Distance-Regular Graphs of Hamming Type

KAZUMASA NOMURA

Tokyo Ikashika University, Kounodai, Ichikawa, 272 Japan

Communicated by the Managing Editors

Received March 2, 1988

It is shown that for any distance-regular graph \( \Gamma \) with \( c_2 = 2, c_3 = 3 \), and \( a_2 = 2a_1 \) \( (a_1 \neq 2) \), there exists a covering \( \theta: H(n, q) \to \Gamma \) where \( q = a_1 + 2, n = b_0/(a_1 + 1) \). By using this fact, we shall give a characterization of the Hamming scheme \( H(n, q) \). These results are a generalization of J. Rifa i Coma's theorem which asserts the same fact for \( a_1 = 0 \).

1. INTRODUCTION

In this paper we shall consider distance-regular graphs with the intersection array

\[
\begin{bmatrix}
0 & 1 & 2 & \cdots & e-1 & e & \cdots & * & \cdots & *
\\
0 & \lambda & 2\lambda & \cdots & (e-1)\lambda & * & \cdots & * & \cdots & * \\
0 & * & * & \cdots & * & * & \cdots & 0
\end{bmatrix},
\]

where * takes an arbitrary value under the restriction that each column sum is \( k \). A \( H(e, \lambda, k) \)-graph is a distance-regular graph the intersection array of which takes the above form. Note that the Hamming graph \( H(n, q) \) is a \( H(e, \lambda, k) \)-graph with \( e = n, \lambda = q - 2, k = n(q - 1) \).

In [11], we have proved the following result about the local structure of a \( H(3, \lambda, k) \)-graph.

THEOREM 1. Let \( \Gamma \) be a \( H(3, \lambda, k) \)-graph \( (\lambda \neq 0, 2) \). Then \( \Gamma_1(u) \cap \Gamma_1(v) \) is a clique for every edge \( uv \) in \( \Gamma \).

In this paper we shall prove the following results which are generalizations of J. Rifa i Coma's theorem [13].

THEOREM 2. Let \( \Gamma \) be a \( H(3, \lambda, k) \)-graph \( (\lambda = 2) \). Then there exists a covering \( \theta: H(n, q) \to \Gamma \) where \( q = \lambda + 2, n = k/(\lambda + 1) \).
Theorem 3. Let $\Gamma$ be a $H(e, \lambda, k)$-graph with $\lambda \neq 2$ and $e \geq 3$. Let $\theta: H(n, q) \rightarrow \Gamma$ be a covering where $q = \lambda + 2$, $n = k/(\lambda + 1)$. Let $u_0$ be a vertex in $\Gamma$ and put $C = \theta^{-1}(u_0)$. Then the code $C$ has the following properties.

(i) $C$ is $e$-error correcting.

(ii) The covering radius of $C$ is $d$, where $d$ is the diameter of $\Gamma$.

(iii) $C$ is completely regular.

In [13], J. Rifa i Coma proved the same results for $\lambda = 0$ and classified all $H(e, 0, k)$-graphs with diameter $d \geq 3$, $e \geq d - 1$.

In the case $e = d$, the code $C$ in Theorem 3 is a perfect code in $H(n, q)$. Perfect $e$-codes have been studied by many authors (e.g., Bannai [1], Best [4], Reuvers [12], Hong [9, 10]). It is known that there is no non-trivial perfect $e$-code for $e \geq 3$, $q \geq 3$ (see [10]). So we get the following corollary.

Corollary. Let $\Gamma$ be a $H(d, \lambda, k)$-graph with diameter $d$, $d \geq 3$, $\lambda \neq 0, 2$. Then $\Gamma$ is isomorphic to $H(n, q)$ with $q = \lambda + 2$, $n = k/(\lambda + 1)$.

In the case $e = d - 1$, $C$ is a uniformly packed code studied by Van Tilborg [14].

Remark 1. Theorem 2 for $\lambda = 0$ has been obtained by Brouwer [7]. Theorem 2 for $\lambda > 0$ was conjectured by Bannai [2].

Remark 2. There exists a distance-regular graphs whose intersection array coincides with $H(n, q)$ but is not isomorphic to $H(n, q)$ in the case $\lambda = 2$ (see [8]). So the assumption $\lambda \neq 2$ in the above theorems is needed.

