Some properties of semi-$E$-convex functions

Xiusu Chen

Department of Mathematics, Chongqing Institute of Commerce, Chongqing 400067,
People’s Republic of China

Received 21 August 2001
Submitted by M.A. Noor

Abstract

In this paper, we show that Theorems 4.2, 4.3 and 4.6 in [Youness, J. Optim. Theory Appl. 102 (1999) 439–450] are incorrect by giving some counterexamples. We introduce a new class of semi-$E$-convex function and discuss some its basic properties.

Keywords: $E$-convex sets; $E$-convex functions; Semi-$E$-convex function; Counterexamples

1. Introduction

The concepts of $E$-convex sets and $E$-convex functions were introduced in [1], which has some important applications in various branches of mathematical sciences, see [2–4]. Yang [5] gave some examples to show that some results in [1] are incorrect. In the notes, counterexamples for Theorems 4.2, 4.3 and 4.6 in [1] are given. A new concept of semi-$E$-convex function is defined, and its properties are discussed with them. We have also corrected the results of [1].

Definition 1 [1]. A set $M \subseteq \mathbb{R}^n$ is said to be $E$-convex iff there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(1 - \lambda)E(x) + \lambda E(y) \in M, \text{ for each } x, y \in M \text{ and } 0 \leq \lambda \leq 1.$$
**Definition 2** [1]. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is said to be \( E \)-convex on a set \( M \subseteq \mathbb{R}^n \) iff there is a map \( E : \mathbb{R}^n \to \mathbb{R}^n \) such that \( M \) is a \( E \)-convex set and
\[
 f \left( \lambda E(x) + (1 - \lambda) E(y) \right) \leq \lambda f \left( E(x) \right) + (1 - \lambda) f \left( E(y) \right),
\]
for each \( x, y \in M \) and \( 0 \leq \lambda \leq 1 \).

It is easy to obtain the following lemma:

**Lemma.** Assume map \( E : \mathbb{R}^n \to \mathbb{R}^n \) such that \( E(M) \) is convex and \( E(M) \subseteq M \). Then a function \( f : \mathbb{R}^n \to \mathbb{R} \) is \( E \)-convex on a set \( M \subseteq \mathbb{R}^n \) iff \( f \) is convex on \( E(M) \).

Let us consider the following programming problem:

\[
(P) \quad \text{Min } f(x), \quad \text{s.t. } x \in M = \{ x \in \mathbb{R}^n : g_i(x) \leq 0, \ i = 1, 2, \ldots, m \},
\]
where \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g_i : \mathbb{R}^n \to \mathbb{R}, \ i = 1, 2, \ldots, m \), are \( E \)-convex function on \( \mathbb{R}^n \).

Youness [1] gives the following results (corresponding to Theorems 4.2, 4.3 and 4.6 in [1]):

**Theorem 1.** Assume that \( E(M) \) is convex and \( \bar{x} \) is a solution of the following problem:

\[
(P_E) \quad \text{Min}(f \circ E)(x), \quad \text{s.t. } x \in M.
\]
Then, \( E(\bar{x}) \) is a solution of problem \((P)\).

**Theorem 2.** Let \( E(M) \) be a convex set. If \( x^0 = E(z^0) \in E(M) \) is a local minimum of the problem \((P)\) on \( M \), then \( x^0 \) is global minimum of problem \((P)\) on \( M \).

**Theorem 3.** The set of optimal solutions of problem \((P)\) is convex.

The above three theorems (Theorems 1–3) are not correct; the three corresponding counterexamples are present in the following.

### 2. Counterexamples

**Example 1.** Give \( g_i : \mathbb{R}^2 \to \mathbb{R}, \ i = 1, 2, 3 \), and \( f : \mathbb{R}^2 \to \mathbb{R} \) as \( g_1(x, y) = x - 2 \), \( g_2(x, y) = y - 3 \), \( g_3(x, y) = -x \), \( f(x, y) = (x - 1)^2 + (y - 1)^2 \), define a map \( E : \mathbb{R}^2 \to \mathbb{R}^2 \) as \( E(x, y) = (0, y - x) \), where \( f, g_i \) are \( E \)-convex on \( \mathbb{R}^2, i = 1, 2, 3 \), consider the following problem:

\[
(P^1) \quad \text{Min } f(x, y), \quad \text{s.t. } (x, y) \in M,
\]
where
\[ M = \{(x, y) \in \mathbb{R}^2 : g_i(x, y) \leq 0, \ i = 1, 2, 3\}, \quad E(M) = \{(0, y) : y \leq 3\} \]
is convex and the corresponding problem:
\[(P_E^1) \quad \text{Min}(f \circ E)(x, y) \quad \text{s.t.} \ (x, y) \in M.\]
The point \( z^0 = (1, 2) \) is an optimal solution of problem \((P_E^1)\), but \( E(z^0) = (0, 1) \) is not a solution of problem \((P_1)\), its optimal solution is the point \((1, 1)\). It shows that Theorem 1 is not true.

