0. Introduction

Representation theory (or theory of modules) has always played an important role for understanding rings, groups, and C*-algebras. So it is a natural question to ask how much information about such objects one can obtain if one knows their representation theory.

Morita and Bass made this precise for rings. Two rings, $A$ and $B$, are said to be Morita equivalent iff their categories of left modules are equivalent. It is shown in [5] that this is equivalent to the existence of a Morita context for $A$ and $B$, i.e. two bimodules $A P_B, B Q_A$ and surjective bimodule homomorphisms from $A P \otimes_B Q_A$ onto $A A_A$, respectively $B Q \otimes_A P_B$ onto $B B_B$ which fulfill a certain transitivity law. Several authors studied Morita equivalence in other contexts, Knauer and Baschewsly for monoids [4], [22], Lin for coalgebras [23], Pareigis in a general categorical framework for monoidal categories [26].

Of course this notion will depend on the chosen category of modules. An interesting example is the following: Let $A$ be a monoid. An $A$-module is a set with a left action of $A$ on it. One can define two monoids to be Morita equivalent iff their categories of unitary modules are equivalent, or iff their categories of all (also nonunitary) modules are equivalent. In the first case, there are Morita equivalent monoids which are not isomorphic [22]. In the second case, two monoids are Morita equivalent if and only if they are isomorphic [4].

Rieffel laid the groundwork for Morita equivalence of C*-algebras and $W^*$-algebras. He introduced two notions for C*-algebras. Two C*-algebras are called strongly Morita equivalent iff there exists an imprimitivity bimodule for them ([30], Def. 6.10), resembling a Morita context for rings. Two C*-algebras are called Morita equivalent iff their categories of nondegenerate *-representations are
equivalent. As of now, results on the second notion are formulated mainly in terms of the enveloping $W^*$-algebras.

In this paper we want to study Morita equivalence for a reasonably large class of $C^*$-algebras, the nuclear $C^*$-algebras. We will see that there is an abundance of pairs of $C^*$-algebras which are Morita equivalent but not strongly Morita equivalent. In Section 1, we recall the basic definitions from [29], discuss some elementary properties of Morita equivalence of $C^*$-algebras and relate it to Morita equivalence of rings. Section 2 is devoted to a classification of type I $C^*$-algebras up to Morita equivalence. This will give a new characterization of type I $C^*$-algebras, namely as the ones which are Morita equivalent to commutative $C^*$-algebras. It is also shown how the $T$-Borel structure on the spectrum is related to the representation theory. Finally, we mention a classification of continuous trace algebras up to strong Morita equivalence. In Section 3 we discuss Morita equivalence for AF and nuclear $C^*$-algebras. In particular we show that all separable, non-type I, nuclear $C^*$-algebras are Morita equivalent. It follows that a separable $C^*$-algebra is nuclear if and only if it is Morita equivalent to an AF algebra. Two AF algebras are Morita equivalent iff their spectra are Mackey–Borel isomorphic. These facts depend on work of Connes [11] and Elliott [16]. We also mention the relationship of strong Morita equivalence of AF algebras and $K$-theory of AF algebras.

1. Notation and preliminaries

1.1. The basic references for $C^*$- and $W^*$-algebras are [3], [12], [13], [27], [32], [33]. For more information on Morita equivalence of $C^*$-algebras consult [7], [29], [30], and on categories [25].

Let us recall some notation from [29]. Let $A$ be a $C^*$-algebra. A non-degenerate $\ast$-representation on a Hilbert space will be called a Hermitian $A$-module. If $V, W$ are Hermitian $A$-modules, then $\text{Hom}_A(V, W)$ will be the Banach space of bounded $A$-homomorphisms from $V$ to $W$. So we get a category $A$-Hermod consisting of the Hermitian $A$-modules as objects and the elements of $\text{Hom}_A(V, W)$ as morphisms from $V$ to $W$. This category is a $\ast$-category, i.e. it has an involution which assigns to each $f \in \text{Hom}_A(V, W)$ its adjoint $f^\ast \in \text{Hom}_A(W, V)$. Now let $N$ be a $W^*$-algebra. A normal unital $\ast$-representation of $N$ on a Hilbert space will be called a normal $N$-module. The full subcategory of $N$-Hermod consisting of normal $N$-modules will be called $N$-Normod.

For a $C^*$-algebra $A$, let $n(A)$ be the enveloping $W^*$-algebra. As Banach space, $n(A)$ is isomorphic to $A^{**}$, the bidual Banach space of $A$ [12, 12.1.3]. It is characterized by the universal property that every $\ast$-representation of $A$ can be uniquely extended to a normal $\ast$-representation of $n(A)$ [12, Prop. 12.1.5]. Under this correspondence the non-degenerate $\ast$-representations of $A$ correspond to the unital normal $\ast$-representations of $n(A)$. Since any $C^*$-algebra can be faithfully represented as an algebra of operators on a Hilbert space, this provides a left adjoint of the forgetful
functor from $W^\ast$-alg (the category of $W^\ast$-algebras with normal $\ast$-homomorphisms) to $C^\ast$-alg (the category of $C^\ast$-algebras with $\ast$-homomorphisms). The above bijection establishes an isomorphism between $A$-Hermod and $n(A)$-Normod.

Let $(A_i)_{i \in I}$ be a family of $C^\ast$-algebras. Let $\Pi A_i = \{(a_i)_{i \in I} | a_i \in A_i \text{ and } \sup \|a_i\| < \infty\}$. Then $\Pi A_i$ is a $C^\ast$-algebra in a canonical way (and a $W^\ast$-algebra, if all $A_i$'s are $W^\ast$-algebras) and provides a product of $(A_i)_{i \in I}$ in $C^\ast$-alg (resp. $W^\ast$-alg) in the categorical sense. The sub-$C^\ast$-algebra of $\Pi A_i$ consisting of the $(a_i)$ vanishing at infinity (i.e. for $\varepsilon > 0$ there exists $K \subseteq I$, $K$ finite, such that $\|a_i\| < \varepsilon$ for $i \in K$) is denoted by $Z A_i$. Note that $Z A_i$ is not a coproduct for $(A_i)$ in the categorical sense. If $A_i = A$ for all $i \in I$, we write $\Pi A$, $\Sigma A$. Let $(V_i)_{i \in I}$ be a family of Hermitian $A$-modules. The Hilbert direct sum $\bigoplus V_i$ is a Hermitian $A$-module with the obvious coordinate-wise action of $A$. Note also that $\bigoplus V_i$ is the coproduct of $(V_i)$ only in $A$-Hermod, the subcategory of $A$-Hermod whose morphisms are the $A$-module homomorphisms of norm smaller than or equal to 1.

