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Applied Mathematics Letters 19 (2006) 215–222
Applied Mathematics Letters
www.elsevier.com/locate/aml

Impulsive partial neutral differential equations

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Received 12 April 2004; received in revised form 18 March 2005; accepted 15 April 2005

Abstract

In this work we study the existence and regularity of mild solutions for impulsive first order partial neutral functional differential equations with unbounded delay.

Keywords: Impulsive differential equations; Neutral differential equations; Semigroups of bounded linear operators

1. Introduction

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last few decades. The literature related to ordinary neutral functional differential equations is very extensive and we refer the reader to [1,2] and the references therein. Partial neutral functional differential equations with delay have been studied in [3–7].

On the other hand, the theory of impulsive differential equations has become an important area of investigation in recent years, stimulated by their numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc. Related to this matter we mention [8,9] and the references in these works.

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2005.04.005

In this work we study the existence and regularity of mild solutions for a class of abstract neutral functional differential equations (ANFDE) with unbounded delay and impulses described in the form

$$\frac{d}{dt}(x(t) + F(t, x_t)) = Ax(t) + G(t, x_t), \qquad t \in I = [0, \sigma], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

$$\triangle x(t_i) = I_i(x_{t_i}), \qquad i = 1, \ldots, n, \tag{1.3}$$

where A is the infinitesimal generator of an analytic semigroup of linear operators, $(T(t))_{t\geq 0}$, on a Banach space $(X, \|\cdot\|)$; the history $x_t : (-\infty, 0] \to X, x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; F, G, I_i, are appropriate functions; $0 < t_1 < \cdots < t_n < \sigma$ are prefixed points and $\triangle\xi(t)$ is the jump of a function ξ at t, which is defined by $\triangle\xi(t) = \xi(t^+) - \xi(t^-)$.

Ordinary impulsive neutral differential systems have been studied recently in several papers; see [10–14]. The existence of solutions for impulsive partial ANFDE is an untreated topic and that is the motivation of this work. In relation with this last remark, we consider it important to observe that some systems similar to (1.1)–(1.3) are studied in [13,14]. However, in these works the authors impose some severe assumptions on the semigroup generated by A which imply that the underlying space X has finite dimension; see [7] for details. As a consequence, the systems studied in these works are really ordinary and not partial.

In this work, $A : D(A) \subset X \to X$ is the infinitesimal generator of an analytic semigroup of linear operators $(T(t))_{t\geq 0}$ on $X, 0 \in \rho(A)$, and $\widetilde{M}$ is a constant such that $\|T(t)\| \leq \widetilde{M}$ for every $t \in I$. The notation $(-A)^\alpha, \alpha \in (0, 1)$, stands for the fractional power of A and X_α is the domain of $(-A)^\alpha$ endowed with the graph norm. For literature relating to semigroup theory, we suggest [15].

We say that a function $u(\cdot)$ is a normalized piecewise continuous function on $[\mu, \tau]$ if u is piecewise continuous, and left continuous on $(\mu, \tau]$. We denote by $\mathcal{PC}([\mu, \tau]; X)$ the space formed by the normalized piecewise continuous functions from $[\mu, \tau]$ to X. The notation $\mathcal{PC}$ stands for the space formed by all functions $u \in \mathcal{PC}([0, \mu]; X)$ such that $u(\cdot)$ is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, \ldots, n$. In this work, $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is the space $\mathcal{PC}$ endowed with the norm $\|x\|_{\mathcal{PC}} = \sup_{s\in I} \|x(s)\|$. It is clear that $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.

In this work we will employ an axiomatic definition for the phase space $\mathcal{B}$ which is similar to that used in [16]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ to X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and verifying the following axioms:

(A) If $x : (-\infty, \mu + b] \to X, b > 0$, is such that $x_\mu \in \mathcal{B}$ and $x|_{[\mu,\mu+b]} \in \mathcal{PC}([\mu, \mu + b] : X)$, then for every $t \in [\mu, \mu + b]$ the following conditions hold:
  (i) x_t is in $\mathcal{B}$,
  (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,
  (iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \mu) \sup\{\|x(s)\| : \mu \leq s \leq t\} + M(t - \mu)\|x_\mu\|_{\mathcal{B}}$,
  where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(B) The space $\mathcal{B}$ is complete.

