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Quadratic form theory over preordered von Neumann-regular rings

M. Dickmann [a], F. Miraglia [b,*]

[a] *Équipe de Logique Mathématique Université de Paris VII, and Projet Topologie et Géométrie Algébriques, Institut de Mathématiques de Jussieu, Paris, France*

[b] *Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, C.P. 66.281 (Ag. Cidade de São Paulo), 05311-970 S. Paulo, S.P., Brazil*

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Abstract

To each pair, $\langle R, T\rangle$, consisting of a unitary commutative von Neumann-regular ring, R, where 2 is a unit and T is a preorder on R, we associate a reduced special group, $G_T(R)$, which faithfully reflects quadratic form theory, modulo T, over free R-modules and then show, using the representation of R as the ring of global sections of its affine scheme, together with results from [M. Dickmann, F. Miraglia, On quadratic forms whose total signature is zero mod 2^n. Solution to a problem of M. Marshall, Invent. Math. 133 (1998) 243–278; M. Dickmann, F. Miraglia, Lam's Conjecture, Algebra Colloq. 10 (2003) 149–176; M. Dickmann, F. Miraglia, Algebraic K-theory of special groups, J. Pure Appl. Algebra 204 (2006) 195–234], that $G_T(R)$ satisfies a powerful K-theoretic property, the [SMC]-property. From this we conclude that quadratic form theory modulo T over free R-modules verifies Marshall's signature conjecture, Lam's conjecture, as well as a reduced version of Milnor's Witt ring conjecture.

Keywords: Preordered rings; von Neumann-regular rings; Special groups; Rings with many units; Algebraic theory of quadratic forms; Algebraic K-theory of rings; Marshall's signature conjecture; Milnor's Witt ring conjecture

The main result of this paper is that, if R is a commutative von Neumann-regular ring in which 2 is a unit, then the reduced theory of quadratic forms with invertible coefficients in R,

* Corresponding author.
E-mail addresses: dickmann@logique.jussieu.fr (M. Dickmann), miraglia@ime.usp.br (F. Miraglia).

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modulo a proper preorder T, satisfies Marshall's signature conjecture and Milnor's Witt ring conjecture (for precise statements, see Section 1 below). The proof takes place in the setting of special groups (abbreviated SG), presented in [4] (see also Section 2 of [3]), and uses the K-theory of those structures, developed in [5] and [8]. Since von Neumann-regular rings (hereafter called vN-rings) are a natural generalization of fields (intuitionistically, *they are fields*) our result does generalize results known in the latter case. Although not explicitly stated, we observe that, by Lemma 6.1, pp. 172–173 of [5], our result also proves Lam's Conjecture for *formally real* vN-rings (i.e., those in which -1 is not a sum of squares) where 2 is invertible.

To a pair $\langle R, T\rangle$ as above, we associate a reduced special group (abbreviated RSG), $G_T(R) = R^*/T^*$ (R^* = units of R). A result from [7] (Theorem 3.16, pp. 17–18) shows that, under these conditions—in fact, under the considerably more general situation where R is a ring with many units—$G_T(R)$ faithfully reflects the reduced theory of quadratic forms modulo T, over free R-modules.

The technique used to prove the stated result can be summarized as follows:

(1) Marshall's signature conjecture was proved in [3] for Pythagorean fields, and in [5] for formally real fields modulo an arbitrary (proper) preorder. For this kind of fields, modulo sums of squares, the problem was posed by Lam in 1976. Our proofs use the theory of SGs, depending on results of Voevodsky (and of Orlov–Vishik–Voevodsky in the latter case), to conclude.
(2) For fields of characteristic zero, Milnor's Witt ring conjecture is a celebrated result of Voevodsky's.
(3) An analysis of our proofs in the field case shows that, in fact, we establish the validity of a powerful K-theoretic property—the [SMC] property—which implies Marshall's signature conjecture. This property was explicitly formulated in [5] (Definition 4.3, p. 168), but occurs without a name already in [3] (Corollary 6.5, p. 275). The [SMC] property asserts, in the abstract context of RSGs, the analog of injectivity of Milnor's "multiplication by $\ell(-1)$" map at each level of the graded mod 2 K-theory ring. Using results in [3] and [5] we prove below (Lemma 1.2) that the [SMC] property is *equivalent*, for arbitrary RSGs, to the *conjunction* of Marshall's signature conjecture and Milnor's Witt ring conjecture.
(4) In view of the foregoing observations, our efforts are directed at proving the [SMC] property for the RSG $G_T(R)$ associated to a pair $\langle R, T\rangle$ as above. To achieve this we use the well-known representation (originally due to Pierce [17]) of a vN-ring R as the ring of global sections of a (pre-)sheaf of rings over the Boolean space $\mathrm{Spec}(R)$ whose stalks are fields (this representation is just the affine scheme of R). The presence of a (proper) preorder T on R forces at least one of the stalks to be preordered by the corresponding image of T. By considering a suitable quotient of R the situation gets reduced to the case where *all* the stalks are (properly) preordered (Proposition 6.8). By Theorem 6.4 of [5] the [SMC] property holds, then, for the RSG associated to each stalk of the sheaf representation of R. Having previously established that:

(i) the special group construction induces a (pre-)sheaf of RSGs on $\mathrm{Spec}(R)$ (Theorem 6.7), and
(ii) the RSG of global sections of a sheaf over a Boolean space whose stalks have the [SMC] property is also [SMC] (Theorem 7.1(c)),

we conclude that its SG of global sections—which is just $G_T(R)$—has the [SMC] property.

Crucial to these arguments are the facts that:

(iii) The K-theory functor on special groups is geometrical (Proposition 2.7(1)), and
(iv) In any presheaf of first-order structures the ring of sections embeds in the product of the stalk structures (Proposition 3.2(c)).

Some ingredients of our proof are valid in more general contexts, and so it seemed appropriate to register them, with moderate extra effort, at that level of generality.

In Sections 2 and 3 we review some notions and prove a few results about geometrical functors and presheaves of first-order theories repeatedly used in the rest of the paper. Some of the basic material on which these proofs rely call for a revision, as the classical literature on sheaf theory deals with presheaves of *algebraic* structures, where no relation other than equality is present in the language; this is briefly done in Section 8.A.

Section 4 deals with rings with many units, a class of rings previously considered in the literature, larger than that of vN-rings. Under mild additional assumptions—namely that 2 is a unit and that all residue fields have cardinality $\geqslant 7$—quadratic form theory via special groups faithfully reflects, for this class of rings, quadratic form theory over free modules ([7], Theorem 3.16). Furthermore, under these conditions, we adapt the K-theory in [11] to our setting, showing that the ensuing mod 2 K-theory is isomorphic to the K-theory of the associated SG (Theorem 4.12). This section also contains a model-theoretic criterion for a subring to inherit the property of having many units (Proposition 4.3).

In Section 5 we prove the basic properties of preorders in vN-rings, emphasizing their connection with the presheaf representation of rings of this type.

Our main result in Section 6 proves that the functor assigning to each preordered ring a (suitable fragment of a) reduced special group is a geometrical functor (Proposition 6.4). The notion of a proto-SG singles out those axioms satisfied by the SG construction as applied to *arbitrary* preordered rings (8.9). In the case of preordered vN-rings this construction produces a full-fledged special group.

In Section 7 we prove the result mentioned in item (ii) above, and obtain our main result Theorem 7.2. As already mentioned, Section 8 summarizes background material that may help the reader to follow the main arguments in the paper.

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1. The [SMC] property for special groups

1.1. Notation and remarks. Let $G = \langle G, \equiv_G, -1\rangle$ be a special group (SG) and write D_G for the representation relation in G.

(1) The K-theory of G, introduced in [5], is the graded $\mathbb{F}_2$-algebra, $k_*G = \langle \mathbb{F}_2, k_1G, \ldots, k_nG, \ldots\rangle$, constructed as follows:

$*$ k_1G is G written additively, that is, we fix an **isomorphism**

$$\lambda : G \to k_1G, \quad \text{with } \lambda(ab) = \lambda(a) + \lambda(b).$$

In particular, $\lambda(1)$ is the zero of k_1G and k_1G has exponent 2, i.e., for $a \in G$, $\lambda(a) = -\lambda(a)$;

* k_*G is the quotient of the graded tensor algebra

$$\langle \mathbb{F}_2, k_1G, \ldots, \underbrace{k_1G \otimes \cdots \otimes k_1G}_{n\ \text{times}}, \ldots \rangle$$

over $\mathbb{F}_2$, by the ideal generated by $\{\lambda(a)\lambda(ab)\colon a \in D_G(1,b)\}$. Thus, for each $n \geqslant 2$, k_nG is the quotient of the n-fold tensor product $k_1G \otimes \cdots \otimes k_1G$ over $\mathbb{F}_2$, by the subgroup consisting of finite sums of elements of the type $\lambda(a_1)\cdots\lambda(a_n)$, where for some $1 \leqslant i \leqslant n-1$ and $b \in G$, we have $a_{i+1} = a_i b$ and $a_i \in D_G(1,b)$. An element of the type $\lambda(x_1)\cdots\lambda(x_n)$ is called a **generator** of k_nG.
* There is a graded ring morphism of degree 1, $\lambda(-1)\,(\cdot)\colon k_nG \to k_{n+1}G$, taking $\eta \in k_nG$ to $\lambda(-1)\eta \in k_{n+1}G$. A special group is [**SMC**] if for all $n \geqslant 1$, multiplication by $\lambda(-1)$ is an injection. Any [SMC] special group is *reduced* ([5, Lemma 6.2, p. 173]).
* A SG-morphism, $f\colon G \to H$, induces a morphism of degree 0 of graded $\mathbb{F}_2$-algebras

$$f_*\colon k_*G \to k_*H,$$

$f_* = \{f_n\colon n \geqslant 0\}$, where $f_0 = Id_{\mathbb{F}_2}$ and for $n \geqslant 1$, $f_n\colon k_nG \to k_nH$ is the unique group morphism whose value on generators is given by $f_n(\lambda(a_1)\cdots\lambda(a_n)) = \lambda(f(a_1))\cdots\lambda(f(a_n))$.

(2) Let $W(G)$ be the Witt ring of G and let $I(G)$ be the fundamental ideal in $W(G)$. For $n \geqslant 0$, set

$$\overline{I^n}(G) = I^n(G)/I^{n+1}(G),$$

where $I^0(G) = W(G)$. The sequence, $W_g(G) = \langle \mathbb{F}_2, \ldots, \overline{I^n}(G), \ldots \rangle$ is, as usual, the *graded Witt ring of* G. In [5] we constructed a graded ring morphism

$$s_* = (s_n)_{n \geqslant 0}\colon k_*(G) \to W_g(G),$$

such that for each $n \geqslant 0$, the following diagram is commutative, where $\otimes 2$ indicates product by the Pfister form $2 = \langle 1, 1 \rangle$:

$$\begin{tikzcd} k_n(G) \arrow[r, "\otimes\lambda(-1)"] \arrow[d, "s_n"'] & k_{n+1}(G) \arrow[d, "s_{n+1}"] \\ \overline{I^n}(G) \arrow[r, "\otimes 2"'] & \overline{I^{n+1}}(G) \end{tikzcd} \qquad \text{(D)}$$

The special group G is [MWRC], i.e., satisfies **Milnor's Witt Ring Conjecture**, if s_n is an isomorphism for all $n \geqslant 0$; it is shown in [5] that this holds for $n \leqslant 2$.

(3) G is [MC] if it satisfies **Marshall's signature conjecture**, that is, for all $n \geqslant 1$ and all forms, φ, over G, if the total signature of φ is congruent to zero mod 2^n, then $\varphi \in I^n(G)$; any such group must be *reduced*. For a detailed account of this property, see [3] and [6].

The relation between properties [SMC], [MC] and [MWRC] is described by

Lemma 1.2. *If G is a reduced special group, then*

$$G \text{ is [SMC]} \quad \textit{iff} \quad G \text{ is [MC] } \textit{and} \text{ [MWRC]}.$$

Proof. By Theorem 5.1 in [3], G is [MC] iff the map "multiplication by $2 = \langle 1, 1\rangle$" from $\overline{I^n}(G)$ to $\overline{I^{n+1}}(G)$ is injective. Hence, if G is [MWRC] and [MC], the commutative diagram (D) above entails that multiplication by $\lambda(-1)$ is injective in all degrees, that is, G is [SMC]. Conversely, by Corollary 4.2 in [5], every [SMC]-group is [MWRC] and once again the commutativity of diagram (D) above entails that multiplication by $\langle 1, 1\rangle$ in the graded Witt ring of G is injective in all degrees. Another application of Theorem 5.1 in [3] guarantees that G is [MC]. □

2. Geometric theories and functors

We assume the reader is familiar with first-order languages, their structures and morphisms. Standard references are [1] and [14]. For the convenience of the reader, we recall:

Definition 2.1. Let L be a first-order language with equality.

Let A, B be L-structures, let $f : A \to B$ be a map and let $\varphi(v_1, \ldots, v_n)$ be a formula of L in the free variables $\bar{v} = \langle v_1, \ldots, v_n\rangle$. For $\bar{a} = \langle a_1, \ldots, a_n\rangle \in A^n$, write $f(\bar{a})$ for $\langle f(a_1), \ldots, f(a_n)\rangle \in B^n$.