1.2. For a $C^\ast$-algebra $A$ let Prim($A$) be the space of primitive ideals, endowed with the Jacobson topology [12, Ch. 3]. The spectrum of $A$, i.e. the set of unitary equivalence classes of irreducible representations of $A$, is denoted by $\hat{A}$. The quasi-spectrum of $A$, i.e. the set of quasi-equivalence classes of factor representations of $A$ [12, Ch. 7], is denoted by $\check{A}$. The Borel structure on $\hat{A}$ generated by the Jacobson topology is called the $T$-Borel structure. For separable $C^\ast$-algebras we have an additional Borel structure on $\hat{A}$, namely the Mackey–Borel structure [12, 3.8]. It is finer than the $T$-Borel structure [12, 3.8.3], but both are the same for separable type I $C^\ast$-algebras [12, 4.6.1].

1.3. Two $W^\ast$-algebras $M, N$ are said to be Morita equivalent iff there is an equivalence of $N$-Normod with $M$-Normod implemented by functors which preserve the involution on these categories (see 1.1). Such an equivalence will automatically be normal [29, Prop. 7.3]. Two equivalent conditions are:

(1) There exists an $M$–$N$-equivalence bimodule $X$, i.e. a complex vector space which is an $M$–$N$-bimodule with $M$- and $N$-valued inner products satisfying certain conditions [29, Def. 7.5].

(2) There exist faithful, normal representations for $M$ and $N$ with isomorphic commutants [29, p. 92, footnote].

We write: $M \sim N$.

1.4. Two $C^\ast$-algebras $A$ and $B$ are said to be Morita equivalent iff there is a (normal) involution preserving equivalence between $A$-Hermod and $B$-Hermod. In view of 1.1 this is equivalent to: $n(A) \sim n(B)$. We write: $A \sim B$.

1.5. For $C^\ast$-algebras, there is a stronger version of Morita equivalence. Two $C^\ast$-algebras are said to be strongly Morita equivalent iff there is an $A$–$B$-imprimitivity bimodule, i.e. a complex vector space which is an $A$–$B$-bimodule having
A-and B-valued inner products satisfying certain conditions [30,6.10]. We write: 
\( A \sim B \). It is shown in [30, Theorem 6.23] that \( A \sim B \) implies \( A \sim B \).

1.6. Let \( A, B \) be \( C^* \)-algebras such that \( A \sim B \).

(a) In [29, 8.1] it is shown that Center(\( n(A) \)) \( \cong \) Center(\( n(B) \)).

(b) Any normal \( * \)-equivalence preserves direct sums [29,4.9] and unitary equivalence. Therefore irreducible modules are preserved. Factor representations and quasi-equivalence of representations are preserved by [12,5.3.1(i)]. Thus it follows: 
\( |\mathcal{A}| = |\mathcal{B}|, \quad |\tilde{\mathcal{A}}| = |\tilde{\mathcal{B}}| \).

(c) It will be shown in Sections 2 and 3 that \( A \) is of type I if and only if \( B \) is, and that \( A \) is nuclear if and only if \( B \) is.

Morita equivalence does not preserve the properties of being a continuous trace algebra, of being an algebra of compact operators and of being CCR.

1.7. If \( A \) and \( B \) are strongly Morita equivalent \( C^* \)-algebras, then \( \mathcal{A} \) is homeomorphic to \( \mathcal{B} \) by [30,6.27]. Thus properties of the spectrum (endowed with the Jacobson topology) will be invariants under strong Morita equivalence. Let us mention some properties which depend only on the topology on the spectrum.

**Proposition.** If \( A \) and \( B \) are \( C^* \)-algebras with homeomorphic spectra, then it follows:

(a) There is a lattice isomorphism between the lattices of closed two sided ideals of \( A \) and \( B \).

(b) Prim(\( A \)) is homeomorphic to Prim(\( B \)).

(c) Center(\( M(A) \)) \( \cong \) Center(\( M(B) \)), where \( M(A) \) is the double centralizer algebra of \( A \) [27,3.12]. In particular, if \( A \) and \( B \) are unital, it follows Center(\( A \)) \( \cong \) Center(\( B \)).

(d) If in addition, \( A \) and \( B \) are of type I, then \( A \) is CCR if and only if \( B \) is CCR, and \( A \) is an algebra of compact operators if and only if \( B \) is.

**Proof.** (a) By [12,3.2.2] there is a bijection between the lattice of closed two-sided ideals of \( A \) and the lattice of open sets of \( A \).

(b) Recall first that for a topological space \( Y \), \( T_0(Y) \) is defined as \( Y/\sim \), endowed with the quotient topology, where \( \sim \) is the following equivalence relation. For \( x, y \in Y \): \( x \sim y \) iff for every open set \( U \) in \( Y \): \( x \in U \) if and only if \( y \in U \). It follows from [12,3.1.3,3.1.5] that: Prim(\( A \)) \( \cong \) \( T_0(A) \).

(c) By the Dauns–Hoffmann Theorem [27,4.4.8] there is an isomorphism between the algebra of bounded continuous functions on Prim(\( A \)) and the center of \( M(A) \). Thus it follows by (b) that Center(\( M(A) \)) \( \cong \) Center(\( M(B) \)). If \( A \) and \( B \) are unital, then \( A = M(A) \) and \( B = M(B) \).

(d) By [12,4.7.15], a type I \( C^* \)-algebra is CCR if and only if the spectrum is a \( T_1 \)-space. A type I \( C^* \)-algebra is an algebra of compact operators (also called dual algebra) if and only if it has discrete spectrum [12,10.10.6]. These facts immediately prove the statements of the proposition. \( \square \)
It should be remarked that under the stronger assumption of the existence of an $A-B$-imprimitivity bimodule statement, (a) is contained in [31, Th. 3.2], and statement (d) is a consequence of [20, Prop. 8.6].

1.8. Now let us consider the relationship to Morita equivalence of rings. Since we do not have an appropriate definition of Morita equivalence for rings without identity, we only look at the unital case. The `only if'-part of the following theorem was suggested to us by Marc Rieffel.

**Theorem.** Let $A$ and $B$ be $C^*$-algebras with identity. Then $A$ and $B$ are strongly Morita equivalent as $C^*$-algebras if and only if they are Morita equivalent as rings.

**Proofs.** Assume first that $A$ and $B$ are strongly Morita equivalent. Let $AXB$ be an imprimitivity bimodule, and $B_\tilde{X}_A$ its dual [30, 6.17]. Then the $A$- and $B$-valued inner products on $X$ induce bimodule homomorphisms $\phi, \psi$ from $AX \otimes_B \tilde{X}_A$ to $AA$, resp. from $B \tilde{X} \otimes_A X_B$ to $BB$. Condition 1 of [30, 6.10] yields that $(X, \tilde{X}, A, B, \phi, \psi)$ is a Morita context in the sense of [5]. Since we assume that the ranges of the inner products have norm dense linear span in $A$ and $B$, it follows that $\phi(X \otimes_B \tilde{X})$ is dense in $A$, and that $\psi(\tilde{X} \otimes_A X)$ is dense in $B$. The open ball around $1_A$ with radius 1 consists entirely of invertible elements. Therefore, $\phi(X \otimes_B \tilde{X})$ contains an invertible element. Since it is an ideal in $A$, it contains $1_A$, so it is equal to $A$. The same holds for $\psi(\tilde{X} \otimes_A X)$. Thus $\phi$ and $\psi$ are surjective. By [5, Th. 1], $A$ and $B$ are Morita equivalent as rings.

