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The S-curvature of homogeneous Randers spaces ☆

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ABSTRACT

In this paper, we give an explicit formula of the S-curvature of homogeneous Randers spaces and prove that a homogeneous Randers space with almost isotropic S-curvature must have vanishing S-curvature. As an application, we obtain a classification of homogeneous Randers space with almost isotropic S-curvature in some special cases. Some examples are also given.

1. Introduction

The notion of Randers spaces was introduced in 1941 by G. Randers in his research on general relativity [10]. They are most closely related to Riemannian metrics among the class of Finsler spaces. Let M be a connected smooth manifold and α be a Riemannian metric on M. Then a Randers metric on M with the underlying Riemannian metric α is a Finsler metric of the form $F=\alpha+\beta$, where β is a smooth 1-form on M satisfying $\|\beta\|_x<1$, $\forall x\in M$, here $\|\beta\|$ denote the length of the form under the Riemannian metric α. A Randers space (M,F) is called homogeneous if the group of isometries $I(M,F)$ of (M,F) acts transitively on M (see [3]). As pointed out in [4], a homogeneous Randers space can be written as a coset space G/H with a G-invariant Randers metric $F=\alpha+\beta$, where both the Riemannian metric α and the form β are invariant under the action of G. In particular, the Lie algebra of G, $\mathfrak{g}$, has a decomposition

$$\mathfrak{g}=\mathfrak{h}+\mathfrak{m}\quad \text{(direct sum of subspaces)}$$

such that $\mathrm{Ad}(h)(\mathfrak{m})\subset\mathfrak{m}$, $h\in H$. Identifying $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the origin o, we get an H-invariant inner product $\langle,\rangle$ on $\mathfrak{m}$. Moreover, the form β corresponds to an invariant vector field $\tilde{X}$ on G/H, which can be generated by an $\mathrm{Ad}(H)$-invariant vector in $\mathfrak{m}$ with length <1, see [4] and Section 3 below.

It is an important problem to compute the geometric quantities, particularly the curvatures of homogeneous spaces. J. Milnor used the formula of the sectional curvature of left invariant Riemannian metric on a Lie group to study the curvature properties of such spaces and obtained some interesting results [8]. The formula of sectional curvature of a homo-

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geneous Riemannian manifolds was used to classify homogeneous Riemannian manifolds with negative curvature (see [5]) or positive curvature (see [13]).

In this paper, we will give a formula for the S-curvature of a homogeneous Randers space. The notion of S-curvature of a Finsler space was introduced by Z. Shen in [11]. It is a quantity to measure the rate of change of the volume form of a Finsler space along the geodesics. S-curvature is a non-Riemannian quantity, or in other words, any Riemannian manifold has vanishing S-curvature. It is shown in [11] that the Bishop–Gromove volume comparison theorem is true for a Finsler space with vanishing S-curvature. We now recall the definition of this important quantity. Let V be an n-dimensional real vector space and F be a Minkowski norm on V. For a basis $\{b_i\}$ of V, let

$$\sigma_F = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n | F(y^i b_i) < 1\}},$$

where Vol means the volume of a subset in the standard Euclidean space $\mathbb{R}^n$ and B^n is the open ball of radius 1. This quantity is generally dependent on the choice of the basis $\{b_i\}$. But it is easily seen that

$$\tau(y) = \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma_F}, \quad y \in V - \{0\}$$

is independent of the choice of the basis where (g_{ij}) is the fundamental tensor of F. $\tau = \tau(y)$ is called the distortion of (V, F). Now let (M, F) be a Finsler space. Let $\tau(x, y)$ be the distortion of the Minkowski norm F_x on $T_x(M)$. For $y \in T_x(M) - \{0\}$, let $\sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot\sigma(0) = y$. Then the quantity

$$S(x, y) = \frac{d}{dt}\big[\tau\big(\sigma(t), \dot\sigma(t)\big)\big]\Big|_{t=0}$$

is called the S-curvature of the Finsler space (M, F). A Finsler space (M, F) is said to have almost isotropic S-curvature if there exists a smooth function $c(x)$ on M and a closed 1-form η such that

$$S(x, y) = (n+1)\big(c(x)F(y) + \eta(y)\big), \quad x \in M, \ y \in T_x(M).$$

If in the above equation $\eta = 0$, then (M, F) is said to have isotropic S-curvature. If $\eta = 0$ and $c(x)$ is a constant, the (M, F) is said to have constant S-curvature.

The purpose of this paper is to give a formula for the S-curvature of a homogeneous Randers space. In literature, there is an explicit formula for S-curvature in a local standard coordinate system by Z. Shen (see [2]). However, for a homogeneous Randers space there should have a formula which does not use local coordinate system. This is our main result in this paper. We will also use the formula to study the properties of S-curvature of homogeneous Randers spaces. In particular, we prove that a homogeneous Randers space with almost isotropic S-curvature must have vanishing S-curvature. As an application, we obtain a classification of homogeneous Randers space with almost isotropic S-curvature in some special cases.

