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Journal of Algebra 299 (2006) 447–455

JOURNAL OF
Algebrawww.elsevier.com/locate/jalgebra

Abstract commensurators of braid groups[☆]

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Received 14 March 2005

Available online 18 October 2005

Communicated by Michel Broué

Abstract

Let B_n be the braid group on $n \geq 4$ strands. We show that the abstract commensurator of B_n is isomorphic to $\text{Mod}(S) \times (\mathbb{Q}^\times \times \mathbb{Q}^\infty)$, where $\text{Mod}(S)$ is the extended mapping class group of the sphere with $n + 1$ punctures.

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Keywords: Braid group; Abstract commensurator

1. Introduction

Artin's *braid group on n strands*, denoted B_n , is the group defined by the following presentation:

$$\langle \sigma_1, \dots, \sigma_{n-1}: \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for all } i, \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1 \rangle.$$

[☆] Both authors are supported by NSF postdoctoral fellowships.

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This group also has a topological interpretation, from which it gets its name (see, e.g., [3]). We denote the center of B_n by Z ; it is infinite cyclic, generated by $z = (\sigma_1 \cdots \sigma_{n-1})^n$. The goal of this paper is to characterize all isomorphisms between finite index subgroups of B_n .

The *abstract commensurator* $\text{Comm}(G)$ of a group G is the group of equivalence classes of isomorphisms of finite index subgroups of G :

$$\text{Comm}(G) = \{ \Phi : \Gamma \xrightarrow{\cong} \Delta : \Gamma, \Delta \text{ finite index subgroups of } G \} / \sim$$

where $\Phi \sim \Phi'$ if there is a finite index subgroup Γ' of G such that $\Phi|_{\Gamma'} = \Phi'|_{\Gamma'}$. The product of elements of $\text{Comm}(G)$ represented by $\Phi_1 : \Gamma_1 \rightarrow \Delta_1$ and $\Phi_2 : \Gamma_2 \rightarrow \Delta_2$ is an element represented by the isomorphism $\Phi_2 \circ \Phi_1|_{\Phi_1^{-1}(\Delta_1 \cap \Gamma_2)}$. A simple example is $\text{Comm}(\mathbb{Z}^n) \cong \text{GL}_n(\mathbb{Q})$.

The *extended mapping class group* $\text{Mod}(S)$ of a surface S is the group of isotopy classes of homeomorphisms of S :

$$\text{Mod}(S) = \pi_0(\text{Homeo}^\pm(S)).$$

Main Theorem. *Suppose $n \geq 4$, and let S be the sphere with $n + 1$ punctures. Then we have:*

$$\text{Comm}(B_n) \cong \text{Mod}(S) \times (\mathbb{Q}^\times \times \mathbb{Q}^\infty).$$

We say that two groups G and G' are *abstractly commensurable* if they have isomorphic finite index subgroups. In this case, it follows from the definition that $\text{Comm}(G) \cong \text{Comm}(G')$. A special case of this is when G' is itself a finite index subgroup of G .

Thus, the main theorem also gives the abstract commensurator of all finite index subgroups of B_n . We now mention two such subgroups of general interest. The *pure braid group on n strands* PB_n is the kernel of the natural map from B_n to the symmetric group on $\{1, \dots, n\}$ which sends σ_i to the transposition switching i and $i + 1$. The Artin group $A(B_{n-1})$ is isomorphic to the finite index subgroup of B_n generated by $\sigma_1^2, \sigma_2, \dots, \sigma_{n-1}$.

Corollary 1. *Suppose $n \geq 4$, and let S be the sphere with $n + 1$ punctures. Then we have:*

$$\text{Comm}(PB_n) \cong \text{Comm}(A(B_{n-1})) \cong \text{Mod}(S) \times (\mathbb{Q}^\times \times \mathbb{Q}^\infty).$$

The factor of $\text{Mod}(S)$ in the main theorem comes from the following theorem of Charney and Crisp, which is a corollary of a theorem of Korkmaz [5,13].