Conversely, let $A$ and $B$ be Morita equivalent as rings. It follows from [5] that there exists a finitely generated, projective module $AM_B$ which is a generator in Mod-$B$, the category of right $B$-modules over $B$, such that: $A \cong \text{Hom}_B(M, M)$. By the dual basis lemma [5, p. 7], we can assume that there exists an $n \in N$ such that $M_B \subseteq B^n$, and that there is a $B$-submodule $N$ such that: $M_B \subseteq N_B$.

Therefore, there exists a $B$-homomorphism $P$ which is a projection from $B^n$ onto $M$, and it follows that

$$PM_n(B)P \cong \text{Hom}_B(M, M).$$

Clearly, $P$ can be viewed as an idempotent in $M_n(B)$. (Note that we do not know if $P$ is self-adjoint.) Now by [21, Th. 26], there exists a self-adjoint projection $Q$ such that: $PM_n(B) = QM_n(B)$, and by [21, Th. 15]

$$PM_n(B)P \cong QM_n(B)Q.$$

The latter algebra is a hereditary sub-$C^*$-algebra of $M_n(B)$, ring isomorphic to $A$ by the preceding considerations. By [32, Th. 4.1.20], there exists also a $*$-isomorphism between them. If we can show that $Q$ is a full projection in $M_n(B)$, then $A$ will be $*$-isomorphic to a full corner of $M_n(B)$. Thus $A$ and $B$ will be strongly Morita equivalent (see the first paragraph in the proof of Theorem 1.1 of [7]).

Since $PM_n(B) = QM_n(B)$, it is enough to show that $P$ is not contained in any
proper two-sided ideal of \( M_n(B) \). Note first that all two-sided ideals of \( M_n(B) \) are of the form \( M_n(I) \), where \( I \) is a two-sided ideal of \( B \). Assume that \( P \in M_n(I) \), for some \( I \). Since \( P \) is a projection onto \( M \), it follows that all elements of \( M \) are of the form \( \sum x_i \), where \( (x_i) \) is the canonical basis for \( B^n \). Since \( M \) is a generator for \( \text{Mod}-B \), [5, Lemma 1, p. 6] yields

\[
B = \sum u(M) \quad \text{where} \quad u \in \text{Hom}_B(M, B) = M^*.
\]  

(\*)

Now each \( u \in M^* \) extends to an element of \( (B^n)^* \), by setting \( u \) zero on \( N \). So \( u(e_i) \) makes sense. From (\*) we conclude that \( 1_B \) is a sum of elements of the form \( u(e_i x_i) \), where \( x_i \in I \) and \( u \in (B^n)^* \). But \( u(e_i x_i) = u(e_i) x_i \), therefore \( I = B \).

The last theorem is not true if \( A \) and \( B \) are just Morita equivalent as \( C^* \)-algebras. By [29, 8.19], \( C([0, 1]) \), the algebra of continuous functions on \([0, 1]\), is Morita equivalent to \( C(S^1) \). They cannot be Morita equivalent as rings, because then they have to be isomorphic as rings by [5, Th. 1(7)]. But then they are also \( * \)-isomorphic [32, Th. 4.1.20]. Therefore their spectra, namely \([0, 1]\) and \( S^1 \), are homeomorphic, which is a contradiction.

1.9. Furthermore, strong Morita equivalence is closely related to stable isomorphism. Two \( C^* \)-algebras \( A \) and \( B \) are said to be stably isomorphic iff \( A \otimes K(H) \cong B \otimes K(H) \), where \( K(H) \) is the \( C^* \)-algebra of compact operators on a separable Hilbert space. The following is shown in [7]:

(a) If \( A \) and \( B \) are stably isomorphic, then \( A \) and \( B \) are strongly Morita equivalent. If \( A \) and \( B \) have countable approximate identities, then the converse holds.

(b) There are pairs of \( C^* \)-algebras which are strongly Morita equivalent but not stably isomorphic.

2. Morita equivalence and type I \( C^* \)-algebras

2.1. Let \( A \) be a \( C^* \)-algebra. By the universal property of \( n(A) \) (see 1.1), \( A \) is a type I \( C^* \)-algebra if and only if \( n(A) \) is a type I \( W^* \)-algebra. Rieffel shows in [29, 8.10, 8.11] that type I \( W^* \)-algebras are exactly the ones which have the same \( W^* \)-representation theory as commutative \( W^* \)-algebras.

Theorem. \( A \) \( W^* \)-algebra is of type I if and only if it is Morita equivalent to a commutative \( W^* \)-algebra. Any type I \( W^* \)-algebra is Morita equivalent to its center. So two type I \( W^* \)-algebras are Morita equivalent if and only if their centers are isomorphic.

2.2. Type I \( C^* \)-algebras share a number of similarities with commutative \( C^* \)-algebras: All representations can be decomposed into direct sums of multiplicity free representations [3, Ch. II, 2]. In the separable case, there is a one-to-one cor-
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respondence between equivalence classes of multiplicity free representations and equivalence classes of measures on the spectrum [3, Ch. IV]. The following theorem shows how type I and commutative C*-algebras are related:

Theorem. Let $A$ be a C*-algebra. Then $A$ is of type I if and only if $A$ is Morita equivalent to a commutative C*-algebra. The spectrum of the latter can be chosen to be equal to $\hat{A}$, endowed with a certain topology which is finer than the Jacobson topology on $\hat{A}$.

Proof. If $A$ is Morita equivalent to a commutative C*-algebra, say $B$, then $n(A)$ is Morita equivalent to a commutative $W^*$-algebra, namely $n(B)$. Therefore, $n(A)$ is a type I $W^*$-algebra by 2.1. Thus $A$ is a type I C*-algebra.

Conversely, let $A$ be a type I C*-algebra. According to 2.1, we have to show that $\text{center}(n(A))$ is isomorphic to $n(C_0(Y))$ for some locally compact Hausdorff space $Y(C_0(Y)$ denotes the C*-algebra of continuous functions on $Y$ vanishing at infinity). If $A$ is a continuous trace algebra, then $\hat{A}$ is a locally compact Hausdorff space by [12, 4.5.3], and $n(C_0(\hat{A}))$ is isomorphic to the center of $n(A)$ by [28, 6.3]. So our assertion holds for continuous trace algebras.

Thus it is enough to show that any type I C*-algebra is Morita equivalent to a continuous trace algebra. In view of [12, 4.5.3 and 4.5.4], a direct sum of continuous trace algebras is a continuous trace algebra again. Now, if $A$ is a type I C*-algebra then there exists an ascending chain of closed two sided ideals $(I_\alpha)$ indexed by the ordinals $0 \leq \alpha \leq \alpha_0$ having the following properties:

(i) $I_0 = 0$, $I_{\alpha_0} = A$.