(a) f **preserves φ** if for all $\bar{a} \in A^n$, $A \models \varphi[\bar{a}] \Rightarrow B \models \varphi[f(\bar{a})]$; f **reflects φ** if the reverse implication holds.
(b) If f is a L-morphism, we say that ***A* is positively existentially closed in *B* along *f*** if f reflects all positive existential L-formulas. Whenever A is a substructure of B and f is the inclusion, we say that ***A* is positively existentially closed in *B***.
(c) Let ***L*-mod** be the category of L-structures and L-morphisms. If Σ is a set of sentences in L, write **Σ-mod** for the subcategory of ***L*-mod** whose objects are the models of Σ.
(d) A formula of L, in the free variables $\bar{t}$, is **geometrical** if it is the negation of an atomic formula **or** a formula of the form $\forall\bar{v}(\varphi_1(\bar{v}; \bar{t}) \to \exists\bar{w}\varphi_2(\bar{v}; \bar{w}; \bar{t}))$, where φ_1, φ_2 are positive and quantifier-free. A **geometrical theory in *L*** is a theory possessing a set of geometrical axioms.
(e) A formula in L is **positive primitive (pp-formula)** if it is of the form $\exists\bar{v}\varphi(\bar{v}; \bar{t})$, where φ is a conjunction of atomic formulas.

Example 2.2.

(a) The theory of groups of exponent 2 is geometrical. Write **2-Grp** for the category of groups of exponent 2.
(b) The theory of unitary commutative rings ($1 \neq 0$) is geometrical. Write **UCR** for the category of unitary commutative rings.
(c) The theory of special groups and of reduced special groups are both geometrical theories. The axioms for special groups (see Definition 1.2, [4], for details) include, in addition to the axioms [SG0]–[SG3] and [SG5] for proto-special groups (see 6.3, below), the sentences
[SG4]: $\forall a, b, c, d((\langle a, b\rangle \equiv_G \langle c, d\rangle) \Rightarrow (\langle a, -c\rangle \equiv_G \langle -b, d\rangle))$;
[SG6]: The isometry of forms of dimension 3 is transitive,

all geometrical sentences. For reducibility, we add

* $1 \neq -1$; [red]: $\forall a((\langle a,a\rangle \equiv_G \langle 1,1\rangle) \Rightarrow a = 1)$.

2.3. Reminder. We assume familiarity with the notions and the basic properties of inductive systems of first-order structures over a right-directed partially ordered set (i.e., $\forall i, j \in I$, $\exists k \in I$ such that $i, j \leqslant k$; hereafter called a **rd-poset**) and of colimits (a.k.a. inductive limits) of such systems. Our notation for these objects is standard and we write

$$M = \varinjlim \mathcal{M} = \varinjlim_{i \in I} \mathcal{M}_i$$

to indicate that M is an inductive limit of $\mathcal{M}$. If $\mathcal{M}, \mathcal{N} : \langle I, \leqslant\rangle \to$ **L-mod** are inductive systems of first-order structures, $\mathcal{M} = \langle \mathcal{M}_i; \{\mu_{ij}\colon i \leqslant j \text{ in } I\}\rangle$, $\mathcal{N} = \langle \mathcal{N}_i; \{\nu_{ij}\colon i \leqslant j \text{ in } I\}\rangle$, then:

* A **dual cone** over $\mathcal{M}$ is a system $\langle A, \{\alpha_i\colon i \in I\}\rangle$, where A is a L-structure and $\alpha_i : \mathcal{M}_i \to A$ are L-morphisms, such that for all $i \leqslant j$ in I, $\alpha_j \circ \mu_{ij} = \alpha_i$;
* A **morphism**, $\eta : \mathcal{M} \to \mathcal{N}$, is a family of L-morphisms, $\eta = \{\mathcal{M}_i \xrightarrow{\eta_i} \mathcal{N}_i\colon i \in I\}$, such that for all $i \leqslant j$ in I, we have $\eta_j \circ \mu_{ij} = \nu_{ij} \circ \eta_i$.

For ready reference we recall:

Fact 2.4. *Let $\mathcal{M} : \langle I, \leqslant\rangle \to L$-mod be an inductive system of L-structures.*

(a) *A dual cone over $\mathcal{M}$, $\langle M, \{\mu_i\colon i \in I\}\rangle$, is* (isomorphic to) $\varinjlim \mathcal{M}$ *iff it verifies*:
  (1) $M = \bigcup_{i \in I} \mu_i(\mathcal{M}_i)$;
  (2) *If $\varphi(v_1, \ldots, v_n)$ is an atomic formula in L, $i \in I$, and $\langle s_1, \ldots, s_n\rangle \in \mathcal{M}_i^n$, then*

$$M \models \varphi\big[\mu_i(s_1), \ldots, \mu_i(s_n)\big] \Rightarrow \begin{cases} \exists k \in I \text{ such that } k \geqslant i \text{ and} \\ \mathcal{M}_k \models \varphi[\mu_{ik}(s_1), \ldots, \mu_{ik}(s_n)]. \end{cases}$$

(b) *Let $\psi(v_1, \ldots, v_n)$ be a disjunction of geometric formulas in L and let $M = \varinjlim \mathcal{M}$. For $i \in I$, let $\bar{s} \in \mathcal{M}_i^n$ and $S_\psi = \{k \in I\colon k \geqslant i$ and $\mathcal{M}_k \models \psi[\mu_{ik}(\bar{s})]\}$. If S_ψ is cofinal in I, then $M \models \psi[\mu_i(\bar{s})]$.*
(c) *(Colimit of morphisms) Let $\mathcal{N} = \langle \mathcal{N}_i; \{\nu_{ij}\colon i \leqslant j \text{ in } I\}\rangle$ be an inductive system of L-structures over I and let $\eta = \{\eta_i\colon i \in I\}$ be a morphism from $\mathcal{M}$ to $\mathcal{N}$. Then, there is a unique L-morphism, $\varinjlim \eta : \varinjlim \mathcal{M} \to \varinjlim \mathcal{N}$, such that $(\varinjlim \eta) \circ \mu_i = \nu_i \circ \eta_i$, for all $i \in I$.*

Remark 2.5. 2.4(b) shows that colimits of models of a geometrical theory are also models of that theory. Moreover, it is straightforward that a geometrical theory is preserved under the product of a non-empty family of its models.

Definition 2.6. A covariant functor, $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$, $\mathcal{D}$ are categories, is **geometrical** if it preserves finite products and right-directed colimits, i.e., if these constructions exist in $\mathcal{C}$, then they exist in $\mathcal{D}$ and F takes one to the other. (We shall mostly use this in the case where $\mathcal{C}$, $\mathcal{D}$ are categories of models of first-order theories.)

Here are some examples of geometrical functors. Others will arise in the sections that follow.

Proposition 2.7. *The following are geometrical functors*:

(1) *The K-theory functor of special groups, that is, for each $n \geqslant 0$, the functor from* **SG** *to* **2-Grp**, *the category of groups of exponent* 2, *given by*

$$\begin{cases} G \longmapsto k_n G; \\ f : G \to H \longmapsto f_n : k_n G \to k_n H; \end{cases}$$

(2) *The Witt-ring functor, $W : \mathbf{SG} \to \mathbf{AWR}$ (the category of abstract Witt rings,* [16]), *which to each special group associates its Witt ring* (see [4], 1.25, pp. 19–20);

(3) *The graded Witt-ring functor, i.e., for each $n \geqslant 0$, the functor from* **SG** *to* **2-Grp**, *given by*

$$\begin{cases} G \longmapsto I^n G; \\ f : G \to H \longmapsto W_n(f) : I^n G \to I^n H, \end{cases}$$

where, for

$$\varphi = \sum_{i=1}^{m} \bigotimes_{j=1}^{n} \langle 1, a_{ij} \rangle \in I^n(G),$$

$$W_n(f)\big(\varphi \bmod I^{n+1}(G)\big) = \left(\sum_{i=1}^{m} \bigotimes_{j=1}^{n} \big\langle 1, f(a_{ij}) \big\rangle \right) \bmod I^{n+1}(H).$$

Proof. For (1), Theorems 4.5 and 5.1 in [8] guarantee the preservation of right-directed colimits and of finite products, respectively. In [2] it is shown that the functor W in (2) is, in fact, an equivalence of categories. Item (3) follows from Proposition 3.1, Theorem 3.3 and Proposition 3.5 in [6]. □

3. Presheaves of first-order structures

In this section we follow the lead set by [9] concerning sheaves of *relational* structures. As a rule the existing literature—e.g., [12,13,18] and [15]—deals with sheaves of *algebraic* structures (such as groups, rings, vector spaces, etc.) where only operation and constant symbols occur, besides equality. However, we need to deal with the more general case where relations other than equality are also present. Though the basic results in this case do not differ significantly from those of the case of purely algebraic structures, some care has to be exerted. In order to keep our arguments in focus, we have included in Section 8.A a summary of results on sheaves of relational structures that we will need, insofar these differ from the classical ones. All notation employed in this section is described in 8.1, 8.2, 8.5 and 8.6.

Definition 3.1. Let $\mathcal{B}$ be a basis for the topological space X and let $\mathfrak{A}$ be a presheaf basis of L-structures over $\mathcal{B}$. If $\varphi(v_1, \ldots, v_n)$ is a formula of L, $\bar{s} = \langle s_1, \ldots, s_n \rangle \in |\mathfrak{A}|^n$ and $\mathfrak{A}_x$ is the stalk of $\mathfrak{A}$ at x, define

$$\mathfrak{v}_{\mathfrak{A}}\big(\varphi(\bar{s})\big) = \left\{ x \in \bigcap_{i=1}^{n} E s_i \colon \mathfrak{A}_x \models \varphi[s_{1x}, \ldots, s_{nx}] \right\},$$

called the **Feferman–Vaught value of φ at $\bar{s}$**. Whenever $\mathfrak{A}$ is clear from context, it will be omitted from the notation. In general, $\mathfrak{v}(\varphi(\bar{s}))$ **is not** an open set in X. Moreover, in view of item (b) in 8.5, for all $\bar{s} \in |\mathfrak{A}|^n$, $\mathfrak{v}_{\mathfrak{A}}(\varphi(\bar{s})) = \mathfrak{v}_{c\mathfrak{A}}(\varphi(\bar{s}))$, where $c\mathfrak{A}$ is the completion of $\mathfrak{A}$ over X.

Proposition 3.2. *Let $\mathcal{B}$ be a basis for the space X and let $\mathfrak{A}$ be a presheaf basis of L-structures over $\mathcal{B}$. Let $\varphi(v_1, \dots, v_n)$ be a L-formula and let $\bar{s} = \langle s_1, \dots, s_n\rangle \in |\mathfrak{A}|^n$. Set $E\bar{s} = \bigcap_{i=1}^n Es_i$.*

(a) *If φ is positive and quantifier free, then*

$$\mathfrak{v}\big(\varphi(\bar{s})\big) = \bigcup\big\{V \in \mathcal{B}\colon\ V \subseteq E\bar{s}\ \text{ and }\ \mathfrak{A}(V) \models \varphi[s_{1|V}, \dots, s_{n|V}]\big\}.$$

In particular, $\mathfrak{v}(\varphi(\bar{s}))$ is an open set in X (not necessarily in $\mathcal{B}$).

(b) *If φ is a conjunction of atomic formulas, then for all $U \in \mathcal{B}$,*

$$U \subseteq \mathfrak{v}\big(\varphi(\bar{s})\big)\ \Rightarrow\ \mathfrak{A}(U) \models \varphi[s_{1|U}, \dots, s_{n|U}].$$

(c) *For $U \in \mathcal{B}$, let $\Gamma(U) = \prod_{x\in U} \mathfrak{A}_x$ be the product L-structure of stalks of $\mathfrak{A}$ at the points of U (cf. Definition 8.6). Then, the map $\gamma^U : \mathfrak{A}(U) \to \Gamma(U)$, given by $\gamma^U(s) = \langle s_x\rangle_{x\in U}$ is a L-embedding, and hence preserves and reflects all quantifier-free L-formulas.*

(d) *Suppose X is Hausdorff and that $\mathcal{B}$ is a Boolean algebra (BA) of clopens in X. If $U \in \mathcal{B}$ is* compact *and $\mathfrak{A}$ is* finitely complete *over U, then γ^U reflects geometric sentences with parameters in $\mathfrak{A}(U)$ (cf. Definitions* 8.2(d) *and* 2.1(d)*).*

Proof. (a) If $\psi_1(v_1, \dots, v_n)$ and $\psi_2(v_1, \dots, v_n)$ are L-formulas and $\bar{s} \in |\mathfrak{A}|^n$, it is clear that

$$\mathfrak{v}\big([\psi_1 \wedge \psi_2](\bar{s})\big) = \mathfrak{v}\big(\psi_1(\bar{s})\big) \cap \mathfrak{v}\big(\psi_2(\bar{s})\big) \quad \text{and} \quad \mathfrak{v}\big([\psi_1 \vee \psi_2](\bar{s})\big) = \mathfrak{v}\big(\psi_1(\bar{s})\big) \cup \mathfrak{v}\big(\psi_2(\bar{s})\big), \quad (\mathfrak{v})$$

and so, it is enough to verify the statement for atomic formulas. Suppose $\varphi(v_1, \dots, v_n)$ is an atomic L-formula and $\bar{s} \in |\mathfrak{A}|^n$. By Lemma 8.8(a), if $\mathfrak{A}_x \models \varphi[s_{1x}, \dots, s_{nx}]$, there is $V \in \mathcal{B}_x$ with $V \subseteq E\bar{s}$ and $\mathfrak{A}(V) \models \varphi[s_{1|V}, \dots, s_{n|V}]$. For $y \in V$, the germ maps, $\alpha_{Vy} : \mathfrak{A}(V) \to \mathfrak{A}_y$ are L-morphisms, and hence preserve atomic formulas. But this entails the displayed equality in (a), as needed.

