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# The composite Euler method for stiff stochastic differential equations

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**Abstract**

In this paper we present the composite Euler method for the strong solution of stochastic differential equations driven by $d$-dimensional Wiener processes. This method is a combination of the semi-implicit Euler method and the implicit Euler method. At each step either the semi-implicit Euler method or the implicit Euler method is used in order to obtain better stability properties. We give criteria for selecting the semi-implicit Euler method or the implicit Euler method. For the linear test equation, the convergence properties of the composite Euler method depend on the criteria for selecting the methods. Numerical results suggest that the convergence properties of the composite Euler method applied to nonlinear SDEs is the same as those applied to linear equations. The stability properties of the composite Euler method are shown to be far superior to those of the Euler methods, and numerical results show that the composite Euler method is a very promising method. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

In this paper we consider numerical methods for the strong solution of stochastic differential equations (SDEs) driven by $d$-dimensional Wiener processes

$$\mathrm{d}y(t) = f(t, y(t))\,\mathrm{d}t + \sum_{j=1}^{d} g_j(t, y(t))\,\mathrm{d}W_j(t), \quad y(t_0) = y_0, \quad t \in [t_0, T], \tag{1}$$

where $f(t, y(t))$ is the drift coefficient, $g_j(t, y(t))$ is the diffusion coefficient and $W_j(t)$ is the standard Wiener process whose increment $\Delta W_j(t) = W_j(t+\Delta t) - W_j(t)$ is a Gaussian random variable $N(0, \Delta t)$.

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When $d = 1$, the SDEs driven by one Wiener process are given by

$$dy(t) = f(t, y(t))\,dt + g(t, y(t))\,dW(t), \quad y(t_0) = y_0, \quad t \in [t_0, T]. \tag{2}$$

There are similar relationships between the numerical methods for ordinary differential equations (ODEs) and those for SDEs. You can find these relationships in, for example, the analysis of Runge–Kutta methods [2,4,7,14], order conditions [2,3,12], multistep methods [1,8] and adaptive step control techniques [9]. There are a number of ways of developing numerical methods for SDEs. The first way is to construct new methods from scratch, the second way is to take existing methods for ODEs and apply them to (1). For example, for solving the ODE

$$y' = f(t, y(t)), \quad y(t_0) = y_0, \quad t \in [t_0, T], \tag{3}$$

we have the explicit Euler method

$$y_{n+1} = y_n + hf(t_n, y_n)$$

and the implicit Euler method

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}),$$

where $t_n = t_0 + nh\,(n = 0, 1, \ldots, N)$ and $h = (T - t_0)/N$. Corresponding to these Euler methods, there are three Euler methods for the SDE (2), namely

(1) the explicit Euler method

$$y_{n+1} = y_n + f(t_n, y_n)h + g(t_n, y_n)\Delta W_n,$$

(2) the semi-implicit Euler method

$$y_{n+1} = y_n + f(t_{n+1}, y_{n+1})h + g(t_n, y_n)\Delta W_n,$$

(3) and the implicit Euler method

$$y_{n+1} = y_n + f(t_{n+1}, y_{n+1})h + g(t_{n+1}, y_{n+1})\Delta W_n,$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$.

Due to the specific nature of the Wiener process, considerable complications arise in a number of straightforward attempts to generalize methods for ODEs. One of the main difficulties in developing numerical methods for SDEs arises when considering stiffness.

Applying the explicit Euler methods for ODEs to the linear test equation

$$y' = \lambda y, \quad Re(\lambda) < 0$$

gives

$$y_{n+1} = (1 + \lambda h)y_n,$$

and the explicit Euler method is stable if $|1 + \lambda h| < 1$, or if $\lambda h \in B(-1, 1)$ (the circle of radius 1 centred on $(-1, 0)$). Applying the implicit Euler method to the linear test equation, gives

$$y_{n+1} = \frac{1}{1 - \lambda h} y_n$$

and the implicit Euler method is stable if $|1/(1 - \lambda h)| < 1$, that is stable for any $h > 0$. Clearly the implicit Euler method can cope with stiffness (where the eigenvalues of (3) have widely varying negative real part) whereas the explicit Euler method cannot.

Consider now what happens in the stochastic case. Applying the implicit Euler method to the linear test equation (interpreted in the Itô sense),

$$\mathrm{d}y = ay\,\mathrm{d}t + by\,\mathrm{d}W(t), \quad y_0 = y(t_0) \tag{4}$$

gives

$$y_{n+1} = R(h, \Delta W_n) y_n,$$

where

$$R(h, \Delta W_n) = \frac{1}{1 - ah - b\Delta W_n}.$$

As $\Delta W_n$ may take any value in $(-\infty, \infty)$, $|R|$ can become unbounded for any $a$, $b$ and $h$. For problems which are stiff in both the deterministic and stochastic components, the implicit Euler method is not appropriate. In addition, the numerical solution of the implicit Euler method converges to the exact solution of the corresponding right-point SDE but not that of the Itô SDE [6].

Thus in this paper, we introduce a modified method to improve upon the stability properties of the Euler methods. The difficulty with the implicit Euler method is that $R(h, \Delta W_n)$ may approach infinity if the generated random number $\Delta W_n$ is in the neighbourhood of $(1 - ah)/b$. In this case another method can be used such as the semi-implicit Euler method. This leads to the following composite method

$$y_{n+1} = \begin{cases} \dfrac{1 + b\Delta W_n}{1 - ah} y_n & \text{under certain condition,} \\[2ex] \dfrac{1}{1 - ah - b\Delta W_n} y_n & \text{else.} \end{cases}$$

We call this method the composite Euler method and it is a combination of the semi-implicit Euler method and the implicit Euler method, given by

$$y_{n+1} = y_n + f(t_{n+1}, y_{n+1})h + [\lambda_n g(t_n, y_n) + (1 - \lambda_n) g(t_{n+1}, y_{n+1})]\Delta W_n, \tag{5}$$

where $\lambda_n \in [0, 1]$ and is determined in every step. It becomes the semi-implicit Euler method when $\lambda_n \equiv 1$, or the implicit Euler method when $\lambda_n \equiv 0$. We can obtain different composite Euler methods if different criteria for choosing $\lambda_n$ are used.

There have been other attempts in the literature in order to improve the stability properties of numerical methods for stiff SDEs. For example, Milshtein et al. [13] present balanced implicit methods by introducing a modified implicit diffusion coefficient, given by

$$y_{n+1} = y_n + f(t_n, y_n)h + \sum_{j=1}^{d} g_j(t_n, y_n)\Delta W_{nj} + C_n(y_n - y_{n+1}),$$

where

$$C_n = c_0(t_n, y_n)h + \sum_{j=1}^{d} c_j(t_n, y_n)|\Delta W_{nj}|.$$

For balanced implicit methods, the type and degree of implicitness can be chosen by appropriate weights $c_0$ and $c_j$ $(j = 1, 2, \ldots, c_d)$. We consider extensions of this idea in later papers.

The outline of the contents of this paper is as follows. From Sections 2–5, we discuss the composite Euler method for SDEs driven by one Wiener process. We give three criteria for selecting $\lambda_n$ and two composite Euler methods in Section 2. In Section 3 we give the mean-square and asymptotic stability regions of these composite Euler methods. The convergence properties of the composite Euler methods are discussed in Section 4. Numerical results for solving one-dimensional SDEs are presented in Section 5. In Section 6 we give criteria for selecting $\lambda_{nj}$ in the composite Euler method for SDEs driven by $d$-dimensional Wiener processes.