(ii) If $\beta$ is a limit ordinal then $I_\beta$ is the norm closure of the union of $I_\alpha$, where $\alpha < \beta$.

(iii) For all $\alpha < \alpha_0$, $I_{\alpha+1}/I_\alpha$ is a continuous trace algebra.

Let $K = \{0 < \alpha < \alpha_0 \mid \alpha$ not a limit ordinal$\}$. We want to show

$$A \sim \sum_{\beta \in K} (I_\beta/I_{\beta-1}).$$

For $\alpha \in K$ let $p_\alpha$ be the open central projection in $n(A)$ corresponding to $I_\alpha$, i.e. the weak closure of $I_\alpha$ equals $p_\alpha n(A)$. Let $q_\alpha = p_\alpha - p_{\alpha-1}$, for $\alpha \in K$. By checking the universal property of $n(A)$ and by the fact that every representation of an ideal $I_\alpha$ in $A$ extends to a representation of $A$, we can see

$$n(I_\alpha) \equiv p_\alpha n(A), \quad n(A/I_\alpha) \equiv (1 - p_\alpha)n(A).$$

So it follows for all $\alpha \in K$ $n(I_\alpha/I_{\alpha-1}) \equiv (1 - p_{\alpha-1})p_\alpha n(A) = q_\alpha n(A)$.

The $q_\alpha$'s are central orthogonal projections with $\sum q_\alpha = 1$; so they constitute a direct product decomposition of $n(A)$. Therefore

$$n(A) \equiv \prod_{\alpha \in K} q_\alpha n(A) \equiv \prod_{\alpha \in K} n(I_\alpha/I_{\alpha-1}) \equiv n\left(\sum_{\alpha \in K} I_\alpha/I_{\alpha-1}\right).$$
This shows $A \sim \sum_{\alpha \in K} I_\alpha / I_{\alpha - 1}$, and the latter algebra is a continuous trace algebra.

Finally, what is the spectrum of this algebra? For $\alpha$, let $F_\alpha = \{ \pi \in \hat{A} \mid \pi(I_\alpha) = 0 \}$. Then $F_\alpha$ is closed in the Jacobson topology on $\hat{A}$ by [12, 3.1]. For $\alpha \in K$, let $Y_\alpha = F_{\alpha - 1} / F_{\alpha}$. As $Y_\alpha$ is the intersection of an open and a closed subset of a locally compact space, it is locally compact again and isomorphic to the spectrum of $I_\alpha / I_{\alpha - 1}$ by [12, 3.2.1]. Let $Y$ be the disjoint union of he $Y_\alpha$, $\alpha \in K$, endowed with the direct sum topology. Then clearly

$$C_0(Y) \cong \sum C_0(Y_\alpha),$$

and the spectrum of $\sum I_\alpha / I_{\alpha - 1}$ is homeomorphic to $Y$. This proves the last part of the theorem. $\square$

2.3. For separable type I $C^*$-algebras, Morita equivalence is related to the $T$-Borel structure on the spectrum.

**Proposition.** The following conditions are equivalent for separable type I $C^*$-algebras:

(1) $A \sim B$.

(2) $\hat{A} \cong \hat{B}$ (with respect to $T$- or Mackey–Borel structure).

(3) $|\hat{A}| = |\hat{B}|$.

**Proof.** (2) and (3) are equivalent, since the spectrum of $A$ and $B$ is a standard Borel space [3, pp. 69, 87]. The Borel structure of such spaces is determined by their cardinality. Since a $*$-equivalence preserves irreducible representations and unitary equivalence, (3) follows from (1). Let $Y_A$, $Y_B$ be the spaces corresponding to $A$ and $B$ as in the proof of Theorem 2.2. Assume that (3) holds. Then it follows: $|Y_A| = |Y_B|$. Because of the separability of $A$ and $B$, $Y_A$ and $Y_B$ are standard Borel spaces. By [29, 8.19], $C_0(Y_A)$ is Morita equivalent to $C_0(Y_B)$, thus $A$ and $B$ are Morita equivalent. So (1) follows from (3). $\square$

2.4. To what extent does 2.3 hold in general? First, the cardinality is not sufficient in general.

**Example.** Let $\mathbb{R}_e$ be equal to $\mathbb{R}$ with the euclidean topology, and $\mathbb{R}_d$ be equal to $\mathbb{R}$ with the discrete topology. Then $C_0(\mathbb{R}_e)$ and $C_0(\mathbb{R}_d)$ are not Morita equivalent. Clearly, $C_0(\mathbb{R}_e)$ has purely continuous representations, i.e. ones which do not have any irreducible subrepresentations. In contrast, $C_0(\mathbb{R}_d)$ is just the direct sum of a continuum of copies of $\mathcal{C}$ and therefore an algebra of compact operators by [3, Th. 1.4.5], thus all representations are direct sums of irreducible ones. All these properties are preserved by any $*$-equivalence.

2.5. On the other hand, Morita equivalence of two $C^*$-algebras does not imply that their spectra are $T$-Borel isomorphic.

**Example.** Let $I = [0, 1]$, and let $L$ be the so called long line, i.e. the ordered set
So $x \in [0, 1]$ in the lexicographic order, with its smallest element deleted, and then endowed with the corresponding order topology [24, pp. 85, 159, 259]. Here, $S_\Omega$ denotes the set of countable ordinals. It is uncountable and has the same cardinality as the continuum by the continuum hypothesis. It is easy to see that $L$ is Borel isomorphic to $Z$, where $Z$ is the disjoint union of a continuum of copies of $I$ endowed with the Borel structure coming from the direct sum topology. Thus $L$ is not countably separated (a Borel space is called countably separated iff there exists a countable family of Borel subsets $X_n$ such that for $x \neq y$, there exists an $n \in \mathbb{N}$ satisfying: $x \notin X_n$, $y \notin X_n$), whereas $I$ is. Since being countably separated is invariant under Borel isomorphisms, $L$ and $I$ are not Borel isomorphic.

Yet we have the following proposition.

**Proposition.** $C(I)$ and $C_0(L)$ are Morita equivalent.

**Proof.** We have to show: $n(C(I)) \sim_w n(C_0(L))$. It follows that from the argument in the proof of Theorem 2.2 that $n(C_0(L)) \sim_w n(C_0(Z)) = \prod C n(C(I))$, where $C$ is an index set with the cardinality of the continuum.

Since $|C \times C| = |C|$, the assertion will follow if we can show that $n(C(I))$ is isomorphic to a direct product of a continuum of copies of itself. It is a direct product of $W^*$-algebras of the form $L^\infty(I; \mu)$, where $\mu$ is a finite Borel measure on $I$. We can also assume that $\mu$ is either atomic or purely continuous. In the first case, $L^\infty(I; \mu)$ is isomorphic to an at most countable direct product of copies of $C$. In the second case, all $L^\infty(I; \mu)$ are isomorphic [33, Ch. III, Th. 1.22].