It is interesting to consider whether the method of this paper can be used to get a formula for the S-curvature of a general homogeneous Finsler space. The local coordinate system used here is evidently convenient for the computation. However, difficulty arises when one tries to get the coefficients of the Chern connection since these quantities are functions on the slit tangent bundle, not on the manifold. Some new method should be found to achieve this goal.

2. The Levi-Civita connection of homogeneous Riemannian manifolds

Let $(G/H, \alpha)$ be a homogeneous Riemannian manifold. Then G/H is a reductive homogeneous manifold in the sense of Nomizu [9], i.e., the Lie algebra of G has a decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \tag{2.1}$$

such that $\mathrm{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m}, \forall h \in H$. We can identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$. Let $\langle , \rangle$ be the corresponding inner product on $\mathfrak{m}$. We now deduce some results concerning the Levi-Civita connection of $(G/H, \alpha)$ which will be useful to compute the S-curvature of homogeneous Randers spaces.

In literature, there are several versions of the formula of the connection for Killing vector fields. Since we are interesting in the differential of (left) invariant vector fields on G/H, we adopt the formula in [7]. Given $v \in \mathfrak{g}$, we can define a one-parameter transformation group φ_t, $t \in \mathbb{R}$ of G/H by

$$\phi_t(gH) = \big(\exp(tv)g\big)H, \quad g \in G.$$

Then φ_t generates a vector field on G/H which is a Killing vector field (this is called the fundamental vector field generated by v in [7]). We denote this vector field by $\hat{v}$. The following formula is a direct consequence of the formula in [7], vol. 2, page 201, see also [13].

$$\langle \nabla_{\hat{v}_1} \hat{v}_2|_o, w\rangle = \frac{1}{2}\big(-\big\langle [v_1, v_2]_\mathfrak{m}, w\big\rangle + \big\langle [w, v_1]_\mathfrak{m}, v_2\big\rangle + \big\langle [w, v_2]_\mathfrak{m}, v_1\big\rangle\big), \quad v_1, v_2, w \in \mathfrak{m}, \tag{2.2}$$

where $o = H$ is the origin of the coset space and $[v_1, v_2]_\mathfrak{m}$ denote the projection of $[v_1, v_2]$ to $\mathfrak{m}$ corresponding to the decomposition (2.1).

To apply the formula (2.2) to our study, we need to deduce some formula for the connection in a local coordinate system. Let $u_1, u_2, \ldots, u_n$ be an orthonormal basis of $\mathfrak{m}$ with respect to $\langle , \rangle$. Then by [6] there exists a neighborhood U of o in G/H such that the mapping

$$(\exp x^1 u_1 \exp x^2 u_2 \cdots \exp x^n u_n)H \mapsto (x^1, x^2, \ldots, x^n) \tag{2.3}$$

defines a local coordinate system on U. Now we compute the coordinate vector fields $\frac{\partial}{\partial x^i}$. Let $gH = (x^1, x^2, \ldots, x^n) \in U$. Then

$$\begin{aligned}
\left.\frac{\partial}{\partial x^i}\right|_{gH} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\exp x^1 u_1 \cdots \exp x^{i-1} u_{i-1} \exp\big(t + x^i\big) u_i \exp x^{i+1} u_{i+1} \cdots \exp x^n u_n\big)\big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\exp x^1 u_1 \cdots \exp x^{i-1} u_{i-1} \exp t u_i \exp -x^{i-1} u_{i-1} \cdots \exp -x^1 u_1 \cdot gH)|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \exp t\big(e^{x^1 \operatorname{ad} u_1} \cdots e^{x^{i-1} \operatorname{ad} u_{i-1}}(u_i)\big) \cdot gH|_{t=0}.
\end{aligned}$$

Denote

$$v_i = e^{x^1 \operatorname{ad} u_1} \cdots e^{x^{i-1} \operatorname{ad} u_{i-1}}(u_i).$$

We have

$$\left.\frac{\partial}{\partial x^i}\right|_{gH} = \hat{v}_i|_{gH}. \tag{2.4}$$

Next we compute the Levi-Civita connection of α under the above coordinate system on U. Let $u_{n+1}, \ldots, u_m$ be a basis of $\mathfrak{h}$. Then we can write

$$v_i = \sum_{j=1}^m f_{ij} u_j,$$

where f_{ij}, $j = 1, 2, \ldots, n$ are functions of $x^1, \ldots, x^{i-1}$. Hence

$$\frac{\partial}{\partial x^i} = \sum_{j=1}^m f_{ij} \hat{u}_j.$$

Therefore we have

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} &= \nabla_{\frac{\partial}{\partial x^i}} \left(\sum_{l=1}^m f_{jl} \hat{u}_l\right) \\
&= \sum_{l=1}^m \left(\frac{\partial f_{jl}}{\partial x^i}\right) \hat{u}_l + \sum_{l=1}^m f_{jl} \nabla_{\frac{\partial}{\partial x^i}} \hat{u}_l \\
&= \sum_{l=1}^m \left(\frac{\partial f_{jl}}{\partial x^i}\right) \hat{u}_l + \sum_{k,l=1}^m f_{ik} f_{jl} \nabla_{\hat{u}_k} \hat{u}_l.
\end{aligned}$$