## 2. The composite Euler method

In this section we consider the composite Euler method (5) for the SDE (2). First, we discuss criteria for choosing $\lambda_n$ in the composite Euler method when solving the linear test equation (4). Applying the composite Euler method to (4), gives

$$y_{n+1} = y_n + ahy_{n+1} + [\lambda_n b y_n + (1-\lambda_n) b y_{n+1}]\Delta W_n,$$

that is

$$y_{n+1} = \frac{1+\lambda_n q I_n}{1-p-(1-\lambda_n)qI_n} y_n, \tag{6}$$

where $p = ah$, $q = b\sqrt{h}$, $\Delta W_n = \sqrt{h} I_n$ and $I_n$ is the $n$th realization of $I$, the standard normal random variable $\mathrm{N}(0,1)$.

The general principle for selecting $\lambda_n$ is to ensure

$$\lim_{N\to\infty} \prod_{n=0}^{N-1} \frac{1+\lambda_n q I_n}{1-p-(1-\lambda_n)qI_n}$$

converges to 0 as fast as possible when the underlying problem is itself asymptotically stable. Since the solution of (4) (interpreted in the Itô sense) is

$$y(t) = \mathrm{e}^{(a-(1/2)b^2)t + bW(t)} y_0,$$

the problem is asymptotically stable whenever $Re(a - \frac{1}{2}b^2) < 0$. In this paper it is assumed that $a < 0$ and $b > 0$.

A simple criterion is that the implicit Euler method is used when $I_n < 0$. In this case

$$\left| \frac{1}{1-p-qI_n} \right| < 1.$$

**Criterion 1.** *For solving SDEs, a simple criterion for selecting $\lambda_n$ in the composite Euler method is given by*

$$\lambda_n^{(1)} = \begin{cases} 0, & I_n < 0, \\ 1, & I_n \geqslant 0. \end{cases}$$

The composite Euler method with Criterion 1 is called the composite Euler method of type 1.

Let

$$f(\lambda) = \frac{1 + qI_n\lambda}{1 - p - (1-\lambda)qI_n},$$

we now consider the criterion for choosing the optimal value of $\lambda_n$. It is easy to obtain

$$\min_{\lambda\in[0,1]} |f(\lambda)| = \min\{|f(0)|, |f(1)|\}.$$

Let

$$Q_{n0} = f(0) = \frac{1}{1 - p - qI_n}, \quad Q_{n1} = f(1) = \frac{1 + qI_n}{1 - p},$$

then the criterion for selecting the optimal $\lambda_n$ is given below.

**Criterion 2.** *For the linear test equation, the criterion for selecting the optimal $\lambda_n$ in the composite Euler method is given by*

$$\lambda_n^{(2)} = \begin{cases} 0, & |Q_{n0}| < |Q_{n1}|, \\ 1, & |Q_{n0}| \geqslant |Q_{n1}|. \end{cases}$$

Now, we consider the criterion for selecting $\lambda_n$ in the composite Euler method when solving the scalar nonlinear SDE (2). Assuming $y_{n+1}, y^*_{n+1}$ are numerical solutions of (5) with $y_n$, $y^*_n$, respectively, then

$$y_{n+1} = y_n + f(t_{n+1}, y_{n+1})h + [\lambda_n g(t_n, y_n) + (1-\lambda_n)g(t_{n+1}, y_{n+1})]\sqrt{h}I_n,$$

$$y^*_{n+1} = y^*_n + f(t_{n+1}, y^*_{n+1})h + [\lambda_n g(t_n, y^*_n) + (1-\lambda_n)g(t_{n+1}, y^*_{n+1})]\sqrt{h}I_n$$

and the following approximate result holds:

$$y_{n+1} - y^*_{n+1} \approx \frac{(1 + \lambda_n(\partial g/\partial y)\sqrt{h}I_n)|_{y=y_n}}{(1 - (\partial f/\partial y)h + (\lambda_n - 1)(\partial g/\partial y)\sqrt{h}I_n)|_{y=y_{n+1}}}(y_n - y^*_n)$$

$$\approx \left.\frac{1 + \lambda_n(\partial g/\partial y)\sqrt{h}I_n}{1 - (\partial f/\partial y)h + (\lambda_n - 1)(\partial g/\partial y)\sqrt{h}I_n}\right|_{y=y_n}(y_n - y^*_n).$$

Let

$$R_{n0} = \left.\frac{1}{1 - (\partial f/\partial y)h - (\partial g/\partial y)\sqrt{h}I_n}\right|_{y=y_n}, \quad R_{n1} = \left.\frac{1 + (\partial g/\partial y)\sqrt{h}I_n}{1 - (\partial f/\partial y)h}\right|_{y=y_n},$$

then Criterion 3 is similar to Criterion 2.

**Criterion 3.** *The criterion for selecting $\lambda_n$ in the composite Euler method for solving a scalar nonlinear SDE is given by*

$$\lambda_n^{(3)} = \begin{cases} 0, & |R_{n0}| < |R_{n1}|, \\ 1, & |R_{n0}| \geqslant |R_{n1}|. \end{cases}$$

The composite Euler method with Criterion 2 or 3 is called the composite Euler method of type 2.

Now, we consider the numerical solution of the scalar nonlinear equation (5). This resulting nonlinear equation can be solved by fixed-point iteration, Newton–Raphson iteration, line search or other iterative methods. Two factors should be taken into account when using such an iteration method.

The first factor is that we should consider the convergence properties of iterative methods for nonlinear equations and the stability properties of numerical methods for SDEs together, especially when $\min\{|R_{n0}|, |R_{n1}|\} > 1$. The second factor is that the stability properties of numerical methods for SDEs vary from step to step because of the uncertainty of the stochastic component, and so it may be necessary to obtain convergence by varying the stepsize. In this paper, the nonlinear equation (5) is solved by Newton–Raphson iteration.

## 3. Stability properties of the composite Euler method

Applying a numerical scheme to the linear test equation (4), the numerical scheme is represented by

$$y_{n+1} = R(h, a, b, \Delta W_n)y_n = R(h, a, b, \sqrt{h}I_n)y_n. \tag{7}$$

Saito and Mitsui [16] introduce the definition of mean-square (MS) stability.

**Definition 1.** A numerical scheme is said to be MS-stable for $h, a, b$ if

$$\bar{R}(h, a, b) = E(R^2(h, a, b, \sqrt{h}I)) < 1.$$

$\bar{R}(h, a, b)$ is called the MS-stability function.

The MS-stability function of the semi-implicit Euler method is

$$\bar{R}_1 = \frac{1 + q^2}{(1 - p)^2}$$

and that of the implicit Euler method does not exist.