There is a continuum of pairwise singular atomic measures on $I$, and also a continuum of pairwise singular purely continuous measures on $I$. (This is most easily seen by considering that $I$ and $I \times I$ are Borel isomorphic and choosing continuous measures on $\{a\} \times I$, $a \in I$.) By [3, Th. 2.2.2], all corresponding representations are pairwise disjoint. Therefore, by [3, Prop. 2.1.4], the corresponding weak closures are mutually direct summands in $n(C(I))$. Thus $n(C(I))$ is a direct product of a continuum of copies of $C$ and $L^\infty(I; \mu)$ where $\mu$ is some finite purely continuous measure on $I$. This shows that $n(C(I))$ is isomorphic to a direct product of copies of itself. \[ \square \]

**2.6. Lemma.** Let $Y$ and $Y'$ be two locally compact spaces. Then $C_0(Y)$ and $C_0(Y')$ are Morita equivalent if and only if $M(Y)$ is isometric isomorphic to $M(Y')$. ($M(Y)$ denotes the Banach space of finite regular Borel measures on $Y$.)

**Proof.** If $M(Y)$ is isometric isomorphic to $M(Y')$, then $n(C_0(Y))$ is isometric isomorphic to $n(C_0(Y'))$ (as Banach spaces). The latter spaces are also commutative unital $C^*$-algebras, so isomorphic to $C(Y_0)$ respectively $C(Y'_0)$, for some compact spaces $Y_0$ and $Y'_0$. By Stone's Theorem (see [1]; it states: Let $X$ and $Y$ be compact spaces such that $C(X)$ and $C(Y)$ are isometrically isomorphic. Then $X$ and $Y$ are homeomorphic), $Y_0$ and $Y'_0$ are homeomorphic. Therefore $n(C_0(Y))$ and $n(C_0(Y'))$
are isomorphic as $W^*$-algebras, thus Morita equivalent. The converse follows from the fact that $n(C(Y))$ and $n(C(Y'))$ are isomorphic, if they are Morita equivalent, and from the uniqueness of preduals of $W^*$-algebras – see [32, p. 29].

From that we conclude the following: There is a bijection between Morita equivalence classes of type I $C^*$-algebras and isometric isomorphism classes of Banach spaces of finite regular Borel measures on locally compact spaces given by

\[
\text{class of } A \quad \longrightarrow \quad \text{class of } M(Y_A), \quad Y_A \text{ locally compact, constructed as in (2.2)}
\]

\[
\text{class of } M(Y) \quad \longrightarrow \quad \text{class of } C_0(Y)
\]

2.7. Which $C^*$-algebras are strongly Morita equivalent to commutative $C^*$-algebras? Without proofs, we mention several results which have been shown independently by Phil Green and the author. See [35].

(a) A $C^*$-algebra $B$ is strongly Morita equivalent to a commutative $C^*$-algebra if and only if it is a CCR-algebra with Hausdorff spectrum, and the corresponding continuous field of $C^*$-algebras defined by $B$ is associated with a continuous field of Hilbert spaces (for definitions see [12, Ch. 10]).

(b) If $A$ is strongly Morita equivalent to $B$, and $A$ is a continuous trace algebra, then also $B$ is a continuous trace algebra.

(c) In the case that the spectrum $X$ is paracompact, strong Morita equivalence corresponds exactly to the Dixmier–Douady classification of locally trivial fields of elementary $C^*$-algebras (see [12, 10.8.4]). According to [12, 10.7], to each continuous trace algebra $A$ with paracompact spectrum $X$ there corresponds an element $\delta(A) \in H^3(X, \mathbb{Z})$.

Now it holds:

Let $A$ and $B$ continuous trace algebras with paracompact spectrum. Then $A$ and $B$ are strongly Morita equivalent if and only if $\check{A} \cong \check{B} \cong X$ and $\delta(A) = \delta(B)$.

(d) Furthermore, by using [12, 10.10.10], the set of strong Morita equivalence classes of continuous trace algebras with paracompact spectrum $X$ can be given a natural group structure. This group is isomorphic to $H^3(X, \mathbb{Z})$. The class of $C_0(X)$ is an identity element in this group.

3. Morita equivalence and nuclear $C^*$-algebras

3.1. As far as representation theory is concerned (in particular decomposition and disintegration of representations), the type I $C^*$-algebras are the best behaved $C^*$-algebras. Section 2 established that they have exactly the same representation theory as commutative $C^*$-algebras.

As far as tensor products of $C^*$-algebras are concerned, it is natural to consider nuclear $C^*$-algebras. Let $A$ and $B$ be two $C^*$-algebras. A $C^*$-norm on $A \otimes B$, their
algebraic tensor product, is an algebra norm \( \nu \) satisfying \( \nu(z^2) = \nu(z^*z) \) for all \( z \in A \otimes B \). Then \( \nu \) is a cross norm, i.e. \( \nu(x \otimes y) = |x| \cdot |y| \) for all \( x \in A \) and \( y \in B \). A C*-algebra is called nuclear if and only if there is only one C*-norm on \( A \otimes B \). This class of C*-algebras is closed under inductive limits. Note that this does not hold for type I C*-algebras. Furthermore, all type I C*-algebras are nuclear. Nuclearity of a C*-algebra can also be characterized by certain properties of the enveloping \( W^* \)-algebra. This suggests that it might be invariant under Morita equivalence (3.2).

A linear map \( \phi \) between C*-algebras \( A \) and \( B \) is called completely positive iff all maps \( \text{Id}_n \otimes \phi \) from \( M_n \otimes A \) to \( M_n \otimes B \) are positive. A C*-algebra (or \( W^* \)-algebra) \( A \) is called injective iff for any two C*-algebras \( B \subseteq C \) and any completely positive contraction \( \phi \) from \( B \) to \( A \), there is a completely positive contraction \( \tilde{\phi} \) from \( C \) to \( A \) extending \( \phi \).

A \( W^* \)-algebra \( N \) on a separable Hilbert space is called hyperfinite iff there exists an increasing sequence of finite dimensional \( W^* \)-subalgebras whose union generates \( N \). According to Elliott [16], a \( W^* \)-algebra on any Hilbert space is called approximately finite dimensional (AF) iff any finite number of elements can be approximated arbitrarily closely in the *-ultra-strong topology by elements of a finite dimensional sub-\( W^* \)-algebra. This class is closed under tensor products and direct products.

Connes showed in [11] that for factors on a separable Hilbert space, injectivity and being AF are equivalent. Using this Choi and Effros could prove in [8] that a C*-algebra is nuclear if and only its enveloping \( W^* \)-algebra is injective.

