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On degenerations of modules

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Abstract

We generalize a result of Zwara concerning the degeneration of modules over Artinian algebras to that over general algebras. In fact, let R be any algebra over a field and let M and N be finitely generated left R-modules. Then, we show that M degenerates to N if and only if there is a short exact sequence of finitely generated left R-modules $0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0$ such that the endomorphism ψ on Z is nilpotent. We give several applications of this theorem to commutative ring theory.

1. Introduction

In the present paper we shall give a necessary and sufficient condition for degeneration of modules (Theorem 2.2). To be more precise, let R be any algebra over a field and let M and N be finitely generated left R-modules. We shall prove that M degenerates to N if and only if there is a short exact sequence of finitely generated left R-modules

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0$$

such that the endomorphism ψ on Z is nilpotent.

This is obviously a generalization of a theorem of G. Zwara [5], who proves this equivalence for a finite dimensional algebra over a field. In that case the nilpotency condition for ψ is not necessary.

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We shall give a precise definition of degenerations, more precisely degenerations along discrete valuation rings, and discuss it to prove the above necessary and sufficient condition in Section 2. We should notice there that our choice of Z appearing in the short exact sequence is different from Zwara's.

In the previous paper [3], we consider a distinct type of degeneration that we call a degeneration along an affine line to distinguish it from the degeneration along a discrete valuation ring. In Section 3, we shall prove that a degeneration along a discrete valuation ring implies a degeneration along an affine line.

As applications of the theorem to commutative ring theory, we comment several remarks in Section 4. For example, we can show that the "G-dimension 0" property for a module is an open property as well as the "maximal Cohen–Macaulay" property.

2. Degenerations along DVR

In this section k always denotes a field and R is a k-algebra. Note that R may not be commutative nor Noetherian.

Definition 2.1. For finitely generated left R-modules M and N, we say that M degenerates to N along a discrete valuation ring, or N is a degeneration of M along a DVR, if there is a discrete valuation ring (V, tV, k) that is a k-algebra (where t is a prime element) and a finitely generated left $R \otimes_k V$-module Q which satisfies the following conditions:

(1) Q is flat as a V-module;
(2) $Q/tQ \cong N$ as a left R-module;
(3) $Q[1/t] \cong M \otimes_k V[1/t]$ as a left $R \otimes_k V[1/t]$-module.

The following theorem is the main theorem of this paper, which gives a perfect condition for the degeneration along a DVR.

Theorem 2.2. *The following conditions are equivalent for finitely generated left R-modules M and N:*

(1) *N is a degeneration of M along a DVR.*
(2) *There is a short exact sequence of finitely generated left R-modules*

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

such that the endomorphism ψ on Z is nilpotent, i.e., $\psi^n = 0$ for $n \gg 1$.

Proof. (1) $\Rightarrow$ (2). Suppose that there are a discrete valuation ring (V, tV, k) that is a k-algebra and a finitely generated left $R \otimes_k V$-module Q such that Q is V-flat, $Q/tQ \cong N$ and $Q[1/t] \cong M \otimes_k V[1/t]$.

First of all we note that $M \otimes_k V$ is a finitely generated $R \otimes_k V$-submodule of $M \otimes_k V[1/t] \cong Q[1/t]$ and hence that $M \otimes_k V \subseteq (1/t^n)Q$ for a large n. Replacing Q with $(1/t^n)Q$ if necessary, we may assume that $M \otimes_k V \subseteq Q$.

Now we claim that

(i) $M \otimes_k V$ is a direct summand of Q as a left R-module.

In fact, the natural inclusion $V \to V[1/t]$ is a splitting monomorphism of k-modules, since k is a field. It follows that the map $M \otimes_k V \to M \otimes_k V[1/t] \cong Q[1/t]$ is a splitting monomorphism of left R-modules. Restricting the splitting map onto Q, we see that the natural inclusion $M \otimes_k V \to Q$ is also a splitting monomorphism of left R-modules.

We also have the following direct decomposition:

(ii) $M \otimes_k V = M \oplus (M \otimes_k tV)$ as a left R-module.