The MS-stability function of the composite Euler method is

$$\bar{R}_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1 + \lambda(x)qx)^2}{(1 - p - (1 - \lambda(x))qx)^2}\, \mathrm{e}^{-x^2/2}\, \mathrm{d}x, \tag{8}$$

where the criterion function $\lambda(x)$ is defined according to the corresponding Criterion 1 or 2, given by

$$\text{Criterion 1:} \quad \lambda^{(1)}(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0; \end{cases} \tag{9}$$

$$\text{Criterion 2:} \quad \lambda^{(2)}(x) = \begin{cases} 1, & \left|\dfrac{1 + qx}{1 - p}\right| < \left|\dfrac{1}{1 - p - qx}\right|, \\ 0, & \left|\dfrac{1 + qx}{1 - p}\right| \geqslant \left|\dfrac{1}{1 - p - qx}\right|. \end{cases} \tag{10}$$

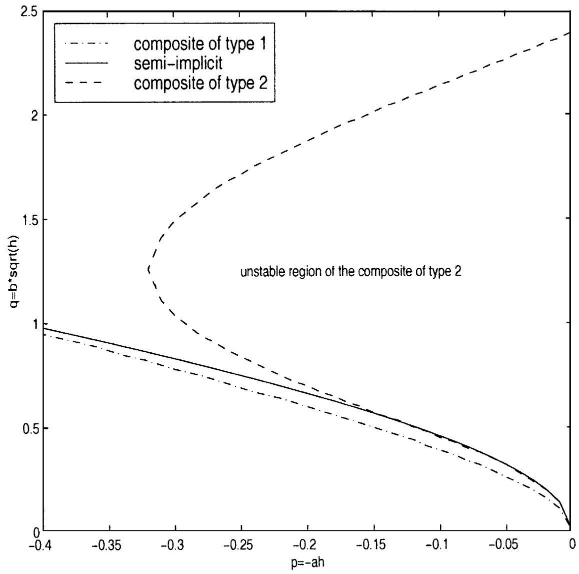


Fig. 1. MS-stability regions of the Euler methods.

It appears not to be possible to find a simple analytical expression of the integral in (8). Thus, we use the composite Trapezoidal rule to compute the integral in (8). The integral interval $(-\infty,\infty)$ is approximated by $[-10,10]$ as the magnitude of the integrand in (8) is very small when $|x|>10$. Fig. 1 gives the MS-stability regions of the semi-implicit Euler method and the composite Euler method of types 1 and 2. The MS-stability regions of the semi-implicit Euler method and the composite Euler method of type 1 are those under the plotted line in Fig. 1. The MS-unstable region of the composite Euler method of type 2 is indicated in Fig. 1. The region outside the indicated region is the MS-stable region of the composite Euler method of type 2. The MS-stability region of the composite Euler method of type 1 is a little smaller than that of the semi-implicit Euler method, while the MS-stability property of the composite Euler method of type 2 is much better than that of the semi-implicit Euler method.

The next important stability definition that must be considered is asymptotic stability. Saito and Mitsui [15] give the definition of $T$-stability to measure asymptotic stability and give two examples concerning the asymptotic stability regions of the Euler–Maruyama scheme for weak solutions of SDEs. We have extended the definition of $T$-stability from weak solutions to strong solutions [5]. For a given number $l$, discretize the interval $[-M_1,M_1]$ as

$$-M_1=x_0<x_1<\cdots<x_l=M_1.$$

Let

$$p_i=\int_{x_{i-1}}^{x_i}\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}\,\mathrm{d}x,\quad u_i\in(x_{i-1},x_i),\quad i=1,2,\ldots,l,$$

then the standard Gaussian random variable $I$ can be approximated by a discrete random variable $U$ with distribution

| $U$ | $u_1$ | $u_2$ | $\dots$ | $u_l$ |
|---|---|---|---|---|
| $p_i$ | $p_1$ | $p_2$ | $\dots$ | $p_l$ |

Assuming $m$ random numbers are generated, there are $\lceil mp_i \rceil$ numbers in the interval $[x_{i-1}, x_i]$ and it is assumed that these numbers are all equal to $u_i$. We can compute

$$T^m = \prod_{i=1}^{l} |R(h, a, b, \sqrt{h}u_i)|^{\lceil mp_i \rceil}$$

or the average

$$T = T(h, a, b) = \prod_{i=1}^{l} |R(h, a, b, \sqrt{h}u_i)|^{p_i}, \tag{11}$$

and say the numerical scheme is $T$-stable if $T < 1$.

Higham [10] studies the asymptotic stability property of a numerical method applied to the linear test equation. It can be proved that the definition of asymptotic stability [10] is equivalent to that of $T$-stability [5]. We can use Lemma 7 in [10] to decide whether the $T$-stability region of a numerical method exists or not and then use formula (11) to plot the $T$-stability region of the numerical method.

The definition of $T$-stability is based on a very large sample size. In practical computation, the number of timesteps is not very large, so the convergence rate of $T^n \to 0$ should be considered. In order to obtain a stable numerical result in practical computations, the following $T(A)$-stability definition is introduced to consider a stricter condition for $T$-value [5].

**Definition 2.** The numerical scheme is said to be $T(A)$-stable if

$$T(h, a, b) < A,$$

where $0 < A < 1$.

Fig. 2 gives the $T(A)$-stability regions of the composite Euler method of type 2. The composite Euler method of type 2 is $T(1)$-stable for all $q$ when $p \leqslant 0$.

Now, we consider the $T(A)$-stability properties of the composite Euler method of type 1 with an equidistant mesh

$$-M_1 = x_0 < x_1 < \cdots < x_{l+1} < \cdots < x_{2l} < x_{2l+1} = M_1.$$

Thus $x_{l+1} = 0$ and $x_i$ and $-x_i$ all are grid points. Let

$$f_i(p, q) = \frac{1 + qx_i}{1 - p} \times \frac{1}{1 - p + qx_i}, \quad x_i > 0,$$

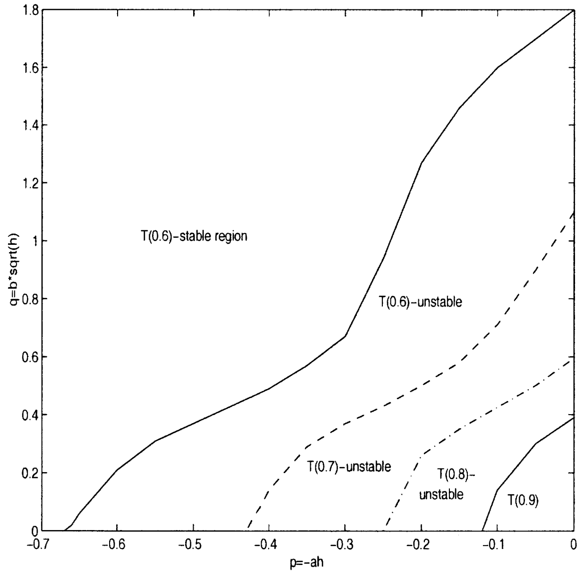


Fig. 2. $T(A)$-stability regions of the composite Euler method of type 2.

then $f_i(p,q)$ is a monotonously increasing function with variables $p$ and $q$, and

$$\lim_{p\to-\infty} f_i(p,q) = 0, \quad \lim_{q\to+\infty} f_i(p,q) = \frac{1}{1-p}.$$

The $T$-value, given by

$$T^{2l+1}(p,q) = \frac{1}{1-p}\prod_{i=l+1}^{2l+1} f_i(p,q)$$

is a monotonously increasing function with variables $p$ and $q$, and

$$\lim_{p\to-\infty} T(p,q) = 0, \quad \lim_{q\to+\infty} T(p,q) = \sqrt{\frac{1}{1-p}}.$$

The $T(A)$-stability properties of the composite Euler method of type 1 are not as good as those of the composite Euler method of type 2 but is better than those of the Euler methods. As an example, the $T(0.8)$-stability region of the composite Euler method of type 1 is presented in Fig. 3. It is $T(0.8)$-stable for any $q$ when $p < -0.57$.