Since by the symmetry of the Levi-Civita connection we have

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0,$$

we only need to compute $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$ for $i \geqslant j$. Since f_{jl} are the functions of $x^1, \ldots, x^{j-1}$, we have $\frac{\partial f_{jl}}{\partial x^i} = 0$, for $i \geqslant j$. Thus

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k,l=1}^m f_{ik} f_{jl} \nabla_{\hat{u}_k} \hat{u}_l, \quad i \geqslant j.$$

By the definition of f_{ij} we easily see that $f_{ij}(0, 0, \ldots, 0) = \delta_{ij}$. Thus

$$\left.\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\right)\right|_o = (\nabla_{\hat{u}_i} \hat{u}_j)|_o, \quad i \geqslant j.$$

Let Γ_{ij}^k be the Christoffel symbols of the connection under the coordinate system, i.e.,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

Then we have

$$\Gamma_{ij}^k(o)\frac{\partial}{\partial x^k}\bigg|_o = (\nabla_{\hat{u}_i}\hat{u}_j)|_o, \quad i \geqslant j.$$

By (2.4) we see that $\frac{\partial}{\partial x^k}|_o = \hat{v}_k|_o = u_k$. Thus,

$$\Gamma_{ij}^l(o) = \langle \Gamma_{ij}^k(o)u_k, u_l\rangle = \langle \nabla_{\hat{u}_i}\hat{u}_j, \hat{u}_l\rangle|_o, \quad i \geqslant j. \tag{2.5}$$

By (2.2), we have

$$\Gamma_{ij}^l(o) = \frac{1}{2}\big(-\langle [u_i, u_j]_{\mathfrak{m}}, u_l\rangle + \langle [u_l, u_i]_{\mathfrak{m}}, u_j\rangle + \langle [u_l, u_j]_{\mathfrak{m}}, u_i\rangle\big), \quad i \geqslant j. \tag{2.6}$$

Sometimes it is convenient to express the formula (2.6) using the structure constants of the Lie algebras. For $1 \leqslant i, j \leqslant m$, let

$$[u_i, u_j] = \sum_{k=1}^m C_{ij}^k u_k.$$

The constants C_{ij}^k are called the structure constants of the Lie algebra $\mathfrak{g}$ with respect to the basis $u_1, \dots, u_n, u_{n+1}, \dots, u_m$. Using the structure constants, we can write (2.6) as

$$\Gamma_{ij}^l = \Gamma_{ji}^l = \frac{1}{2}\big(-C_{ij}^l + C_{il}^j + C_{jl}^i\big), \quad i \geqslant j. \tag{2.7}$$

Now we give some applications of the formulae (2.2)–(2.6) to Randers spaces. By [4], there is a one-to-one correspondence between the invariant Randers metrics on G/H with the underlying Riemannian metric α and the G-invariant vector fields on G/H with length < 1. Further, the G-invariant vector fields on G/H are one-to-one corresponding to the set

$$V = \big\{u \in \mathfrak{m} \mid \mathrm{Ad}(h)X = X,\ \forall h \in H\big\}.$$

Hence the invariant Randers metrics are one-to-one corresponding to the set

$$V_1 = \big\{u \in V \mid \langle u, u\rangle < 1\big\}.$$

Let u be an non-zero element in V_1. Then u corresponds via the Riemannian metric α to the G-invariant 1-form β such that $\beta(y) = \langle y, u\rangle$, $y \in \mathfrak{m}$. The corresponding Randers metric is then defined by $F = \alpha + \beta$. Select an orthonormal basis $u_1, u_2, \dots, u_n$ of $\mathfrak{m}$ such that $u_n = \frac{u}{|u|}$ and define the local coordinate system as in (2.3). Since $\tilde{u}$ is invariant under the action of G, we have

$$\tilde{u}|_{gH} = \mathrm{d}\tau_g(u),$$

where τ_g is the diffeomorphism of G/H defined by $g_1H \mapsto gg_1H$. Thus

$$\begin{aligned}\tilde{u}|_{gH} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\tau_g\big(\exp(tu)H\big)\big)\big|_{t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big(\exp x^1u_1 \exp x^2u_2 \cdots \exp\big(x^n + ct\big)u_n\big)H\big|_{t=0}\\ &= c\frac{\partial}{\partial x^n}\bigg|_{gH},\end{aligned}$$

where $c = |u| < 1$.

Our first application is the following

Proposition 2.1. *The Randers metric $F = \alpha + \beta$ is of Berwald type if and only if*

$$\langle [u, v]_{\mathfrak{m}}, v\rangle = 0, \quad \langle [v, w]_{\mathfrak{m}}, u\rangle = 0, \quad \forall v, w \in \mathfrak{m}.$$

Proof. By [1], F is of Berwald type if and only if β is parallel with respect to α, or equivalently, the invariant vector field $\tilde{u}$ is parallel with respect to α, i.e., $\Gamma_{ni}^l = \Gamma_{in}^l = 0$, for $i, l = 1, 2, \dots, n$. By the invariance of $\tilde{u}$, we only need to check at the origin o. By (2.6), this is equivalent to

$$-\langle [u_n, u_i]_{\mathfrak{m}}, u_l\rangle + \langle [u_l, u_i]_{\mathfrak{m}}, u_n\rangle + \langle [u_l, u_n]_{\mathfrak{m}}, u_i\rangle = 0, \quad i, l = 1, 2, \dots, n. \tag{2.8}$$

