COMPUTING AN INTERIOR POINT FOR INEQUALITIES USING LINEAR OPTIMIZATION (*)

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ABSTRACT

The problem of finding the middle of a feasible region defined by solutions to a set of linear inequalities is considered. The solution of this problem is formulated as a primal-dual pair of linear optimization problems whose solutions can be obtained using linear programming computations.

INTRODUCTION

We consider here the problem of solving linear inequalities. These are written in matrix-vector notation as

$$A\underset{\sim}{x} \leqslant \underset{\sim}{b}, \qquad \underset{\sim}{x} \geqslant \underset{\sim}{0}, \qquad (A_{m \times n},\ \underset{\sim}{x}_{1 \times n},\ \underset{\sim}{b}_{1 \times m}) \tag{1}$$

There are many ways to solve equation (1) for $\underset{\sim}{x}$ that involve reposing it as a linear optimization problem. Such techniques are well-known and are a part of the classical literature on linear programming [2], [3]. Einarsson [1] has a problem of the form of equation (1) but with an additional requirement. Each inequality is to be satisfied as a *strict* inequality. The value of the smallest nonnegative slack variable

$$s_i = b_i - \sum_{j=1}^{n} a_{ij}\, x_j$$

is to be a maximum. If the row vectors of the matrix A are scaled to each have length one, this defines the *middle* of the feasible region.

The purpose of this note is to show that Einarsson's requirement can be stated easily using linear optimization theory.

THE PRIMAL-DUAL FORMULATION FOR THE MIDDLE POINT

To formulate a primal problem statement for Einarsson's requirements we introduce the two constant vectors

$$\underset{\sim}{e}_{1 \times m} = (1, \ldots, 1)^T, \quad \underset{\sim}{u}_{1 \times (n+1)} = (0, \ldots, 0, 1)^T$$

and a nonnegative scalar variable

$$s = \min_{1 \leqslant i \leqslant m} s_i, \quad (s_i = b_i - \sum_{j=1}^{n} a_{ij} x_j \geqslant 0)$$

Define the vector

$$\hat{\underset{\sim}{x}}_{1 \times (n+1)} = (\underset{\sim}{x}^T, s)^T \geqslant \underset{\sim}{0} \tag{2}$$

The primal problem for solving equation (1) for a point in the middle of the feasible region is

maximize $\underset{\sim}{u}^T \hat{\underset{\sim}{x}}$ $(= s)$

subject to $[A : \underset{\sim}{e}]\, \hat{\underset{\sim}{x}} \leqslant \underset{\sim}{b}, \ \hat{\underset{\sim}{x}} \geqslant \underset{\sim}{0}$ (3)

The dual of the problem of equation (3) is (Ref. [3], p. 124),

minimize $\underset{\sim}{b}^T \underset{\sim}{v}$

subject to $\begin{bmatrix} A^T \\ \underset{\sim}{e}^T \end{bmatrix} \underset{\sim}{v} \geqslant \underset{\sim}{u}, \ \underset{\sim}{v} \geqslant \underset{\sim}{0}$ (4)

Nonnegative slack (or excess) variables can be introduced into either of the problems given in equations (3) or (4) to yield a standard linear programming problem [3].

If the dual problem of equation (4) is solved, its dual coefficients provide a solution of the primal problem of equation (3). While either problem can be used to obtain a solution, the dual formulation of equation (4) has a real advantage for $m \gg n$. The reason for this is that the revised simplex algorithm will be solving a sequence of (essentially) n by n linear algebraic systems in equation (4). The problem of equation (3) requires a sequence of m by m linear algebraic systems, probably entailing a lot more work if $m \gg n$.

It is interesting to note that such inequality problems as Einarsson's can give rise to linear programming problems that are *naturally* unbounded. In fact whenever there exists any solution of equation (1), $\tilde{\underset{\sim}{x}} \geqslant \underset{\sim}{0}$, and any n-vector $\tilde{\underset{\sim}{t}}$ satisfying $A\tilde{\underset{\sim}{t}} < \underset{\sim}{0}$ then the primal problem of equation (3) will be unbounded.

In case some of the components, x_i, of the vector $\underset{\sim}{x}$ are free to take on either sign, one can introduce the positive and negative parts

$$x_i = x_i' - x_i'', \quad (x_i', x_i'' \geqslant 0),$$

into the primal problem of equation (3). This process leads again to a standard linear programming problem

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after slack (or excess) variables are introduced into equations (3) or (4).

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