## 4. Convergence properties

The solution of SDE (2) can be written as

$$y(t) = y_0 + \int_{t_0}^{t} f(s,y(s))\,\mathrm{d}s + \int_{t_0}^{t} g(s,y(s))\,\mathrm{d}W(s),$$

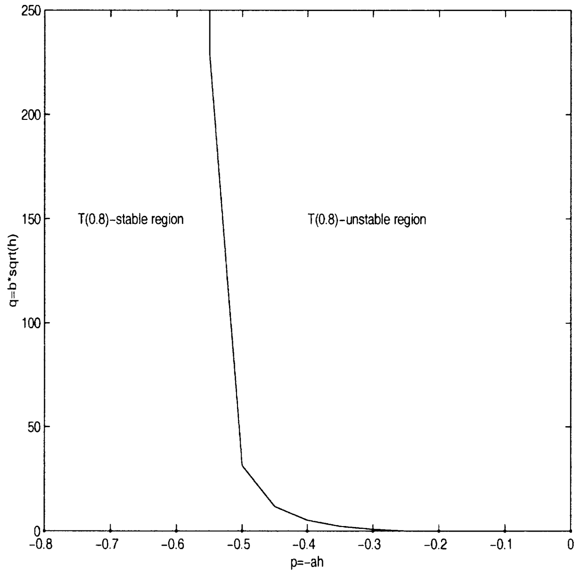


Fig. 3. $T(A)$-stability regions of the composite Euler method of type 1.

where the integral $\int_0^t g(s, y(s))\,\mathrm{d}W(s)$ is a stochastic integral. This integral can be calculated by the limit of the approximating sums. Giving an equidistant discretization of interval $[0, T]$,

$$t_0 < t_1 < \cdots < t_N = T, \quad t_n = t_0 + \frac{(T - t_0)n}{N}, \quad n = 0, 1, \ldots, N \tag{12}$$

and letting $\xi_n = \theta t_n + (1 - \theta)t_{n-1}$ ($\theta \in [0, 1]$), the stochastic integral is defined as the limit (in the mean-square sense), as $N \to \infty$, of the approximating sums

$$\sum_{n=1}^{N} g(y(\xi_n))(W(t_n) - W(t_{n-1})).$$

Unlike Riemann integrals, the values of stochastic integrals depend on the choice of $\theta$. For example, the stochastic integral of the Wiener process is given by

$$\int_{t_0}^{T} W(s)\,\mathrm{d}W(s) = \frac{1}{2}(W^2(T) - W^2(t_0)) + \left(\theta - \frac{1}{2}\right)(T - t_0).$$

We have the following three types of stochastic integrals and corresponding SDEs:

- The Itô integrals when $\theta = 0$. The corresponding Itô SDEs are the equations using the usual notation (2);
- The Stratonovich integrals when $\theta = \frac{1}{2}$. The corresponding Stratonovich SDEs are denoted by

$$\mathrm{d}y(t) = f_1(y(t))\,\mathrm{d}t + g(y(t)) \circ \mathrm{d}W(t).$$

- The backward integrals when $\theta = 1$ [17]. The corresponding right-point SDEs [6] are denoted by

$$\mathrm{d}y(t) = f_2(y(t))\,\mathrm{d}t + g(y(t)) \bullet \mathrm{d}W(t).$$

The relationships between these SDEs are

$$f_1(y(t)) = f(y(t)) - \tfrac{1}{2}g'(y(t))g(y(t)),$$

$$f_2(y(t)) = f(y(t)) - g'(y(t))g(y(t)).$$

It can be proved that the numerical solutions of the implicit Euler method for solving right-point SDEs converge to the exact solutions of the right-point SDEs with order $O(h^{1/2})$ [6].

Now, we consider the convergence properties of the composite Euler method applying to the linear test equation (6), given by

$$\begin{aligned} y_N &= \frac{1+\lambda_{N-1}b\Delta W_{N-1}}{1-ah-(1-\lambda_{N-1})b\Delta W_{N-1}}y_{N-1} \\ &= y_0\prod_{n=0}^{N-1}\frac{1+\lambda_n b\Delta W_n}{1-ah-(1-\lambda_n)b\Delta W_n}. \end{aligned}$$

Let

$$P_N = \prod_{n=0}^{N-1}\frac{1+\lambda_n b\Delta W_n}{1-ah-(1-\lambda_n)b\Delta W_n},$$

then

$$\ln P_N = \sum_{n=0}^{N-1}\ln(1+\lambda_n b\Delta W_n) - \sum_{n=0}^{N-1}\ln(1-ah-(1-\lambda_n)b\Delta W_n),$$

$$\ln(1+\lambda_n b\Delta W_n) = \lambda_n b\Delta W_n - \tfrac{1}{2}(\lambda_n b\Delta W_n)^2 + \tfrac{1}{3}(\lambda_n b\Delta W_n\xi_{n1})^3, \quad 0<\xi_{n1}<1,$$

$$\begin{aligned} -\ln(1-ah-(1-\lambda_n)b\Delta W_n) &= ah+(1-\lambda_n)b\Delta W_n + \tfrac{1}{2}(ah+(1-\lambda_n)b\Delta W_n)^2 \\ &\quad + \tfrac{1}{3}[(ah+(1-\lambda_n)b\Delta W_n)\xi_{n2}]^3, \quad 0<\xi_{n2}<1. \end{aligned}$$

Finally, we have that

$$\ln P_N = a(t_N-t_0) + b(W(t_N)-W(t_0)) + \frac{1}{2}\sum_{n=0}^{N-1}(1-2\lambda_n)(b\Delta W_n)^2 + R_N.$$

It can be proved that $R_N$ converges to zero in mean-square sense, namely

$$\lim_{N\to\infty} E(R_N^2) = 0.$$

For Criterion 1 and the corresponding criterion function $\lambda_n^{(1)}$ (9), we have that

$$E(\lambda_n^{(1)}b^2(\Delta W_n)^2) = \int_{-\infty}^{\infty}\lambda_n^{(1)}(x)b^2x^2\frac{1}{\sqrt{2\pi h}}\,e^{-x^2/2h}\,dx = \int_0^{\infty}b^2x^2\frac{1}{\sqrt{2\pi h}}\,e^{-x^2/2h}\,dx = \frac{1}{2}b^2h,$$

$$\lim_{N\to\infty} E\left[\left(\frac{1}{2}\sum_{n=0}^{N-1}(1-2\lambda_n^{(1)})(b\Delta W_n)^2\right)^2\right] = 0.$$

This means that the numerical solutions of the composite Euler method of type 1 converge to the exact solution of the Stratonovich linear test equation, namely

$$\lim_{N\to\infty} E[(y_N - y_0 e^{a(T-t_0)+b(W(T)-W(t_0))})^2] = 0.$$

For Criterion 2 and the corresponding criterion function $\lambda_n^{(2)}$ (10), we have the following result.