This is obvious from the direct decomposition $V = k \oplus tV$ as a k-module. From (i) and (ii) we obtain the following isomorphism of left R-modules:

(iii) $Q/(M \otimes_k tV) \cong M \oplus Q/(M \otimes_k V)$.

We should note that

(iv) $Q/(M \otimes_k V)$ is finitely generated as a left R-module.

Actually, since $(M \otimes_k V)[1/t] = Q[1/t]$, we see that $t^n Q \subseteq M \otimes_k V$ for a large integer n. Therefore $Q/(M \otimes_k V)$ is a finitely generated left module over $R \otimes_k V/(t^n)$. Noting that $R \otimes_k V/(t^n)$ is finitely generated as a left R-module, we conclude that $Q/(M \otimes_k V)$ is finitely generated over R.

Now we consider a left R-module homomorphism

$$f : Q/(M \otimes_k V) \longrightarrow Q/(M \otimes_k tV),$$

which is defined by $f(x + M \otimes_k V) = tx + M \otimes_k tV$ for any $x \in Q$. And we claim that

(v) f is a monomorphism and $\mathrm{Coker}(f) \cong N$.

To prove this, we should note that t is a nonzero divisor on Q. Consequently, if $f(x + M \otimes_k V) = 0$, then $tx \in M \otimes_k tV = t(M \otimes_k V)$ and we have $x \in M \otimes_k V$. This implies that f is a monomorphism. Since the image of f is $tQ/(M \otimes_k tV)$, it is easy to see that the cokernel of f is isomorphic to $Q/tQ \cong N$.

Taking the direct decomposition (iii) into account, we may describe the monomorphism f as follows:

$$f = \begin{pmatrix} \phi \\ \psi \end{pmatrix} : Q/(M \otimes_k V) \to M \oplus Q/(M \otimes_k V).$$

Thus denoting $Q/(M \otimes_k V)$ by Z, we have the following exact sequence of finitely generated left R-modules:

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0.$$

It remains to prove that

(vi) ψ is a nilpotent endomorphism on Z.

To prove this, let $\pi : Q/(M \otimes_k tV) \to Q/(M \otimes_k V)$ be the natural projection induced from the decomposition (iii). Then it is easy to see that the morphism ψ is identical with the composition $\pi \cdot f : Q/(M \otimes_k V) \to Q/(M \otimes_k V)$ that is just a homothety by the element t. Since $(M \otimes_k V)[1/t] = Q[1/t]$ and since Q is a finitely generated $R \otimes_k V$-module, we have $t^n(Q/(M \otimes_k V)) = 0$ for a large integer n. Therefore we see that $\psi^n = 0$ as desired.

(2) ⇒ (1). Suppose that there is an exact sequence of finitely generated left R-modules

$$0 \to Z \xrightarrow{f=\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

such that ψ is nilpotent. Considering a trivial exact sequence

$$0 \to Z \xrightarrow{j=\binom{0}{1}} M \oplus Z \to M \to 0,$$

we shall combine these two exact sequences along a $[0, 1]$-interval. More precisely, let V be the discrete valuation ring $k[t]_{(t)}$, where t in an indeterminate over k, and consider a left $R \otimes_k V$-homomorphism

$$g = j \otimes t + f \otimes (1-t) = \begin{pmatrix} \phi \otimes (1-t) \\ 1 \otimes t + \psi \otimes (1-t) \end{pmatrix} : Z \otimes_k V \to (M \oplus Z) \otimes_k V.$$

First of all, we claim that

(i) g is a monomorphism.