Remark 3. In the proof of the above theorems, except Theorem 3(iii), we shall use only local properties of $\Gamma$; i.e., we do not need the full distance-regularity of $\Gamma$.

Some elementary definitions about distance-regular graphs and codes are given in Section 2. More precise descriptions can be found in [1]. In the proof of the above theorem, we shall use intersection diagrams of distance-regular graphs. For the definition and elementary properties of intersection diagrams, see [6].

2. SOME TERMINOLOGY

Let $\Gamma$ be a connected graph with valency $k$ and diameter $d$. Let $\delta$ denote the usual metric on $\Gamma$. For vertices $u, v$ in $\Gamma$ and for non-negative integers $r, s$, we define

\[ \Gamma_r(u) = \{ x \in \Gamma | \delta(x, u) = r \}, \]
\[ D'_s(u, v) = \Gamma_r(u) \cap \Gamma_s(v). \]
If the size of $D^t(u, v)$ depends only on the distance between $u$ and $v$ rather than the individual vertices, $\Gamma$ is said to be distance-regular. If $\Gamma$ is distance-regular, the parameters $p^t_r = |D^t(u, v)|$, where $t = \partial(u, v)$, are called the intersection numbers of $\Gamma$. In particular we put

$$a_r = p^t_{1,r}, \quad b_r = p^t_{1,r+1}, \quad c_r = p^t_{1,r-1}.$$ 

Clearly $c_0 = a_0 = b_0 = 0$, $b_0 = k$, $c_1 = 1$, $a_r + b_r + c_r = k$ hold. The following array of intersection numbers is called the intersection array of $\Gamma$:

$$\begin{bmatrix}
0 & 1 & c_2 & \cdots & c_{d-1} & c_d \\
0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\
k & b_1 & b_2 & \cdots & b_{d-1} & 0
\end{bmatrix}.$$ 

Let $A_i (0 \leq i \leq d)$ be $n \times n$-matrix $(n = |\Gamma|)$, indexed by $\Gamma$, whose $uv$ entry is equal to 1 if $\partial(u, v) = i$, and 0 if otherwise. $A_i$ is called the $i$th adjacency matrix. We have $A_0 = I$, and $A = A_1$ is the usual adjacency matrix of $\Gamma$. Let $A$ be the linear subspace of $M_n(\mathbb{C})$ spanned by $A_0, A_1, \ldots, A_d$. Then $A$ becomes a subalgebra of $M_n(\mathbb{C})$ and is called the adjacency algebra of $\Gamma$.

The Hamming graph $H(n, q)$ is a graph with vertex set $V = \mathbb{Q}^n$, where $Q = \{0, 1, 2, \ldots, q-1\}$. Two vertices $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ are adjacent if $\# \{i \mid a_i \neq b_i\} = 1$. $H(n, q)$ has the following intersection array:

$$\begin{bmatrix}
0 & 1 & 2 & \cdots & i & \cdots & n \\
0 & q-2 & 2(q-2) & \cdots & i(q-2) & \cdots & n(q-2) \\
n(q-1) & (n-1)(q-1) & (n-2)(q-1) & \cdots & (n-i)(q-1) & \cdots & 0
\end{bmatrix}.$$ 

A subset $C$ of $V$ is called a code of length $n$ over the alphabet $Q$. A code is $e$-error correcting if $\partial(a, b) > 2e$ holds for any $a, b \in C, a \neq b$. The covering radius of $C$ is the minimum number $\rho$ which satisfies $\bigcup_{a \in C} B_\rho(a) = V$, where $B_\rho(a) = \bigcup_{0 \leq i \leq \rho} \Gamma_i(a)$. An $e$-error correcting code $C$ is perfect if its covering radius is $e$. A code $C$ is completely regular if $|\Gamma_i(a) \cap C|$ depends only on $\partial(a, C)$ and $i$ rather than individual vertices $a \in V$. The weight $wt(a)$ of $a = (a_1, \ldots, a_n)$ is the number of non-zero $a_i$'s, i.e., $wt(a) = \partial(0, a)$ where $0 = (0, \ldots, 0)$.