Example: The phase space $\mathcal{PC}_r \times L^2(g, X)$.

Let $g : (-\infty, -r] \to \mathbb{R}$ be a positive function verifying the conditions (g-6) and (g-7) of [16]. This means that $g(\cdot)$ is Lebesgue integrable on $(-\infty, -r)$ and that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that $g(\xi + \theta) \leq \gamma(\xi)g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. Let $\mathcal{B} := \mathcal{PC}_r \times L^2(g; X), r \geq 0$, be the space formed of all classes of functions $\varphi : (-\infty, 0] \to X$ such that $\varphi|_{[-r,0]} \in \mathcal{PC}([-r, 0], X), \varphi(\cdot)$ is Lebesgue measurable on $(-\infty, -r]$ and $g(\cdot) \mid \varphi(\cdot)|^p$ is Lebesgue integrable on $(-\infty, -r]$. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} := \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p \mathrm{d}\theta\right)^{1/p} + \sup_{\theta\in[-r,0]} \|\varphi(\theta)\|.$$

From the proof of [16, Theorem 1.3.8] it follows that $\mathcal{B}$ is a phase space which verifies the axioms (A) and (B) of our work. Moreover, when $r = 0$, $H = 1$, $M(t) = \gamma(-t)^{\frac{1}{2}}$ and $K(t) = 1 + (\int_{-t}^{0} g(\tau)\mathrm{d}\tau)^{\frac{1}{2}}$ for $t \geq 0$.

Example: The phase space $\mathcal{PC}_g(X)$.

As usual, we say that a function $\varphi : (-\infty, 0] \to X$ is normalized piecewise continuous if the restriction of φ to any interval $[-r, 0]$ is a normalized piecewise continuous function.

Let $g : (-\infty, 0] \to [1, \infty)$ be a continuous function which satisfies the conditions (g-1), (g-2) of [16]. We denote by $\mathcal{PC}_g(X)$ the space formed by the normalized piecewise continuous functions φ such that $\frac{\varphi}{g}$ is bounded on $(-\infty, 0]$ and by $\mathcal{PC}_g^0(X)$ the subspace of $\mathcal{PC}_g(X)$ formed by the functions $\varphi : (-\infty, 0] \to X$ such that $\frac{\varphi(\theta)}{g(\theta)} \to 0$ as $\theta \to -\infty$. It is easy to see that $\mathcal{PC}_g(X)$ and $\mathcal{PC}_g^0(X)$ endowed with the norm $\|\varphi\|_{\mathcal{B}} := \sup_{\theta\in(-\infty,0]} \frac{\|\varphi(\theta)\|}{g(\theta)}$ are phase spaces in the sense considered in this work.

2. Existence results

To study the existence of solutions of (1.1)–(1.3), we always assume that the next condition holds.

H_1 The functions $G, F : I \times \mathcal{B} \to X$ and $I_i : \mathcal{B} \to X, i = 1, \ldots, n$, are continuous and satisfy the following conditions:

(i) For every $x : (-\infty, \sigma] \to X$ such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$, the function $t \to G(t, x_t)$ is strongly measurable and the function $t \to F(t, x_t)$ belongs to $\mathcal{PC}$.

(ii) There are positive constants $\beta \in (0, 1), L_F, L_G, L_i, i = 1, \ldots, n$, such that F is X_β-valued, $(-A)^\beta F : I \times \mathcal{B} \to X$ is continuous and

$$\|G(t, \psi_1) - G(t, \psi_2)\| \leq L_G\|\psi_1 - \psi_2\|_{\mathcal{B}}, \qquad \psi_i \in \mathcal{B}, t \in I,$$
$$\|(-A)^\beta F(t, \psi_1) - (-A)^\beta F(t, \psi_2)\| \leq L_F\|\psi_1 - \psi_2\|_{\mathcal{B}}, \qquad \psi_i \in \mathcal{B}, t \in I,$$
$$\|I_i(\psi_1) - I_i(\psi_2)\| \leq L_i\|\psi_1 - \psi_2\|_{\mathcal{B}}, \qquad \psi_i \in \mathcal{B}.$$