To prove this, let S be a multiplicatively closed subset of $k[t]$ consisting of all the elements of the form $1 + tp(t)$ $(p(t) \in k[t])$. Note that, since $V = S^{-1}k[t]$, we have $X \otimes_k V = S^{-1}X[t]$ for any left R-module X. Therefore, it is enough to prove that the mapping

$$g' = tj + (1-t)f : Z[t] \to M[t] \oplus Z[t]$$

is a monomorphism of left $R[t]$-modules, since $g = S^{-1}g'$. To show this, let $\zeta = \sum_{i=r}^{s} z_i t^i$ $(z_r \neq 0)$ be a nonzero element of $Z[t]$. Then we have

$$g'(\zeta) = \sum_{i=r}^{s} \bigl(j(z_i)t^{i+1} + f(z_i)(1-t)t^i\bigr),$$

where the coefficient of t^r of the right-hand side is $f(z_r)$ that is nonzero, since f is an injection. Thus we show that $g'(\zeta) \neq 0$, hence that g' is a monomorphism.

Now we denote the cokernel of the monomorphism g by Q, which is a finitely generated left $R \otimes_k V$-module and there is an exact sequence

$$0 \to Z \otimes_k V \xrightarrow{g} (Z \otimes_k V) \oplus (M \otimes_k V) \to Q \to 0.$$

We claim that

(ii) Q is flat over V and $Q/tQ \cong N$.

To prove this, we note that $g \otimes_V V/tV = f$ that is an injection. Since $Z \otimes_k V$ and $M \otimes_k V$ are flat over V, it follows that $\mathrm{Tor}_1^V(Q, V/tV) = 0$ and therefore Q is flat over V. Furthermore, we see that $Q/tQ = \mathrm{Coker}(g \otimes_V V/tV) = \mathrm{Coker}(f) \cong N$ as a left R-module.

Now it remains to prove the following claim to complete the proof.

(iii) $Q[1/t] \cong M \otimes_k V[1/t]$ as a left $R \otimes_k V$-module.

The variable t is of course a unit in the center of the ring $R \otimes_k V[1/t]$. Thus the morphism $g \otimes_V V[1/t]$ is essentially the same as the morphism

$$Z \otimes_k V[1/t] \xrightarrow{\binom{s\phi}{1+s\psi}} M \otimes_k V[1/t] \oplus Z \otimes_k V[1/t],$$

where $s = (1-t)/t \in V[1/t]$. Note that $s\psi : Z \otimes_k V[1/t] \to Z \otimes_k V[1/t]$ is nilpotent as well as ψ, hence $1 + s\psi$ is an automorphism on $Z \otimes_k V[1/t]$. Thus we have an isomorphism $Q[1/t] \cong M \otimes_k V[1/t]$ from the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z \otimes_k V[1/t] & \xrightarrow{\binom{s\phi}{1+s\psi}} & M \otimes_k V[1/t] \oplus Z \otimes_k V[1/t] & \longrightarrow & Q[1/t] & \longrightarrow & 0 \\
 & & \| & & \downarrow \begin{pmatrix} 1 & -s\phi(1+s\psi)^{-1} \\ 0 & (1+s\psi)^{-1} \end{pmatrix} & & \downarrow & & \\
0 & \longrightarrow & Z \otimes_k V[1/t] & \xrightarrow[\binom{0}{1}]{} & M \otimes_k V[1/t] \oplus Z \otimes_k V[1/t] & \longrightarrow & M \otimes_k V[1/t] & \longrightarrow & 0.
\end{array}$$

This completes the proof of the theorem. □

Remark 2.3. As G. Zwara has shown in [5], if R is an Artinian k-algebra, then the following conditions are equivalent for finitely generated left R-modules M and N:

(1) N is a degeneration of M along a DVR.
(2′) There is a short exact sequence of finitely generated left R-modules

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0.$$

Note here that we need not the nilpotency assumption for ψ. Actually, suppose that there is such an exact sequence as in the condition (2′). Then, since $\mathrm{End}_R(Z)$ is an Artinian ring, we decompose Z as $Z' \oplus Z''$ and according to this decomposition, we can describe ψ as

$$\begin{pmatrix} \psi' & 0 \\ 0 & \psi'' \end{pmatrix} : Z' \oplus Z'' \to Z' \oplus Z'',$$

where ψ' is an isomorphism and ψ'' is nilpotent (Fitting theorem). Therefore, we obtain an exact sequence of the following type:

$$0 \to Z'' \xrightarrow{\binom{\phi''}{\psi''}} M \oplus Z'' \to N \to 0,$$

such that ψ'' is nilpotent. In this way, our theorem contains the theorem of Zwara.