**Theorem 1.** *For the composite Euler method of type 2, the following result is true*:

$$\lim_{h\to 0} P\{\lambda_n^{(2)}(\Delta W_n) = 1\} = 1.$$

**Proof.** For any given $\varepsilon > 0$, find a large number $M$ to satisfy

$$P\{|I_n| > M\} = \frac{\varepsilon}{2},$$

where $I_n \sim \mathrm{N}(0,1)$ and $\Delta W_n = \sqrt{h} I_n$. Let events $A$, $B$ and $C$ satisfy

$$A = \{|Q_{n1}| < |Q_{n0}|\}, \quad B = \{|I_n| \leqslant M\}, \quad C = \{|I_n| > M\},$$

respectively, then we have that

$$P\{\lambda_n^{(2)}(\Delta W_n) = 1\} = P\{|Q_{n1}| < |Q_{n0}|\} = P\{AB\} + P\{AC\}.$$

For given $a < 0$, $b > 0$ and $\varepsilon$, find a stepsize $h_0$ satisfying

$$P\{0 < I_n < -\frac{a}{b}\sqrt{h_0}\} = \frac{\varepsilon}{2}.$$

Supposing that the step size $h$ is small enough to ensure $b\sqrt{h}M < 1$ and $h < h_0$, then the following result is true:

$$0 < \frac{1 + b\sqrt{h}x}{1 - ah} < \frac{1}{1 - ah - b\sqrt{h}x}$$

when $-M < x < 0$ or $-(a/b)\sqrt{h} < x < M$. Thus,

$$P\{AB\} = P\{-M < I_n < 0\} + P\left\{-\frac{a}{b}\sqrt{h} < I_n < M\right\} > 1 - \varepsilon.$$

So we can prove this theorem as

$$1 > P\{\lambda_n^{(2)}(\Delta W_n) = 1\} > P\{AB\} > 1 - \varepsilon. \qquad \square$$

For Criterion 2 and the corresponding criterion function $\lambda_n^{(2)}$ (10), we have that

$$\lim_{h\to 0} E\left(\sum_{n=0}^{N-1} \lambda_n^{(2)} b^2 (\Delta W_n)^2\right) = \lim_{h\to 0} \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} \lambda_n^{(2)}(x) b^2 x^2 \frac{1}{\sqrt{2\pi h}} e^{-x^2/2h}\, dx$$

$$= \lim_{h\to 0} \sum_{n=0}^{N-1} \int_{-\infty}^{\infty} b^2 x^2 \frac{1}{\sqrt{2\pi h}} e^{-x^2/2h}\, dx = b^2 (T - t_0)$$

and

$$\lim_{N\to\infty} E\left[\left(\frac{1}{2}\sum_{n=0}^{N-1}(1-2\lambda_n^{(2)})(b\Delta W_n)^2-\left(-\frac{1}{2}b^2(T-t_0)\right)\right)^2\right]=0.$$

This shows that the numerical solutions of the composite Euler method of type 2, applied to the linear test equation, converge to the exact solution of the Itô linear test equation, namely

$$\lim_{N\to\infty} E[(y_N-y_0\mathrm{e}^{(a-(1/2)b^2)(T-t_0)+b(W(T)-W(t_0))})^2]=0.$$

The above analysis shows that the composite Euler method of type 2 has the same convergence properties as that of the semi-implicit Euler method, and is nearly a semi-implicit Euler method when the stepsize is small. For the medium stepsize which is often used in practical computation, a small portion of the implicit Euler method is used in order to improve the stability properties. In this case we obtain numerical results with acceptable accuracy (see numerical results in the next section).

## 5. Numerical results

Numerical results for solving SDEs driven by one Wiener process are reported in this section. Denoting $y_N^{(i)}$ and $y^{(i)}(t_N)$ as the numerical solution and exact solution at step point $t_N$ in $i$th simulation, respectively, we use means of absolute errors $M$ and rates $R$, defined by

$$M=\frac{1}{K}\sum_{i=1}^{K}|y_N^{(i)}-y^{(i)}(t_N)|,\quad R=\frac{M}{\sqrt{h}},$$

to measure the accuracy and the convergence properties of the composite Euler method.

The first test equation is the linear test equation with $a=-1$ and $b=1$

$$\begin{aligned}&\text{Itô's form}: \quad \mathrm{d}y=-\mathrm{d}t+\mathrm{d}W(t),\\ &\text{Stratonovich's form}: \quad \mathrm{d}y=-\mathrm{d}t+\mathrm{d}\circ W(t),\end{aligned}\qquad y(0)=1,\quad t\in[0,3]$$

with exact solutions

$$\text{Itô's form}: \quad y(t)=\mathrm{e}^{-(3/2)t+W(t)},$$

$$\text{Stratonovich's form}: \quad y(t)=\mathrm{e}^{-t+W(t)}.$$

Table 1 gives numerical results of the composite Euler method of types 1 and 2. The data is obtained by $K=5000$ simulations. The numerical solutions of the composite Euler method of type 1 converge to the exact solution of the Stratonovich linear test equation, while those of the composite Euler method of type 2 converge to exact solution of the Itô linear test equation. For the composite Euler method of type 2, the percentages in Table 1 are used to indicate the portion of the semi-implicit Euler method which is used in all $K$ simulations. The percentage approaches 100% when the stepsize $h$ approaches zero.

Applying a numerical method to nonlinear SDEs, we may obtain stable or unstable solutions in different simulations. The main reason for obtaining unstable solutions is the convergence property of Newton–Raphson iteration for solving nonlinear equation with unknown variable $y_{n+1}$ in every

Table 1
Numerical results of linear test equation with $b=1$

| $1/h$ | Composite of type 1 | | | | Composite of type 2 | | | | % |
|---|---|---|---|---|---|---|---|---|---|
| | Itô | | Stratonovich | | Itô | | Stratonovich | | |
| | $M$ | $R$ | $M$ | $R$ | $M$ | $R$ | $M$ | $R$ | |
| $2^4$ | 0.1466 | 0.58 | 0.0636 | 0.25 | 0.01532 | 0.061 | 0.1801 | 0.72 | 90.11 |
| $2^5$ | 0.1438 | 0.81 | 0.0456 | 0.25 | 0.00981 | 0.055 | 0.1706 | 0.96 | 92.98 |
| $2^6$ | 0.1490 | 1.19 | 0.0381 | 0.30 | 0.00624 | 0.049 | 0.1751 | 1.40 | 95.05 |
| $2^7$ | 0.1457 | 1.64 | 0.0233 | 0.26 | 0.00408 | 0.046 | 0.1607 | 1.81 | 96.50 |
| $2^8$ | 0.1573 | 2.51 | 0.0291 | 0.30 | 0.00279 | 0.044 | 0.1712 | 2.74 | 97.49 |
| $2^9$ | 0.1667 | 3.77 | 0.0147 | 0.33 | 0.00218 | 0.049 | 0.1776 | 4.01 | 98.23 |
| $2^{10}$ | 0.1770 | 5.66 | 0.0111 | 0.35 | 0.00153 | 0.049 | 0.1839 | 5.88 | 98.76 |

step. We can expect an accurate approximation in a step if the stability property of the numerical method for SDE and the convergence property of the iterative method for the nonlinear equation are all satisfactory in this step. When solving nonlinear test equations, we just consider means and rates of the stable solutions in this paper.