Remark 2.1. Let $x : (-\infty, \sigma] \to X$ be such that $x_0 = \varphi$ and $x|_I \in \mathcal{PC}$, and assume that H_1 holds. From the continuity of $s \to AT(t - s)$ in the uniform operator topology on $[0, t)$ and the estimate

$$\|AT(t - s)F(s, x_s)\| = \|(-A)^{1-\beta}T(t - s)(-A)^\beta F(s, x_s)\| \leq \frac{C_{1-\beta}}{(t - s)^{1-\beta}}\|(-A)^\beta F(s, x_s)\|,$$

it follows that the function $\theta \to AT(t - \theta)F(\theta, x_\theta)$ is integrable on $[0, t)$ for every $t > 0$. Proceeding similarly we can assert that the function $s \to T(t - s)G(s, x_s)$ is integrable on $[0, t]$ for every $t > 0$.

Next, we introduce the concepts of mild and strong solutions of (1.1)–(1.3).

Definition 2.1. A function $u : (-\infty, \sigma] \to X$ is a mild solution of the impulsive abstract Cauchy problem (1.1)–(1.3) if $u_0 = \varphi$; $u(\cdot)_{|I} \in \mathcal{PC}$ and

$$\begin{aligned} u(t) = {} & T(t)(\varphi(0) + F(0, \varphi)) - F(t, u_t) - \int_0^t AT(t-s)F(s, u_s)\mathrm{d}s \\ & + \int_0^t T(t-s)G(s, u_s)\mathrm{d}s + \sum_{0<t_i<t} T(t-t_i)[I_i(u_{t_i}) + \triangle F(t_i, u_{t_i})], \qquad t \in I. \end{aligned} \tag{2.1}$$

Definition 2.2. A function $u : (-\infty, \sigma] \to X$ is a strong solution of (1.1)–(1.3) if $u_0 = \varphi$; $u(\cdot)_{|I} \in \mathcal{PC}$; the functions $u(t)$ and $F(t, u_t)$ are differentiable a.e. on I with derivatives $u'(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}F(t, u_t)$ in $L^1(I; X)$; Eq. (1.1) is verified a.e. on I and (1.3) is valid for every $i = 1, \ldots, n$.

Let $u(\cdot)$ be a strong solution and assume that H_1 holds. From the semigroup theory, we get

$$u(t) = T(t)\varphi(0) - \int_0^t T(t-s)\frac{\mathrm{d}}{\mathrm{d}t}F(s, u_s)\mathrm{d}s + \int_0^t T(t-s)G(s, u_s)\mathrm{d}s, \qquad t \in [0, t_1),$$

which implies that

$$\begin{aligned} u(t_1^-) = {} & T(t_1)(\varphi(0) + F(0, \varphi)) - F(t_1^-, u_{t_1^-}) - \int_0^{t_1} AT(t-s)F(s, u_s)\mathrm{d}s \\ & + \int_0^{t_1} T(t-s)G(s, u_s)\mathrm{d}s. \end{aligned}$$

By using that $u(\cdot)$ is a solution of (1.1) on (t_1, t_2) and that $u(t_1^+) = u(t_1^-) + I_1(u_{t_1})$, we see that

$$\begin{aligned} u(t) = {} & T(t-t_1)(u(t_1^+) + F(t_1^+, u_{t_1^+})) - F(t, u_t) - \int_{t_1}^t AT(t-s)F(s, u_s)\mathrm{d}s \\ & + \int_{t_1}^t T(t-s)G(s, u_s)\mathrm{d}s \\ = {} & T(t)(\varphi(0) + F(0, \varphi)) - F(t, u_t) - \int_0^t AT(t-s)F(s, u_s)\mathrm{d}s + \int_0^t T(t-s)G(s, u_s)\mathrm{d}s \\ & + T(t-t_1)\left[I_1(u_{t_1}) + \triangle F(t_1, u_{t_1})\right], \qquad t \in (t_1, t_2). \end{aligned}$$

Reiterating these procedure, we can conclude that $u(\cdot)$ is also a mild solution of (1.1)–(1.3).

Remark 2.2. Our definition of a mild solution is different from that introduced in [13,14].

In the following result, the main result of this work, $K_\sigma = \sup_{s \in I} K(s)$.