Setting $i = l$ in (2.8) we get

$$\langle [u_n, u_i]_{\mathfrak{m}}, u_i\rangle = 0, \quad i = 1, 2, \dots, n. \tag{2.9}$$

Now it is easily seen that (2.9) is equivalent to

$$\langle [u_n, u_i]_{\mathfrak{m}}, u_l\rangle + \langle [u_n, u_l]_{\mathfrak{m}}, u_i\rangle = 0, \quad i, l = 1, 2, \dots, n. \tag{2.10}$$

Combining (2.8), (2.9) and (2.10) we complete the proof. □

Remark. The condition (2.8) was already presented in [4], but a detailed proof was not given there.

3. The *S*-curvature

Now we are ready to compute the S-curvature of F. Since $(G/H, F)$ is homogeneous, we only need to compute at the origin $o = H$. Let $(U, (x^1, x^2, \dots, x^n))$ be the local coordinate system as in Section 2. According to the formula of the S-curvature in local coordinate systems [2], we need to compute the following quantities at the origin: I. $e_{00} = e_{ij} y^i y^j$, where $e_{ij} = r_{ij} + b_i s_j + b_j s_i$, $r_{ij} = \frac{1}{2}(b_{i;j} + b_{j:i})$ and the b_i's are defined by $\beta = b_i dx^i$. Further, $s_i = b_j s^j{}_i$ and $s^i{}_j$ is defined by $s^i{}_j = a^{ih} s_{hj}$, where $s_{ij} = \frac{1}{2}(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i})$ and (a^{kl}) is the inverse matrix of (a_{ij}); II. $s_0 = s_i y^i$; III. $\rho_0 = \rho_{x^i} y^i$, where $\rho = \ln\sqrt{1 - \|\beta\|}$ and $\|\beta\|$ is the length of the form β with respect to α. The quantity of type III is easy. In fact $\rho_{x^i} = 0$ for any i, since β, as an invariant form on G/H, has constant length. Therefore $\rho_0 = 0$. Next we compute e_{00} and s_0.

First, since

$$b_i = \beta\left(\frac{\partial}{\partial x^i}\right) = \left\langle \tilde{u}, \frac{\partial}{\partial x^i}\right\rangle = c\left\langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i}\right\rangle,$$

we have

$$\begin{aligned}\frac{\partial b_i}{\partial x^j} &= c\frac{\partial}{\partial x^j}\left\langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i}\right\rangle\\ &= c\left(\left\langle \nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i}\right\rangle + \left\langle \frac{\partial}{\partial x^n}, \nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}\right\rangle\right).\end{aligned} \tag{3.1}$$

Hence at the origin we have (here we use the symmetry of the connection: $\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} - \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = [\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}] = 0$)

$$s_{ij}(o) = \frac{1}{2}c\left(\left\langle \nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right\rangle - \left\langle \nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle\right)\bigg|_o.$$

By (2.2)–(2.6), we have

$$s_{ij}(o) = \frac{1}{2}c\langle [u_i, u_j]_{\mathfrak{m}}, u_n\rangle. \tag{3.2}$$

Since at the origin we have $(a_{ij}) = I_n$, we get

$$s^i{}_j(o) = a^{ik}(o)s_{kj}(o) = \sum_{k=1}^{n} \delta_{ik}s_{kj}(o) = s_{ij}(o).$$

Therefore

$$s_i(o) = b_l(o)s^l{}_i(o) = cs^n{}_i(o) = cs_{ni}(o).$$

Thus for $y = y^i u_i \in \mathfrak{m}$, we have

$$\begin{aligned}s_0(y) = y^l s_l(o) = cy^l s_{nl}(o) &= \frac{1}{2}c^2 y^l\langle [u_n, u_l]_{\mathfrak{m}}, u_n\rangle\\ &= \frac{1}{2}\langle [cu_n, y^l u_l]_{\mathfrak{m}}, cu_n\rangle\\ &= \frac{1}{2}\langle [u, y]_{\mathfrak{m}}, u\rangle.\end{aligned} \tag{3.3}$$

Next we compute r_{ij}. Suppose $i \geqslant j$. Then we have

$$\begin{aligned}r_{ij}(o) &= \frac{1}{2}(b_{i;j} + b_{j;i})|_o\\ &= \frac{1}{2}\left(\frac{\partial b_i}{\partial x^j} - b_l\Gamma^l_{ji} + \frac{\partial b_j}{\partial x^i} - b_l\Gamma^l_{ij}\right)\bigg|_o\\ &= \frac{1}{2}\left(\frac{\partial b_i}{\partial x^j} + \frac{\partial b_j}{\partial x^i}\right)\bigg|_o - c\Gamma^n_{ij}(o).\end{aligned}$$