Theorem 1.1. *Suppose $n \geq 4$, and let S the sphere with $n + 1$ punctures. Then we have:*

$$\text{Comm}(B_n/Z) \cong \text{Mod}(S).$$

This theorem relies on the classical fact that B_n/Z is isomorphic to the finite index subgroup of $\text{Mod}(S)$ consisting of orientation preserving elements that fix a single given puncture (see, e.g., [5]). Thus, there is a natural homomorphism $\text{Mod}(S) \rightarrow \text{Comm}(B_n/Z) \cong$

$\text{Comm}(\text{Mod}(S))$, as $\text{Mod}(S)$ acts on itself by inner automorphisms. Korkmaz’s theorem is that this map is surjective, and Charney and Crisp’s contribution is that the map is injective (this is implicit in the work of Ivanov [12]).

Other braid groups. Our proof does not hold for $n = 3$, as there is no analog of Theorem 1.1. Indeed, PB_3/Z is isomorphic to the free group F_2 , and $\text{Comm}(F_2)$ contains $\text{Aut}(F_n)$ for all $n > 0$. Also, note $B_2 \cong \mathbb{Z}$ and $B_1 = 1$.

Historical background. We think of $\text{Comm}(B_n)$ as describing “hidden automorphisms” of B_n (compare Neumann and Reid [14] and Farb and Weinberger [10]). In that sense, our main result is a generalization of the theorem of Dyer and Grossman that $\text{Out}(B_n) \cong \mathbb{Z}/2\mathbb{Z}$ [7]. Recently, Charney and Crisp proved that $\text{Out}(A(B_{n-1})) \cong (\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$ [5], and Bell and Margalit proved that $\text{Out}(PB_n) \cong \mathbb{Z}^N \rtimes (\Sigma_n \times \mathbb{Z}/2\mathbb{Z})$, where $N = \binom{n}{2} - 1$ and Σ_n is the symmetric group on n letters [1]. Charney and Crisp also showed that the abstract commensurator of any finite type Artin group (e.g., B_n) contains an infinitely generated abelian subgroup. Very recently, Crisp has determined the abstract commensurators of certain 2-dimensional Artin groups [6].

As explained, there is a close connection between braid groups and mapping class groups. Ivanov proved that $\text{Comm}(\text{Mod}(S))$ is isomorphic to $\text{Mod}(S)$ for most surfaces S [12]; the genus zero case is Theorem 1.1. Building on Ivanov’s ideas, $\text{Mod}(S)$ was also shown to be the abstract commensurator of two other important subgroups of $\text{Mod}(S)$: the *Torelli subgroup* by Farb and Ivanov [9], and the so-called *Johnson kernel* by Brendle and Margalit [4]. A similar phenomenon exists with the related group $\text{Out}(F_n)$; it is a result of Farb and Handel that $\text{Comm}(\text{Out}(F_n))$ is isomorphic to $\text{Out}(F_n)$ when $n \geq 4$ [8].

The notion of an abstract commensurator can also be viewed as a generalization of the *commensurator* of a subgroup $H < G$; this is the subgroup of G consisting of those elements g for which $H \cap gHg^{-1}$ has finite index in both H and gHg^{-1} . Godelle, Paris, and Rolfsen have studied commensurators of subgroups of Artin groups [11,15,16].

Remark. Some of the ingredients in our proof can be viewed as generalizations of facts about automorphism groups (see Section 3), and are likely well known to others familiar with commensurators, though we have found no references for them.

Outline of proof. First we find a group \mathcal{G} which is abstractly commensurable to B_n , and which is a direct product over its center, Z .

Proposition 1. $\mathcal{G} = \hat{\mathcal{G}} \times Z$.