The second test equation is a nonlinear SDE, given by

$$\begin{aligned} &\text{Itô's form}: \quad \mathrm{d}y = a^2 y(1+y^2)\,\mathrm{d}t + a(1+y^2)\,\mathrm{d}t, \\ &\text{Stratonovich's form}: \quad \mathrm{d}y = a(1+y^2)\circ \mathrm{d}t, \end{aligned} \qquad y(0)=1, \quad t\in[0,2].$$

The exact solution of the second test equation is given by [11]

$$y = \tan(aW(t) + \arctan y_0). \tag{13}$$

The third nonlinear test equation is given by

$$\begin{aligned} &\text{Itô's form}: \quad \mathrm{d}y = -(\alpha+\beta^2 y)(1-y^2)\,\mathrm{d}t + \beta(1-y^2)\,\mathrm{d}W(t), \\ &\text{Stratonovich's form}: \quad \mathrm{d}y = -\alpha(1-y^2)\,\mathrm{d}t + \beta(1-y^2)\circ \mathrm{d}W(t). \end{aligned} \qquad y(0)=0.5, \quad t\in[0,3].$$

The exact solution of the third test equation is given by [11]

$$y(t) = \frac{(1+y_0)\exp(-2\alpha t + 2\beta W_t) + y_0 - 1}{(1+y_0)\exp(-2\alpha t + 2\beta W_t) + 1 - y_0}. \tag{14}$$

For the second and third test equations, we apply the composite Euler method of type 1 to the Stratonovich test equations and the composite Euler method of type 2 to the Itô test equations. Tables 2 and 3 give means and rates of these composite Euler methods for solving the second and third test equations, respectively. All of the data in this table are based on 1000 simulated trajectories. For the composite Euler method of type 2, the percentages in Tables 2 and 3 are used to indicate the portion of the semi-implicit Euler method which is used in all 1000 simulations.

The numerical results in Tables 2 and 3 suggest that the convergence properties of the composite Euler method, applied to nonlinear SDEs, are the same as those applied to linear equations.

Next, we show that the stability properties of the composite Euler method of type 2 is better than those of the Euler methods. First, we consider the Itô linear test equation with $a=-5$ and

Table 2
Numerical results for the second test equation

| $1/h$ | Stratonovich form (type 1) | | | | Itô form (type 2) | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $a=0.1$ | | $a=0.3$ | | $a=0.1$ | | | $a=0.3$ | | |
| | $M$ | $R$ | $M$ | $R$ | $M$ | $R$ | % | $M$ | $R$ | % |
| $2^4$ | 0.0115 | 0.046 | 0.0423 | 0.16 | 0.00981 | 0.039 | 98.03 | 0.0628 | 0.25 | 89.16 |
| $2^5$ | 0.0089 | 0.050 | 0.0376 | 0.21 | 0.00659 | 0.037 | 98.60 | 0.0548 | 0.31 | 92.59 |
| $2^6$ | 0.0068 | 0.054 | 0.0350 | 0.28 | 0.00492 | 0.039 | 98.95 | 0.0470 | 0.37 | 94.10 |
| $2^7$ | 0.0042 | 0.048 | 0.0333 | 0.37 | 0.00333 | 0.037 | 99.29 | 0.0383 | 0.43 | 95.87 |
| $2^8$ | 0.0033 | 0.052 | 0.0276 | 0.44 | 0.00234 | 0.037 | 99.48 | 0.0328 | 0.52 | 97.20 |
| $2^9$ | 0.0020 | 0.045 | 0.0234 | 0.52 | 0.00162 | 0.036 | 99.63 | 0.0283 | 0.64 | 97.92 |
| $2^{10}$ | 0.0015 | 0.046 | 0.0212 | 0.49 | 0.00113 | 0.036 | 99.74 | 0.0234 | 0.74 | 98.31 |

Table 3
Numerical results for the third test equation

| $1/h$ | Stratonovich form (type 1) | | | | Itô form (type 2) | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $b=0.1$ | | $b=1$ | | $b=0.1$ | | | $b=1$ | | |
| | $M$ | $R$ | $M$ | $R$ | $M$ | $R$ | % | $M$ | $R$ | % |
| $2^4$ | $3.9\cdot10^{-4}$ | $1.5\cdot10^{-3}$ | 0.0226 | 0.0906 | $5.1\cdot10^{-4}$ | $2.0\cdot10^{-3}$ | 50.48 | 0.0211 | 0.0855 | 96.54 |
| $2^5$ | $1.6\cdot10^{-4}$ | $8.9\cdot10^{-4}$ | 0.0193 | 0.1094 | $2.6\cdot10^{-4}$ | $1.4\cdot10^{-3}$ | 53.77 | 0.0175 | 0.0988 | 96.39 |
| $2^6$ | $5.1\cdot10^{-5}$ | $4.1\cdot10^{-4}$ | 0.0153 | 0.1227 | $1.2\cdot10^{-4}$ | $9.9\cdot10^{-4}$ | 60.72 | 0.0108 | 0.0868 | 97.29 |
| $2^7$ | $7.5\cdot10^{-6}$ | $8.5\cdot10^{-5}$ | 0.0130 | 0.1475 | $6.5\cdot10^{-5}$ | $7.3\cdot10^{-4}$ | 69.00 | 0.0087 | 0.0981 | 97.92 |
| $2^8$ | $9.7\cdot10^{-6}$ | $1.6\cdot10^{-4}$ | 0.0096 | 0.1533 | $3.4\cdot10^{-5}$ | $5.5\cdot10^{-4}$ | 76.70 | 0.0059 | 0.0939 | 98.46 |
| $2^9$ | $1.4\cdot10^{-5}$ | $3.2\cdot10^{-4}$ | 0.0072 | 0.1629 | $1.8\cdot10^{-5}$ | $4.0\cdot10^{-4}$ | 83.80 | 0.0049 | 0.1113 | 98.78 |
| $2^{10}$ | $1.4\cdot10^{-5}$ | $4.4\cdot10^{-4}$ | 0.0051 | 0.1625 | $8.7\cdot10^{-6}$ | $2.7\cdot10^{-4}$ | 87.80 | 0.0035 | 0.1114 | 99.13 |

Table 4
The $T$-values of the methods with $p=-0.25$

| | $q=0.2$ | $q=0.6$ | $q=1.0$ | $q=1.6$ | $q=2.4$ |
|---|---|---|---|---|---|
| Semi-implicit method | 0.7831 | 0.6428 | 0.6500 | 0.8156 | 1.1085 |
| Composite method of type 2 | 0.7805 | 0.6395 | 0.5974 | 0.5604 | 0.4854 |

different $b$. The linear test equation is solved with a fixed step size $h=0.05$, so $p=-0.25$. Table 4 lists some $T$-values, calculated by (11), of the semi-implicit Euler method and the composite Euler method of type 2 with $p=-0.25$ and different $q$. In the first column of Table 5, we give means of absolute errors with $q=0.6$. The two methods are all stable in this case. The accuracy of the numerical results of these two methods is nearly the same order. In the second column of Table 5 we give numerical results with $q=1.6$. The composite method is stable, but we cannot get stable results by the semi-implicit Euler method although $p$ and $q$ are in the $T(0.9)$-stability region of the semi-implicit Euler method. This is in accord with the definition of $T(A)$-stability. We can obtain