**Example 2.** Give \( g_i : \mathbb{R}^2 \to \mathbb{R}, \ i = 1, 2, 3, \) as \( g_1(x, y) = -x - 2, g_2(x, y) = x - 3, g_3(x, y) = y - 3 \), define a map \( E : \mathbb{R}^2 \to \mathbb{R}^2 \) as \( E(x, y) = (0, y) \), and \( f : \mathbb{R}^2 \to \mathbb{R} \)
\[ f(x, y) = \begin{cases} (y - 1)^2, & -2 \leq x \leq 2, \\ (y - 1)^2 - (2 - x)^2, & \text{otherwise,} \end{cases} \]
where \( f, g_i \) are \( E \)-convex on \( \mathbb{R}^2 \), \( i = 1, 2, 3 \). Consider the following problem \((P)_1\):
\[ \text{Min}_{(x, y) \in M} f(x, y), \]
where \( M = \{(x, y) \in \mathbb{R}^2 : g_i(x, y) \leq 0, \ i = 1, 2, 3\}, \) and \( E(M) \) is convex, the following set
\[ X = \{(x, y) \mid 1 \leq x \leq 2 \text{ and } 0 \leq y \leq 3 - x \text{ or } 2 < x \leq 3 \text{ and } 0 \leq y \leq x - 1 \} \]
\[ \cup \{(x, 0) \mid 0 \leq x \leq 1\} \]
is the optimal solution set of problem \((P)_1\), and it is not convex. It shows that Theorem 3 is not true.

**Example 3.** Give \( g_i : \mathbb{R}^2 \to \mathbb{R}, \ i = 1, 2, 3, \) as \( g_1(x, y) = -x, g_2(x, y) = x - 3, g_3(x, y) = y - 2 \), define a map \( E : \mathbb{R}^2 \to \mathbb{R}^2 \) as \( E(x, y) = (0, y) \), and \( f : \mathbb{R}^2 \to \mathbb{R} \)
\[ f(x, y) = \begin{cases} 0 & 1 \leq x \leq 2 \text{ and } 0 \leq y \leq 3 - x, \\ x + y - 3 & 1 \leq x < 2 \text{ and } 3 - x \leq y \leq 2, \\ y - x + 1 & 2 \leq x \leq 3 \text{ and } x - 1 \leq y \leq 2, \\ |y| & 1 \leq x \leq 3 \text{ and } y < 0 \text{ or } y > 2, \\ |y| & x < 1 \text{ or } x > 3, \end{cases} \]
where \( f, g_i \) are \( E \)-convex on \( \mathbb{R}^2 \), \( i = 1, 2, 3 \). Consider the following problem:
\[(P) \quad \text{Min}_{(x, y) \in M} f(x, y), \quad \text{s.t.} \ (x, y) \in M, \]
where \( M = \{(x, y) \in \mathbb{R}^2 : g_i(x, y) \leq 0, \ i = 1, 2, 3\}, \) and \( E(M) \) is convex, the following set
\[ X = \{(x, y) \mid 1 \leq x \leq 2 \text{ and } 0 \leq y \leq 3 - x, \text{ or } 2 < x \leq 3 \text{ and } 0 \leq y \leq x - 1 \} \]
is the optimal solution set of problem \((P)\), and it is not convex. It shows that Theorem 3 is not true.
3. Semi-$E$-convex function and main results

Now, we introduce a new concept of semi-$E$-convex function, and discuss its properties. By them, we correct the above three results as the Theorems 4–7. In the following, we consider the problem

\[ \text{(P)} \quad \text{Min } f(x), \quad \text{s.t. } x \in M = \{ x \in \mathbb{R}^n : g_i(x) \leq 0, \ i = 1, 2, \ldots, m \}, \]

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are numerical functions on $\mathbb{R}^n$, $M$ is $E$-convex.