(b) By the first equality in ($\mathfrak{v}$) above, it suffices to verify the statement for an atomic L-formula, φ. Suppose $U \subseteq_o \mathfrak{v}(\varphi(\bar{s}))$, with $U \in \mathcal{B}$. Then, for each $x \in U$, $\mathfrak{A}_x \models \varphi[s_{1x}, \dots, s_{nx}]$. By Lemma 8.8(a), there is $V \in \mathcal{B}_x$, with $V \subseteq E\bar{s}$, such that $\mathfrak{A}(V) \models \varphi[s_{1|V}, \dots, s_{n|V}]$. Let $V_x = V \cap U$; note that $V_x \in \mathcal{B}_x$. Moreover, since the restriction maps are L-morphisms, we also have

$$\mathfrak{A}(V_x) \models \varphi[s_{1|V_x}, \dots, s_{n|V_x}]. \quad \text{(I)}$$

Thus, we get a covering of U in $\mathcal{B}$, $\{V_x\colon\ x \in U\}$, with the property in (I). It now follows from the extensionality condition [ext] in Definition 8.2(a), that $\mathfrak{A}(U) \models \varphi[\bar{s}_{|U}]$, as desired.

(c) This is a consequence of item (b), upon verifying that γ^U reflects and preserves atomic L-formulas. Indeed, if $\psi(v_1, \dots, v_n)$ is an atomic L-formula and $\bar{t} = \langle t_1, \dots, t_n\rangle \in \mathfrak{A}(U)^n$, then the facts that the maps α_{Ux}, $x \in U$, are L-morphisms and that $\Gamma(U)$ has the product L-structure, immediately entail $\Gamma(U) \models \varphi[\gamma^U(\bar{t})]$. Conversely, if this relation holds, then $\mathfrak{v}(\varphi(\bar{t})) = U \in \mathcal{B}$, and item (b) guarantees that $\mathfrak{A}(U) \models \varphi[t_1, \dots, t_n]$.

(d) Let $L_{\mathfrak{A}(U)}$ be the language L augmented by constants from $\mathfrak{A}(U)$. We first show that γ^U reflects positive existential $L_{\mathfrak{A}(U)}$-sentences. It is well-known that a positive existential formula is logically equivalent to a disjunction of pp-formulas (as in 2.1(e)). Hence, it suffices to verify the statement for pp-sentences in $L_{\mathfrak{A}(U)}$. To simplify exposition, we shall also assume that such a pp-sentence has only one existential quantifier, i.e., it is of the form $\exists v\psi(v; t_1, \ldots, t_n)$, where ψ is a conjunction of atomic formulas in $L_{\mathfrak{A}(U)}$, whose parameters from $\mathfrak{A}(U)$ are $t_1, \ldots, t_n$. The reader will readily realize that the method extends straightforwardly to the general case. Moreover, write γ for the L-embedding γ^U (see (c)).

Suppose $\Gamma(U) \models \exists v\psi(v)[\gamma(\bar{t})]$; because $\Gamma(U)$ has the product L-structure, for every $x \in U$, $\mathfrak{A}_x \models \exists v\psi(v)[t_{1x}, \ldots, t_{nx}]$. Therefore, for each $x \in U$ there is $z_x \in \mathfrak{A}_x$ such that

$$\mathfrak{A}_x \models \psi[z_x; t_{1x}, \ldots, t_{nx}].$$

By Lemma 8.8(a), there is $V_x \in \mathcal{B}_x \subseteq U$ and $z(x) \in \mathfrak{A}(V_x)$ such that

$$\mathfrak{A}(V_x) \models \psi\bigl[z(x); t_{1|V_x}, \ldots, t_{n|V_x}\bigr]. \tag{II}$$

Since U is compact, there is a finite collection, $\{x_1, \ldots, x_m\} \subseteq U$ such that $\{V_{x_j}\colon 1 \leqslant j \leqslant m\}$ cover U. Since $\mathcal{B}$ is a BA, a standard argument yields disjoint clopens, $V_j \in \mathcal{B}$, $1 \leqslant j \leqslant m$, such that

$$V_j \subseteq V_{x_j} \quad \text{and} \quad U = \bigcup_{j=1}^{m} V_j. \tag{III}$$

Let $Z = \{z(x_j)_{|V_j}\colon 1 \leqslant j \leqslant m\}$; since their extents are disjoint, with union U, and $\mathfrak{A}(U)$ is finitely complete, there is $z \in \mathfrak{A}(U)$ such that $z_{|V_j} = z(x_j)_{|V_j}$, $1 \leqslant j \leqslant m$. Moreover, since $V_j \subseteq V_{x_j}$, (II) and the fact that ψ is a conjunction of atomic formulas entail

$$\text{For all } 1 \leqslant j \leqslant m, \quad \mathfrak{A}(V_j) \models \psi[z_{|V_j}; t_{1|V_j}, \ldots, t_{n|V_j}]. \tag{IV}$$

Now (III), (IV) and Remark 8.3(a) imply that $\mathfrak{A}(U) \models \psi[z; t_1, \ldots, t_n]$, i.e., $\exists v\psi(v; t_1, \ldots, t_n)$ holds in $\mathfrak{A}(U)$, as needed. To complete the proof, suppose that $\sigma(t_1, \ldots, t_n)$ is a geometric $L_{\mathfrak{A}(U)}$-sentence. If σ is the negation of an atomic sentence, reflection follows immediately from the fact that γ is an L-embedding. Let $\sigma(\bar{t})$ be a $L_{\mathfrak{A}(U)}$-sentence of the form $\forall\bar{v}(\varphi_1(\bar{v}; \bar{t}) \to \exists\bar{y}\varphi_2(\bar{v}, \bar{y}; \bar{t}))$, with φ_1, φ_2 positive and quantifier-free. Let $\bar{s} = \langle s_1, \ldots, s_n\rangle \in \mathfrak{A}(U)^n$ and suppose $\mathfrak{A}(U) \models \varphi_1[\bar{s}; \bar{t}]$. Since γ is a L-embedding, we have $\Gamma(U) \models \varphi_1[\gamma(\bar{s}); \gamma(\bar{t})]$; since $\sigma(\bar{t})$ holds in $\Gamma(U)$, it follows that $\Gamma(U) \models \exists\bar{y}\varphi_2[\gamma(\bar{s}), \bar{y}; \gamma(\bar{t})]$ and so, the fact that $\mathfrak{A}(U)$ is positively existentially closed in $\Gamma(U)$ along γ guarantees that $\exists\bar{y}\varphi_2(\bar{s}, \bar{y}; \bar{t})$ holds in $\mathfrak{A}(U)$, as needed. □

We now have

Theorem 3.3. *Let L, $L^\sharp$ be first-order languages with equality and let Σ, $\Sigma^\sharp$ be geometrical theories in L and $L^\sharp$, respectively. Let X be a Boolean space and let $\mathcal{B}$ be the Boolean algebra of clopens in X. Let $\mathfrak{A}: \mathcal{B} \to$ $\boldsymbol{\Sigma}$-**mod** be a finitely complete presheaf basis of models of Σ. If $F: \boldsymbol{\Sigma}$-**mod** $\to \boldsymbol{\Sigma}^\sharp$-**mod** is a geometrical functor, then*

(a) *$F \circ \mathfrak{A} : \mathcal{B} \to \boldsymbol{\Sigma}^\sharp$-**mod** is a finitely complete presheaf basis of models of $\Sigma^\sharp$.*
(b) *For all $x \in X$, the stalk of $F \circ \mathfrak{A}$ at x is $(F \circ \mathfrak{A})_x = F(\mathfrak{A}_x)$.*

Proof. Item (b) follows immediately from (a) and the fact that F preserves right-directed colimits. For (a), let $U \in \mathcal{B}$ and let $\overline{V} = \{V_1, \ldots, V_n\}$ be a disjoint clopen covering of U. Since $\mathfrak{A}$ is extensional and finitely complete, Proposition 8.4(a) guarantees that, for $1 \leqslant j \leqslant n$, the diagram below left is commutative, with $\alpha(U; \overline{V})$ a L-isomorphism:

$$
\begin{tikzcd}
\mathfrak{A}(U) \arrow[rr, "\alpha(U;\overline{V})"] \arrow[dr, "\cdot|V_j"'] & & \prod_{j=1}^n \mathfrak{A}(V_j) \arrow[dl, "p_j"] \\
& \mathfrak{A}(V_j) &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(\mathfrak{A}(U)) \arrow[rr, "F(\alpha(U;\overline{V}))"] \arrow[dr, "\cdot|F(V_j)"'] & & \prod_{j=1}^n F(\mathfrak{A}(V_j)) \arrow[dl, "F(p_j)"] \\
& F(\mathfrak{A}(V_j)) &
\end{tikzcd}
$$

Since F preserves finite products, $F(p_j)$ is the jth coordinate projection and the diagram above right is commutative, for $1 \leqslant j \leqslant n$. Moreover, $F(\alpha(U; \overline{V}))$ is clearly a $L^\sharp$-isomorphism. By the equivalence in Proposition 8.4(a), $F \circ \mathfrak{A}$ is an extensional, finitely complete presheaf basis of models of $\Sigma^\sharp$. $\square$

4. Rings with many units

In this section we first give a model-theoretic criterion for a subring to inherit the property of having many units, and then show that if A is a ring with many units, the ring-theoretic analog the Milnor's K-theory of fields, introduced in [11], when reduced mod 2, is canonically isomorphic to the K-theory of the special group naturally associated to A in [7]. To begin, we recall

Definition 4.1. Let R be a ring.

(a) A polynomial $f \in R[X_1, \ldots, X_n]$ has **local unit values relative to maximal ideals** if for all maximal ideals $\mathfrak{m}$ in R, there is $\bar{u} \in R^n$ such that $f(\bar{u}) \notin \mathfrak{m}$. Similarly, one defines the notion f having local unit values relative to prime ideals in R.
(b) R is a **ring with many units** if for all $f \in R[X_1, \ldots, X_n]$, if f has local unit values relative to maximal ideals, then there is $\bar{y} \in R^n$ such that $f(\bar{y})$ is a unit in R.

Remark 4.2. Since every maximal ideal is prime and all (proper) prime ideals are contained in a maximal ideal, a ring R has many units iff for all $f(X_1, \ldots, X_n) \in R[X_1, \ldots, X_n]$,

$$
\begin{array}{l} f \text{ has local unit values relative} \\ \text{to all prime ideals in } R \end{array} \quad \Rightarrow \quad \begin{array}{l} \exists \bar{z} = \langle z_1, \ldots, z_n \rangle \in R^n \text{ such that} \\ f(\bar{z}) \text{ is a unit in } R. \end{array}
$$

Examples of rings with many units are semi-local rings, arbitrary products of rings with many units and more generally, the ring of global sections of a sheaf of rings over a partitionable space, whose stalks are rings with many units. In particular, the ring of global sections of a sheaf of rings over a Boolean space, whose stalks are rings with many units, is a ring with many units. The reader can find more information, as well as the proof of these results in [7], where it is also shown that, under mild assumptions, the RSGs associated to rings of this type faithfully represent the quadratic form theory over free modules (Theorems 3.15 and 3.16, [7]).

Proposition 4.3. *Let R be a ring with many units.*

(a) *If S is a positively existentially closed subring of R, then S is a ring with many units.*
(b) *If e is an idempotent in R, then $Re = \{ae\colon a \in R\}$, a ring with identity e, also has many units.*
(c) *Let T be a ring and let $f : T \to R$ be a map that preserves addition, multiplication and* 0.[1] *If T is positively existentially closed in R along f, then T has many units.*

Proof. (a) We shall use the equivalence noted in Remark 4.2. Let $f(X_1, \ldots, X_n)$ be a polynomial with coefficients in S, that has local unit values relative to all prime ideals in S. If P is a prime ideal in R, then $Q = P \cap S$ is a prime ideal in S and so there is $\bar{x} \in S^n$ such that $f(\bar{x}) \notin Q$. Because f has coefficients in S, it is clear that $f(\bar{x}) \in S$. Hence, $f(\bar{x})$ cannot be in P. Thus, f has local unit values relative to all prime ideals in R. Since R has many units, there is $\bar{r} \in R^n$, such that $f(\bar{r})$ is a unit in R. Now consider the sentence φ given by

$$\exists x_1 \cdots x_n \exists u \big(u \cdot f(x_1, \ldots, x_n) = 1\big).$$

Because f has coefficients in S, φ is a pp-sentence of the language of rings with parameters in S. Since S is positively existentially closed in R and $R \models \varphi$, the same is true in S and so f has unit values in S, as needed.