By (3.1) and (2.2)–(2.6) we have

$$\frac{1}{2}\left(\frac{\partial b_i}{\partial x^j} + \frac{\partial b_j}{\partial x^i}\right)\bigg|_o = -\frac{1}{2}c\langle [u_i, u_j]_{\mathfrak{m}}, u_n\rangle, \quad i \geqslant j. \tag{3.4}$$

Combining (2.6) with (3.4) we get

$$r_{ij}(o) = -\frac{1}{2}c\big(\langle [u_n, u_i]_{\mathfrak{m}}, u_j\rangle + \langle [u_n, u_j]_{\mathfrak{m}}, u_i\rangle\big), \quad i \geqslant j. \tag{3.5}$$

Note that r_{ij} is symmetric with respect to the indices i, j and the right hand of (3.5) is also symmetric with respect to i, j. We conclude that (3.5) is also valid for $i \leqslant j$. On the other hand, a direct computation shows that

$$b_i s_j + b_j s_i|_o = \begin{cases} 0, & \text{for } 0 \leqslant i, j \leqslant n-1, \\ \frac{1}{2}c^3\langle [u_n, u_i]_{\mathfrak{m}}, u_n\rangle, & \text{for } 1 \leqslant i \leqslant n-1,\ j = n, \\ \frac{1}{2}c^3\langle [u_n, u_j]_{\mathfrak{m}}, u_n\rangle, & \text{for } i = n,\ 1 \leqslant j \leqslant n-1, \\ 0, & \text{for } i = j = n. \end{cases}$$

Consequently

$$\begin{aligned} e_{00}(y) &= r_{ij}(o)y^i y^j + (b_i s_j + b_j s_i)|_o y^i y^j \\ &= -\frac{1}{2}c\big(\langle [u_n, u_i]_{\mathfrak{m}}, u_j\rangle + \langle [u_n, u_j]_{\mathfrak{m}}, u_i\rangle\big)y^i y^j + \sum_{j=1}^{n-1} c^3\langle [u_n, u_j]_{\mathfrak{m}}, u_n\rangle y^j y^n \\ &= -\frac{1}{2}\big(\langle [cu_n, y^i u_i]_{\mathfrak{m}}, y^j u_j\rangle + \langle [cu_n, y^j u_j]_{\mathfrak{m}}, y^i u_i\rangle\big) + \sum_{j=1}^{n}\langle [cu_n, u_j y^j]_{\mathfrak{m}}, cu_n\rangle c y^n \\ &= -\langle [u, y]_{\mathfrak{m}}, y\rangle + \langle [u, y]_{\mathfrak{m}}, u\rangle\langle y, u\rangle \\ &= \langle [u, y]_{\mathfrak{m}}, \langle y, u\rangle u - y\rangle, \end{aligned}$$

where we have used the skew-symmetry of the Lie brackets: $[u_n, u_n] = 0$ and the facts that $cu_n = u$, $y^n = \langle y, u_n\rangle$. Finally we obtain the formula of the S-curvature:

$$S(o, y) = (n+1)\left\{\frac{e_{00}(y)}{2F(y)} - \big(s_o(y) - \rho_0(y)\big)\right\} = \frac{n+1}{2}\left\{\frac{\langle [u, y]_{\mathfrak{m}}, \langle y, u\rangle u - y\rangle}{F(y)} - \langle [u, y]_{\mathfrak{m}}, u\rangle\right\}.$$

We summarize our computation as the following

Theorem 3.1. *Let $(G/H, \alpha)$ be a homogeneous Riemannian manifold and suppose that the Lie algebra $\mathfrak{g}$ of G has a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with $\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m}$. Then there is a one-to-one correspondence between the G-invariant Randers metric on G/H with the underlying Riemannian metric α and the set*

$$V_1 = \big\{u \in \mathfrak{m} \mid \alpha(u) < 1, \mathrm{Ad}(h)(u) = u, \forall h \in H\big\}.$$

For a fixed $u \in V_1$, the corresponding Randers metric F has the S-curvature

$$S(o, y) = \frac{n+1}{2}\left\{\frac{\langle [u, y]_{\mathfrak{m}}, \langle y, u\rangle u - y\rangle}{F(y)} - \langle [u, y]_{\mathfrak{m}}, u\rangle\right\}, \quad y \in \mathfrak{m},$$

where $o = H$ is the origin of G/H and we have identified the tangent space $T_o(G/H)$ with $\mathfrak{m}$.

From the above formula, we can see that $S(u) = S(-u) = 0$. Therefore, for a homogeneous non-Riemannian Randers space, the S-curvature has at least two (non-zero) zero points at any tangent space of the manifold.

4. Some applications

In this section we give some applications of the results in Section 2. Shen and Xing proved that a Randers metric is of almost isotropic curvature if and only if it is of isotropic S-curvature. They have also characterized Randers metrics with isotropic S-curvature (see [12]). The following theorem shows that the S-curvature of a homogeneous Rander spaces possesses more special properties.