**Definition 3.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be semi-$E$-convex on a set $M \subseteq \mathbb{R}^n$ iff there is a map $E : \mathbb{R}^n \to \mathbb{R}^n$ such that $M$ is an $E$-convex set and

\[
 f(\lambda E(x) + (1 - \lambda) E(y)) \leq \lambda f(x) + (1 - \lambda) f(y),
\]

for each $x, y \in M$ and $0 \leq \lambda \leq 1$.

**Proposition 1.** If function $f : \mathbb{R}^n \to \mathbb{R}$ is semi-$E$-convex on an $E$-convex set $M \subseteq \mathbb{R}^n$ then $f(E(x)) \leq f(x)$ for each $x \in M$.

**Proof.** Since $f$ is semi-$E$-convex on an $E$-convex set $M \subseteq \mathbb{R}^n$, then for any $x, y \in M$ and $0 \leq \lambda \leq 1$, we have $\lambda E(x) + (1 - \lambda) E(y) \in M$ and

\[
 f(\lambda E(x) + (1 - \lambda) E(y)) \leq \lambda f(x) + (1 - \lambda) f(y).
\]

Thus for $\lambda = 1$, $f(E(x)) \leq f(x)$ is derived. \qed

**Remark 1.** An $E$-convex function on an $E$-convex set is not necessary a semi-$E$-convex function.

**Example 4.** Let $E : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $E(x, y) = (1 + x, y)$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x, y) = x^2 + y^2$, $\mathbb{R}^2$ is an $E$-convex set. $f$ is $E$-convex function on set $M$. Since $f(E(0, 0)) = 1 > f(0, 0) = 0$, then from Proposition 1, it follows that $f$ is not a semi-$E$-convex function on set $M$.

**Proposition 2.** If function $f_i : \mathbb{R}^n \to \mathbb{R}$ is semi-$E$-convex and bounded from above on an $E$-convex set $M \subseteq \mathbb{R}^n$, then, the function $f = \sup_{i \in I} f_i(x)$ is semi-$E$-convex on $M$.

**Proposition 3.** If the functions $f_i : \mathbb{R}^n \to \mathbb{R}$ are all semi-$E$-convex on an $E$-convex set $M \subseteq \mathbb{R}^n$, $i = 1, 2, \ldots, k$, with the same map $E$, then the function $h(x) = \sum_{i=1}^k a_i f_i(x)$ is semi-$E$-convex on $M$ for $a_i \geq 0$, $i = 1, 2, \ldots, k$.

**Proposition 4.** If function $f : \mathbb{R}^n \to \mathbb{R}$ is semi-$E$-convex on an $E$-convex set $M \subseteq \mathbb{R}^n$, then for any real number $\alpha \in \mathbb{R}$, the level set $K_\alpha = \{ x \mid x \in M, \ f(x) \leq \alpha \}$ is $E$-convex.
Proof. For any \(x, y \in K_\alpha\) and \(0 \leq \lambda \leq 1\), then, \(f(x) \leq \alpha\), \(f(y) \leq \alpha\). Since \(f\) is semi-\(E\)-convex on an \(E\)-convex set \(M \subseteq \mathbb{R}^n\), then, \(\lambda E(x) + (1 - \lambda)E(y) \in M\), and
\[
f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(x) + (1 - \lambda) f(y) \leq \alpha,
\]
i.e., \((1 - \lambda)E(x) + \lambda E(Y) \in K_\alpha\), hence, \(K_\alpha\) is \(E\)-convex. 

Remark 2. The converse of Proposition 4 is not true. We give an example of a function \(f(x)\), whose level set \(K_\alpha = \{x \mid x \in M, f(x) \leq \alpha\}\) is \(E\)-convex, but the function \(f(x)\) is not a semi-\(E\)-convex.

Example 5. Let \(E : \mathbb{R} \to \mathbb{R}\) be defined as
\[
Ex = \begin{cases} 
1, & 1 \leq x \leq 4, \\
1 + \frac{2}{\pi} \arctan(1 - x), & x < 1, \\
2 + \frac{4}{\pi} \arctan(x - 4), & x > 4.
\end{cases}
\]
The set \(\mathbb{R}\) is an \(E\)-convex set, and let the function \(f : \mathbb{R} \to \mathbb{R}\) be defined by
\[
f(x) = \begin{cases} 
2, & \text{if } x < 1, \text{ or } x > 4, \\
x - 3, & \text{if } 1 \leq x < 2, \\
3 - x, & \text{if } 2 \leq x \leq 3, \\
x - 3, & \text{if } 3 < x \leq 4.
\end{cases}
\]
Then the level sets
\[
K_\alpha = \begin{cases} 
\mathbb{R}, & \text{if } \alpha \geq 2, \\
[1, 4], & \text{if } 1 < \alpha < 2, \\
[1, 2) \cup [3 - \alpha, 3 + \alpha], & \text{if } 0 \leq \alpha < 1, \\
[1, 2), & \text{if } \alpha < 0,
\end{cases}
\]
are always \(E\)-convex. Since
\[
f\left(\frac{1}{2}E(1) + \frac{1}{2}E(5)\right) = f(2) = 1 > \frac{1}{2} f(1) + \frac{1}{2} f(5) = 0,
\]
then the function \(f(x)\) is not semi-\(E\)-convex on the \(E\)-convex set \(\mathbb{R}\), and \(f(x)\) is not convex on \(\mathbb{R}\), either.