(b) Let $\alpha(\overline{X}) \in Re[X_1, \ldots, X_n]$ be a polynomial. Observe that for $\bar{a} \in R^n$

$$e\alpha(\bar{a}) = \alpha(\bar{a}) = \alpha(a_1 e, \ldots, a_n e), \tag{I}$$

since for a monomial $(c_\nu e) X_1^{\nu_1} \cdots X_n^{\nu_n}$ in α, $(c_\nu e) a_1^{\nu_1} \cdots a_n^{\nu_n} = (c_\nu e)(a_1 e)^{\nu_1} \cdots (a_n e)^{\nu_n}$. Suppose α has local unit values with respect to all prime ideals in Re (cf. 4.2), and consider

$$\beta(\overline{X}) = \alpha(\overline{X}) + (1 - e) \in R[X_1, \ldots, X_n].$$

Let Q be a (proper) prime ideal in R; since $e(1 - e) = 0$, we have two possibilities:

(i) $e \in Q$: In view of (I), for all $\bar{b} \in R^n$, $\beta(\bar{b}) = e\alpha(\bar{b}) + (1 - e) \notin Q$ (otherwise $1 - e \in Q$, and Q would not be a proper ideal);
(ii) $e \notin Q$: $P = Q \cap Re$ is a proper prime ideal in Re and so there is $\bar{a} \in R^n$ such that $\alpha(\bar{a}) = \alpha(a_1 e, \ldots, a_n e) \notin P$. Because $1 - e \in Q$, we conclude that $\beta(\bar{a}) = \alpha(\bar{a}) + (1 - e) \notin Q$, otherwise $\alpha(\bar{a})$ would belong to $Q \cap Re = P$.

We have just shown that β has local unit values with respect to all prime ideals in R. Since R has many units, there is $u \in R$ and $\bar{c} \in R^n$ such that

$$1 = u\beta(\bar{c}) = u\big(\alpha(\bar{c}) + (1 - e)\big) = u\alpha(c_1 e, \ldots, c_n e) + u(1 - e).$$

Multiplying this equation by e, we get $e = (ue)\ \alpha(c_1 e, \ldots, c_n e)$, and $\alpha(\bar{c})$ is a unit in Re, as needed.

[1] So f is a morphism with respect to the language of rings *without* identity.

(c) Since $e = f(1)$ is an idempotent in R and f identifies T with a positively existentially closed subring of Re, the conclusion follows from items (a) and (b). $\square$

Remark 4.4. Since positive existential formulas are geometrical (2.1(d)), 3.2(d), 4.3 and the fact that products of rings with many units have many units, imply that the ring of global sections of a sheaf of rings over a Boolean space whose stalks are many unit rings, is also a ring with many units.

We now adapt to our purposes a condition introduced in [11] (p. 29):

Definition 4.5. Let A be a ring and let $m \geqslant 1$ be an integer. We say that

(a) A satisfies [H1-m] ($A \models$ [H1-m]) if for all $n \geqslant 2$ and all $1 \leqslant k \leqslant m$, if $\{f_1, \ldots, f_k\}$ is a family of surjective linear forms over the free A-module A^n, there is $v \in A^n$ such that $f_j(v) \in A^*$, $1 \leqslant j \leqslant k$.
(b) A satisfies [H1] if $A \models$ [H1-m] for all $m \geqslant 1$.

It is mentioned in the Examples given on page 33 of [11] that all semilocal rings whose residue fields are infinite verify [H1]. Generalizing this observation we have

Proposition 4.6. *Let $m \geqslant 2$ be an integer. If A is a ring with many units, whose residue fields all have cardinality $\geqslant m$, then $A \models$ [H1-m].*

Proof. We start with the following

Fact 4.7. *If F is a field of cardinality $\geqslant m$, then $F \models$ [H1-m]. In particular, infinite fields verify [H1].*

Proof. By induction on $m \geqslant 1$. Clearly, *any ring* verifies [H1-1]. Assume the result true for m, that F has at least $m + 1$ elements and that $\{f_1, \ldots, f_k\}$, $1 \leqslant k \leqslant m + 1$, are surjective linear forms from F^n to F. If $k \leqslant m$, the induction hypothesis immediately implies the desired result. So, assume $k = m + 1$. The induction hypothesis yields $v \in F^n$ such that $f_j(v) \neq 0$ (i.e., $f_j(v) \in F^*$), $1 \leqslant j \leqslant m$. If $f_{m+1}(v) \neq 0$, we are done. Otherwise, select w such that $f_{m+1}(w) \neq 0$ and consider the set

$$A = \left\{ f_j(w)/f_j(v)\colon\ 1 \leqslant j \leqslant m \right\}.$$

Since A has at most m elements and F has at least $m + 1$ elements, there is $\lambda \in F \setminus A$. Now consider $x = w - \lambda v \in F^n$; then,

$$f_{m+1}(x) = f_{m+1}(w) \neq 0 \quad \text{and,} \quad \text{for } 1 \leqslant j \leqslant m, \quad f_j(x) = f_j(w) - \lambda f_j(v) \neq 0,$$

because $\lambda \notin A$, establishing Fact 4.7.

Now let A be a ring with many units, whose residue fields all have at least m elements, and let $\{f_1, \ldots, f_k\}$ be surjective linear forms from A^n to A ($k \leqslant m$). Let $\{e_1, \ldots, e_n\}$ be the canonical basis of A^n and set, for $1 \leqslant j \leqslant k$ and $1 \leqslant l \leqslant n$, $a_{jl} =_{\text{def}} f_j(e_l)$. Now, let

$$p(X_1, \ldots, X_n) = \prod_{j=1}^{k} \sum_{l=1}^{n} a_{jl} X_l = \prod_{j=1}^{k} f_j. \tag{I}$$

If $\mathfrak{m}$ is a maximal ideal in A and $1 \leqslant j \leqslant k$, the form f_j naturally induces a *surjective* linear form, $f_j/\mathfrak{m}$, from $(A/\mathfrak{m})^n$ to $A/\mathfrak{m}$, given by

$$x/\mathfrak{m} = (x_1/\mathfrak{m}, \ldots, x_n/\mathfrak{m}) \longmapsto f_j(x)/\mathfrak{m}.$$

Indeed, if $x_l - y_l \in \mathfrak{m}$, $1 \leqslant l \leqslant n$, then, with notation as in (I),

$$f_j(x) - f_j(y) = f_j(x - y) = \sum_{l=1}^{n} a_{jl}(x_l - y_l) \in \mathfrak{m},$$

and $f_j/\mathfrak{m}$ is well defined. It is clear that $f_j/\mathfrak{m}$ is surjective. By Fact 4.7, there is $v/\mathfrak{m} \in (A/\mathfrak{m})^n$ such that $[f_j/\mathfrak{m}](v/\mathfrak{m}) \neq 0$, that is, $f_j(v) \notin \mathfrak{m}$, for all $1 \leqslant j \leqslant k$. Since $\mathfrak{m}$ is a prime ideal, (I) entails

$$p(v) = \prod_{j=1}^{k} f_j(v) \notin \mathfrak{m},$$

and thus $p(X_1, \ldots, X_n)$ has local unit values in A. Hence, there is $x \in A^n$ such that $p(x) \in A^*$. But this immediately implies that $f_j(x) \in A^*$, $1 \leqslant j \leqslant k$, ending the proof. $\square$

We now present a mod 2 K-theory of rings, patterned after the construction in Section 3 of [11]. Let A be a ring. We set $K_0 A = \mathbb{Z}$ and let $K_1 A$ be A^* written additively, that is, we fix an isomorphism

$$l : A^* \to K_1 A, \qquad \text{such that } l(ab) = l(a) + l(b),\ \forall a, b \in A^*.$$

Then, Milnor's K-theory of A is the graded ring (Definition 3.2, p. 47, [11])

$$K_* A = \langle \mathbb{Z}, K_1 A, \ldots, K_n A, \ldots \rangle,$$

obtained as the quotient of the graded tensor algebra over $\mathbb{Z}$,

$$\langle \mathbb{Z}, K_1 A, \ldots, \underbrace{K_1 A \otimes \cdots \otimes K_1 A}_{n \text{ times}}, \ldots \rangle$$

by the ideal generated by $\{l(a) \otimes l(b)\colon a, b \in A^* \text{ and } a + b = 1 \text{ or } 0\}$. Hence, for each $n \geqslant 2$, $K_n A$ is the quotient of the n-fold tensor product over $\mathbb{Z}$, $K_1 A \otimes \cdots \otimes K_1 A$, by the subgroup consisting of sums of generators $l(a_1) \otimes \cdots \otimes l(a_n)$, such that for some $1 \leqslant i \leqslant n - 1$, $a_i + a_{i+1} = 1$ or 0. As usual, we shall write the generators in $K_n A$ as $l(a_1) \cdots l(a_n)$, omitting the tensor operation. As a consequence of (the proof of) Proposition 3.2.3 in [11] (p. 48) and Proposition 4.6 we have

Lemma 4.8. *Let A be a ring with many units whose residue fields all have at least 7 elements. Then, K_*A is the graded ring obtained as the quotient of the graded tensor algebra over $\mathbb{Z}$,*

$$\langle \mathbb{Z}, K_1A, \ldots, \underbrace{K_1A \otimes \cdots \otimes K_1A}_{n \text{ times}}, \ldots \rangle$$

by the graded ideal generated by $\{l(a)l(b)\colon a, b \in A^* \text{ and } a + b = 1\}$.

Proof. By Proposition 3.2.3 in [11], the result holds for rings satisfying [H1] in 4.5(b). However, an analysis of the proof shows that what is needed is [H1-6], and the desired conclusion follows from 4.6. □

Definition 4.9. If A is a ring, we define the **mod 2 K-theory of A**, as the graded ring

$$k_*A = \langle k_0A, k_1A, \ldots, k_nA, \ldots \rangle =_{\text{def}} K_*A/2K_*A,$$

that is, for each $n \geqslant 0$, k_nA is the quotient of K_nA by the subgroup $\{2\eta \in K_nA\colon \eta \in K_nA\}$.

We have $k_0A = \mathbb{F}_2$ and $k_1A \approx A^*/A^{*^2}$, via an isomorphism still denoted by l. A generator in k_nA will be written $l(a_1)\cdots l(a_n)$. Clearly, k_nA is a group of exponent 2, i.e., $\eta + \eta = 0$, for all $\eta \in k_nA$.

Lemma 4.10. *If A is a ring verifying* [H1-6]*, then for all b, a, $a_1, \ldots, a_n \in A^*$ and all permutations σ of $\{1, \ldots, n\}$*

(a) *In k_2A, $l(a)l(-a) = 0$.*
(b) *In k_2A, $l(a)l(-1) = l(a)^2$.*
(c) *In k_2A, $l(a)l(b) = l(b)l(a)$.*
(d) *In k_nA, $l(a_1)\cdots l(a_n) = l(a_{\sigma(1)})\cdots l(a_{\sigma(n)})$.*
(e) *If $t_1, \ldots, t_n \in A^*$, then in k_nA, $l(t_1^2a_1)\cdots l(t_n^2a_n) = l(a_1)\cdots l(a_n)$.*

Proof. (a) The proof of Proposition 3.2.3 in [11] shows that if A verifies [H1-6], then $l(a)l(-a) = 0$ in K_2A and so the same is true in k_2A.

(b) From (a) we get $0 = l(a)l(-a) = l(a)[l(-1) + l(a)] = l(a)l(-1) + l(a)^2$. Since k_2A is a group of exponent two, the conclusion follows.

(c) From (a) and (b) we get

$$\begin{aligned} 0 = l(-ab)l(ab) &= \big[l(-a) + l(b)\big]\big[l(a) + l(b)\big] = l(b)l(a) + l(-a)l(b) + l(b)^2 \\ &= l(b)l(a) + \big[l(-1) + l(a)\big]l(b) + l(b)^2 = l(b)l(a) + l(a)l(b), \end{aligned}$$

and so, since k_2 is a group of exponent two, we obtain $l(a)l(b) = l(b)l(a)$, as needed.

Item (c) implies that the conclusion in (d) holds for all transpositions $(i, i + 1)$. Since the symmetric group is generated by these transpositions, the full statement in (d) follows immediately.

For item (e), note that for $t, a \in A^*$, $l(t^2a) = 2l(t) + l(a) = l(a)$, since $2l(t) = 0$ in k_1A. □

Our next order of business is to connect the mod 2 K-theory of a ring with many units satisfying certain conditions with the K-theory of the special group naturally associated to it, as in 8.9.

