Syst*. **3** (1980), 93−104.
16. J. C. Kelly, Bitopological spaces, *Proc. London Math. Soc*. **13** (1963), 71−89.
17. R. D. Kopperman, Asymmetry and duality in topology, *Topology Appl*. **66** (1995), 1−39.
18. R. Lowen, A comparison of different compactness notions in fuzzy topological spaces, *Math. Anal. Appl*. **64** (1978), 446−454.
19. S. R. Malghan and S. S. Benchalli, On fuzzy topological spaces, *Glas. Mat. Ser. III* **16**(36) (1981), 313−325.
20. N. Nakajima, Generalized fuzzy sets, *Fuzzy Sets Syst*. **32** (1989), 307−314.
21. S. Romaguera and J. Marin, On the bitopological extension of the Bing metrization theorem, *J. Austral. Math. Soc. Ser. A* **44** (1988), 233−241.
22. A. H. Stone, Paracompactness and product spaces, *Bull. Amer. Math. Soc*. **53** (1948), 977−982.
23. Guo-Jun Wang, Theory of topological molecular lattices, *Fuzzy Sets Syst*. **47** (1992), 351−376.