Proposition 5. Suppose the function \(f : \mathbb{R}^n \to \mathbb{R}\) is \(E\)-convex on an \(E\)-convex set \(M \subseteq \mathbb{R}^n\). Then \(f\) is semi-\(E\)-convex on set \(M\) iff \(f(Ex) \leq f(x)\) for each \(x \in M\).

Proof. Suppose the function \(f : \mathbb{R}^n \to \mathbb{R}\) is \(E\)-convex on an \(E\)-convex set \(M\) and \(f(Ex) \leq f(x)\) for each \(x \in M\), then for any \(x, y \in M\) and \(0 \leq \lambda \leq 1\), we have
\[
f(\lambda E(x) + (1 - \lambda)E(y)) \leq \lambda f(E(x)) + (1 - \lambda) f(E(y)) \leq \lambda f(x) + (1 - \lambda) f(y).
\]
Hence, \(f\) is semi-\(E\)-convex on the \(E\)-convex set \(M\).

The other hand is derived from the Proposition 1. 

Remark 3. From Proposition 5, it follows that $E$-convex function $f$ on an $E$-convex set $M$ with the property $f(E(x)) \leq f(x)$ for each $x \in M$ is semi-$E$-convex on set $M$, but the converse is not true. The following example gives a semi-$E$-convex function, which is not $E$-convex.

Example 6. Let $E : \mathbb{R} \to \mathbb{R}$ be defined as

$$E_x = \begin{cases} 
1, & 1 \leq x \leq 4, \\
1 + \frac{2}{\pi} \arctan(1 - x), & x < 1, \\
2 + \frac{4}{\pi} \arctan(x - 4), & x > 4. 
\end{cases}$$

The set $\mathbb{R}$ is an $E$-convex set, and let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 
7, & \text{if } x < 1, \text{ or } x > 4, \\
x - 3, & \text{if } 1 \leq x < 2, \\
3 - x, & \text{if } 2 \leq x \leq 3, \\
x - 3, & \text{if } 3 < x \leq 4. 
\end{cases}$$

The function $f(x)$ is semi-$E$-convex on the $E$-convex set $\mathbb{R}$.

Since

$$f\left(\frac{1}{2}E(1) + \frac{1}{2}E(5)\right) = f(2) = 1 > \frac{1}{2}f(E(1)) + \frac{1}{2}f(E(5))$$

$$= \frac{1}{2}f(1) + \frac{1}{2}f(3) = -1,$$

then the function $f(x)$ is not $E$-convex on the $E$-convex set $\mathbb{R}$.

Definition 4. The mapping $f : \mathbb{R}^n \to \mathbb{R}$ is said to be quasi-semi-$E$-convex on a set $M \subseteq \mathbb{R}^n$, if

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leq \max\{f(x), f(y)\},$$

for each $x, y \in M, \lambda \in [0, 1]$ such that $\lambda E(x) + (1 - \lambda)E(y) \in M$.

Proposition 6. Let $M \subseteq \mathbb{R}^n$ is $E$-convex set. Then the function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi-semi-$E$-convex if and only if the level set $K_\alpha = \{x \mid x \in M, \ f(x) \leq \alpha\}$ is $E$-convex for each $\alpha \in \mathbb{R}$.

Proof. Let $f$ is quasi-semi-$E$-convex on $E$-convex set $M$. Thus, for any $x, y \in K_\alpha$, and $\lambda \in [0, 1]$, we have $\lambda E(x) + (1 - \lambda)E(y) \in M, \ f(x) \leq \alpha, \ f(y) \leq \alpha$ and

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leq \max\{f(x), f(y)\} \leq \alpha.$$

It follows that $\lambda E(x) + (1 - \lambda)E(y) \in K_\alpha$ and the set $K_\alpha$ is $E$-convex.