We then define the transvection subgroup $\text{Tv}(\mathcal{G})$ of $\text{Comm}(\mathcal{G})$ and show this group splits off as a semidirect factor.

Proposition 2. $\text{Comm}(\mathcal{G}) \cong \text{Mod}(S) \ltimes \text{Tv}(\mathcal{G})$.

To understand the structure of $\text{Tv}(\mathcal{G})$, we define the subgroup \mathcal{H} of simple transvections and show this group splits from $\text{Tv}(\mathcal{G})$ as a semidirect factor.

Proposition 3. $\text{Tv}(\mathcal{G}) \cong \mathbb{Q}^\times \rtimes \mathcal{H}$.

Finally, we use the notion of a divisible group to describe \mathcal{H} .

Proposition 4. $\mathcal{H} \cong \mathbb{Q}^\infty$.

2. The proof

Let $n \geq 4$ be fixed. We start by finding a group \mathcal{G} which is abstractly commensurable to B_n , and which splits over its center. A priori, this is an easier group to work with than B_n , and as mentioned earlier, it has an isomorphic abstract commensurator.

Length homomorphism. We will make use of the *length homomorphism* $L : B_n \rightarrow \mathbb{Z}$, which is defined by $\sigma_i \rightarrow 1$ for all i . Note that L is indeed a homomorphism, and that $L(z) = n(n - 1)$.

Proposition 1. B_n is abstractly commensurable to the external direct product

$$\mathcal{G} = \hat{\mathcal{G}} \times Z$$

where $\hat{\mathcal{G}}$ is a finite index subgroup of B_n/Z .

Proof. Let \mathcal{K} be the kernel of the composition

$$B_n \xrightarrow{L} \mathbb{Z} \rightarrow \mathbb{Z}/n(n - 1)\mathbb{Z}$$

where the latter map is reduction modulo $n(n - 1)$.

Since $Z < \mathcal{K}$, we have:

$$1 \rightarrow Z \rightarrow \mathcal{K} \rightarrow \mathcal{K}/Z \rightarrow 1. \tag{1}$$

We can view the restriction of L to \mathcal{K} as a projection to Z by defining a map $\mathcal{K} \rightarrow Z$ via $g \mapsto z^{L(g)/n(n-1)}$; this is a splitting for the sequence since $L(z) = n(n - 1)$. Thus, \mathcal{K} is isomorphic to the external direct product which we denote $\mathcal{G} = \hat{\mathcal{G}} \times Z \cong \mathcal{K}$ where $\hat{\mathcal{G}} = \mathcal{K}/Z$. \square

As an abuse of notation, we will identify $\hat{\mathcal{G}}$ and Z with their images in \mathcal{G} .

Virtual center of \mathcal{G} . We will use the fact that the structure of \mathcal{G} with respect to its center is preserved under passage to finite index subgroups:

Lemma 2.1. *If Γ is any finite index subgroup of \mathcal{G} , then $Z(\Gamma) = \Gamma \cap Z$.*

Via the isomorphism of Proposition 1, this lemma is part of the proof of Theorem 1.1; it is equivalent to the statement that the map from $\text{Mod}(S)$ to $\text{Comm}(\text{Mod}(S))$ is injective.

Transvections. We define the *transvection subgroup* $\text{Tv}(\mathcal{G})$ of $\text{Comm}(\mathcal{G})$ by the following short exact sequence (compare with [5]):

$$1 \rightarrow \text{Tv}(\mathcal{G}) \rightarrow \text{Comm}(\mathcal{G}) \rightarrow \text{Comm}(\hat{\mathcal{G}}) \rightarrow 1. \tag{2}$$

That $\text{Comm}(\mathcal{G})$ surjects onto $\text{Comm}(\hat{\mathcal{G}})$ follows directly from the fact that $\mathcal{G} = \hat{\mathcal{G}} \times Z$.

Proposition 2. $\text{Comm}(\mathcal{G}) \cong \text{Mod}(S) \times \text{Tv}(\mathcal{G})$.