### 3.2. Proposition

**Being nuclear is invariant under Morita equivalence of C*-algebras.**

**Proof.** Let \( A, B \) be C*-algebras, \( A \sim B \) and \( A \) nuclear. By 1.3 there exist faithful, normal representations of \( n(A) \) and \( n(B) \) such that \( n(A)' \) is isomorphic to \( n(B)' \). Now, \( n(A) \) is injective, therefore \( n(A)' \) is injective by [11, Prop. 6.4(a)]. Since injectivity is invariant under isomorphisms, \( n(B)' \) is injective. Thus it follows that \( n(B) = n(B)' \) is injective, i.e. \( B \) is nuclear. \( \square \)

### 3.3. For any \( W^* \)-algebra \( N \), let \( N_1 \) be the type I direct summand. Analogously one defines \( M_{1,2}, M_{1,3}, M_{1,2,\omega}, M_{1,3,\omega} \).

**Lemma.** (1) *Given two families \((M_j)_{j \in J}, (N_j)_{j \in J} \) of \( W^* \)-algebras such that \( M_j \sim_w N_j \) for all \( j \in J \). Then \( \Pi M_j \sim_w \Pi N_j \).*

(2) *Given two \( W^* \)-algebras \( M \) and \( N \), then \( M \sim_w N \) if and only if \( M_{1} \sim_w N_{1}, M_{1,2} \sim_w N_{1,2} \) and \( M_{1,3} \sim_w N_{1,3} \).*

**Proof.** Let \( X_j \) be \( M_j - N_j \)-equivalence bimodules. They can be endowed with a natural seminorm \( | | \) (see [29, Def. 3.1]). Then \( X = \{(x_j)_{j \in J} | \sup |x_j| < \infty \} \) together with the obvious pointwise operations of \( \Pi M_j \) and \( \Pi N_j \) on \( X \) and the
\[ \text{For (2), the full subcategory of } \mathcal{M}\text{-}\text{Normod consisting of those normal } \mathcal{M}\text{-modules } V \text{ such that } \text{Hom}_{\mathcal{M}}(V, V) \text{ is of pure type I is equivalent to } \mathcal{M}_1\text{-}\text{Normod (in the same way for type II and III). Since any } \ast\text{-equivalence preserves these subcategories, (2) follows.} \]

3.4. A C*-algebra \( A \) is called approximately finite dimensional (AF) iff there is an increasing sequence of finite dimensional sub-C*-algebras whose union is norm-dense in \( A \) (see [6]). A C*-algebra \( A \) is called uniformly hyperfinite (UHF) iff there is an increasing sequence of finite dimensional factors \( A_n \) with the same identity such that the union of the \( A_n \) is norm-dense in \( A \) (see [14]). In [16] Elliott shows that for two UHF-algebras \( A, B \), \( n(A) \) is isomorphic to \( n(B) \). His classification of the AF direct summands of \( n(A) \) for any separable C*-algebra \( A \) is crucial for the proof of this fact. The main features are:

(A) If \( A \) is not of type I, then the properly infinite AF direct summand of \( n(A) \) is unique up to isomorphism [16, Theorem 4.1], and it is isomorphic to a product of a continuum of copies of itself [16, 3.4, 3.5]. It contains any properly infinite AF W*-algebra with separable predual as direct summand [16, Th. 2.1].

(B) It follows from [16, 3.5, 3.6, 3.7, 4.1, 4.3]: For \( t = I_1, I_2, \ldots, I_\infty, \Pi_1 \), let \( r(t) \) be the number of minimal direct summands of \( n(A)_t \), and \( F(t) \) be the unique injective factor of type \( t \) (see [11]). (Also \( r(t) \) is equal to the number of disjoint factor representations of type \( t \).)

(i) If \( r(t) \) is countable, \( n(A)_t \equiv \Pi_{r(t)} F(t) \).

(ii) If \( r(t) = c = \text{continuum} \), \( n(A)_t \equiv (\Pi_c F(t)) \oplus (\Pi_c F(t) \otimes C) \), where \( C \) is the (unique up to isomorphism) commutative W*-algebra on a separable Hilbert space without minimal projections.

Applying this classification we show:

**Theorem.** For any separable nuclear C*-algebra \( A \), there is a separable AF C*-algebra \( B \) such that \( n(A) \cong n(B) \).

**Proof.** First, we show that \( n(A) \) is approximately finite dimensional. Because of the separability of \( A \), we can assume that \( n(A) \) is a direct product of W*-algebras on separable Hilbert spaces. We also can assume that each of these summands is of pure type I, II_1, II_\infty, III.

Since the class of AF W*-algebras is closed under direct products and tensor products, it follows from the structure theory of type I W*-algebras that \( n(A)_1 \) is AF.

Consider now a direct summand of type II_1. One can write it as direct integral of factors on separable Hilbert spaces [13]. According to [11, Prop. 6.5] and [13, Ch. II], almost all factors in this decomposition will be injective, type II_1 factors. Consequently, almost all of them will be isomorphic to the unique injective type II_1 factor \( R \) by [11, Th. 1]. Using [13, Ch. II, Prop. 3.3], one concludes that
this direct summand is isomorphic to $R \otimes C$, where $C$ is a commutative $W^*$-algebra on a separable Hilbert space, thus it is AF.

Now consider a direct summand of type II$_1$ (or III). Again we can represent it as a direct integral of factors on a separable Hilbert space. Almost all of them will be of type II$_1$ (respectively III) and injective, therefore also AF by [11, Th. 6]. Because of [19], being AF and hyperfiniteness are equivalent for properly infinite $W^*$-algebras. So almost all these factors are hyperfinite $W^*$-algebras. By [34, Th. 2], a direct integral of hyperfinite $W^*$-algebras is hyperfinite. Hence this direct summand is hyperfinite, so it is AF. It follows that $n(A)$ is AF.

Now, we can apply the results of Elliott which we have mentioned before the statement of the theorem. The AF $C^*$-algebra $B$ will be constructed as a countable direct sum of separable AF $C^*$-algebras which generate the $n(A)_t$'s, $t = I_1, I_2, \ldots, I_{\infty}, I_1$ and II$_{\infty}$. We define:

- $C_i$ = the commutative $C^*$-algebra of dimension $i$.
- $C_{\infty}$ = the $C^*$-algebra of sequences in $C$ converging to zero.
- $C_c = C(Y)$, where $Y$ is the Cantor discontinuum.

All of these are separable AF $C^*$-algebras.

We use the same notation as in the remarks before the theorem. For $i = 1, 2, \ldots, \infty$, let $r_i = r(I_i)$. We define for $i$:

$$A_i = \begin{cases} C_n \otimes F(I_i) & \text{if } i < \infty, \\ C_{\infty} \otimes K(H) & \text{if } i = \infty, \end{cases}$$

where $H$ is an infinite dimensional separable Hilbert space,

$$B_1 = \sum_{i=1}^{\infty} A_i \oplus A_{\infty}.$$ 

Now, set $r = r(I_1)$. If $A$ is of type I, there are no type II representations at all. If $A$ is not of type I, then $A$ has a type II factor representation (see [12, Ch. 9]). Of course, also in this case, $r$ might be zero (e.g. $A = D \otimes K(H)$, where $D = M_2^{\infty}$, the Glimm algebra). Let us define now: $B = B_1 \oplus B_2$, where

$$B_2 = \begin{cases} 0 & \text{if } A \text{ is of type I}, \\ C_c \otimes D & \text{if } r > 0, \\ K(H) \otimes D & \text{if } r = 0 \text{ and } A \text{ is non-type I}. \end{cases}$$

Examining (B) we can conclude

$$n(A_i) = n(A)_{I_i}, \text{ for } i = 1, 2, \ldots, \infty$$

and

$$n(B_2)_{I_{I_1}} = n(A)_{I_{I_1}}.$$

Since by (A) the properly infinite AF direct summand is always the same whenever $A$ is not of type I, it follows that the properly infinite, continuous direct summands of $n(A)$ and $n(B_2)$ are isomorphic. Because for any family of $C^*$-algebras $(B_i)$ it holds:

$$n(\Sigma B_i) \equiv \Pi n(B_i)$$
one concludes: \( n(B) \equiv n(A) \). Since \( B \) is a countable direct sum of separable AF \( C^* \)-algebras, it is a separable AF \( C^* \)-algebra too. 