Lemma 4.11. *Let A be a ring with many units, whose residue fields all have at least 7 elements. Let $a, b, a_1, \ldots, a_n \in A^*$, with $a \in D_A(1, b)$. If $a_i = a$ and $a_j = ab$ for some $1 \leqslant i \neq j \leqslant n$, then $l(a_1) \cdots l(a_n) = 0$ in $k_n A$.*

Proof. Let A be a ring as in the statement. It is noted in the proof of Theorem 3.16 in [7] that A satisfies the following property (therein called [w2t], cf. 3.11, p. 16):

$$\forall u, v, w \in A^*,\ w \in D_A(u, v) \Rightarrow \exists p, q \in A^* \text{ such that } w = p^2 u + q^2 v. \tag{$\natural$}$$

Hence, since $a \in D_A(1, b)$, there are $p, q \in A^*$ such that $a = p^2 + q^2 b$. Hence,

$$1 = (p^2/a^2)a + (q^2/a^2)ba = (p/a)^2 a + (q/a)^2 ab,$$

and so, the definition of $k_* A$ and 4.10(e) yield $l(a)l(ab) = 0$ in $k_2 A$. The general statement follows immediately from 4.10(d). $\square$

Theorem 4.12. *Let A be a ring with many units such that $2 \in A^*$ and whose residue fields all have at least 7 elements. Then, $G(A) = \langle G(A), \equiv, -1 \rangle$ (see 8.9) is a special group. Moreover, the rules $\alpha_0 = Id_{\mathbb{F}_2}$ and $\alpha_n : k_n A \to k_n G(A)$, defined on generators by $\alpha_n(l(a_1) \cdots l(a_n)) = \lambda(\bar{a}_1) \cdots \lambda(\bar{a}_n)$, for $n \geqslant 1$, determine a graded ring isomorphism between the mod 2 K-theory of A and the K-theory of the special group $G(A)$.*

Proof. The fact that $G(A)$ is a special group is established in Theorem 3.16 of [7]. Now, the proof of Theorem 2.5 in [5], yielding an analogous result for fields of characteristic $\neq 2$, with Lemma 4.11 in the role of Lemma 2.4 of [5], applies, *ipsis litteris*, to show that $\alpha = \{\alpha_n\colon n \geqslant 0\}$ is a graded ring isomorphism between $k_* A$ and $k_* G(A)$. $\square$

5. Preorders on von Neumann-regular rings

5.1. The Boolean algebra of idempotents. Let R, S be rings.

(a) Let $B(R) = \{e \in R\colon e^2 = e\}$ be the set of idempotents in R. With the operations

$$e \wedge f = ef \quad \text{and} \quad e \vee f = e + f - ef,$$

$\langle B(R), \wedge, \vee, 0, 1 \rangle$ is a Boolean algebra (BA), where the complement of e is $1 - e$. Note that for $e, f \in B(R)$, $e \leqslant f \Leftrightarrow ef = e \Leftrightarrow e \vee f = f$.

If $f : R \to S$ is a ring-morphism, then $B(f) =_{\mathrm{def}} f_{|B(R)}$ is a BA-morphism from $B(R)$ to $B(S)$; it is clear that this correspondence preserves composition and identity. Hence, we have a covariant functor from the category **UCR** (2.2(b)) to **BA**, the category of BAs.

If $e \in B(R)$, the principal ideal $(e) = Re$ is a ring, whose unit is e.

(b) For $e \in B(R)$, let $\varphi_{1e} : R \to Re$, be the ring morphism given by $\varphi_{1e}(a) = ae$. If $f \leqslant e$, write φ_{ef} for $(\varphi_{1f})_{|Re} : Re \to Rf$; since $ef = f$, we have $\varphi_{ef}(ae) = af$. Note that

(1) $\varphi_{ee} = Id_{Re}$;
(2) For $h \leqslant f \leqslant e$, $\varphi_{eh} = \varphi_{fh} \circ \varphi_{ef}$.

(c) $e, f \in B(R)$ are **disjoint** or **orthogonal** if $ef = 0$; thus, $ef = 0 \Leftrightarrow f \leqslant 1 - e \Leftrightarrow e \leqslant 1 - f$. Clearly, if e and f are disjoint, then $e \vee f = e + f$.

(d) A family $\{f_1, \ldots, f_n\} \subseteq B(R)$ is a **covering** of $e \in B(R)$ if $e = \bigvee_{i=1}^n f_i$. A covering $\{f_1, \ldots, f_n\}$ of e is an **orthogonal decomposition of e** if the f_j are pairwise disjoint. In this case we have

$$e = \sum_{j=1}^n f_j = \bigvee_{j=1}^n f_j.$$

(e) An orthogonal decomposition of $e \in B(R)$, $\{f_j \in B(R)\colon 1 \leqslant j \leqslant n\}$ induces a decomposition into a direct sum of rings, $Re = \bigoplus_{j=1}^n Rf_j$, defined by $ae \longmapsto \sum_{j=1}^n af_j$.

Proposition 5.2. *If R is a ring and $e \in B(R)$, let $\{f_j \in B(R)\colon 1 \leqslant j \leqslant n\}$ be a covering of e.*

(a) *There is an orthogonal decomposition of e, $\{e_j \in B(R)\colon 1 \leqslant j \leqslant n\}$, with $e_j \leqslant f_j$ for $1 \leqslant j \leqslant n$. Such an orthogonal decomposition is said to be* ***subordinate to the covering*** *$\{f_j\colon 1 \leqslant j \leqslant n\}$.*

(b) *If $a, b \in R$, then $ae = be \Leftrightarrow$ for all i $1 \leqslant i \leqslant n$, $af_i = bf_i$.*

(c) *Let $a_1, \ldots, a_n \in R$ be such that for all $1 \leqslant j, k \leqslant n$, $a_k f_k f_j = a_j f_j f_k$. Then, there is $a \in R$ such that $af_j = a_j f_j$, for all $1 \leqslant j \leqslant n$.*

(d) *If R_e is the ring of fractions of R with respect to e, then*
(1) *The map $\lambda_e : Re \to R_e$, given by $\lambda_e(re) = re/1$ is a ring isomorphism*;
(2) *For $f \leqslant e$, the map $\gamma_{ef} : R_e \to R_f$, given by $\gamma_{ef}(re/1) = rf/1$, is a ring morphism, such that the following diagram is commutative*

$$\begin{array}{ccc} Re & \xrightarrow{\lambda_e} & R_e \\ {\scriptstyle \varphi_{ef}}\downarrow & & \downarrow{\scriptstyle \gamma_{ef}} \\ R_f & \xrightarrow[\lambda_f]{} & R_f \end{array}$$

where $\varphi_{ef} : Re \to Rf$ is as in 5.1(b).

Proof. (a) The required orthogonal decomposition is given by: $e_1 = f_1$, and for $2 \leqslant j \leqslant n$, $e_j = f_j(1 - f_1)(1 - f_2) \cdots (1 - f_{j-1})$.

(b) It suffices to prove ($\Leftarrow$). Let $\{e_j\colon 1 \leqslant j \leqslant n\}$ be an orthogonal decomposition of e, subordinate to $\{f_j\colon 1 \leqslant j \leqslant n\}$. For $1 \leqslant j \leqslant n$, $ae_j = ae_j f_j = bf_j e_j = be_j$, and so, $ae = a\sum_{j=1}^n e_j = be$.

(c) Let $\{e_j\colon 1 \leqslant j \leqslant n\}$ be an orthogonal decomposition of e, subordinate to $\{f_j\colon 1 \leqslant j \leqslant n\}$ and set $a = \sum_{j=1}^n a_j e_j$. Then, for $1 \leqslant k \leqslant n$,

$$af_k = \sum_{j=1}^n a_j e_j f_k = \sum_{j=1}^n a_j e_j f_j f_k = \sum_{j=1}^n a_k e_j f_j f_k = a_k f_k \sum_{j=1}^n e_j = a_k f_k e = a_k f_k.$$

(d) (1) Clearly, λ_e is an injective ring morphism. For $x \in R$, since $e(x - xe) = 0$, in R_e we have

$$x/e = x/1 = xe/1, \tag{I}$$

and λ_e is also surjective, whence an isomorphism. Item (2) is straightforward. □

Definition 5.3. A ring R is **von Neumann-regular (vN-ring)** if every principal ideal is generated by an idempotent. Thus, if $a \in R$, there is $e \in B(R)$ such that $(a) = (e)$. Equivalently,

$$\forall a \in R \exists e \in B(R) \text{ and } \exists b \in R \text{ such that } ae = a \text{ and } ab = e. \tag{[vN]}$$

We refer to e as **the idempotent associated** to a (clearly, it is unique). Yet another formulation of von Neumann regularity of a ring R, is to require that every element of R be divisible by its square. A vN-ring is also called **absolutely flat**, being precisely the rings with the property that all modules are flat. Write **vN** for the category of vN-rings and ring morphisms.

The following lemma, whose proof is omitted, summarizes the basic facts concerning vN-rings.

5.4. Notation. In the sequel, $Z(a)$ stands for $\{P \in \mathrm{Spec}(R)\colon a \notin P\}$, where $\mathrm{Spec}(R)$ is the Zariski spectrum of R, and $a \in R$.

Lemma 5.5. *Let R be a vN-ring and let $e \in B(R)$.*

(a) *All prime ideals in R are maximal and the map $P \in \mathrm{Spec}(R) \overset{\tau}{\longmapsto} P \cap B(R)$ is a natural bijective correspondence between $\mathrm{Spec}(R)$ and the maximal ideals in the Boolean algebra $B(R)$.*

(b) *Let P be a prime ideal in R and let R_P be the localization of R at P. If $P \in Z(e)$, let*
  (1) *$\lambda_{eP} : R_e \to R_P$, be given by $\lambda_{eP}(x/e) = x/e$;*
  (2) *$\varphi_{eP} : Re \to R/P$, be given by $\varphi_{eP}(re) = r/P$;*
  (3) *$\lambda_P : R/P \to R_P$, be given by $\lambda_P(x/P) = x/1$.*
  Then, λ_{eP}, φ_{eP} are surjective ring morphisms, λ_P is an isomorphism and diagram (I) *below is commutative,*

$$\begin{tikzcd} Re \arrow[r, "\lambda_e"] \arrow[d, "\varphi_{eP}"'] \arrow[dr, phantom, "\mathrm{(I)}"] & R_e \arrow[d, "\lambda_{eP}"] \\ R/P \arrow[r, "\lambda_P"'] & R_P \end{tikzcd} \qquad \begin{tikzcd} Re \arrow[rr, "\varphi_{ef}"] \arrow[dr, "\varphi_{eP}"'] & \mathrm{(II)} & Rf \arrow[dl, "\varphi_{fP}"] \\ & R/P & \end{tikzcd}$$

  where λ_e is the isomorphism in 5.2(d)(1). *Moreover, if $f \in B(R)$ is such that $P \in Z(f)$ and $f \leqslant e$, then diagram* (II) *above is commutative.*

(c) *With the Zariski topology,* $\mathrm{Spec}(R)$ *is a Boolean space, with a basis of clopens,*

$$\mathcal{Z} = \bigl\{Z(e) \subseteq \mathrm{Spec}(R)\colon\ e \in B(R)\bigr\}$$

that is a Boolean algebra isomorphic to $B(R)$ *by the map* $e \in B(R) \longmapsto Z(e) \in \mathcal{Z}$. *Moreover,*

(1) *The map* $\mathfrak{r}$ *in* (a) *is a homeomorphism between* $\mathrm{Spec}(R)$ *and* (*the maximal ideal version of*) *the Stone space of* $B(R)$;

(2) *For all* $P \in \mathrm{Spec}(R)$, *the filter* $\mathcal{Z}_P = \{Z(e) \in \mathcal{Z}\colon\ P \in Z(e)\}$ *of clopen neighborhoods of* P *is order-isomorphic to the ultrafilter* $\{e \in B(R)\colon\ e \notin P\}$ *in* $B(R)$.

(d) *If* I *is an ideal of* R, *then* R/I *is a vN-ring and* $\mathrm{Spec}(R/I)$ *is naturally homeomorphic to the set* $V(I) =_{\mathrm{def}} \{P \in \mathrm{Spec}(R)\colon\ I \subseteq P\}$, *with the topology induced by* $\mathrm{Spec}(R)$. *In particular,* Re *is a vN-ring, with* $\mathrm{Spec}(Re) = Z(e)$.

With these preliminaries, we state

Proposition 5.6. *Let* R *be a vN-ring and let* $\mathcal{Z}$ *be the Boolean algebra of clopens in* $\mathrm{Spec}(R)$.

(a) *The assignments*

$$\begin{cases} Z(e) \in \mathcal{Z} \longmapsto Re, \\ Z(f) \subseteq Z(e) \longmapsto \varphi_{ef} : Re \to Rf, \end{cases} \tag{$\mathfrak{R}$}$$

constitute a presheaf basis of vN-rings over $\mathcal{Z}$, *denoted by* $\mathfrak{R}$, *with the following properties*:

(1) $\mathfrak{R}$ *is finitely complete over all* $Z(e) \in \mathcal{Z}$;

(2) *Notation as in* 5.1(b) *and* 5.5(b.3), *for each* $P \in \mathrm{Spec}(R)$, *the colimit of the inductive system* $\langle Re; \{\varphi_{ef}\colon\ f \leqslant e\}, e, f \notin P\rangle$, *is* $\langle R/P; \{\varphi_{eP}\colon\ e \notin P\}\rangle$. *Hence, the stalk of* $\mathfrak{R}$ *at* P *is the field* R/P, *i.e.,* $\mathfrak{R}_P = R/P = \varinjlim_{e \notin P} Re$.

(b) *The completion of* $\mathfrak{R}$ (cf. 8.5), $c\mathfrak{R}$, *is* (*naturally isomorphic to*) *the affine scheme of* R. *Moreover, for all* $e \in B(R)$, $c\mathfrak{R}(Z(e)) = Re$; *in particular, the ring of global sections of* $c\mathfrak{R}$ *is precisely* R.