Proof. Since $\hat{\mathcal{G}}$ is a finite index subgroup of B_n/Z , Theorem 1.1 gives $\text{Comm}(\hat{\mathcal{G}}) \cong \text{Mod}(S)$. We define a splitting for (2) by sending an element of $\text{Comm}(\hat{\mathcal{G}})$ represented by $\Psi : \hat{\Gamma} \rightarrow \hat{\Delta}$ to the element of $\text{Comm}(\mathcal{G})$ represented by $\Psi \times 1 : \hat{\Gamma} \times Z \rightarrow \hat{\Delta} \times Z$. \square

Simple transvections. There is a homomorphism θ from $\text{Tv}(\mathcal{G})$ to \mathbb{Q}^\times which measures the action on Z . Indeed, given an element of $\text{Tv}(\mathcal{G})$ represented by $\Phi : \Gamma \rightarrow \Delta$, and any $z^q \in \Gamma$, we must have $\Phi(z^q) = z^p$ for some nonzero p (by Lemma 2.1); define $\theta([\Phi])$ to be p/q . This is a well-defined homomorphism; we call its kernel the group \mathcal{H} of *simple transvections*:

$$1 \rightarrow \mathcal{H} \rightarrow \text{Tv}(\mathcal{G}) \xrightarrow{\theta} \mathbb{Q}^\times \rightarrow 1. \tag{3}$$

Proposition 3. $\text{Tv}(\mathcal{G}) \cong \mathbb{Q}^\times \times \mathcal{H}$.

Proof. There is a splitting of (3): given $p/q \in \mathbb{Q}^\times$, where $p, q \in \mathbb{Z}$, let $\Phi : \hat{\mathcal{G}} \times \langle z^q \rangle \rightarrow \hat{\mathcal{G}} \times \langle z^p \rangle$ be the transvection which is the identity on the first factor, and sends z^q to z^p . \square

Subgroup structure. Given finite index subgroups $\hat{\Gamma} < \hat{\mathcal{G}}$ and $Z_0 < Z$, by an abuse of notation, we identify the external direct product $\hat{\Gamma} \times Z_0$ with its image under the obvious inclusion $\hat{\Gamma} \times Z_0 < \hat{\mathcal{G}} \times Z$. The next lemma says that we can always choose subgroups of this type as domains for representatives of commensurators.

Lemma 2.2. *If Γ is a finite index subgroup of \mathcal{G} , then Γ has a finite index subgroup Γ' of the form:*

$$\Gamma' = \hat{\Gamma}' \times Z(\Gamma)$$

where $\hat{\Gamma}'$ is a finite index subgroup of $\hat{\mathcal{G}}$.

Proof. By Lemma 2.1, $Z(\Gamma) < Z$. Let Γ' be the kernel of the composition

$$\Gamma \hookrightarrow \mathcal{G} \rightarrow Z \rightarrow Z/Z(\Gamma)$$

where the latter two maps are the obvious projections. Denote the composition of the first two maps by π . Then the short exact sequence

$$1 \rightarrow Z(\Gamma) \rightarrow \Gamma' \rightarrow \Gamma'/Z(\Gamma) \rightarrow 1$$

has a splitting $\Gamma' \rightarrow Z(\Gamma)$ given by $g \mapsto \pi(g)$ with kernel $\hat{\Gamma}' < \hat{\mathcal{G}}$, and the lemma follows. \square

Transvections and cohomology. In order to get a clearer picture of \mathcal{H} , we need a description of elements of $\text{Tv}(\mathcal{G})$. Recall that for a group G , we have $H^1(G, \mathbb{Z}) \cong \text{Hom}(G, \mathbb{Z})$.