3.5. The following proposition is a consequence of Theorem 3.4.

**Proposition.** A separable \( C^* \)-algebra is nuclear if and only if it is Morita equivalent to an AF \( C^* \)-algebra.

**Proof.** If \( A \) is nuclear, then it is Morita equivalent to an AF \( C^* \)-algebra by 3.4. The converse holds by 3.2, since any (also non-separable) AF \( C^* \)-algebra is nuclear. 

The question arises if the last proposition holds without separability assumption. It might be possible since Choi and Effros show in [8, Prop. 5]: Every nuclear \( C^* \)-algebra is an inductive limit of separable nuclear \( C^* \)-algebras.

3.6. **Corollary.** (1) For any two separable \( C^* \)-algebras with uncountable spectrum, it holds \( n(A)_1 \sim n(B)_1 \).

(2) For any separable, non-type I \( C^* \)-algebra \( A \) and any separable, nuclear \( C^* \)-algebra \( B \), it holds \( A \sim A \oplus B \).

**Proof.** (1) By [29, 7.11], all type I factors are Morita equivalent as \( W^* \)-algebras. We use the same notation as in the proof of theorem 3.4. By assumption, there exist \( i, j \in \{1, 2, \ldots, \infty\} \), such that \( A \) (resp. \( B \)) has uncountably many, pairwise disjoint factor representations of type \( I_i \) (resp. \( I_j \)). Then it follows:

\[
\begin{align*}
n(A)_i & \equiv (\Pi_i F(I_i)) \oplus (\Pi_i F(I_i) \otimes C), \\
n(B)_j & \equiv (\Pi_j F(I_j)) \oplus (\Pi_j F(I_j) \otimes C).
\end{align*}
\]

By (3.3), \( n(A)_i \sim n(B)_j \), and for each \( k \in \{1, 2, \ldots, \infty\} \)

\[
n(A)_i \sim n(A)_i \oplus n(A)_{i_k}.
\]

So it follows inductively that \( n(A)_i \sim n(A)_1 \); the same holds for \( B \). Therefore \( n(A)_1 \sim n(B)_1 \).

(2) This follows immediately from 3.3, the proof of 3.4, and statement (1).

3.7. **Theorem.** All non-type I, separable, nuclear \( C^* \)-algebras are Morita equivalent.

**Proof.** Since any separable, non-type I \( C^* \)-algebras \( A \) and \( B \) have uncountable spectra, it is enough to consider the type II and type III parts of \( n(A) \) and \( n(B) \) (3.6). By paragraph (A) in 3.4, we have to consider only the type II parts. We use the notation of 3.4. Here \( L(H) \) denotes the \( W^* \)-algebra of all bounded linear operators on a separable, infinite dimensional Hilbert space \( H \).

\( R \otimes L(H) \) and \( R \otimes L(H) \otimes C \) are properly infinite AF \( W^* \)-algebras with separable
predual; so by [16, Th. 2.1], they are isomorphic to direct summands of \( n(A) \) and \( n(B) \). Because of that and since the properly infinite AF direct summand of \( n(A) \), respectively \( n(B) \), is isomorphic to a product of a continuum of copies of itself [16, 3.4], it follows that

\[
\begin{align*}
n(A)_{\text{II}_\text{I}} &\equiv n(A)_{\text{II}_\text{I}} \oplus \Pi_J(R \otimes L(H)) \\
&\equiv n(A)_{\text{II}_\text{I}} \oplus \Pi_J(R \otimes L(H)) \oplus \Pi_J(R \otimes L(H) \otimes C),
\end{align*}
\]

for any countable index set \( J \).

By using this and considering the fact that \( M \sim M \otimes L(H) \) for any \( W^*-\)algebra \( M \) (see [29, 8.6]), and the description of \( n(A)_{\text{II}_\text{I}} \) in 3.4, conclude

\[
\begin{align*}
n(A)_{\text{II}_\text{I}} \sim n(A)_{\text{II}_\text{I}} \oplus (A)_{\text{II}_\text{I}} = n(A)_{\text{II}_\text{I}}.
\end{align*}
\]

Since \( n(A)_{\text{II}_\text{I}} \equiv n(B)_{\text{II}_\text{I}} \) and \( n(A)_{\text{III}} \equiv n(B)_{\text{III}} \), it follows that \( A \sim B \). \( \square \)

### 3.8. In Section 2 we have seen that certain Borel structures on the spectrum of a C*-algebra are related to its representation theory. So 3.6 (2) corresponds to the fact that for any AF C*-algebra \( A \) and any non-type I C*-algebra \( B, \hat{A} \) is Mackey Borel isomorphic to a Borel subset of \( \hat{B} \) (this follows from [17]). On the other hand, 3.7 corresponds to the fact that the Mackey-Borel structure on the spectrum of a separable, non-type I AF C*-algebra is always the same (see [17]). An immediate consequence of these facts is:

**Corollary.** (1) Two separable AF C*-algebras are Morita equivalent if and only if their spectra are Mackey-Borel isomorphic.

(2) Two separable nuclear C*-algebras with Mackey- or T-Borel isomorphic spectra are Morita equivalent.

**Proof.** (1) follows from 2.4, 3.7, and the above mentioned theorem of Elliott that the spectra of all non-type I AF C*-algebras are Mackey-Borel isomorphic.

(2) follows from 3.7, since the T- and Mackey-Borel structure on the spectrum of a non-type I C*-algebra is not countably separated (Glimm's Theorem, see [27, 6.8.7]). \( \square \)

There are a lot of examples of Morita equivalent AF C*-algebras with spectra which are not T-Borel isomorphic. For example, \( D \sim D \oplus D \), where \( D \) is the Glimm algebra. The T-Borel structure on \( \hat{D} \) is trivial, whereas the T-Borel structure on \( D \oplus D \) is not. This example also shows that the property of being simple is not preserved under Morita equivalence.