By the proof of (2) $\Rightarrow$ (1) of the theorem, we get the following result as a corollary.

Corollary 2.4. *Suppose that M degenerates to N along a DVR. Then as a discrete valuation ring V we can always take the ring $k[t]_{(t)}$.*

Remark 2.5. Assume that there is an exact sequence of finitely generated left R-modules

$$0 \to N' \xrightarrow{p} M \xrightarrow{q} N'' \to 0.$$

Then it is easy to see that M degenerates to $N' \oplus N''$ along a DVR. In fact, we have only to notice that there is an exact sequence

$$0 \to N' \xrightarrow{\binom{p}{0}} M \oplus N' \xrightarrow{\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)} N'' \oplus N' \to 0,$$

where the mapping $\psi : N' \to N'$ is the zero mapping, hence nilpotent.

3. Degenerations along affine lines

We have considered a different kind of degenerations in the previous paper [3], mainly for maximal Cohen–Macaulay modules over a commutative Cohen–Macaulay local ring. To distinguish it from the degenerations defined in Definition 2.1, we make the following definition.

Definition 3.1. In this definition we assume that k is an algebraically closed field to identify the affine line with k. And let R be a k-algebra as before. For finitely generated left R-modules M and N, we say that M degenerates to N along an affine line, or N is a degeneration of M along an affine line, if there is a finitely generated left module Q over $R \otimes_k k[t]$ which satisfies the following conditions:

(1) Q is flat as a $k[t]$-module.
(2) For any $c \in k$, let us denote $Q/(t-c)Q$ by Q_c, which is a finitely generated left R-module. Then, $Q_0 \cong N$ as a left R-module.
(3) There is a non-empty Zariski open subset U of $\mathbb{A}_k^1 \cong k$ such that if $c \in U$, then $Q_c \cong M$ as a left R-module.

Compare this with [3, Definition 4.1].

Now we prove an implication between degenerations in the following theorem. But one should remark that the converse implication does not hold in general. See Remark 4.2.

Theorem 3.2. *Assume that k is an algebraically closed field and that R is a left Noetherian k-algebra. Let M and N be finitely generated left R-modules. If M degenerates to N along a DVR, then M degenerates to N along an affine line.*

Proof. Suppose that M degenerates to N along a DVR. As we have shown in Corollary 2.4, we may take $V = S^{-1}k[t]$ as a discrete valuation ring, where $S = \{1 + tp(t) \mid p(t) \in k[t]\}$. Thus, there is a finitely generated $S^{-1}R[t]$-module Q such that Q is flat over V, $Q/tQ \cong N$ and $Q[1/t] \cong M \otimes_k S^{-1}k[t, t^{-1}]$. Now take a finitely generated $R[t]$-submodule Q' of Q so that $S^{-1}Q' = Q$.

First of all, we claim that

(i) Q' is flat over $k[t]$.

To show this, it is enough to prove that the multiplication map by $p(t)$ on Q' is an injection for any nonzero element $p(t) \in k[t]$. Since Q is flat over $S^{-1}k[t]$, the multiplication by $p(t)$ on Q is either an injection or a bijection. Since Q' is a submodule of Q, the claim follows.

Recall that for an element $c \in k$, the R-module Q'_c is defined to be $Q'/(t-c)Q'$. Next we claim that

(ii) $Q'_0 \cong N$ as an R-module.

To show this note that $N \cong Q/tQ \cong S^{-1}(Q'/tQ') = S^{-1}Q'_0$ by our choice. Since Q'_0 is an $R[t]$-module but annihilated by t, any element of S acts on Q'_0 as the identity. Thus we have $N \cong S^{-1}Q'_0 = Q'_0$.

To finish the proof we are proving that

(iii) there is a non-empty open subset $U \subset \mathbb{A}_k^1 \cong k$ such that $Q'_c \cong M$ for any $c \neq 0$ in U.