Proof. (a) Item (d) in Lemma 5.5, together with relations (1) and (2) in 5.1(b), show that $\mathfrak{R}$ is a contravariant functor from $\langle \mathcal{Z}, \subseteq^{op}\rangle$ to the category of vN-rings. Since each $Z(e)$ is compact clopen, the extensionality of $\mathfrak{R}$ and its finite completeness over $Z(e)$ follow immediately from items (b) and (c) of Proposition 5.2, respectively. It remains to prove (2). By 2.4(a), it must be shown, in view of the definition of the maps φ_{ef} and φ_{eP} (5.5(b)), that:

* $R/P = \bigcup_{e \notin P} \varphi_{eP}(Re)$, which is clear from 5.5(b);
* For all $e \notin P$ and $xe \in Re$, $x/P = 0 \Rightarrow \exists f \leqslant e$ such that $f \notin P$ and $xf = 0$.

Let h be the idempotent associated to x; since $x \in P$, the same is true of h, whence, $1 - h \notin P$. Set $f = e(1-h)$; then, $f \leqslant e$, $f \notin P$ and $xf = xe(1-h) = xhe(1-h) = 0$, as needed.

(b) If R is a ring, the classical presheaf basis associated to R, whose completion is its affine scheme, is the contravariant functor from $\mathcal{Z} = \{Z(a)\colon\ a \in R\}$ to **UCR** (cf. 2.2(b)), defined by

(i) $Z(a) \in \mathcal{Z} \longmapsto R_a$, the ring of fractions of R with respect to a;

(ii) If $Z(a) \subseteq Z(b)$, then $a^n = ub$, for some $n \geqslant 1$ and $u \in R$, whence, b is invertible in R_a, $1/b = u/a^n$.

By the universal property of rings of fractions, there is a *unique* ring morphism, $\rho_{ba}: R_b \to R_a$, given, for $r \in R$, by $\rho_{ba}(r/b^m) = ru^m/a^{nm}$, and this definition is independent of the parameters $n \geqslant 1$ and u. The presheaf basis so defined is complete over any $Z(a)$, $a \in R$. Now, if R is a vN-ring, then

(iii) For all $a \in R$, $Z(a) = Z(e)$, where e is the idempotent associated to a;
(iv) For $f \leqslant e$, the ring morphisms ρ_{ef} are precisely the γ_{ef} of Proposition 5.2(d.2). Indeed, in this case we have $ef = f$ and so, recalling equality (I) in the proof of 5.2(d.1), we obtain, for $r \in R$, $\rho_{ef}(r/e^m) = \rho_{ef}(re/1) = rf/e = ref/1 = rf/1 = \gamma_{ef}(re/1)$.
(v) For all e in $B(R)$, the maps λ_e of 5.2(d.1) are isomorphisms, making the diagram displayed in 5.2(d.2) commutative.

From (i)–(v), we conclude that the presheaf basis constructed in part (a) above is **isomorphic** to the classical presheaf basis associated to the affine scheme of R, and so their completions must also be isomorphic. That $c\mathfrak{R}(Z(e)) = Re$ follows from (a.1) and the remarks in 8.5(a). □

Proposition 5.6 shows that every vN-ring is represented as the ring of global sections of a sheaf of vN-rings over a Boolean space, whose stalks are fields, in fact, the residue fields at its maximal ideals. The converse of this statement is also true; this correspondence, originally due to Pierce, can be found in [17]. We shall now deal with preorders in vN-rings.

5.7. Definition and notation. Let R be a ring and let S be a subset of R.

(a) We write S^* for the set of units in S. In particular, R^* is the (multiplicative) group of units in R.
(b) As usual, a **preorder** in a ring R is a set $T \subseteq R$ closed under addition and multiplication, and containing R^2. T is **proper** if $T \neq R$; if $2 \in \dot{R}$, this is equivalent to $-1 \notin T$.

If $P \in \mathrm{Spec}(R)$, and T is a preorder of R, let $T/P =_{\mathrm{def}} \{a/P \in R/P\colon a \in T\}$ be the **preorder induced by T** on the quotient R/P.

(c) A vN-ring, R, is **strongly formally real** if for all $P \in \mathrm{Spec}(R)$, R/P is a formally real field.
(d) A preorder T of a vN-ring R is **strict** if for all $P \in \mathrm{Spec}(R)$, T/P is a *proper* preorder of the residue field R/P.

Lemma 5.8. *Let R be a vN-ring and let T be a preorder of R.*

(a) *2 is a unit in $R \Leftrightarrow$ all residue fields of R have characteristic $\neq 2$.*
(b) *If 2 is a unit in R, then for all $f \in B(R)$, the following are equivalent*:
  (1) *For all $0 \neq e \leqslant f$, Te is a proper preorder of Re*;
  (2) *For all $P \in Z(f)$, T/P is a proper preorder of R/P.*
(c) *If 2 is a unit in R, $e \in B(R)$ and $a \in R$, the following are equivalent*:
  (1) *For all $P \in Z(e)$, $a/P \in T/P$*;
  (2) *$ae \in T$.*

Proof. a) It suffices to prove ($\Leftarrow$); if 2 is not a unit in R, it would be contained in a maximal ideal, P, and the field R/P would have characteristic 2.

(b) (1) $\Rightarrow$ (2): Assume (2) false, and let $P \in Z(f)$ be so that $-1 \in T/P$, i.e., there is $t \in T$, such that the germ $(t+1)_P = (t+1)/P = 0$. By Lemma 8.8(a), there is $e \leqslant f$ such that $e \notin P$ and

$$(t+1)_{|Z(e)} = (t+1)e = te + e = 0.$$

But this means that $-e \in Te$, and so Te is not proper in Re (e is the identity of Re).

(2) $\Rightarrow$ (1): If for some $\emptyset \neq Z(e) \subseteq Z(f)$, Te is improper, then, since $2_{|Z(e)}$ is a unit in Re (by (a)), we have $-e \in Te$, or equivalently, $(t+1)e = 0$, for some $t \in T$. If $P \in Z(e)$, then $t+1 \in P$, that is, $-1 \in T/P$, violating (2).

(c) One should keep in mind that $B(R) \subseteq T$, since every idempotent is a square.

(1) $\Rightarrow$ (2): By (1), for each $P \in Z(e)$, there is $t^P \in T$ such that $(t^P)/P = a/P$ holds in R/P. Compactness, Proposition 5.6(2) and Lemma 8.8(a) yield idempotents $f_1, \ldots, f_n \leqslant e$ and $t_1, \ldots, t_n \in T$, such that

$$\bigvee_{i=1}^{n} f_i = e \quad \text{and} \quad t_i f_i = a f_i,\ 1 \leqslant i \leqslant n.$$

Let $\{e_i\colon 1 \leqslant i \leqslant n\}$ be a orthogonal decomposition of e subordinate to $\{f_i\colon 1 \leqslant i \leqslant n\}$, and set $x = \sum_{i=1}^{n} t_i e_i$. Since $t_i,\ e_i \in T$, it is clear that $x \in T$. Moreover, for $1 \leqslant i \leqslant n$, we have $t_i e_i = t_i f_i e_i = a f_i e_i = a e_i$, wherefrom it follows, summing over i, that $ae = x \in T$, as needed.

(2) $\Rightarrow$ (1): From $e(1-e) = 0$ it follows that $e \notin P \Leftrightarrow e/P = 1/P$. Hence $ae \in T$, and $e \notin P$ entail $ae/P = a/P \in T/P$. $\square$

Corollary 5.9. *If R is a vN-ring in which 2 is a unit, the following are equivalent*:

(1) *For all $e \in B(R)$, Re is a formally real ring*;
(2) *For every $P \in \mathrm{Spec}(R)$, R/P is a formally real field.*

Proof. Just apply Lemma 5.8(b) to the preorder $T = \Sigma R^2$. $\square$

Lemma 5.10. *Let R be a vN-ring in which 2 is a unit and let T be a proper preorder of R. With notation as in Proposition 5.6, the assignments*

$$\begin{cases} \quad Z(e) \in \mathcal{Z} \longmapsto Te, \\ Z(f) \subseteq Z(e) \longmapsto (\varphi_{ef})_{|Te} \colon Te \to Tf, \end{cases} \tag{$\mathfrak{T}$}$$

constitute a finitely complete presheaf basis $\mathfrak{T}$ of preorders over $\mathcal{Z}$, such that for all $P \in \mathrm{Spec}(R)$,

$$\varinjlim \bigl\langle Te;\ \bigl\{(\varphi_{ef})_{|Te}\colon f \leqslant e\bigr\}, e, f \notin P\bigr\rangle = \bigl\langle T/P;\ \{(\varphi_{eP})_{|Te}\colon e \notin P\}\bigr\rangle, \tag{$\mathfrak{T}_P$}$$

that is, $\mathfrak{T}_P$ is the preorder T/P of field R/P.

Proof. Clearly, $\mathfrak{T}$ is a contravariant functor from $\mathcal{Z}$ to the category of sets. The extensionality of $\mathfrak{T}$ follows immediately from that of $\mathfrak{R}$, because for all $e \in B(R)$, $\mathfrak{T}(Z(e)) = Te \subseteq \mathfrak{R}(Z(e))$.

To check finite completeness, let $\{f_1, \ldots, f_n\} \subseteq B(R)$ and let $\{a_j f_j \in Tf_j\colon 1 \leqslant j \leqslant n\}$ be a compatible set of sections in $\mathfrak{T}$; this means:

$$\text{For all } 1 \leqslant i,\ j \leqslant n, \quad a_j f_j f_i = a_i f_i f_j. \tag{I}$$

Let $\{e_1, \ldots, e_n\}$ be an orthogonal decomposition of $f = \bigvee_{i=1}^n f_j$, subordinate to $\{f_1, \ldots, f_n\}$, as in 5.2(a), and consider

$$z = \sum_{i=1}^{n} a_i e_i.$$

Then, $z = zf \in Tf$, and for all $1 \leqslant j \leqslant n$, (I) and the fact that $e_k f_k = e_k$, yield

$$zf_j = \sum_{i=1}^{n} a_i e_i f_j = \sum_{i=1}^{n} a_i e_i f_i f_j = \sum_{i=1}^{n} a_j e_i f_i f_j = a_j f_j \sum_{i=1}^{n} e_i = a_j f_j f = a_j f_j,$$

and hence z is the gluing of $\{a_j f_j \in Tf_j\colon 1 \leqslant j \leqslant n\}$ in $\mathfrak{T}$. To establish $(\mathfrak{T}_P)$ we have to show:

(A) $T/P = \bigcup_{e \notin P} \varphi_{eP}(Te)$;
(B) For $e \notin P$, $\varphi_{eP}(ae) = a/P \in T/P \Rightarrow \exists f \leqslant e$ such that $\varphi_{ef}(ae) = af \in Tf$.

(A) is clear, since $T/P = \varphi_{1P}(T)$ and $e/P = 1/P$ for $e \notin P$, imply $\varphi_{eP}(re) = r/P = \varphi_{1P}(r)$, for all $r \in R$. The argument for (B) is similar to that in the proof of 5.6(a). Let $a \in R$ be such that $a - t \in P$ for some $t \in T$. Let h be the idempotent associated to $a - t$; then $h \in P$ and so $1 - h \notin P$. Take $f = e(1 - h)$; then, $e \geqslant f \notin P$ and $(a - t)f = (a - t)he(1 - h) = 0$, showing that $af = tf \in Tf$, as needed. $\square$

6. The presheaf of special groups of a preordered vN-ring

Before presenting the presheaf basis of the title we shall make some general observations, that will help to organize the exposition and may apply to more general situations. The setting is as follows:

Definition 6.1.

(a) A **preordered ring (p-ring)** is a pair $\langle A, T\rangle$ such that
[pr 1]: A is a ring, such that $2 \in A^*$;
[pr 2]: T is a (not necessarily proper) preorder of A.
In case the preorder T is improper, i.e., $T = A$, the pair $\langle A, A\rangle$ will be called the **trivial p-ring**; else, $\langle A, T\rangle$ is called **proper**.
(b) A **morphism of p-rings**, $f : \langle A, T\rangle \to \langle A', T'\rangle$, is a ring morphism, $f : A \to A'$, such that $f(T) \subseteq T'$. Let **p-Ring** be the category of p-rings and their morphisms.

Remark 6.2. The language of p-rings is $L = \langle +, \cdot, 0, 1, -1, T\rangle$, i.e., the first-order language of unitary rings, with an additional *unary* predicate, T, interpreted as a preorder (and, of course, equality). Besides atomic formulas of the type $\tau_1 = \tau_2$, where τ_i are terms ($i = 1, 2$), we also have $\tau_1 \in T$. Clearly, the theory of p-rings is geometrical (2.1(d)).

A construction similar to that of the case of (preordered) fields associates to each p-ring $\langle A, T\rangle$ the group of exponent two $G_T(A) = A^*/T^*$, endowed with a four-place relation $\equiv_T$, in such a way that the L-structure $\langle G_T(A), \equiv_T, -1\rangle$ verifies the axioms [SG 0]–[SG 3] and [SG 5] for special groups, as well as the axiom of reduction; cf. [4], Definition 1.2. For the reader's convenience, this construction is briefly summarized in Section 8.B. Note that this construction is functorial (Lemma 8.14). However, under these very general conditions, axioms [SG 4] and [SG 6] for special groups may fail. These considerations suggest the following

Definition 6.3. (a) A **proto special group** (**π-SG**), is a triple, $G = \langle G, \equiv_G, -1\rangle$, consisting of

* A group, G, of exponent two, written multiplicatively (and so its identity is 1);
* A distinguished element, -1, in G (we write $-x$ for $-1 \cdot x$, $\forall x \in G$);
* A binary relation $\equiv_G$ on $G \times G$, satisfying the axioms [SG 0]–[SG 3] and [SG 5] in 8.11(a).