### 3.9. So clearly the converse direction of 3.8 (2) does not hold for the T-Borel structure. But does it hold for the Mackey-Borel structure? One would have to show: Two separable, nuclear, non-type I C*-algebras have Mackey-Borel
isomorphic spectra. According to 3.4 this is not implausible, since Theorem 3.4 yields: For any separable nuclear, non-type I \( C^* \)-algebra \( A \) there exists a (non-type I) AF \( C^* \)-algebra \( B \) such that \( A^* \) is isometric isomorphic to \( B^* \) (by using the uniqueness of preduals of \( W^* \)-algebras).

Let \( U \) be the universal UHF \( C^* \)-algebra with 'generalized integer' \( 2 \infty 3 \infty 5 \infty \cdots \) (see [14,3.1,3.2] and [15, p. 435]). It follows from [17] that \( \hat{U} \) is a Borel subset of \( \hat{A} \). \( \hat{U} \) would be Mackey–Borel isomorphic to \( \hat{A} \), if one could show: \( \hat{A} \) is Mackey–Borel isomorphic to a Borel subset of \( \hat{U} \). Analogously to [17] one would have to show that there is a subalgebra \( C \) of \( U \) and a surjective homomorphism from \( C \) onto \( A \) with the properties of Theorem 2 in [17].

One way of doing that could be the construction of a complete order injection [9] \( \psi \) from \( A \) into \( U \) and an application of Theorem 4.1 in [9] (which states: For an unital complete order injection \( \psi \) from \( A \) into \( B \), there exists a \( * \)-homomorphism \( \phi \) from \( C^*(\psi(A)) \) onto \( A \) such that \( \phi \circ \psi = \text{Id}_A \)). Maybe the following refinement of the completely positive approximation property which also characterizes nuclearity (see [10, Th. 3.1]) will be useful in this context:

A \( C^* \)-algebra is nuclear if and only if the diagrams of completely positive contractions

\[
\begin{array}{c}
U \\
\sigma \downarrow \quad \tau \\
A \\
\text{Id} \quad A
\end{array}
\]

approximately commute in the point-norm topology.

3.10. We have not discussed the case of separable non-nuclear \( C^* \)-algebras. Since for separable \( C^* \)-algebras all representations are direct integrals of factor representations, Morita equivalence of separable, non-nuclear \( C^* \)-algebras should certainly be closely related to classification of factors of type II and III. We include a necessary criterium for Morita equivalence which should prove useful in discussing non-nuclear \( C^* \)-algebras.

**Proposition.** Let \( A \) and \( B \) be two separable \( C^* \)-algebras such that \( A \sim B \). Then \( A \) and \( B \) have the same type III factor representations, i.e. for any type III factor representation \( \pi \) of \( A \) there exists a type III factor representation \( \sigma \) of \( B \) such that the weak closures of \( \pi(A) \) and \( \sigma(B) \) are isomorphic.

**Proof.** By [12,5.3.7], we can assume that \( \pi \) acts on a separable Hilbert space. Let \( P \) be the minimal central projection in \( n(A) \) corresponding to \( \pi \), i.e. the weak closure of \( \pi(A) \) equals \( Pn(A) \). Assume that \( X \) is an \( n(A) \sim n(B) \)-equivalence bimodule, and that \( J \) is the weak closure of \( \text{span}(\langle PX, PX \rangle_{n(B)}) \). Then \( PX \) is a \( w-\text{cl}(\pi(A)) \sim J \)-equivalence bimodule. Note that \( J \) is a direct summand in \( n(B) \), since it is a weakly closed ideal in \( n(B) \) (see [29, proof of 7.6] and [32, Sect. 1.10]). An application of
[29, 2.5 and 8.4] shows that \( J \) is also a type III factor. Thus, \( J \) determines a factor representation \( \sigma \) of \( B \) such that \( w-cl(\sigma(B)) = J \). By [12, 5.3.7] (each factor representation of a separable \( C^* \)-algebra is quasi-equivalent to a factor representation on a separable Hilbert space), \( J \) is a factor on a separable Hilbert space. But two type III \( W^* \)-algebras on separable Hilbert spaces are isomorphic if they are Morita equivalent ([29, Cor. 8.16]). This shows that \( w-cl(\pi(A)) \equiv w-cl(\sigma(B)) \).

3.11. We conclude this section with a discussion of strong Morita equivalence of AF \( C^* \)-algebras and its relationship to \( K \)-theory.

By 1.9, separable AF \( C^* \)-algebras, \( A \) and \( B \), are strongly Morita equivalent if and only if \( A \otimes K(H) \cong B \otimes K(H) \). Since \( K_0(A) \cong K_0(A \otimes K(H)) \) (for a treatment of \( K \)-theory of \( C^* \)-algebras see [15]), and since the generating, hereditary subset in \( K_0(A \otimes K(H))^+ \) for \( A \otimes K(H) \) is the whole positive cone of \( K_0(A) \), it follows from the classification theorem [15, 3.2]: \( A \sim B \) if and only if \( K_0(A) \cong K_0(B) \) as ordered groups.

Now let us consider the special case of UHF \( C^* \)-algebras. To each isomorphism class of them there corresponds exactly one ‘generalized integer’ \( 2^{\alpha(2)}3^{\alpha(3)}5^{\alpha(5)} \ldots \) (see [14, Sect. 3]). By the description of the \( K \)-groups of UHF \( C^* \)-algebras given there, it is easy to see that for two given UHF \( C^* \)-algebras with corresponding generalized integers \( 2^{\alpha(2)}3^{\alpha(3)}5^{\alpha(5)} \ldots \), resp. \( 2^{\beta(2)}3^{\beta(3)}5^{\beta(5)} \ldots \), it holds: Strong Morita equivalence of them is equivalent to:

1. \( \alpha(i) = \beta(i) \) except for a finite number of indices \( i \);
2. \( \alpha(i) = \infty \) if and only if \( \beta(i) = \infty \).

In contrast, all UHF \( C^* \)-algebras are Morita equivalent, and in fact, have isomorphic enveloping \( W^* \)-algebras [16, 4.5]. Any of these equivalences between UHF \( C^* \)-algebras will preserve weak containment [12, 3.4], since they are simple. Also approximate equivalence [2, Th. 5] will be preserved, because: Two representations of a simple non-type I \( C^* \)-algebra are approximately equivalent iff the Hilbert spaces on which they act have the same dimension. For UHF \( C^* \)-algebras, any \( * \)-equivalence preserves the dimension of the underlying Hilbert space of a representation, since any representation is a direct sum of representations with underlying, separable, infinite dimensional Hilbert spaces; and direct sums are preserved by Morita equivalence ([29, Prop. 4.9]).

Finally, being AF is invariant under strong Morita equivalence, since: Let \( A, B \) be separable, \( A \sim B \), and let \( A \) be an AF \( C^* \)-algebra. Then \( A \otimes K(H) \cong B \otimes K(H) \). Thus \( B \otimes K(H) \) is AF containing \( B \) as hereditary sub-\( C^* \)-algebra. By [18, Th. 3.11], any hereditary sub-\( C^* \)-algebra of an AF algebra is AF too. Therefore, \( B \) is AF.

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