Note that there is an isomorphism of $S^{-1}R[t]$-modules; $S^{-1}Q'[1/t] = Q[1/t] \cong M \otimes_k S^{-1}k[t, t^{-1}]$. Therefore, it follows from the next lemma that one can choose an element $q(t) \in S$ such that

$$Q'[1/t]_{q(t)} \cong M \otimes_k k\big[t, t^{-1}\big]_{q(t)}$$

as an $R[t]_{q(t)}$-module. Now define an open subset $U \subset k$ as $\{c \in k \mid q(c) \neq 0\}$. Then, for $c \neq 0 \in U$, a mapping $h : k[t, t^{-1}]_{q(t)} \to k$ defined by $h(p(t)) = p(c)$ $(p(t) \in k[t, t^{-1}]_{q(t)})$ is a well-defined k-algebra map. Taking tensor product of this $k[t, t^{-1}]_{q(t)}$-algebra k with the above, we have

$$Q'[1/t]_{q(t)} \otimes_{k[t,t^{-1}]_{q(t)}} k \cong \left(M \otimes_k k\left[t, t^{-1}\right]_{q(t)}\right) \otimes_{k[t,t^{-1}]_{q(t)}} k$$

and this shows $Q'_c \cong M$ as desired. □

It remains to prove the following lemma that is well-known and easily proved. We leave its proof to the reader as an exercise.

Lemma 3.3. *Let A be a left Noetherian ring and let S be a multiplicatively closed subset in the center of A. For finitely generated left A-modules X and Y, if $S^{-1}X$ is isomorphic to $S^{-1}Y$ as a left $S^{-1}A$-module, then there is an element $s \in S$ such that $X[1/s]$ is isomorphic to $Y[1/s]$ as a left $A[1/s]$-module.*

4. Remarks for commutative Noetherian algebras

In the rest of the paper, we assume that R is a commutative Noetherian algebra over a field k. In this case we have the following result as a corollary of Theorem 2.2.

Corollary 4.1. *Suppose that M and N are R-modules of finite length. Then the following conditions are equivalent*:

(1) *N is a degeneration of M along a DVR.*
(2) *There is an exact sequence*

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

where Z is also a module of finite length.

In particular, if M degenerates to N along a DVR, then we must have an equality of the lengths; $\ell_R(M) = \ell_R(N)$.

Proof. (1) ⇒ (2). Suppose that M degenerates to N along a DVR. Then by virtue of Theorem 2.2, there is an exact sequence

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

where ψ is a nilpotent endomorphism. Now let $\mathfrak{p}$ be any prime ideal of R that is not a maximal ideal. Since M and N are of finite length, we see that $M_{\mathfrak{p}} = N_{\mathfrak{p}} = 0$. Thus taking a localization at $\mathfrak{p}$ of the above exact sequence, we see that $\psi_{\mathfrak{p}} : Z_{\mathfrak{p}} \to Z_{\mathfrak{p}}$ is an isomorphism. Since $\psi_{\mathfrak{p}}$ is nilpotent as well, we conclude that $Z_{\mathfrak{p}} = 0$. This is true for any non-maximal prime ideal $\mathfrak{p}$, and thus Z is of finite length. In this case, it follows from the exact sequence that $\ell_R(M) + \ell_R(Z) = \ell_R(N) + \ell_R(Z)$, hence $\ell_R(M) = \ell_R(N)$.

(2) $\Rightarrow$ (1). Suppose that there is such an exact sequence as in (2). Note that $\mathrm{End}_R(Z)$ is an Artinian algebra, hence by the completely same manner as in Remark 2.3 we may have an exact sequence of the same type but with the nilpotent ψ. Therefore, M degenerates to N along a DVR by Theorem 2.2. □

Remark 4.2. There is an example where the opposite direction of the implication in Theorem 3.2 does not hold.