Table 5
Average errors with $p = -0.25$

| | $q = 0.6$ | | $q = 1.6$ | |
|---|---|---|---|---|
| | Semi-implicit | Composite (type 2) | Semi-implicit | Composite (type 2) |
| $t_1 = 1$ | 0.0152 | 0.0160 | 34.66 | 0.0177 |
| $t_1 = 2$ | $2.01 \cdot 10^{-4}$ | $1.29 \cdot 10^{-4}$ | 477.08 | $5.74 \cdot 10^{-5}$ |
| $t_1 = 3$ | $1.53 \cdot 10^{-6}$ | $1.78 \cdot 10^{-6}$ | $1.03 \cdot 10^{3}$ | $1.10 \cdot 10^{-7}$ |
| $t_1 = 4$ | $1.73 \cdot 10^{-8}$ | $8.59 \cdot 10^{-9}$ | $1.81 \cdot 10^{5}$ | $4.61 \cdot 10^{-9}$ |
| $t_1 = 5$ | $2.69 \cdot 10^{-10}$ | $5.21 \cdot 10^{-11}$ | $1.74 \cdot 10^{4}$ | $4.80 \cdot 10^{-14}$ |

Table 6
The percentages of the stable solutions

| | $\beta = 2.2$ | $\beta = 2.6$ | $\beta = 2.8$ | $\beta = 3$ | $\beta = 3.2$ |
|---|---|---|---|---|---|
| Semi-implicit | 99.0 | 86.8 | 70.5 | 45.3 | 17.5 |
| Composite (type 2) | 98.7 | 91.7 | 84.5 | 75.0 | 62.0 |

stable results by the semi-implicit Euler method at $t_1 = 15$ with the same $p$ and $q$. In this case, the number of time steps is 300.

For nonlinear SDEs, we consider numerical solutions of the third test equation (Itô's form) with $h = 0.05$, $\alpha = -10$ and different $\beta$. This approach is different from that in practical computation. In practical computation we would choose a suitable step size $h$ for the given $\alpha$ and $\beta$ in order to get numerical solutions with satisfied accuracy.

Table 6 gives percentages of stable solutions obtained by the semi-implicit Euler method and the composite Euler method of type 2. All of the data in this table are based on 5000 simulated trajectories. We can see the stability property of the composite Euler method of type 2 is much better than that of the semi-implicit method for solving the third test equation.

## 6. SDEs driven by $d$-dimensional Wiener processes

In this section we consider the composite Euler method applied to the Itô SDEs driven by $d$-dimensional Wiener processes (1), leading to the numerical scheme

$$y_{n+1} = y_n + f(t_{n+1}, y_{n+1})h + \sum_{j=1}^{d} [\lambda_{nj} g_j(t_n, y_n) + (1 - \lambda_{nj}) g_j(t_{n+1}, y_{n+1})] \Delta W_{nj}. \tag{15}$$

We will give the criterion for selecting $\lambda_{nj}$ and analyse the stability properties of the composite Euler method (15) for linear scalar multiplicative problems.

Applying the composite Euler method (15) to the linear scalar multiplicative test equation

$$\mathrm{d}y = ay\,\mathrm{d}t + \sum_{j=1}^{d} b_j y\,\mathrm{d}W_j(t) \tag{16}$$

gives

$$y_{n+1} = y_n + a y_{n+1} h + \sum_{j=1}^{d} (\lambda_{nj} b_j y_n + (1 - \lambda_{nj}) b_j y_{n+1}) \Delta W_{nj},$$

so that

$$y_{n+1} = \frac{1 + \sum_{j=1}^{d} \lambda_{nj} q_j I_{nj}}{1 - p - \sum_{j=1}^{d} (1 - \lambda_{nj}) q_j I_{nj}} y_n,$$

where $p = ah$, $q_j = b_j \sqrt{h}$, $\Delta W_{nj} = \sqrt{h} I_{nj}$ and $I_{nj}$ is the $n$th realization of the standard normal random variable $I_j$.

Let

$$Q_{\lambda_{n1} \cdots \lambda_{nd}} = \frac{1 + \sum_{j=1}^{d} \lambda_{nj} q_j I_{nj}}{1 - p - \sum_{j=1}^{d} (1 - \lambda_{nj}) q_j I_{nj}},$$

then similar to the analysis in Section 2, the following result holds, namely

$$\min_{0 \leqslant \lambda_{nj} \leqslant 1} |Q_{\lambda_{n1} \cdots \lambda_{nd}}| = \min_{\lambda_{nj} = 0,1} \{ |Q_{\lambda_{n1} \cdots \lambda_{nd}}| \}.$$

For SDEs driven by two-dimensional Wiener processes, for example, the above result is

$$\min_{0 \leqslant \lambda_1, \lambda_2 \leqslant 1} |Q_{\lambda_1 \lambda_2}| = \min\{ |Q_{00}|, |Q_{01}|, |Q_{10}|, |Q_{11}| \}.$$

In step $n$, if a sequence $\lambda_{01}, \lambda_{02}, \ldots, \lambda_{0d}$ satisfies

$$\min_{\lambda_{nj} = 0,1} \{ |Q_{\lambda_{n1} \cdots \lambda_{nd}}| \} = |Q_{\lambda_{01} \cdots \lambda_{0d}}|, \tag{17}$$

we can choose

$$\lambda_{nj} = \lambda_{0j}, \quad j = 1, 2, \ldots, d. \tag{18}$$

Next, we consider the stability properties of the composite Euler method (15) applied to the SDEs driven by two-dimensional Wiener processes.

For the MS-stability property of the composite Euler method, similar to (8), we can define a function

$$f(p, q_1, q_2, x_1, x_2) = \frac{1 + \lambda_1(x_1) q_1 x_1 + \lambda_2(x_2) q_2 x_2}{1 - p - (1 - \lambda_1(x_1)) q_1 x_1 - (1 - \lambda_2(x_2)) q_2 x_2},$$

where $\lambda_1(x_1)$ and $\lambda_2(x_2)$ are defined by

$$(\lambda_1(x_1), \lambda_2(x_2)) = (\lambda_{01}, \lambda_{02})$$

and $\lambda_{01}$ and $\lambda_{02}$ are determined by (17) and (18). The MS-stability function of the composite Euler method is given by

$$\bar{R}_3 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^2(p, q_1, q_2, x_1, x_2) \frac{1}{2\pi} \mathrm{e}^{-(x_1^2 + x_2^2)/2} \, \mathrm{d}x_1 \, \mathrm{d}x_2$$

and the composite Euler method is MS-stable if $\bar{R}_3 < 1$.

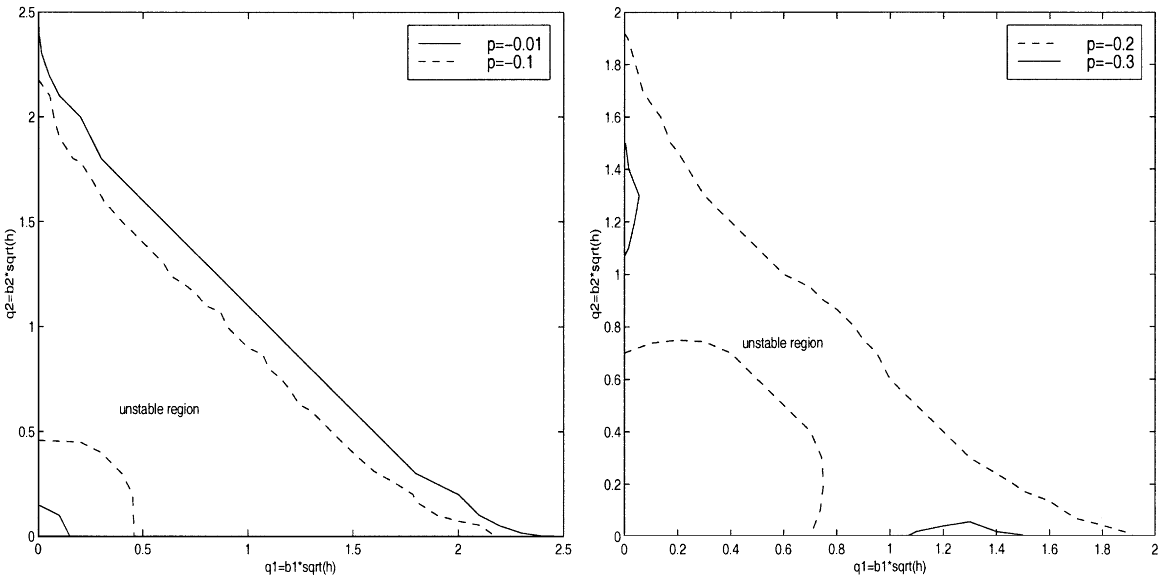


Fig. 4. MS-stability regions of the composite method.