A map $\theta: \Gamma \rightarrow \Gamma'$ between graphs $\Gamma$ and $\Gamma'$ is said to be a local isomorphism if $\theta$ induces an isomorphism

$$\theta|_{\Gamma_i(u)}: \Gamma_i(u) \rightarrow \Gamma_1(\theta(u))$$

for every vertex $u$ in $\Gamma$. A local isomorphism is a covering if it is surjective.
3. Lemmas

First we describe some elementary properties of a covering. A sequence of vertices \( u_0, u_1, \ldots, u_s \) in a graph \( \Gamma \) is said to be a walk if \( u_i \) is adjacent to \( u_{i-1} \) \((0 < i \leq s)\).

**Lemma 1.** Let \( \theta: \Gamma \to \Gamma' \) be a local isomorphism and let \( u_0 \in \Gamma \), \( u'_0 = \theta(u_0) \). Let \( u'_0, u'_1, \ldots, u'_s \) be a walk in \( \Gamma' \) starting at \( u'_0 \). Then there is a walk \( u_0, u_1, \ldots, u_s \) in \( \Gamma \) with \( \theta(u_i) = u'_i \) \((0 \leq i \leq s)\).

**Proof.** Straightforward.  

**Lemma 2.** Every local isomorphism \( \theta: \Gamma \to \Gamma' \) is surjective if \( \Gamma' \) is connected.

**Proof.** Immediate from Lemma 1.  

**Lemma 3.** Let \( \theta: \Gamma \to \Gamma' \) be a covering. Then for \( u, v \in \Gamma \) and \( v' \in \Gamma' \),

\[
\varrho(u, v) \geq \varrho(\theta(u), \theta(v)),
\]

\[
\varrho(u, \theta^{-1}(v')) = \varrho(\theta(u), v').
\]

**Proof.** It can be easily proved by using Lemma 1.  

In the following, we assume \( \Gamma \) is a \( H(3, \lambda, k) \)-graph, \( \lambda \neq 2 \).

**Lemma 4.** Let \( uwwz \) be a 4-cycle in \( \Gamma \). If \( u \) is adjacent to \( w \), then \( v \) is adjacent to \( z \).

**Proof.** \( D_1^1(u, w) \) is a clique by Theorem 1.  

**Lemma 5.** Let \( u \in \Gamma \). Each connected component of \( \Gamma_1(u) \) is a clique of size \( \lambda + 1 \). Therefore \( \Gamma_1(u) \) has \( n = k/(\lambda + 1) \) components.

**Proof.** Straightforward.  

**Lemma 6.** Let \( uwwz \) and \( uww'z' \) be 4-cycles in \( \Gamma \) with \( w \neq w', z \neq z' \), \( \varrho(u, w) = \varrho(u, w') = 2 \). If \( w \) is adjacent to \( w' \), \( z \) is also adjacent to \( z' \).

**Proof.** Put \( D'_s = D'_s(u, v) \). Note that there is a no edge between \( D_1^1 \) and \( D_2^2 \cup D_2^1 \) by Lemma 4. Note that \( w, w' \in D_2^1 \) and \( z, z' \in D_2^1 \). We have \( |\Gamma_1(w) \cap D_2^1| = \lambda \) since there are \( \lambda \) edges from \( w \) to \( \Gamma_1(v) \). Then we get \( |\Gamma_1(w) \cap D_2^2| = 2\lambda - \lambda = \lambda \) since there are \( 2\lambda \) edges from \( w \) to \( \Gamma_2(u) \). By the same reason we have \( |\Gamma_1(z) \cap D_2^1| = \lambda \). But \( \Gamma_1(w) \cap \Gamma_1(z) \) has size \( \lambda \) and it is included in \( D_2^2 \) since there is just one edge from \( w \) to \( D_2^1 \) and there is just one edge from \( z \) to \( D_2^2 \) as \( c_2 = 2 \). This implies \( \Gamma_1(z) \cap D_2^2 = \Gamma_1(w) \cap D_2^2 \).
There is a vertex $x$ ($x \neq w$) which is adjacent to $z, w'$, since $\partial(z, w') = 2$. Assume $x \neq z'$. Then $x$ must be in $D_2^1$. We have $x \in \Gamma_1(z) \cap D_2^1 = \Gamma_1(w) \cap D_2^1$. Then $x$ is adjacent to $w$ and $w'$. Thus we have $\partial(w, w') = 1$, $\partial(x, v) = 2$ in the 4-cycle $xwuw'$, contradicting Lemma 4. So we have $x = z'$.