Conversely, let $M \subseteq \mathbb{R}^n$ is $E$-convex set and $K_\alpha$ is $E$-convex. for each $\alpha \in \mathbb{R}$. We have to show that $f$ is quasi-semi-$E$-convex. For each $x, y \in M, \lambda \in [0, 1]$ such that $\lambda E(x) + (1 - \lambda)E(y) \in M$, let $\alpha = \max\{f(x), f(y)\}$, thus $x \in K_\alpha$. 
y ∈ K_α, since K_α is E-convex set, then λE(x) + (1 − λ)E(y) ∈ K_α, it follows that
\[ f(\lambda E(x) + (1 − \lambda)E(y)) \leq \alpha = \max\{f(x), f(y)\}. \]
Which shows that f is quasi-semi-E-convex.

**Remark 4.** From Proposition 6, it follows that the function f(x) in Example 5 is quasi-semi-E-convex, but f(x) is not semi-E-convex or E-convex on \( \mathbb{R} \).

**Proposition 7.** Let \( g_i : \mathbb{R}^n \to \mathbb{R} \) is quasi-semi-E-convex on \( \mathbb{R}^n \), \( i = 1, 2, \ldots, k \). Then the set \( M = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, \ i = 1, 2, \ldots, k\} \) is E-convex.

**Proof.** From Proposition 6, it follows that the set \( M_i = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0\} \) is E-convex, \( i = 1, 2, \ldots, k \), which implies that the set \( M = \bigcap_{i=1}^{k} M_i \) is E-convex.

**Proposition 8.** Let \( f : \mathbb{R}^n \to \mathbb{R} \). Then a semi-E-convex function f on E-convex set \( M \subseteq \mathbb{R}^n \) is also quasi-semi-E-convex on M.

Given a map \( E : \mathbb{R}^n \to \mathbb{R}^n \), let the mapping \( E \times I : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R} \) to be
\[(E \times I)(x, t) = (E(x), t), \quad \text{for} \ \ (x, t) \in \mathbb{R}^n \times \mathbb{R}.\]

It is easy to show that \( M \subseteq \mathbb{R}^n \) is E-convex, if and only if \( M \times \mathbb{R} \) is E × I-convex.

Now, we define an epigraph of f as follows:
\[ \text{epi}(f) = \{(x, \alpha) : x \in M, \ \alpha \in \mathbb{R}, \ f(x) \leq \alpha\} \]

the following Proposition 9 gives a characterization of a semi-E-convex function in term of its epi(f).

**Proposition 9.** Assume \( M \) is E-convex, then f is semi-E-convex function on M if and only if epi(f) is E × I-convex on M × \( \mathbb{R} \).

**Proof.** Assume that f is semi-E-convex on M. Let (x, \alpha), (y, \beta) ∈ epi(f), \( \lambda \in [0, 1] \), then it follows that \( \lambda E(x) + (1 − \lambda)E(y) \in M \), and
\[ f(\lambda E(x) + (1 − \lambda)E(y)) \leq \lambda f(x) + (1 − \lambda) f(y) \leq \lambda \alpha + (1 − \lambda) \beta, \]
thus
\[(\lambda E(x) + (1 − \lambda)E(y), \lambda \alpha + (1 − \lambda) \beta) \in \text{epi}(f),\]
which implies that epi(f) is E × I-convex on M × \( \mathbb{R} \).

Conversely, let epi(f) is E × I-convex on M × \( \mathbb{R} \). Let x, y ∈ M, \( \lambda \in [0, 1] \), then (x, f(x)) ∈ epi(f), and (y, f(y)) ∈ epi(f). Since epi(f) is E × I-convex on M × \( \mathbb{R} \), we have
\[(\lambda E(x) + (1 − \lambda)E(y), \lambda f(x) + (1 − \lambda) f(y)) \in \text{epi}(f),\]
which implies that
\[ f(\lambda E(x) + (1 - \lambda) E(y)) \leq \lambda f(x) + (1 - \lambda) f(y), \]
this shows that \( f \) is a semi-\( E \)-convex function on \( M \). \( \Box \)

**Definition 5.** The function \( f : \mathbb{R}^n \to \mathbb{R} \) is said to be pseudo-semi-\( E \)-convex on \( E \)-convex set \( M \subseteq \mathbb{R}^n \), if there exists a strictly positive function \( b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) such that
\[ f(x) < f(y) \Rightarrow f(\lambda E(x) + (1 - \lambda) E(y)) \leq f(y) + \lambda(\lambda - 1)b(x, y) \]
for all \( x, y \in M \), and \( \lambda \in (0, 1) \).