Theorem 4.1. *Let $(G/H, F)$ be a homogeneous Randers space where F is defined by a G-invariant Riemannian metric α and $0 \neq u \in V_1$ as in Theorem 3.1. Then (M, F) has almost isotropic S-curvature if and only if it has vanishing S-curvature.*

Proof. We only need to prove the "only if" part. Suppose that F has almost isotropic S-curvature. Then there exists a closed 1-form η on G/H and a function $c(x)$ on G/H such that

$$S(x, y) = (n+1)\big(c(x)F(y) + \eta(y)\big), \quad \forall y \in T(G/H).$$

In particular, at the origin $x = o$ we have

$$\frac{n+1}{2}\left\{\frac{\langle [u,y]_{\mathfrak{m}}, \langle y,u\rangle u - y\rangle}{F(y)} - \big\langle [u,y]_{\mathfrak{m}}, u\big\rangle\right\} = (n+1)\big(c(o)F(y) + \eta(y)\big), \quad \forall y \in \mathfrak{m}. \tag{4.1}$$

Considering the values at $y = u$ and $y = -u$ we get

$$c(o)F(u) + \eta(u) = 0$$
$$c(o)F(-u) - \eta(u) = 0.$$

Thus $c(o)(F(u) + F(-u)) = 0$. Therefore we have $c(o) = 0$. Hence

$$\left\{\frac{\langle [u,y]_{\mathfrak{m}}, \langle y,u\rangle u - y\rangle}{F(y)} - \big\langle [u,y]_{\mathfrak{m}}, u\big\rangle\right\} = 2\eta(y). \tag{4.2}$$

Writing $y = y^i u_i$, where u_i is the orthonormal basis of $\mathfrak{m}$ as in Section 2, we can rewrite (4.2) as

$$\big(2\eta(y) + \big\langle [u,y]_{\mathfrak{m}}, u\big\rangle\big)\sqrt{\sum_{i=1}^{n}(y^i)^2} = \big\langle [u,y]_{\mathfrak{m}}, \langle y,u\rangle u - y\big\rangle - \big(2\eta(y) + \big\langle [u,y]_{\mathfrak{m}}, u\big\rangle\big) \times \langle u,y\rangle.$$

Note that the right hand of the above equation is a polynomial of y^i. This can hold only when

$$2\eta(y) + \big\langle [u,y]_{\mathfrak{m}}, u\big\rangle = 0.$$

Therefore we have

$$\big\langle [u,y]_{\mathfrak{m}}, \langle y,u\rangle u - y\big\rangle = 0, \quad \forall y \in \mathfrak{m}. \tag{4.3}$$

Now the subspace $\mathfrak{m}$ has a decomposition

$$\mathfrak{m} = L(u) + L(u)^{\perp},$$

where $L(u)$ is the span of u. By (4.3), for any $y_2 \in L(u)^{\perp}$ we have

$$\big\langle [u,y_2], y_2\big\rangle = 0.$$

Hence for any $y = y_1 + y_2$, $y_1 \in L(u)$, $y_2 \in L(u)^{\perp}$ we have

$$\begin{aligned}
0 &= \big\langle [u,y]_{\mathfrak{m}}, \langle y,u\rangle u - y\big\rangle \\
&= \big\langle [u,y_2]_{\mathfrak{m}}, \langle y_1,u\rangle u - y_1\big\rangle \\
&= \left\langle [u,y_2]_{\mathfrak{m}}, \langle y_1,u\rangle u - \frac{\langle y_1,u\rangle}{\langle u,u\rangle} u\right\rangle \\
&= \langle y_1,u\rangle \times \left(1 - \frac{1}{\langle u,u\rangle}\right) \times \big\langle [u,y_2]_{\mathfrak{m}}, u\big\rangle.
\end{aligned}$$

Since $\langle u,u\rangle < 1$, the above equality implies that

$$\big\langle [u,y_2]_{\mathfrak{m}}, u\big\rangle = 0, \quad \forall y_2 \in L(u)^{\perp},$$

or equivalently

$$\big\langle [u,y]_{\mathfrak{m}}, u\big\rangle = 0, \quad \forall y \in \mathfrak{m}. \tag{4.4}$$

Combining (4.3) and (4.4), we see that the S-curvature of F vanishes at o. Since $(G/H, F)$ is homogeneous, the S-curvature vanishes everywhere. □

For $y \in \mathfrak{m}$, we define a linear transformation $\mathrm{ad}_{\mathfrak{m}}(y)$ as

$$\mathrm{ad}_{\mathfrak{m}}(y)(v) = [y,v]_{\mathfrak{m}}.$$

The proof of Theorem 4.1 has the following

Corollary 4.2. *The homogeneous Randers space in Theorem 3.1 has almost isotropic S-curvature if and only if $\mathrm{ad}_{\mathfrak{m}}(u)$ is skew-symmetric with respect to the inner product $\langle , \rangle$. In particular, it has vanishing S-curvature if and only if $\mathrm{ad}_{\mathfrak{m}}(u)$ is skew symmetric with respect to the inner product $\langle , \rangle$.*

Proof. If $\mathrm{ad}_{\mathfrak{m}}(u)$ is skew symmetric with respect to $\langle,\rangle$. Then for any $y\in\mathfrak{m}$, we have

$$\big\langle [u,y]_{\mathfrak{m}}, y\big\rangle = 0,$$

and

$$\big\langle [u,y]_{\mathfrak{m}}, u\big\rangle = -\big\langle y, [u,u]_{\mathfrak{m}}\big\rangle = 0.$$