**Lemma 7.** Let $uv$ be an edge in $\Gamma$ and $w \in D_2^2(u, v)$. Then there is just one edge from $w$ to $D_2^1(u, v)$.

**Proof.** Put $D_v^1 = D_v(u, v)^1$. There are two edges $wx, wx'$ from $w$ to $\Gamma_1(u)$. If $x, x' \in D_1^1$, we get $\partial(x, x') = 1$ by Theorem 1, contradicting Lemma 4. So we may assume $x \in D_1^2$. For each vertex $y \in D_1^1$, there is just one vertex $z$ ($\neq u$) which is adjacent to $x, y$. Here $z$ must be in $D^1_v$ since there is no edge between $D_1^1$ and $D_2^1 \cup D_2^2$. But $|D_1^1| = \lambda$ and $|\Gamma_1(x) \cap D_2^2| = \lambda$ as in the proof of Lemma 6. This implies $\partial(y, w) = 1$ for some $y \in D_1^1$.

**Lemma 8.** Let $uvwz$ and $uw'z'$ be 4-cycles with $\partial(w, w') = \partial(z, z') = \partial(u, w) = \partial(u, w') = 2$. Then $\partial(w, z') = 3$.

**Proof.** Clearly $\partial(w, z') \neq 0, 1$. Assume $\partial(w, z') = 2$ and let $x$ be a vertex which is adjacent to $w$ and $z'$. Then $x \in D_2^2(u, v)$. Since $x$ is adjacent to $z'$, we have $\partial(x, w') = 1$ as in the proof of Lemma 6. Then there are two edges $xw, wx'$ from $x$ to $D_2^1$, contradicting Lemma 7.

### 4. Proof of Theorem 2

Let $\Gamma$ be a $H(3, \lambda, k)$-graph ($\lambda \neq 2$) and put $q = \lambda + 2, n = k/(\lambda + 1)$. Note that $n$ is an integer by Lemma 5. Let $Q = \{0, 1, \ldots, q - 1\}, V = Q^n, V_i = \{a \in V | wt(a) = i\}, W_i = \{a \in V | wt(a) \leq i\}$. We shall construct maps

$$\theta_r : W_r \rightarrow \Gamma$$

$(0 \leq r \leq n)$

which satisfy the following conditions for $a, b, b' \in W_r$.

(i) If $a \in W_{r-1}$, then $\theta_r(a) = \theta_{r-1}(a)$.

(ii) If $\partial(a, b) = 1$, then $\partial(\theta_r(a), \theta_r(b)) = 1$.

(iii) If $\partial(a, b) = \partial(a, b') = 1$ and $\partial(b, b') = 2$, then $\partial(\theta_r(b), \theta_r(b')) = 2$.

Fix a vertex $u_0$ in $\Gamma$ and define $\theta(0) = u_0$. Since $\Gamma_1(u_0)$ is isomorphic to $V_1$ by Lemma 5, $\{u_0\} \cup \Gamma_1(u_0)$ is isomorphic to $W_1$. Fix an isomorphism $\theta_1 : W_1 \rightarrow \{u_0\} \cup \Gamma_1(u_0)$. Clearly $\theta_1$ satisfies (i)–(iii).

Now assume $r \geq 2$ and $\theta_{r-1}$ has been defined. For $a \in W_{r-1}$, we put $\theta_r(a) = \theta_{r-1}(a)$. We define $\theta_r(a)$ for $a \in V_r$ as follows.

Take $b_1, b_2 \in V_{r-1}$ ($b_1 \neq b_2$) which are adjacent to $a$. Then $\partial(b_1, b_2) = 2$. 

and there is just one vertex \( c \in V_{r-2} \) which is adjacent to \( b_1 \) and \( b_2 \). Put \( y_1 = \theta_{r-1}(b_1) \), \( y_2 = \theta_{r-1}(b_2) \), \( z = \theta_{r-1}(e) \). Then \( \partial(y_1, y_2) = 2 \) since \( \theta_{r-1} \) satisfies (iii). There is just one vertex \( x (\neq z) \) which is adjacent to \( y_1 \) and \( y_2 \). We define \( \theta_r(a) = x \).