Lemma 2.3. *Suppose $\Phi : \Gamma \rightarrow \Delta$ represents an element of $\text{Tv}(\mathcal{G})$. Then there exists $\phi \in H^1(\Gamma, \mathbb{Z})$ so that Φ is given by:*

$$\Phi(g) = gz^{\phi(g)}.$$

If $[\Phi] \in \mathcal{H}$ and $\Gamma \cong \hat{\Gamma} \times Z(\Gamma)$, we may view ϕ as an element of $H^1(\hat{\Gamma}, \mathbb{Z})$.

By Lemma 2.2, the second statement applies to all elements of \mathcal{H} .

Proof. We define an element $\phi \in H^1(\Gamma, \mathbb{Z})$ by the equation

$$z^{\phi(g)} = g^{-1}\Phi(g).$$

That ϕ is a homomorphism follows from the assumption that Φ is a transvection (in particular, $g^{-1}\Phi(g)$ is central):

$$z^{\phi(gh)} = h^{-1}g^{-1}\Phi(g)\Phi(h) = g^{-1}\Phi(g)h^{-1}\Phi(h) = z^{\phi(g)+\phi(h)}.$$

The first statement follows. The second statement is clear: if $[\Phi] \in \mathcal{H}$, then $\Phi(g) = g$ for all $g \in Z(\Gamma)$; thus $\phi|_{Z(\Gamma)} = 0$, and ϕ descends to $\hat{\Gamma} \cong \Gamma/Z(\Gamma)$. \square

Direct limits. Let I be a directed partially ordered set; that is, I is a partially ordered set with the property that for any $i, j \in I$, there is a $k \in I$ with $i, j \leq k$. A collection of abelian groups $\{G_i\}_{i \in I}$ and homomorphisms $\{f_{ij} : G_i \rightarrow G_j\}_{\{i, j \in I | i \leq j\}}$ forms a *direct system* if: (1) the homomorphism f_{ii} is the identity map for all i ; and (2) given any two homomorphisms f_{ij} and f_{jk} , we have $f_{ik} = f_{jk} \circ f_{ij}$.

The *direct limit* of the direct system (G_i, f_{ij}) , which we denote $\lim G_i$, is the group which satisfies the following universal property: if $\iota_i : G_i \rightarrow G$ is a collection of homomorphisms respecting the homomorphisms f_{ij} , then there is a unique homomorphism $\iota : \lim G_i \rightarrow G$ through which each ι_i factors. It follows that if each ι_i is a monomorphism, then so is ι . In this case, each G_i naturally includes into $\lim G_i$.

Direct limit of cohomology groups. We will consider the direct system of groups $H^1(\hat{\Gamma}_i, \mathbb{Z})$, where $\hat{\Gamma}_i$ ranges over all finite index subgroups of $\hat{\mathcal{G}}$. Since \mathbb{Z} is torsion free, the natural homomorphism $H^1(\hat{\Gamma}_1, \mathbb{Z}) \rightarrow H^1(\hat{\Gamma}_2, \mathbb{Z})$, for any $\hat{\Gamma}_2 < \hat{\Gamma}_1$, is injective; the inclusion is given by restriction of homomorphisms. That (the index set of) $\{\hat{\Gamma}_i\}$ forms a directed partially ordered set is the fact that any two finite index subgroups have a common finite index subgroup (namely, their intersection). Properties (1) and (2) of direct systems are apparent. Thus, $\lim H^1(\hat{\Gamma}_i, \mathbb{Z})$ is defined.

Lemma 2.4. $\mathcal{H} \cong \lim H^1(\hat{\Gamma}_i, \mathbb{Z})$.

Proof. If $\hat{\Gamma}_i$ is any finite index subgroup of $\hat{\mathcal{G}}$, then there is a monomorphism

$$\mathcal{E}_{\hat{\Gamma}_i} : H^1(\hat{\Gamma}_i, \mathbb{Z}) \rightarrow \mathcal{H}$$

given by

$$\phi \mapsto [\Phi : \hat{\Gamma}_i \times Z \xrightarrow{\cong} \Gamma']$$

where $\Phi(g) = gz^{\phi(g)}$.