By Theorem 3.1, F has vanishing S-curvature. On the other hand, if F has almost isotropic S-curvature, then (4.3) and (4.4) hold. Thus

$$\big\langle [u,y]_{\mathfrak{m}}, y\big\rangle = 0.$$

Thus $\mathrm{ad}_{\mathfrak{m}}(u)$ is skew symmetric with respect to $\langle,\rangle$. □

As another application of the above formulae, we have

Proposition 4.3. *Let $(G/H,F)$ be a homogeneous Randers space defined by α and $u\neq 0$ as in Theorem 3.1. Then F is of Douglas type if and only if*

$$\big\langle [v_1,v_2]_{\mathfrak{m}}, u\big\rangle = 0, \quad \forall v_1,v_2\in\mathfrak{m}.$$

Furthermore, if F is of Douglas type and has almost isotropic S-curvature, then F is of Berwald type.

Proof. According to [1], F is of Douglas type if and only if the corresponding 1-form β is closed, i.e., $\beta=0$. Since β is G-invariant, we only need to prove that $\mathrm{d}\beta|_o=0$. Using the local coordinate of (2.2), we see that this is equivalent to

$$s_{ij}(0)=\frac{1}{2}\left(\frac{\partial b_i}{\partial x^j}-\frac{\partial b_j}{\partial x^i}\right)=0, \quad \forall i,j.$$

By (3.1), this is equivalent to

$$\frac{1}{2}c\big\langle [u_i,u_j]_{\mathfrak{m}}, u_n\big\rangle = 0, \quad \forall i,j,$$

that is,

$$\big\langle [v_1,v_2]_{\mathfrak{m}}, u\big\rangle = 0, \quad \forall v_1,v_2\in\mathfrak{m}.$$

This proves the first assertion. If F has almost isotropic S-curvature, then by Corollary 4.2, we have

$$\big\langle [u,v]_{\mathfrak{m}}, v\big\rangle = 0.$$

By Proposition 2.1, if F is in addition of Douglas type, then F must be of Berwald type. □

5. Left invariant Randers metrics on Lie groups and some examples

Let $F=\alpha+\beta$ be a left invariant Randers metric on a connected Lie group G. Then for any $g\in G$ and $y\in\mathfrak{g}=T_e(G)$, we have $F(\mathrm{d}L_g(y))=F(y)$, where L_g is the left translation defined by g. Hence

$$\alpha\big(\mathrm{d}L_g(y)\big)+\beta\big(\mathrm{d}L_g(y)\big)=\alpha(y)+\beta(y). \tag{5.1}$$

Substituting y with $-y$ in (4.1) we get

$$\alpha\big(\mathrm{d}L_g(y)\big)-\beta\big(\mathrm{d}L_g(y)\big)=\alpha(y)-\beta(y). \tag{5.2}$$

Combining (5.1) and (5.2) we have

$$\alpha\big(\mathrm{d}L_g(y)\big)=\alpha(y), \qquad \beta\big(\mathrm{d}L_g(y)\big)=\beta(y).$$

This means that both the Riemannian metric α and the 1-form β are invariant under the left translations of G. The left invariant Riemannian metric corresponds to an inner product $\langle,\rangle$ on $\mathfrak{g}$ and the 1-from β corresponds to a left invariant vector field on G, i.e., an element of $\mathfrak{g}$, with length less than 1. On the other hand, any inner product on $\mathfrak{g}$ defines a left invariant Riemannian metric on G and this metric together with any element of $\mathfrak{g}$ of length less than 1 defines a left invariant Randers metric on $\mathfrak{g}$. By Theorem 3.1 we have

Proposition 5.1. *Let G be a n-dimensional connected Lie group with Lie algebra $\mathfrak{g}$. Let $\langle,\rangle$ be an inner product on $\mathfrak{g}$ and $u\in\mathfrak{g}$ with $\langle u,u\rangle<1$. Then the left invariant Randers metric F on G defined by $\langle,\rangle$ and u has the S-curvature*

$$S(e,y)=\frac{n+1}{2}\left\{\frac{\langle [u,y],\langle y,u\rangle u-y\rangle}{F(y)}-\big\langle [u,y],u\big\rangle\right\}.$$

F has almost isotropic S-curvature if and only if F has vanishing S-curvature if and only if the linear endomorphism $\mathrm{ad}(u)$ of $\mathfrak{g}$ is skew symmetric with respect to the inner product $\langle,\rangle$.

As an application, we classify left invariant Randers metrics with almost isotropic S-curvature on a connected nilpotent Lie group.