We shall show that the vertex \( x \) does not depend on the choice of \( b_1 \) and \( b_2 \). We may assume \( r \geq 3 \). Let \( b_1, b_2, b'_2 \) be three distinct vertices in \( V_{r-1} \) which are adjacent to \( a \). Then we can take \( c, c', d \in V_{r-2} \) and \( e \in V_{r-3} \) with \( \partial(e, c) = \partial(e, c') = \partial(e, d) = 1 \), \( \partial(c, d_1) = \partial(c, b_2) = 1 \), \( \partial(c', b_1) = \partial(c', b'_2) = 1 \), \( \partial(d, b_2) = \partial(d, b'_2) = 1 \). Let \( y_1, y_2, y'_2, z, z', u, v \) be the image of \( b_1, b_2, b'_2, c, c', d, e \) by \( \theta_r \), respectively. Let \( x (\neq z) \) be adjacent to \( y_1 \) and \( y_2 \). Then we have \( \partial(y_1, u) = 3 \) by Lemma 8. But \( \{x, y', z, z'\} \subset D_2(y_1, u) \). This implies \( x = x' \). We have shown that each pair \((b_1, b_2), (b_1, b'_2)\) gives the same vertex \( x \). This is easy enough to prove.

Next we show that \( \theta_r \) satisfies the condition (ii). We may assume \( r \geq 2 \). Let \( a, b \in W_r \), \( \partial(a, b) = 1 \). If \( a \in V_{r-1} \) or \( b \in V_{r-1} \), clearly (ii) holds. Assume \( a, b \in V_r \). We can take \( c, d, e \in V_{r-1} \) and \( f \in V_{r-2} \) with \( \partial(f, c) = \partial(f, d) = \partial(f, e) = 1 \), \( \partial(a, d) = \partial(a, c) = \partial(a, e) = 1 \), \( \partial(d, e) = 1 \). Let \( x, y, z, u, v, w \) be the image of \( a, b, c, d, e, f \) by \( \theta_r \), respectively. Then we can easily show \( \partial(x, y) = 1 \) by using Lemma 6.

We shall show that \( \theta_r \) satisfies (iii). We may assume \( r \geq 2 \). Let \( a, b_1, b_2 \in W_r \), \( \partial(a, b_1) = \partial(a, b_2) = 1 \), \( \partial(b_1, b_2) = 2 \). First assume \( a \in V_r \). If \( b_1, b_2 \in V_{r-1} \), clearly (iii) holds. If \( b_1 \in V_r \) and \( b_2 \in V_{r-1} \), we can easily show (iii) by taking \( c \in V_{r-1} \) and \( d \in V_{r-2} \) with \( \partial(c, a) = \partial(c, b_1) = 1 \), \( \partial(d, b_2) = \partial(d, c) = 1 \). In the case \( b_1, b_2 \in V_r \), take \( c, d \in V_{r-1} \) and \( e \in V_{r-2} \) with \( \partial(c, a) = \partial(c, b_1) = 1 \), \( \partial(d, a) = \partial(d, b_2) = 1 \), \( \partial(d, e) = \partial(d, d) = 1 \).

Next assume \( a \in V_{r-1} \). In case \( b_1, b_2 \in V_r \), take \( c, d \in V_{r-1} \) and \( e \in V_{r-2} \) with \( \partial(e, c) = \partial(e, a) = \partial(e, d) = 1 \), \( \partial(b_1, c) = 1 \), \( \partial(b_2, d) = 1 \) and \( a, c, d \) are distinct. In case \( b_1 \in V_{r-1} \) and \( b_2 \in V_r \), take \( c \in V_{r-1} \) and \( d \in V_{r-2} \) with \( \partial(d, c) = \partial(d, a) = \partial(d, b_1) = 1 \), \( \partial(c, b_2) = 1 \) and \( c \neq a \). We can show this easily for the remaining cases.

Now we get \( \theta_0, \theta_1, \ldots, \theta_n \) which satisfies (i)—(iii). Then \( \theta_n \) is the desired covering.

5. PROOF OF THEOREM 3

Let \( \Gamma \) be a \( H(e, \lambda, k) \)-graph \((\lambda \neq 2, \; e \geq 3)\) with diameter \( d \). Let \( Q = \{0, 1, 2, \ldots, q - 1\}, \; V = Q^n \) where \( q = \lambda + 2, \; n = k/(\lambda + 1) \). Let \( \theta : V \to \Gamma \) be a covering and put \( C = \theta^{-1}(u_0) \) for a fixed vertex \( u_0 \in \Gamma \). We need the following lemma for the proof of Theorem 3.