Moreover, this map respects the inclusions of the aforementioned direct system: if $\hat{\Gamma}_2 < \hat{\Gamma}_1$ then $H^1(\hat{\Gamma}_1, \mathbb{Z}) < H^1(\hat{\Gamma}_2, \mathbb{Z})$ by restriction. Thus $\mathcal{E}_{\hat{\Gamma}_1}$ is the restriction of $\mathcal{E}_{\hat{\Gamma}_2}$, and so the universal property guarantees a well-defined injection

$$\mathcal{E} : \lim H^1(\hat{\Gamma}_i, \mathbb{Z}) \hookrightarrow \mathcal{H}.$$

The inverse of \mathcal{E} is provided by Lemma 2.3, so \mathcal{E} is an isomorphism. \square

Divisible groups. An abelian group G is *divisible* if for any element g of G , and any positive integer q , there is an element h of G with $h^q = g$. The next fact follows from definitions:

Fact 1. Any torsion free divisible group is a vector space over \mathbb{Q} .

Braid groups and free groups. Given the inclusion of B_n/Z into $\text{Mod}(S)$ described in Section 1, it follows from the definition of PB_n that PB_n/Z is isomorphic to the subgroup of $\text{Mod}(S)$ consisting of orientation preserving mapping classes that fix each puncture (note $Z(PB_n) = Z$). Thus, for any $m < n$, there is a surjection $PB_n/Z \rightarrow PB_m/Z(PB_m)$ obtained by “forgetting” $n - m$ of the punctures. In this way, PB_n/Z maps surjectively onto $PB_3/Z(PB_3)$, which is isomorphic to the free group on two letters F_2 (the last statement follows, for example, from the Birman exact sequence [2, Theorem 1.4]). We record this fact for future reference:

Fact 2. If $n \geq 3$, then PB_n/Z surjects onto F_2 .

The following proposition completes the proof of the main theorem.

Proposition 4. $\mathcal{H} \cong \mathbb{Q}^\infty$.

Proof. By Lemma 2.4, we need only prove $\lim H^1(\hat{\Gamma}_i, \mathbb{Z}) \cong \mathbb{Q}^\infty$. By Fact 1, and since $\lim H^1(\hat{\Gamma}_i, \mathbb{Z})$ is countable, it suffices to show that $\lim H^1(\hat{\Gamma}_i, \mathbb{Z})$ is a torsion free divisible group which contains free abelian subgroups of arbitrary rank.

First, $\lim H^1(\hat{\Gamma}_i, \mathbb{Z})$ is torsion free and abelian since each $H^1(\hat{\Gamma}_i, \mathbb{Z})$ has these properties.

To see that $\lim H^1(\hat{\Gamma}_i, \mathbb{Z})$ is a divisible group, let $\phi \in H^1(\hat{\Gamma}, \mathbb{Z}) \subset \lim H^1(\hat{\Gamma}_i, \mathbb{Z})$, and let $q \in \mathbb{Z}$. Consider the subgroup $\hat{\Gamma}_q$ of $\hat{\Gamma}$ that is the kernel of the composition:

$$\hat{\Gamma} \xrightarrow{\phi} \mathbb{Z} \rightarrow \mathbb{Z}/q\mathbb{Z}.$$

Then $\phi|_{\hat{\Gamma}_q}$ maps to $q\mathbb{Z}$. Thus, $\phi = q\phi'$ for some $\phi' \in H^1(\hat{\Gamma}_q, \mathbb{Z})$.