G is **reduced** (abbreviated π-RSG) if $1 \neq -1$ and it satisfies the first equivalence in 8.11(b).

For $a, b, c \in G$, write $c\langle a, b\rangle$ for $\langle ca, cb\rangle$. The product ab is the **discriminant** of $\langle a, b\rangle$.

If $G = \langle G, \equiv_G, -1\rangle$ is a π-SG and $x, y \in G$, define

$$D_G(x, y) = \big\{z \in G\colon \langle z, zxy\rangle \equiv_G \langle x, y\rangle\big\},$$

the set of elements **represented by x and y** in G. Since G has exponent two ($x^2 = 1$, for all x),

(i) By [SG 3], $\langle z, u\rangle \equiv_G \langle x, y\rangle$ entails $u = zxy$;
(ii) [SG 0] and [SG 1] imply $\{x, y\} \subseteq D_G(x, y)$;
(iii) For $x \in G$, $D_G(1, x) = \{z \in G\colon z\langle 1, x\rangle \equiv_G \langle 1, x\rangle\}$.

(b) If $G_i = \langle G_i, \equiv_{G_i}, -1\rangle$ are π-SGs, $i = 1, 2$, a **morphism of π-SGs**, $h : G_1 \to G_2$, is a morphism of the underlying groups, such that $h(-1) = -1$ and

$$\forall a, b, c, d \in G_1,\ \langle a, b\rangle \equiv_{G_1} \langle c, d\rangle \ \Rightarrow\ \big\langle h(a), h(b)\big\rangle \equiv_{G_2} \big\langle h(c), h(d)\big\rangle.$$

Write **π-SG** and **π-RSG** for the categories of π-SGs and π-RSGs, respectively.

(c) If $\langle A, T\rangle$ is a p-ring, $G_T(A) = \langle G_T(A), \equiv_T, -1\rangle$ denotes the **π-SG associated to $\langle A, T\rangle$**, as constructed in 8.9. Note that

* If $\langle A, T\rangle$ is non-trivial, then $G_T(A)$ is a π-RSG;
* If $\langle A, T\rangle$ is trivial, then $G_T(A)$ is the trivial **special group**, $\{1\}$.

In the case that $T = \Sigma A^2$, write $G_{red}(A)$ for $G_T(A)$.

Lemma 8.14 proves that the assignment of its π-SG to any p-ring is functorial. Furthermore,

Proposition 6.4. *With notation as in* 8.14*, the* π-**SG** *functor from* **p-Ring** *to* **π-SG***, given by,*

$$\begin{cases} \langle A, T\rangle \longmapsto G_T(A), \\ \langle A_1, T_1\rangle \xrightarrow{h} \langle A_2, T_2\rangle \longmapsto G_{T_1}(A_1) \xrightarrow{h^\pi} G_{T_2}(A_2) \end{cases}$$

is a geometrical functor.

Proof. Regarding products, it is enough to check that the π-SG functor preserves binary products. If $\langle A_i, T_i\rangle$, $i = 1, 2$, are p-rings, then their product is the p-ring $\langle A, T\rangle = \langle A_1 \times A_2, T_1 \times T_2\rangle$; note that $\langle A, T\rangle$ is trivial iff both components are trivial. Clearly, $p_i : \langle A, T\rangle \to \langle A_i, T_i\rangle$, the canonical coordinate projections, are p-ring morphisms. Moreover, we have $A^* = A_1^* \times A_2^*$, $T^* = T_1^* \times T_2^*$ and

$$\langle x, y\rangle \in D_T\big(\langle 1, 1\rangle, \langle u, v\rangle\big) \text{ iff } x \in D_{T_1}(1, u) \text{ and } y \in D_{T_2}(1, v).$$

It is then straightforward to check that $G_T(A) = G_{T_1}(A_1) \times G_{T_2}(A_2)$, as well as that the projections are precisely p_i^π, $i = 1, 2$, as needed. It remains to check that the π-SG functor preserves right-directed colimits. This is the content of the following

Fact 6.5. *Let $\langle I, \leqslant\rangle$ be a rd-poset and let $\mathcal{A} = \langle\langle A_i, T_i\rangle;\ \{h_{ij}\colon\ i \leqslant j \text{ in } I\}\rangle$ be an inductive system of p-rings and p-ring morphisms. Let $\mathcal{G} = \langle G_{T_i}(A_i);\ \{h_{ij}^\pi\colon\ i \leqslant j \text{ in } I\}\rangle$ be the associated inductive system of π-SGs.*

(a) Let $\langle A;\ \{h_i\colon\ i \in I\}\rangle = \varinjlim A_i$ in the category of rings and set $T = \bigcup_{i\in I} h_i(T_i)$. Then, $\langle A, T\rangle$ is a p-ring, $h_i : \langle A_i, T_i\rangle \to \langle A, T\rangle$ is a morphism of p-rings and

$$\big\langle\langle A, T\rangle;\ \{h_i\colon\ i \in I\}\big\rangle = \varinjlim \mathcal{A} \textit{ in the category of p-rings.}$$

Moreover, $\langle A, T\rangle$ is a trivial p-ring

$$\textit{iff } \mathfrak{t} = \big\{i \in I\colon\ \langle A_i, T_i\rangle \textit{ is a trivial p-ring}\big\} \textit{ is cofinal in } I,$$
$$\textit{iff } \big\{i \in I\colon\ \langle A_i, T_i\rangle \textit{ is a proper p-ring}\big\} \textit{ is not cofinal in } I.$$

(b) $\langle G_T(A);\ \{h_i^\pi\colon\ i \in I\}\rangle = \varinjlim \mathcal{G}$.

Proof. Since $A = \varinjlim_{i\in I} A_i$ in the category of rings (a category of algebraic structures, where equality is the only relation symbol), by 2.4(a) we know that

(1) $A = \bigcup_{i\in I} h_i(A_i)$;
(2) $\forall i \in I$ and $x \in A_i$, $h_i(x) = 0 \Rightarrow \exists k \geqslant i$ such that $h_{ik}(x) = 0$.

We first verify that T is a preorder of A. If $x, y \in T$, there are $i, j \in I$, together with $u \in T_i$ and $v \in T_j$ such that $h_i(u) = x$ and $h_j(v) = y$. Select $q \geqslant i, j$, and consider $w_x = h_{iq}(u)$ and $w_y = h_{jq}(v)$, both in T_q (the h_{ij} are p-ring morphisms). Then, $h_q(w_x) = x$ and $h_q(w_y) = y$, $w_x \overset{+}{\cdot} w_y \in T_q$ and $x \overset{+}{\cdot} y = h_q(w_x \overset{+}{\cdot} w_y) \in T$, showing that $T \overset{+}{\cdot} T \subseteq T$. Similarly, one verifies that $A^2 \subseteq T$, and that $-1 \in T \Leftrightarrow \mathfrak{t} = \{i \in I\colon\ \langle A_i, T_i\rangle$ is the trivial p-ring$\}$ is cofinal in I. Since (up to isomorphism) inductive limits do not change upon restricting the index set to a cofinal subset, if $\mathfrak{t}$ is cofinal in I, it follows that items (a) and (b) in the statement hold true (where for all i, j, h_{ij}^π and h_i^π are the only possible maps from $\{1\}$ to $\{1\}$). If $\mathfrak{t}$ *is not cofinal* in I, then the fact that $\langle I, \leqslant\rangle$ is right-directed immediately implies that its complement is cofinal in I and, as above, we may assume that for all $i \in I$, $\langle A_i, T_i\rangle$ is a *proper* p-ring, which entails that $\langle A, T\rangle$ is also a *proper* p-ring. The very definition of T guarantees that h_i is a p-ring morphism and that the analog of (1) above holds for T and T_i. By 2.4(a), to finish the proof that $\langle A, T\rangle = \varinjlim_{i\in I}\langle A_i, T_i\rangle$ it suffices to check that

$$\forall i \in I,\ \forall x \in A_i,\ h_i(x) \in T \ \Rightarrow\ \exists k \geqslant i \text{ such that } h_{ik}(x) \in T_k \tag{I}$$

(corresponding to 2.4(a.2) for the predicate T). If $h_i(x) \in T$, then there is $j \in I$ and $y \in T_j$ such that $h_j(y) = h_i(x)$. Select $q \geqslant i, j$ and consider $w_x = h_{iq}(x) \in A_q$ and $w_y = h_{jq}(y) \in T_q$. Note that, $h_q(w_x) = h_q(h_{iq}(x)) = h_i(x) = h_j(y) = h_q(h_{jq}(y)) = h_q(w_y) \in T_q$, and so (2) above guarantees that there is $k \geqslant q$ such that $h_{qk}(w_x) = h_{qk}(w_y) \in T_k$. But then

$$h_{ik}(x) = h_{qk}(h_{iq}(x)) = h_{qk}(w_x) = h_{qk}(w_y) \in T_k,$$

as needed to establish (I) and to complete the proof of (a).

(b) Recall our working hypothesis that all $\langle A_i, T_i\rangle$ are proper p-rings. To ease notation write

* $G_i = \langle G_i, \equiv_i, -1\rangle$ for the π-RSG $G_{T_i}(A_i) = \langle G_{T_i}(A_i), \equiv_{T_i}, -1\rangle$ $(i \in I)$;
* $G = \langle G, \equiv_T, -1\rangle$ for $G_T(A) = \langle G_T(A), \equiv_T, -1\rangle$.
* The elements of G_i and G will still be denoted by a^{T_i} and a^T, respectively.

By Lemma 8.14, $\mathcal{G} = \langle G_i;\ \{h^\pi_{ij}\colon i \leqslant j \text{ in } I\}\rangle$ is an inductive system of π-RSGs, $h^\pi_i : G_i \to G$ is a π-SG morphism and the following diagram is commutative, for $i \leqslant j$:

$$\begin{tikzcd} G_i \arrow[rr, "h^\pi_{ij}"] \arrow[dr, "h^\pi_i"'] & & G_j \arrow[dl, "h^\pi_j"] \\ & G & \end{tikzcd}$$

that is, $\mathfrak{G} = \langle G;\ \{h^\pi_i\colon i \in I\}\rangle$ is a dual cone over $\mathcal{G}$. By Fact 2.4(a), to show that $\mathfrak{G} = \varinjlim \mathcal{G}$ we must verify the following conditions:

(A) $G = \bigcup_{i\in I} h^\pi_i(G_i)$;
(B) For all $i \in I$ and $x, y, u, v \in A^*_i$,

$$\text{(B1)} \quad h^\pi_i\big(x^{T_i}\big) = 1 \Rightarrow \exists k \geqslant i \text{ such that } h^\pi_{ik}\big(x^{T_i}\big) = 1;$$

and by Lemma 8.13(d),

$$\text{(B2)} \quad \begin{cases} h_i(xy)^T = h^\pi_i((xy)^{T_i}) = h^\pi_i((uv)^{T_i}) = h_i(uv)^T \text{ and } h_i(xu) \in D_T(1, h_i(uv)) \\ \Downarrow \\ \exists k \geqslant i \text{ such that } h_{ik}(xyuv) \in T_k \text{ and } h_{ik}(xu) \in D_{T_k}(1, h_{ik}(uv)). \end{cases}$$

To establish (A) it suffices to verify that $A^* = \bigcup_{i\in I} h_i(A^*_i)$; once this is shown, we get $T^* = T \cap A^* = \bigcup_{i\in I} h_i(T^*_i)$, and so, $G = A^*/T^* = \bigcup_{i\in I} h^\pi_i(A^*_i/T^*_i)$. Since any ring morphism preserves units, it is enough to check that $A^* \subseteq \bigcup_{i\in I} h_i(A^*_i)$. Suppose $x \in A^*$, i.e., there is $y \in A$ such that $xy = 1$. By equality (1) (at the beginning of the proof), there are $i, j \in I$

and $a \in A_i$, $b \in A_j$ such that $h_i(a) = x$ and $h_j(b) = y$. Select $q \geqslant i, j$ and set $c = h_{iq}(a)$, $d = h_{jq}(b)$. Then, $h_q(c) = x$, $h_q(d) = y$ and we have $h_q(cd) = xy = 1 = h_q(1)$. Item (2) (at the beginning of the proof) applied to $cd - 1$ yields $k \geqslant q$ such that $h_{qk}(cd) = h_{qk}(1) = 1$, that is, $h_{qk}(c) \in A_k^*$. Since, $h_k(c) = x$, the required inclusion is proven.