For example, let $R = k[\![x]\!]$ be the formal power series ring over an algebraically closed field k and let $M = R/(x)$ and $N = R/(x^2)$. Since M and N have distinct lengths, N can never be a degeneration of M along a DVR by Corollary 2.4. On the other hand, consider the $R[t]$-module $Q = R[t]/(x^2 - tx)$. It is easy to see that $\mathrm{Ass}_{R[t]} Q = \{(x), (x - t)\}$, hence any nonzero element of $k[t]$ is a nonzero divisor on Q. This implies that Q is flat over $k[t]$. For any element $c \in k$, note that $Q_c \cong R/(x(x - c))$, hence that $Q_0 \cong R/(x^2)$ and $Q_c \cong R/(x)$ for $c \neq 0$, since $x - c$ is a unit in R. Therefore, from the definition, the module M degenerates to N along an affine line. Note that there is an exact sequence

$$0 \to R \xrightarrow{\binom{1}{x}} R/(x) \oplus R \xrightarrow{(x,-1)} R/(x^2) \to 0,$$

however the endomorphism $R \xrightarrow{x} R$ is not nilpotent.

Remark 4.3. Let R be a Cohen–Macaulay local ring and let M and N be maximal Cohen–Macaulay modules over R. If N is a degeneration of M along a DVR, then there is an exact sequence

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

where ψ is nilpotent. In such a case, we can show that Z is also a maximal Cohen–Macaulay module. Hence, we can take such a short exact sequence inside the category of maximal Cohen–Macaulay modules (cf. [2]).

To show that Z is maximal Cohen–Macaulay, let d be the Krull dimension of R and let K be the residue field of the local ring R. Since $\mathrm{Ext}^i_R(K, M) = \mathrm{Ext}^i_R(K, N) = 0$ for $0 \leqslant i < d$, it follows from the induced long exact sequence of cohomologies that $\mathrm{Ext}^i_R(K, \psi) : \mathrm{Ext}^i_R(K, Z) \to \mathrm{Ext}^i_R(K, Z)$ is an isomorphism for $0 \leqslant i < d$. Since $\mathrm{Ext}^i_R(K, \psi)$ is a nilpotent map as well as ψ, we see that $\mathrm{Ext}^i_R(K, Z) = 0$ for $0 \leqslant i < d$, and this shows that Z is also a maximal Cohen–Macaulay module.

Remark 4.4. Auslander and Bridger [1] give a definition of G-dimension, which we denote by $\text{G-dim}_R M$ for a finitely generated R-module M. In our subsequent work [4], we have shown the following fact:

(1) If there is an exact sequence of finitely generated R-modules

$$0 \to Z \to M \oplus Z \to N \to 0,$$

then we have an inequality $\text{G-dim}_R M \leqslant \text{G-dim}_R N$.

Combining this with Theorem 2.2, we have the following result as a corollary:

(2) Assume that R is a Noetherian commutative algebra over a field k and let M and N be finitely generated R-modules. Suppose that N is a degeneration of M along a DVR. Then the inequality $\text{G-dim}_R M \leqslant \text{G-dim}_R N$ holds.

In particular, if N has G-dimension 0, then so does M in this case. We infer from this that if there is an algebraic set that parameterizes a family of finitely generated R-modules, then the set of points corresponding to modules with G-dimension 0 should form an open subset. By this property, we may say that the property for a module having G-dimension 0 is an 'open' property. This generalizes a well-known fact that the maximal Cohen–Macaulay property for modules over a Gorenstein local ring is an open property.

References

[1] M. Auslander, M. Bridger, Stable module theory, Mem. Amer. Math. Soc. 94 (1969).

[2] Y. Yoshino, Cohen–Macaulay modules over Cohen–Macaulay rings, in: Lecture Notes Ser., vol. 146, Cambridge Univ. Press, Cambridge, 1990.

[3] Y. Yoshino, On degenerations of Cohen–Macaulay modules, J. Algebra 248 (2002) 272–290.

[4] Y. Yoshino, Degeneration and G-dimension of modules, Lecture at Lisbon Conference of Commutative Algebra, June, 2003.

[5] G. Zwara, Degenerations of finite dimensional modules are given by extensions, Compositio Math. 121 (2000) 205–218.