Theorem 2.1. *If* $\Theta = K_\sigma[L_F(\|(-A)^{-\beta}\|(1 + 2n\tilde{M}) + \frac{C_{1-\beta}\sigma^\beta}{\beta}) + \widetilde{M}(\sigma L_G + \sum_{i=1}^n L_i)] < 1$, *then there exists a unique mild solution of* (1.1)–(1.3).

Proof. On the metric space $\mathcal{BPC} = \{u : (-\infty, \sigma] \to X; u_0 = \varphi,\ u|_I \in \mathcal{PC}\}$ endowed with the metric $d(u, v) = \|u - v\|_{\mathcal{PC}}$, we define the operator $\Gamma : \mathcal{BPC} \to \mathcal{BPC}$ by

$$\Gamma x(t) = \begin{cases} \varphi(t), & t \le 0, \\ T(t)(\varphi(0) + F(0,\varphi)) - F(t,x_t) - \int_0^t AT(t-s)F(s,x_s)\mathrm{d}s \\ \quad + \int_0^t T(t-s)G(s,x_s)\mathrm{d}s + \sum_{0<t_i<t} T(t-t_i)[I_i(x_{t_i}) + \triangle F(t_i, x_{t_i})], & t \in I. \end{cases}$$

From Remark 2.1, we know that $s \to T(t-s)G(s,x_s)$ and $s \to AT(t-s)F(s,x_s)$ are integrable on $[0,t)$ for every $t \in I$. Thus, Γ is well defined with values in $\mathcal{PC}$. Let $u, v \in \mathcal{BPC}$. Using that $\|\triangle F(t_i, u_{t_i}) - \triangle F(t_i, v_{t_i})\| \le 2L_F K_\sigma \|(-A)^{-\beta}\| \|(u-v)_{|I}\|_{\mathcal{PC}}$, we get

$$\begin{aligned} \|\Gamma u(t) - \Gamma v(t)\| &\le \|(-A)^{-\beta}\| L_F \|u_t - v_t\|_{\mathcal{B}} + \int_0^t \left[\frac{C_{1-\beta} L_F}{(t-s)^{1-\beta}} + \widetilde{M} L_G \right] \|u_s - v_s\|_{\mathcal{B}} \mathrm{d}s \\ &+ \widetilde{M} \sum_{t_i \le t} (L_i \|u_{t_i} - v_{t_i}\|_{\mathcal{B}} + 2\|(-A)^{-\beta}\| L_F K_\sigma \|(u-v)_{|I}\|_{\mathcal{PC}}) \\ &\le K_\sigma \left[L_F \left(\|(-A)^{-\beta}\|(1 + 2n\widetilde{M}) + \frac{C_{1-\beta}\sigma^\beta}{\beta} \right) + \widetilde{M} \left(\sigma L_G + \sum_{i=1}^n L_i \right) \right] \|(u-v)_{|I}\|_{\mathcal{PC}}, \end{aligned}$$

and hence $d(\Gamma u, \Gamma v) \le \Theta d(u, v)$, which proves that Γ is a contraction on $\mathcal{BPC}$. Thus, there exists a unique mild solution of (1.1)–(1.3). This completes the proof. □

In the next result, for $x \in X$, $\mathcal{X}_x : (-\infty, 0] \to X$ represents the function defined by $\mathcal{X}_x(\theta) = 0$ for $\theta < 0$ and $\mathcal{X}_x(0) = x$, and $(S(t))_{t \ge 0}$ is the family of linear operators defined by $S(t)\psi(\theta) = \psi(0)$ on $[-t, 0]$ and $S(t)\psi(\theta) = \psi(t+\theta)$ on $(-\infty, -t]$. Moreover, for the sake of brevity, we put $t_0 = 0$, $t_{n+1} = \sigma$ and $I_0 = 0$.

Theorem 2.2. *Assume that the hypotheses of Theorem 2.1 are satisfied and let $u(\cdot)$ be the unique mild solution of (1.1)–(1.3). If X is reflexive, $F(I \times \mathcal{B}) \subset D(A)$ and the next condition holds:*

(a) *$u_{t_i} + \mathcal{X}_{I_i(u_{t_i})} \in \mathcal{B}$, $u(t_i) + I_i(u_{t_i}) \in D(A)$ and $t \to S(t)(u_{t_i} + \mathcal{X}_{I_i(u_{t_i})})$ is Lipschitz on $[t_i, t_{i+1}]$ for every $i = 0, \ldots n$,*

then $u(\cdot)$ is a strong solution of (1.1)–(1.3).