**Lemma 9.** Let \( a, b \in V \). If \( \partial(a, b) \leq e \), then \( \partial(a, b) = \partial(\theta(a), \theta(b)) \).
Proof. We proceed by induction on \( r = \delta(a, b) \). We may assume \( r > 1 \). Choose \( c \in V \) as \( \delta(a, c) = r - 1 \) and \( \delta(c, b) = 1 \). Clearly \( \theta \) induces a bijection \( \Gamma_1(c) \to \Gamma_1(\theta(c)) \). Since \( \theta \) preserves distances less than \( r \) by induction, \( \theta \) induces an injection from

\[
S = \Gamma_1(c) \cap (\Gamma_{r-2}(a) \cup \Gamma_{r-1}(a))
\]

into

\[
T = \Gamma_1(\theta(c)) \cap (\Gamma_{r-2}(\theta(a)) \cup \Gamma_{r-1}(\theta(a))).
\]

Since \( S \) and \( T \) have the same size, \( \theta \) maps \( S \) onto \( T \). Therefore \( \theta(b) \) does not belong to \( T \), and this implies \( \theta(b) \in \Gamma_r(a) \).

Proof of (i). Let \( a, b \in C \), \( a \neq b \). Let \( a = a_0, a_1, \ldots, a_s = b \) (\( s = \delta(a, b) \)) be a shortest path connecting \( a \) and \( b \). Put \( u_i = \theta(a_i) \) (\( 1 \leq i \leq s \)). We have \( \delta(u_0, u_s) = \varepsilon \) by Lemma 9. We also have \( \delta(u_i, u_{i+1}) \neq \varepsilon - 1 \) by the same argument as that in the proof of Lemma 9. So we get \( s > 2\varepsilon \).

Proof of (ii). We can easily show (ii) by Lemma 3.

Proof of (iii). Let \( \Theta \) be the \( q^n \times |\Gamma| \)-matrix, indexed by \( V \times \Gamma \), the \((a, \nu)\)-entry \((a \in V, \nu \in \Gamma)\) of which is given by

\[
\Theta_{a, \nu} = \begin{cases} 1 & \text{if } \theta(a) = \nu \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( \bar{A}_i \) and \( A_i \) be the \( i \)th adjacency matrix of \( H(n, q) \) and \( \Gamma \), respectively. Then clearly \( \bar{A}_1 \Theta = \Theta A_1 \) holds. It is well known that the \( i \)th adjacency matrix of a distance-regular graph can be expressed as a polynomial of its adjacency matrix (see [3]). Thus \( \bar{A}_i = v_i(\bar{A}_1) \) for some polynomial \( v_i(X) \). Then we get

\[
\bar{A}_i \Theta = v_i(\bar{A}_1) \Theta = \Theta v_i(A_1).
\]

Here we have

\[
(\bar{A}_i \Theta)_{a, \nu} = |\Gamma_i(a) \cap \theta^{-1}(\nu)|
\]

and

\[
(\Theta v_i(A_1))_{a, \nu} = (v_i(A_1))_{\theta(a), \nu}.
\]

Since \( v_i(A_1) \) is an element of the adjacency algebra of \( \Gamma \), \( |\Gamma_i(a) \cap \theta^{-1}(\nu)| \) depends only on \( \delta(\theta(a), \nu) = \delta(a, \theta^{-1}(\nu)) \).

Remark 4. The above proof of (iii) was proved by A. Munemasa. In this proof, none of the special values of the intersection numbers of \( H(n, q) \)
is needed. Hence the same arguments will do well for a covering \( \theta: \Gamma' \to \Gamma \) between any distance-regular graphs \( \Gamma', \Gamma \) which have the same intersection numbers \( b, (0 \leq i < e) \), \( c, (0 < i \leq e) \).

ACKNOWLEDGMENTS

I express my gratitude to the referees for their comments and suggestions. I also thank E. Bannai, T. Ito, A. Munemasa, and H. Enomoto for their kind suggestions, and J. Rifa i Coma for repeated communications.

REFERENCES

10. Y. Hong, On the nonexistence of nontrivial perfect \( e \)-codes and tight \( 2e \)-designs in Hamming schemes \( H(n, q) \) with \( e \geq 3 \) and \( q \geq 3 \), *Graphs Combin.* 2 (1986), 145–164.