**Proposition 10.** If \( f : \mathbb{R}^n \to \mathbb{R} \) is a semi-\( E \)-convex function on \( E \)-convex set \( M \), then \( f \) is pseudo-semi-\( E \)-convex.

**Proof.** Since \( f(x) < f(y) \) and \( f \) is semi-\( E \)-convex on \( E \)-convex set \( M \), then for all \( x, y \in M \) and \( \lambda \in (0, 1) \), we have
\[
\begin{align*}
\lambda f(x) + (1 - \lambda) f(y) & = f(y) + \lambda(f(x) - f(y)) \\
& < f(y) + \lambda(1 - \lambda)(f(x) - f(y)) \\
& = f(y) + \lambda(\lambda - 1)(f(x) - f(y)) \\
& = f(y) + \lambda(\lambda - 1)b(x, y),
\end{align*}
\]
where \( b(x, y) = f(y) - f(x) > 0 \), the required result. \( \Box \)

**Proposition 11.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a quasi-semi-\( E \)-convex function on \( E \)-convex set \( M \), if \( \phi : \mathbb{R} \to \mathbb{R} \) is a nondecreasing function, then the composite function \( \phi \circ f \) is a quasi-semi-\( E \)-convex on \( M \).

**Proof.** For all \( x, y \in M \), \( \lambda \in (0, 1) \), we have
\[
f(\lambda E(x) + (1 - \lambda) E(y)) \leq \max\{f(x), f(y)\},
\]
thus
\[
\phi \circ f(\lambda E(x) + (1 - \lambda) E(y)) \leq \phi[\max\{f(x), f(y)\}] = \max\{\phi \circ f(x), \phi \circ f(y)\}.
\]
From which it follows that the composite function \( \phi \circ f \) is a quasi-semi-\( E \)-convex on \( M \). \( \Box \)

**Theorem 4.** Assume \( M \) is an \( E \)-convex set \( M \), and \( f(Ex) \leq f(x) \) for each \( x \in M \). \( \bar{x} \) is a solution of the following problem:
\[
(\overline{\mathcal{P}}_E) \quad \text{Min}(f \circ E)(x), \quad \text{s.t.} \ x \in M.
\]
Then, \( E(\bar{x}) \) is a solution of problem \((\overline{\mathcal{P}})\).
Proof. Let \( E(\bar{x}) \) be a nonsolution of problem \((\overline{P})\); then there is \( y \in M \) such that \( f(y) < f(E(\bar{x})) \) then, \( f(E(y)) \leq f(y) < f(E(\bar{x})) \) which contradicts the optimality of \( \bar{x} \) for problem \((PE)\). Hence, \( E(\bar{x}) \) be a solution of problem \((\overline{P})\). □

Theorem 5. Assume function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is semi-\( E \)-convex on an \( E \)-convex set \( M \subseteq \mathbb{R}^n \) and \( \bar{x} \) is a solution of the following problem:

\[
(PE) \quad \min(f \circ E)(x), \quad \text{s.t.} \ x \in M.
\]

Then, \( E(\bar{x}) \) is a solution of problem \((\overline{P})\).

Proof. Follows from Theorem 4. □

Theorem 6. If \( x^0 = E(z^0) \in E(M) \) is a local minimum of the problem \((\overline{P})\) on an \( E \)-convex set \( M \), and \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is \( E \)-convex on the set \( M \), and \( f(E(x)) \leq f(x) \) for each \( x \in M \), then \( x^0 \) is global minimum of problem \((\overline{P})\) on \( M \).

Proof. Let \( x^0 = E(z^0) \in E(M) \) be a nonglobal minimum of the problem \((\overline{P})\) on \( M \), then, there is \( y \in M \) such that \( f(y) < f(x^0) = f(E(z^0)) \), since function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is \( E \)-convex and \( f(E(x)) \leq f(x) \) for each \( x \in M \), it implies that

\[
f(\lambda E(y) + (1 - \lambda)x^0) \\
\leq \lambda f(E(y)) + (1 - \lambda)f(E(z^0)) \leq \lambda f(y) + (1 - \lambda)f(x^0) \leq f(x^0),
\]

for any small \( \lambda \in (0, 1) \), which contradicts the local optimality of \( x^0 \) for problem \((\overline{P})\). Hence, \( x^0 \) is global minimum of problem \((\overline{P})\) on \( M \). □

Theorem 7. Assume function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is strictly semi-\( E \)-convex on an \( E \)-convex set \( M \subseteq \mathbb{R}^n \), i.e.,

\[
f(\lambda E(x) + (1 - \lambda)E(y)) < \lambda f(x) + (1 - \lambda)f(y),
\]

for each \( x, y \in M \), \( x \neq y \), and \( 0 < \lambda < 1 \). Then the global optimal solutions of problem \((\overline{P})\) is unique.