Proposition 5.2. *Let G be a connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $\langle,\rangle$ be an inner product on $\mathfrak{g}$ and $u \in \mathfrak{g}$ with $\langle u,u\rangle < 1$. Then the Randers metric constructed by $\langle,\rangle$ and u has almost isotropic S-curvature (and hence vanishing S-curvature) if and only if $u \in C(\mathfrak{g})$, where $C(\mathfrak{g})$ denotes the center of $\mathfrak{g}$.*

Proof. If u lies in the center of $\mathfrak{g}$, then $\operatorname{ad} u = 0$. Hence it is skew symmetric with respect to $\langle,\rangle$. By Proposition 5.1, the corresponding Randers metric F has vanishing S-curvature. On the other hand, if F has vanishing S-curvature, then $\operatorname{ad} u$ is skew symmetric with respect to $\langle,\rangle$. Thus $\operatorname{ad} u$ is a (complex) semisimple endomorphism of $\mathfrak{g}$. On the other hand, since the Lie algebra $\mathfrak{g}$ is nilpotent, $\operatorname{ad} u$ is also nilpotent [6]. This forces $\operatorname{ad} u = 0$. Hence $u \in C(\mathfrak{g})$. $\square$

Since any non-zero nilpotent Lie algebra has non-zero center, Proposition 5.2 implies that for any inner product on the Lie algebra $\mathfrak{g}$, there exists u in $\mathfrak{g}$ such that F has vanishing S-curvature. However, such Randers metrics may not be of Berwald type as pointed by the next examples.

Example 5.1. A real Lie algebra $\mathfrak{n}$ is called two-step nilpotent if $[\mathfrak{n},\mathfrak{n}] \neq 0$ and $[[\mathfrak{n},\mathfrak{n}],\mathfrak{n}] = 0$. A two-step nilpotent Lie algebra is called non-singular if for any $x \in \mathfrak{n} - C(\mathfrak{n})$, where $C(\mathfrak{n})$ denotes the center of $\mathfrak{n}$, the linear mapping $\operatorname{ad} x : \mathfrak{n} \to C(\mathfrak{n})$ defined by $\operatorname{ad} x(y) = [x,y]$ is surjective. Suppose N is a Lie group with Lie algebra $\mathfrak{n}$ which is non-singular two-step nilpotent. Let $\langle,\rangle$ be an inner product on $\mathfrak{n}$ and $u \in \mathfrak{n}$ such that $\langle u,u\rangle < 1$. According to Proposition 5.2, the corresponding left invariant Randers metric F has almost isotropic (hence vanishing) S-curvature if and only $u \in C(\mathfrak{n})$. Since $\mathfrak{n}$ is non-singular, we have $[\mathfrak{n},\mathfrak{n}] = C(\mathfrak{n})$. Now suppose F is in addition of Berwald type. Then by Proposition 2.1, we have $u \perp [\mathfrak{n},\mathfrak{n}] = C(\mathfrak{n})$. Hence $u = 0$. Thus on a non-singular two-step nilpotent Lie group, there does not exist any left invariant Randers metric which is non-Riemannian and of Berwald type. But there always exists non-Riemannian ones with vanishing S-curvature.

The most important non-singular two-step nilpotent Lie algebras are the Heisenberg Lie algebra. Let $\mathfrak{n}$ be the real vector space with a basis $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$, z, $n \geqslant 1$. Define the brackets as follows:

$$[x_i, x_j] = [y_i, y_j] = 0, \quad [x_i, y_j] = \delta_{ij} z, \quad [x_i, z] = [y_j, z] = 0.$$

Then it is easily seen that $\mathfrak{n}$ is a two-step nilpotent Lie algebra with center spanned by z, called Heisenberg Lie algebra. It is also easily seen that $\mathfrak{n}$ is non-singular. Hence on a Lie group with Heisenberg Lie algebra (i.e., Heiserberg Lie group) there does not exist any non-Riemannian left invariant Randers metric of the Berwald type. But for any inner product $\langle,\rangle$ on $\mathfrak{n}$, set

$$u = c\frac{z}{\sqrt{\langle z,z\rangle}}, \quad |c| < 1.$$

Then the corresponding Randers metric has vanishing S-curvature.

Example 5.2. Let G be connected compact Lie group, $\operatorname{Lie} G = \mathfrak{g}$. Fix a $\operatorname{Ad} G$-invariant inner product $\langle,\rangle$ on $\mathfrak{g}$. The corresponding left invariant Riemannian metric Q is then bi-invariant under the action of G. Hence (G,Q) is a globally symmetric Riemannian space (see [6]). Since $\langle,\rangle$ is $\operatorname{Ad} G$ invariant, for any $u \in \mathfrak{g}$, $\operatorname{ad}(u)$ is skew symmetric with respect to $\langle,\rangle$. Therefore, for any $u \in \mathfrak{g}$ with $\langle u,u\rangle < 1$, the left invariant Randers metric F constructed by $\langle,\rangle$ and u has vanishing S-curvature. Therefore F is of Berwald type if and only if $u \perp [\mathfrak{g},\mathfrak{g}]$. In particular, if G is semisimple, then no such Randers metric can be non-Riemannian and of Berwald type. However, if G is not semisimple, then $\mathfrak{g} \neq [\mathfrak{g},\mathfrak{g}]$. For any $u \in [\mathfrak{g},\mathfrak{g}]^{\perp}$, the corresponding Randers metric must be of Berwald type. This method provides a convenient way to construct globally defined Berwald spaces which are neither Riemannian nor locally Minkowskian (see [1] for the details).

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