Proof. Let $u^i \in C([t_i, t_{i+1}] : X)$, $i = 0, \ldots, n$, be such that $u^i = u$ on $(t_i, t_{i+1}]$. It is easy to prove that $u^i(\cdot)$ is a mild solution, in the sense introduced in [5], of the abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t}(x(t) + F(t, x_t)) = Ax(t) + G(t, x_t), \qquad t \in [t_i, t_{i+1}], \tag{2.2}$$

$$x_{t_i} = u_{t_i} + \mathcal{X}_{I_i(u_{t_i})}. \tag{2.3}$$

From [5, Theorem 3.1], we infer that $u^i(\cdot)$ is a strong solution of (2.2) and (2.3). This means that u^i and $t \to F(t, (u^i)_t)$ are differentiable a.e. on $[t_i, t_{i+1}]$; that $(u^i)'$ and $\frac{\mathrm{d}}{\mathrm{d}t} F(t, u^i_t)$ belong to $L^1([t_i, t_{i+1}]; X)$ and that (2.2) is verified a.e. on $[t_i, t_{i+1}]$. This is enough for concluding that $u(\cdot)$ is a strong solution of (1.1)–(1.3). □

3. Example

Next, we study Example 4.1 of [5] subjected to impulsive conditions. Let $X = L^2([0,\pi])$, $\mathcal{B} = \mathcal{PC}_0 \times L^2(g, X)$, the phase space introduced in Section 1, and $A : D(A) \subset X \to X$ be the operator $Af = f''$ with domain $D(A) := \{f \in X : f'' \in X,\ f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X. The spectrum of A is discrete with eigenvalues $-n^2$, $n \in \mathbb{N}$, and associated normalized eigenvectors $z_n(\xi) := (\frac{2}{\pi})^{1/2}\sin(n\xi)$. Moreover, the set $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X, $T(t)f = \sum_{n=1}^{\infty} e^{-n^2 t}\langle f, z_n\rangle z_n$ for $f \in X$ and $(-A)^{\alpha} f = -\sum_{n=1}^{\infty} n^{2\alpha}\langle f, z_n\rangle z_n$ for $f \in X_\alpha$. It follows from these expressions that $\|T(t)\| \leq e^{-t}$, $\|(-A)^{\frac{1}{2}}T(t)\| \leq \frac{e^{\frac{-t}{2}} t^{-\frac{1}{2}}}{\sqrt{2}}$ for each $t > 0$ and that $\|(-A)^{-1/2}\| = 1$.

Consider the first order neutral differential equation with unbounded delay and impulses:

$$\frac{\partial}{\partial t}\left[u(t,\xi) + \int_{-\infty}^{t}\int_0^{\pi} b(t-s,\eta,\xi)u(s,\eta)\,d\eta\, ds\right] = \frac{\partial^2}{\partial \xi^2}u(t,\xi) + a_0(\xi)u(t,\xi)$$
$$+ a_1(t,\xi) + \int_{-\infty}^{t} a(t-s)u(s,\xi)\,ds, \qquad t \in I = [0,\sigma], \tag{3.1}$$
$$u(t,0) = u(t,\pi) = 0, \qquad t \in I, \tag{3.2}$$
$$u(\tau,\xi) = \varphi(\tau,\xi), \qquad \tau \leq 0, \qquad \xi \in J = [0,\pi], \tag{3.3}$$
$$\triangle u(t_i)(\xi) = \int_{-\infty}^{t_i} p_i(t_i - s)u(s,\xi)\,ds, \qquad \xi \in J = [0,\pi], \tag{3.4}$$

where $0 < t_1 < \cdots < t_n < \sigma$ are prefixed numbers and $\varphi \in \mathcal{B}$. To study this system we will assume that $a_0 : J \to \mathbb{R}$, $a_1 : I \times J \to \mathbb{R}$ are continuous functions and that the following conditions hold:

(i) The functions $\frac{\partial^i b(s,\eta,\xi)}{\partial \xi^i}$, $i = 0, 1$, are measurable on $\mathbb{R} \times J^2$, $b(s,\eta,\pi) = b(s,\eta,0) = 0$, for every $(s,\eta) \in \mathbb{R} \times J$ and $N_0 := \int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi} \frac{1}{g(\tau)}(b(\tau,\eta,\zeta))^2\, d\eta\, d\tau\, d\zeta < \infty$.