Proof. Let \( x_1, x_2 \in M \) be two different global optimal solutions of problem \((\overline{P})\). Then, \( f(x_1) = f(x_2) \). Since \( M \) is \( E \)-convex and \( f \) is strictly semi-\( E \)-convex, then

\[
f(\lambda E(x_1) + (1 - \lambda)E(x_2)) < \lambda f(x_1) + (1 - \lambda)f(x_2) = f(x_1)
\]

for each \( \lambda \in (0, 1) \).

Which contradicts the optimality of \( x_1 \) for problem \((\overline{P})\). Then, the global optimal solution of the problem \((\overline{P})\) is unique. □

Theorem 8. Assume function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is quasi-semi-\( E \)-convex on an \( E \)-convex set \( M \subseteq \mathbb{R}^n \). Let \( \alpha = \min_{x \in M} f(x) \). Then the set \( X = \{ x \in M \mid f(x) = \alpha \} \) of optimal solutions of problem \((\overline{P})\) is \( E \)-convex. If \( f \) is strictly quasi-semi-\( E \)-convex on an \( E \)-convex set \( M \subseteq \mathbb{R}^n \), i.e., \( f(\lambda E(x) + (1 - \lambda)E(y)) < \max\{ f(x), f(y) \} \) for each \( x, y \in M \) and \( x \neq y \). Then the set \( X \) is a singleton.
Proof. For any \( x, y \in X \) and \( 0 \leq \lambda \leq 1 \), then \( x, y \in M \) and \( f(x) = \alpha, f(y) = \alpha \), since function \( f : \mathbb{R}^n \to \mathbb{R} \) is quasi-semi-\( E \)-convex on an \( E \)-convex set \( M \subseteq \mathbb{R}^n \), then,

\[
\lambda E(x) + (1 - \lambda) E(y) \in M \quad \text{and} \quad f\left( \lambda E(x) + (1 - \lambda) E(y) \right) \leq \max\{ f(x), f(y) \} = \alpha,
\]

which implies that \( (1 - \lambda) E(x) + \lambda E(y) \in X \), it follows that \( X \) is \( E \)-convex.

For the second part, assume to the contrary that \( x, y \in X \), and \( x \neq y \), for \( \lambda \in (0, 1) \), then \( (1 - \lambda) E(x) + \lambda E(y) \in M \). Further, since \( f \) is strictly quasi-semi-\( E \)-convex on \( M \), we have \( f(\lambda E(x) + (1 - \lambda) E(y)) < \max\{ f(x), f(y) \} = \alpha \). This contradicts that \( \alpha = \min_{x \in M} f(x) \) and hence the result follows. \( \blacksquare \)

**Theorem 9.** Assume function \( f : \mathbb{R}^n \to \mathbb{R} \) is semi-\( E \)-convex on an \( E \)-convex set \( M \subseteq \mathbb{R}^n \). Then the set of optimal solutions of problem (\( P \)) is \( E \)-convex.

**Proof.** Let \( \bar{x} \) be an optimal solution of problem (\( P \)), and let \( \alpha = f(\bar{x}) \). Let \( X \) be the set of optimal solutions for problem (\( P \)) as follows:

\[
X = \{ x \in M \mid f(x) \leq \alpha \},
\]

for any \( x, y \in X \) and \( 0 \leq \lambda \leq 1 \), since function \( f : \mathbb{R}^n \to \mathbb{R} \) is semi-\( E \)-convex on an \( E \)-convex set \( M \subseteq \mathbb{R}^n \), then,

\[
f\left( \lambda E(x) + (1 - \lambda) E(y) \right) \leq \lambda f(x) + (1 - \lambda) f(y) \leq \alpha.
\]

Thus, \( (1 - \lambda) E(x) + \lambda E(y) \in X \), it follows that \( X \) is \( E \)-convex. \( \square \)

**Theorem 10.** If functions \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g_i : \mathbb{R}^n \to \mathbb{R} \) are quasi-semi-\( E \)-convex on \( \mathbb{R}^n \), \( i = 1, 2, \ldots, m \). Then the set of optimal solutions of problem (\( P \)) is \( E \)-convex.