We now construct free abelian subgroups of arbitrarily large rank in $\lim H^1(\hat{\Gamma}_i, \mathbb{Z})$. Let $\Pi : PB_n/Z \rightarrow F_2$ be the surjection given by Fact 2. We can choose finite index free subgroups $F_k < F_2$ with any rank $k \geq 2$. Thus, since surjections and passages to finite index subgroups both induce inclusions on cohomology, we have the required injections:

$$\mathbb{Z}^k \cong H^1(F_k, \mathbb{Z}) \hookrightarrow H^1(\Pi^{-1}(F_k) \cap \hat{G}, \mathbb{Z}) \hookrightarrow \lim H^1(\hat{\Gamma}_i, \mathbb{Z}). \quad \square$$

3. Generalities

As we have mentioned, given a group Γ , we view $\text{Comm}(\Gamma)$ as a generalization of $\text{Aut}(\Gamma)$. Automorphism groups of central extensions of centerless groups can be understood as follows. First, let G be a group with $Z(G) = 1$, A an abelian group, and

$$1 \rightarrow A \rightarrow \Gamma \rightarrow G \rightarrow 1 \tag{4}$$

a split central extension. This induces a split exact sequence

$$1 \rightarrow \text{tv}(\Gamma) \rightarrow \text{Aut}(\Gamma) \rightarrow \text{Aut}(G) \rightarrow 1. \tag{5}$$

The subgroup $\text{tv}(\Gamma)$ of $\text{Aut}(\Gamma)$ consists of those automorphisms which become trivial upon passing to the quotient G . This group fits into a split exact sequence

$$1 \rightarrow H^1(G, A) \rightarrow \text{tv}(\Gamma) \rightarrow \text{Aut}(A) \rightarrow 1. \tag{6}$$

The inclusion of $H^1(G, A)$ into $\text{tv}(\Gamma)$ is defined by sending any ϕ in $H^1(G, A)$ to the map given by $g \mapsto g\tilde{\phi}(g)$, where $\tilde{\phi}$ is the pullback of ϕ to $H^1(\Gamma, A)$ (compare Lemma 2.3).

If (4) is not split, then we still have sequences (5) and (6), but these need not be exact (although the second map in each is still injective).

When all finite index subgroups of G are centerless, we obtain a completely analogous picture for $\text{Comm}(G)$. In particular, (5) becomes

$$1 \rightarrow \text{Tv}(\Gamma) \rightarrow \text{Comm}(\Gamma) \rightarrow \text{Comm}(G) \rightarrow 1. \tag{5'}$$

The group $\text{Tv}(\Gamma)$ consists of those commensurators which are trivial in $\text{Comm}(G)$. This also determines a sequence analogous to (6):

$$1 \rightarrow \lim H^1(G_i, A) \rightarrow \text{Tv}(\Gamma) \rightarrow \text{Comm}(A) \rightarrow 1, \tag{6'}$$

where G_i ranges over the finite index subgroups of G . When (4) virtually splits, these are also split exact. Otherwise, they need not be exact.

Finally, we have $\lim H^1(G_i, A) \cong \lim H^1(G_i, A/T)$, where T is the torsion subgroup of A . If A is finitely generated and $A/T \cong \mathbb{Z}^m$, then

$$\lim H^1(G_i, A) \cong \mathbb{Q}^{m \cdot vb_1} \quad \text{and} \quad \text{Comm}(A) \cong \text{GL}_m(\mathbb{Q})$$

where $vb_1 \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ is the *virtual first Betti number of G* , i.e., the supremum of the first Betti numbers of finite index subgroups of G . So in this case, we have:

$$\text{Comm}(\Gamma) \cong \text{Comm}(G) \times (\text{GL}_m(\mathbb{Q}) \times \mathbb{Q}^{m \cdot vb_1}).$$

Acknowledgments

We are grateful to Bob Bell, Mladen Bestvina, Joan Birman, Benson Farb, Walter Neumann, and Kevin Wortman for much encouragement and many enjoyable conversations. The second author would like to thank the Mathematics Department of Columbia University for providing a very pleasant and stimulating environment for the visit during which this project was begun.

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