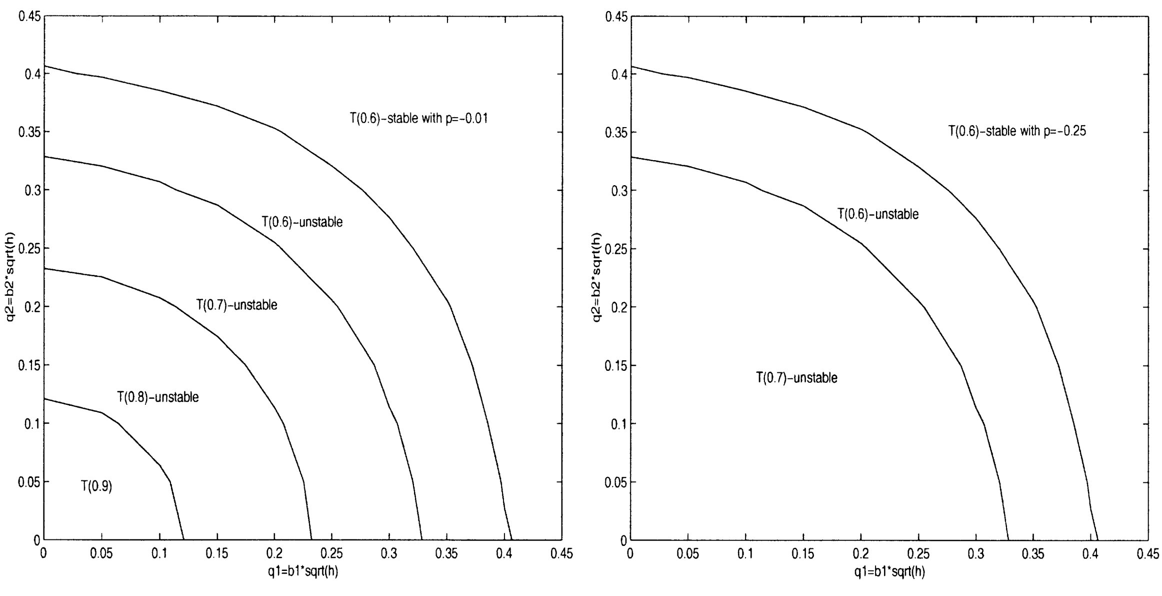


Fig. 5. $T(A)$-stability regions of the composite method.

Fig. 4 gives MS-stability regions of the composite Euler method with different $p$. The MS-unstable regions of the composite Euler method are those which are enclosed by the plotted line and axes. When $p=-0.3$, the MS-unstable region (two seperated regions) is very small. When $p<-0.33$, the composite Euler method is MS-stable for any $q_1$ and $q_2$.

Table 7
The means of the absolute errors with $p = -0.25$

| | Case 1 | | Case 2 | |
|---|---|---|---|---|
| | Semi-implicit | Composite | Semi-implicit | Composite |
| $t_1 = 1$ | $1.03 \cdot 10^{-2}$ | $5.54 \cdot 10^{-3}$ | 0.592 | $1.56 \cdot 10^{-3}$ |
| $t_1 = 2$ | $1.42 \cdot 10^{-4}$ | $5.89 \cdot 10^{-5}$ | 3.024 | $4.19 \cdot 10^{-7}$ |
| $t_1 = 3$ | $1.31 \cdot 10^{-6}$ | $2.65 \cdot 10^{-7}$ | 0.159 | $7.18 \cdot 10^{-10}$ |
| $t_1 = 4$ | $9.61 \cdot 10^{-10}$ | $1.84 \cdot 10^{-10}$ | $1.09 \cdot 10^{-3}$ | $8.99 \cdot 10^{-16}$ |
| $t_1 = 5$ | $2.74 \cdot 10^{-12}$ | $2.8 \cdot 10^{-13}$ | $3.39 \cdot 10^{-6}$ | $8.75 \cdot 10^{-23}$ |

Fig. 5 gives $T(A)$-stability regions of the composite Euler method with $p = -0.01$ and $-0.25$, respectively. The areas enclosed by the solid line and axes are the unstable regions of the composite Euler method with different $A$. The $T(A)$-stability property of the composite Euler method is better if any of $-p$, $q_1$, $q_2$ is larger. For any $q_1, q_2$, the $T$-value is always less than 0.6 when $p < -\frac{2}{3}$.

Table 7 gives two typical cases of numerical results with $p = -0.25$. Here a fixed step size $h = 0.05$ is used. The T-values of the semi-implicit Euler method and the composite Euler method are denoted by $T_{\text{semi}}$ and $T_{\text{composite}}$, respectively.

*Case* 1: $q_1 = 0.4472$, $q_2 = 0.2236$. Then $T_{\text{semi}} = 0.6799$ and $T_{\text{composite}} = 0.6800$. At $t_1 = 5$, the semi-implicit and composite Euler methods are stable as the number of the time steps is 100. The accuracy of these two methods is nearly the same.

*Case* 2: $q_1 = 1$, $q_2 = 0.6633$. Then $T_{\text{semi}} = 0.6902$ and $T_{\text{composite}} = 0.5474$. In this case we can obtain stable results by the semi-implicit and the composite Euler methods at $t_1 = 5$. The accuracy of the composite Euler method is better than that of the semi-implicit Euler method as the $T$-value of the composite method is smaller.

## 7. Conclusions

In this paper, we have constructed the composite Euler method by combining the semi-implicit Euler method and the implicit Euler method. At each step an SDE is solved by either the semi-implicit Euler method or the implicit Euler method according to the characteristics of the SDE. For the linear test equation, the convergence properties of the composite Euler method depend on the criterion for choosing parameter $\lambda_n$. For nonlinear equations, the numerical results suggest that the convergence properties of the composite Euler method applied to the nonlinear SDEs is the same as those applied to linear equations. The theoretical analysis and the numerical results show that the composite Euler method is a very promising method.

Future work should be based on the construction of high-order composite stochastic Runge–Kutta methods. As nearly all of the existing stochastic Runge–Kutta methods are explicit and semi-implicit methods, we should construct the corresponding semi-implicit and/or implicit stochastic Runge–Kutta methods, study the stability properties of these methods and then consider the optimal combination of these methods. These topics will be the subjects of future work.

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