(ii) The functions $a : \mathbb{R} \to \mathbb{R}$, $p_i : \mathbb{R} \to \mathbb{R}$ are continuous, $\int_{-\infty}^{0} \frac{a^2(-\theta)}{g(\theta)} d\theta < \infty$ and $L_i := (\int_{-\infty}^{0} \frac{p_i^2(\theta)}{g(\theta)} d\theta)^{\frac{1}{2}} < \infty$ for all $i = 1, \ldots, n$.

By defining the operators G, F, I_i by

$$F(t,\psi)(\xi) := \int_{-\infty}^{0}\int_0^{\pi} b(-\tau,\eta,\xi)\psi(\tau,\eta)\,d\eta\, d\tau,$$
$$G(t,\psi)(\xi) := a_0(\xi)\psi(0,\xi) + \int_{-\infty}^{0} a(-\tau)\psi(\tau,\xi)\,d\tau,$$
$$I_i(\psi)(\xi) := \int_{-\infty}^{0} p_i(-s)\psi(s,\xi)\,ds,$$

we can model (3.1)–(3.4) as the abstract system (1.1)–(1.3). Moreover, the maps $F(t,\cdot)$, $G(t,\cdot)$, I_i are bounded linear operators, $\|F(t,\cdot)\| \leq N_0^{\frac{1}{2}}$, $\|G(t,\cdot)\| \leq \left(\int_{-\infty}^{0} \frac{a(-\theta)}{g(\theta)} d\theta\right)^{\frac{1}{2}} + \sup_{\xi \in I}|a_0(\xi)|$ for all $t \in I$ and $\|I_i\| \leq L_i$ for every $i = 1, \ldots, n$.

Proposition 3.1. *Assume that the previous conditions are verified. If*

$$\left(1+\left(\int_{-\sigma}^{0} g(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}}\right)\left[N_1^{\frac{1}{2}}(1+2n+\sqrt{2e})+\sigma\left(\left(\int_{-\infty}^{0}\frac{a(-\theta)}{g(\theta)}\mathrm{d}\theta\right)^{\frac{1}{2}}\right.\right.$$
$$\left.\left.+\sup_{\xi\in J}|a_0(\xi)|\right)+\sum_{i=1}^{n}L_i\right]<1,$$

where $N_1 := \int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{g(\tau)}(\frac{\partial}{\partial\zeta}b(\tau,\eta,\zeta))^2\mathrm{d}\eta\mathrm{d}\tau\mathrm{d}\zeta$, *then there exists a unique mild solution,* $u(\cdot)$, *of* (3.1)–(3.4)*. Moreover, if* φ *and* $u(\cdot)$ *verify the conditions in* Theorem 2.2, *the function* $\frac{\partial^2 b(\theta,\eta,\xi)}{\partial\xi^2}$ *is measurable and* $N_2 = \int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{g(s)}(\frac{\partial^2 b(s,\eta,\xi)}{\partial\xi^2})^2\mathrm{d}\eta\mathrm{d}s\mathrm{d}\xi < \infty$, *then* $u(\cdot)$ *is a strong solution.*

Proof. A easy estimation using (i) permits us to prove that F is $X_{\frac{1}{2}}$-valued and that $F : I\times\mathcal{B}\to X_{\frac{1}{2}}$ is continuous. Moreover, $(-A)^{\frac{1}{2}}F(t,\cdot)$ is a bounded linear operator and $\|(-A)^{\frac{1}{2}}F(t,\cdot)\|\le N_1^{\frac{1}{2}}$ for each $t\in I$. Now, the existence of a mild solution is a consequence of Theorem 2.1.

The regularity assertion follows directly from Theorem 2.2, since under these additional conditions, the function $AF : I\times\mathcal{B}\to X$ is well defined and continuous. The proof is completed. □

Acknowledgements

The authors wish to thank the referees for their comments and suggestions. The work of the second author was supported by FONDECYT-CONICYT, Project 1020259 and Project 7020259.

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