**Proof.** From Proposition 7, it follows that \( M \) is \( E \)-convex set. Hence by Theorem 8, the set \( X = \{ x \in M \mid f(x) = \alpha \} \) of optimal solutions of problem (\( P \)) is \( E \)-convex. \( \square \)

**Corollary.** If functions \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g_i : \mathbb{R}^n \to \mathbb{R} \) are semi-\( E \)-convex on \( \mathbb{R}^n \), \( i = 1, 2, \ldots, m \). Then the set of optimal solutions of problem (\( P \)) is \( E \)-convex.

**Theorem 11.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be semi-\( E \)-convex on \( E \)-convex set \( M \), let \( \phi : \mathbb{R} \to \mathbb{R} \) be positively homogeneous nondecreasing function, then the composite function \( \phi \circ f \) is semi-\( E \)-convex on \( M \).

**Proof.** Since \( f \) is semi-\( E \)-convex on \( E \)-convex set \( M \), we have, for all \( x, y \in M \) and \( \lambda \in (0, 1) \),
\[ f\left(\lambda E(x) + (1 - \lambda) E(y)\right) \leq \lambda f(x) + (1 - \lambda) f(y) \quad \Rightarrow \]
\[ \phi \circ f\left(\lambda E(x) + (1 - \lambda) E(y)\right) \leq \phi \circ \left[\lambda f(x) + (1 - \lambda) f(y)\right] \leq \lambda \phi \circ f(x) + (1 - \lambda) \phi \circ f(y) \]
from which it follows that \( \phi \circ f \) is semi-\( E \)-convex on \( M \).

We have the following result, which is similar to that in [2].

**Theorem 12.** Assume function \( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable semi-\( E \)-convex on an \( E \)-convex set \( M \subseteq \mathbb{R}^n \), and \( u \in M \) is a fixed point of the map \( E \), i.e., \( u = E(u) \). Then \( u \in M \) is the minimum of function \( f \) on \( M \) if and only if \( u \in M \) satisfies the inequality
\[
\langle f'(E(u)), E(v) - E(u) \rangle \geq 0, \quad \forall v \in M, \tag{1}
\]
where \( f' \) is the differential of \( f \) at \( E(u) \).

**Proof.** Let \( u \in M \) be a minimum of function \( f \) on \( M \). Then
\[ f(E(u)) \leq f(E(v)), \quad \forall v \in M. \]
Since \( M \) is \( E \)-convex, we have
\[ (1 - \lambda) E(u) + \lambda E(v) \in M, \quad \forall u, v \in M. \]
Since \( f \) is differentiable on \( M \), we have
\[
f(E(u)) \leq f\left((1 - \lambda) E(u) + \lambda E(v)\right) = f\left(E(u) + \lambda (E(v) - E(u))\right) = f(E(u)) + \lambda \langle f'(E(u)), E(v) - E(u) \rangle + 0(\lambda) = f(E(u)) + \lambda \langle f'(E(u)), E(v) - u \rangle + 0(\lambda).
\]
Dividing the above inequality by \( \lambda \) and taking \( \lambda \to 0 \), we have
\[
\langle f'(E(u)), E(v) - E(u) \rangle = \langle f'(u), E(v) - E(u) \rangle \geq 0,
\]
which is the required result (1).

Conversely, let \( u \in M \) satisfy (1). Since \( f \) is semi-\( E \)-convex on \( E \)-convex set \( M \), for all \( u, v \in M, \lambda \in [0, 1], \) we have \((1 - \lambda) E(u) + \lambda E(v) \in M, \) and
\[
f\left(E(u) + \lambda (E(v) - E(u))\right) \leq (1 - \lambda) f(u) + \lambda f(v) = f(E(u)) + \lambda (f(v) - f(u)).
\]
Which implies that
\[ f(v) - f(u) \geq \frac{f(E(u) + \lambda (E(v) - E(u))) - f(E(u))}{\lambda}. \]
Letting \( \lambda \to 0 \), we have
\[ f(v) - f(u) \geq \langle f'(E(u)), E(v) - E(u) \rangle \geq 0. \]
Using (1), which implies that \( f(u) \leq f(v), \forall v \in M \), showing that \( u \in M \) is the minimum of \( f \) on \( M \). □

Acknowledgment

The author is indebted to the referees for valuable comments and suggestions for improvement.

References