The implication (B1) is immediate from (I), because for all $a \in A_i^*$, $h_i^\pi(a^{T_i}) = 1$ iff $h_i(a) \in T$. Note that we have just shown that $G = \varinjlim_{i \in I} G_i$ in the category of groups. It remains to verify (B2); its antecedent means

$$h_i(xyuv) \in T \text{ and } \exists t_1, t_2 \in T \text{ such that } h_i(xu) = t_1 + t_2 h_i(uv).$$

Since $T = \bigcup_{j \in I} h_j(T_j)$ and I is right-directed, a standard argument yields $k \geqslant i$ and representatives b_ℓ of t_ℓ (i.e., $h_\ell(b_\ell) = t_\ell$ for $\ell = 1, 2$) and a of $h_i(xyuv)$ **in $\boldsymbol{T_k}$** so that $h_{ik}(xu) = b_1 + b_2 h_{ik}(uv)$. Hence, $a = h_{ik}(xyuv) \in T_k$ and $h_{ik}(xu) \in D_{T_k}(1, h_{ik}(uv))$, as required. □

Next we discuss presheaf bases of p-rings over Boolean spaces and the presheaf bases of π-SGs that arise from them. We begin with the following

Remark 6.6. Let $\mathcal{B}$ be a basis for the topological space X and let $\mathfrak{P}: \mathcal{B} \to$ **p-Ring**,

$$\begin{cases} U \in \mathcal{B} \longmapsto \langle \mathfrak{P}(U), T(U) \rangle; \\ U \subseteq_o V \longmapsto p_{VU}: \mathfrak{P}(U) \to \mathfrak{P}(V), \end{cases}$$

be a presheaf basis of p-rings over $\mathcal{B}$. The extensionality condition [ext] in Definition 8.2 applies also to the predicate T, that is interpreted as the preorder on each ring of sections. Since the restriction maps are p-ring morphisms, the assignments

$$\begin{cases} U \in \mathcal{B} \longmapsto T(U); \\ U \subseteq_o V \longmapsto (p_{VU})_{|T(V)}: T(V) \to T(U), \end{cases}$$

constitute a presheaf basis of preorders, $\mathfrak{T}$. Hence:

(1) Every presheaf basis of p-rings, $\mathfrak{P}$, comes equipped with a presheaf basis of preorders, $\mathfrak{T}$;
(2) The language of presheaves applies to $\mathfrak{T}$. For instance, for $U \in \mathcal{B}$, we may require that $\mathfrak{T}$ be **finitely complete over $\boldsymbol{U}$**, defined in 8.2(d.1). Note that this *does not* imply that $\mathfrak{P}$ is finitely complete over U, since a finite set of compatible sections in $|\mathfrak{P}|$, outside $|\mathfrak{T}|$, may not have a gluing in $\mathfrak{P}$.

Theorem 6.7. *Let $\mathcal{B}$ be the BA of clopens of the Boolean space X. With notation as in* 6.6*, let $\mathfrak{P}: \mathcal{B} \to$* **p-Ring** *be a presheaf basis of p-rings over $\mathcal{B}$, with associated presheaf of preorders, $\mathfrak{T}$, both of which are assumed to be finitely complete over each $U \in \mathcal{B}$. Let $\mathfrak{G}_\mathfrak{T} = \mathfrak{G}: \mathcal{B} \to$* **$\boldsymbol{\pi}$-SG** *be the composition of $\mathfrak{P}$ with the π-SG functor, i.e.,*

$$\begin{cases} U \in \mathcal{B} \longmapsto \mathfrak{G}(U) = G_{T(U)}(\mathfrak{P}(U)); \\ U \subseteq_o V \longmapsto p_{VU}^\pi: \mathfrak{G}(V) \to \mathfrak{G}(U). \end{cases}$$

Then,

(a) *$\mathfrak{G}$ is a finitely complete presheaf basis of π-SGs over $\mathcal{B}$. For $x \in X$, let $\mathcal{B}_x = \{U \in \mathcal{B}: x \in U\}$ be the filter of clopen neighborhoods of x in X. If $\mathfrak{P}_x = \langle\langle \mathfrak{P}_x, T_x\rangle; \{p_{Ux}: U \in \mathcal{B}_x\}\rangle$ is the stalk of $\mathfrak{P}$ at x, then $\mathfrak{G}_x = \langle G_{T_x}(\mathfrak{P}_x); \{p^{\pi}_{Ux}: U \in \mathcal{B}_x\}\rangle$ is the stalk of $\mathfrak{G}$ at x.*

(b) *The set $\tau_{\text{prop}} = \{x \in X: T_x$ is a proper preorder in $\mathfrak{P}_x\}$ is closed in X. Moreover,*
   (1) *For all $U \in \mathcal{B}$, $U \cap \tau_{\text{prop}} \neq \emptyset \Leftrightarrow \mathfrak{P}(U)$ is a proper p-ring. In particular, if $T(X)$ is a proper preorder in $\mathfrak{P}(X)$, then $\tau_{\text{prop}} \neq \emptyset$;*
   (2) *For all $x \in \tau_{\text{prop}}$, $\mathfrak{G}_x$ is a π-RSG.*

Proof. Since the theories of p-rings and of π-SGs are geometrical and the π-SG functor from **p-Ring** to **π-SG** is geometrical (Proposition 6.4), all statements in (a) are immediate consequences of Theorem 3.3. As for (b), with notation as in Definition 3.1, observe that 5.7(b) implies that $\tau^c_{\text{prop}} = X \setminus \tau_{\text{prop}}$ is the Feferman–Vaught value of the atomic sentence $-1 \in T$ (Proposition 3.2(a)):

$$\tau^c_{\text{prop}} = \mathfrak{v}_{\mathfrak{P}}(-1 \in T) = \bigcup\bigl\{U \in \mathcal{B}: \mathfrak{P}(U) \models -1 \in T(U)\bigr\},$$

whence τ^c_{prop} is open and (1) holds. If $x \in \tau_{\text{prop}}$, (1) entails that for all $U \in \mathcal{B}_x$, $T(U)$ is a *proper* preorder of the ring $\mathfrak{P}(U)$, and (2) follows from the equivalences in item (a) of Fact 6.5. □

Being the ring of global sections of a sheaf of rings whose stalks are fields, Theorem 2.10 in [7] guarantees that any vN-ring, R, is a ring with many units and so, by Proposition 4.3(b), for all $0 \neq e \in B(R)$, the ring Re is also a ring with many units.

By Theorems 3.15 and 3.16 of [7], if A is a ring with many units where $2 \in A^*$, all residue fields of A have at least 7 elements, and T is a proper preorder of A, then the π-SG $G_T(A)$ associated to $\langle A, T\rangle$ is, in fact, a *reduced special group*, that faithfully represents the reduced theory, modulo T, of quadratic forms over free A-modules, with coefficients in A^*. If R is a vN-ring in which 2 is a unit and T is a *strict* preorder of R, then for all $P \in \text{Spec}(R)$, T/P is a proper preorder of the residue field R/P, and so all residue fields of R are formally real. Hence, the results in [7] apply, yielding, in particular, that, **$\mathbf{G}_T(\mathbf{R})$ is a reduced special group whenever $\boldsymbol{T}$ is a strict preorder of $\boldsymbol{R}$**. Proposition 6.8 below, one of the main reduction steps in our argument, will show that, in fact, if T is any **proper** preorder of a vN-ring R in which 2 is a unit, then $G_T(R)$ is a reduced special group.

Henceforth in this section, fix a **proper preordered vN-ring**, $\langle \boldsymbol{R}, \boldsymbol{T}\rangle$, **where $\mathbf{2} \in \boldsymbol{R}^*$**. Note that item (1) in Theorem 6.7(b), together with relation ($\mathfrak{T}_P$) in Lemma 5.10, guarantee that

$$\tau_{\text{prop}} = \bigl\{P \in \text{Spec}(R):\ T/P \text{ is a proper preorder in } R/P\bigr\}$$

is a **non-empty closed set** in Spec(R). Define $I = \bigcap \tau_{\text{prop}}$; clearly, I is an ideal in R. Let $q_I: R \to R/I$ be the canonical quotient morphism. Clearly, 2 is a unit in the vN-ring R/I (5.5(d)). We now have

Proposition 6.8. *With notation as above,*

(a) *For all $P \in \text{Spec}(R)$, $I \subseteq P \Leftrightarrow P \in \tau_{\text{prop}}$. Moreover, if τ_{prop} is endowed with the topology induced by* Spec(R)*, then, $Q \in \text{Spec}(R/I) \longmapsto q_I^{-1}(Q) \in \tau_{\text{prop}}$ is a homeomorphism.*
(b) *T/I is a* ***strict*** *preorder on R/I.*
(c) *For $a \in R$, the following are equivalent*:

(1) $a/I \in T/I$;
(2) $a \in T$.

(d) $q_I^\pi : G_T(R) \to G_{T/I}(R/I)$ *is an* ***isomorphism*** *of reduced special groups.*

Proof. (a) For the first assertion, it suffices to verify ($\Rightarrow$). Suppose $e \in B(R)$ is such that $e \notin P$; hence, $e \notin I$, and its definition yields $Q \in \tau_{\text{prop}}$ such that $e \notin Q$. Hence, every clopen neighborhood of P has non-empty intersection with τ_{prop}; since this set is closed, we get $P \in \tau_{\text{prop}}$, as needed. The equivalence just proven shows, with notation as in 5.5(d), that $V(I) = \tau_{\text{prop}}$; the remaining assertion follows from that same result.

(b) Clearly, T/I is a preorder of R/I; since $\tau_{\text{prop}} \neq \emptyset$, T/I is a proper preorder of R/I. By (a), we may identify $\text{Spec}(R/I)$ with τ_{prop}; if $P \in \tau_{\text{prop}}$, then

$$(R/I)/(P/I) = R/P \quad \text{and} \quad (T/I)/(P/I) = T/P.$$

Since T/P is a proper preorder of R/P, the contention is established.

(c) (1) $\Rightarrow$ (2): If $a/I \in T/I$, there is $t \in T$ such that $a/I = t/I$ and so, $a - t \in P$, for all $P \in \tau_{\text{prop}}$. Let e be the idempotent associated to $a - t$. Then,

(i) From $e(a - t) = a - t$, it follows that $(a - t)(1 - e) = 0$, i.e., $a(1 - e) = t(1 - e)$.
(ii) For all $P \in \tau_{\text{prop}}$, $e \in P$, that is, $Z(e) \cap \tau_{\text{prop}} = \emptyset$. If $Q \in Z(e)$, then $T/Q = R/Q$, whence $a/Q \in T/Q$. Since this holds for all $Q \in Z(e)$, Lemma 5.8(c) guarantees that $ae \in T$.

The last equality in (i) entails $a = t(1 - e) + ae$; because t, $(1 - e)$, $ae \in T$, we get $a \in T$, as needed. That (2) implies (1) is obvious.

(d) Since $q_I : \langle R, T\rangle \to \langle R/I, T/I\rangle$ is a morphism p-rings, Lemma 8.14 guarantees that q_I^π is a morphism of π-SGs; since it is clearly surjective, it will be an isomorphism iff it reflects representation, that is, for $a, b \in R^*$,

$$(a/I)^{T/I} \in D_I\big(1, (b/I)^{T/I}\big) \Rightarrow a^T \in D_T\big(1, b^T\big), \tag{I}$$

where D_I denotes representation in $G_{T/I}(R/I)$. Because the π-groups in question are *reduced* (8.11(b)), (I) implies that q_I^π is injective. The antecedent means that $a/I = (x + yb)/I$, for some $x, y \in T$; consequently, $a - (x + yb)/I \in T/I$ and item (c) entails $a - (x + yb) \in T$. Setting $t = a - (x + yb)$, we have $a = (x + t) + yb$, with $x + t$, $y \in T$, establishing (I). As observed in the paragraphs preceding the statement of this proposition, since T/I is a *strict* preorder on R/I, $G_{T/I}(R/I)$ is, in fact, a reduced special group, and so the same must be true of $G_T(R)$, ending the proof. $\square$

Summarizing, we can state

Corollary 6.9. *Let R be a vN-ring where 2 is a unit and let T be a proper preorder of R. With notation as in* 5.6, 5.10, 6.7 *and* 6.8, *let $\langle \mathfrak{R}, \mathfrak{T}\rangle$ be the presheaf basis of p-rings over $\mathcal{Z}$, associated to $\langle R, T\rangle$. Then,*

(a) $\mathfrak{G} = \mathfrak{G}_{\mathfrak{T}}(\mathfrak{R})$ *is a finitely complete presheaf basis of* ***special groups****, such that*
(1) *For all $P \in \text{Spec}(R)$, the stalk of $\mathfrak{G}$ at P, $\mathfrak{G}_P$, is the special group $G_{T/P}(R/P)$, associated to the preorder T/P of the field R/P;*

(2) $\tau_{\text{prop}} = \{P \in \text{Spec}(R)$: $G_{T/P}(R/P)$ *is a non-trivial RSG*$\}$ *is closed and non-empty in* $\text{Spec}(R)$.

(b) *If T is a* ***strict*** *preorder of R, then for all $e \in B(R)$, $\mathfrak{G}(Z(e)) = G_{Te}(Re)$ is a reduced special group and for all $P \in \text{Spec}(R)$, $\mathfrak{G}_P$ is the* reduced *special group $G_{T/P}(R/P)$.*

Proof. We comment only on the first assertion in (a), since the others follow directly from the preceding discussion. If $0 \neq e$ is an idempotent in R, we have two possibilities:

* Te is a proper preorder of Re: In this case, since Re is a vN-ring in which 2 is a unit, it follows from Proposition 6.8 that $\mathfrak{G}(Z(e)) = G_{Te}(Re)$ is a reduced special group*;*
* $Te = Re$: Here we get $G_{Te}(Re) = \{1\}$, the trivial *special group*.

In any case, $\mathfrak{G}$ is a presheaf of special groups, as stated. $\square$

7. The [SMC] property for preordered vN-rings

In this section we apply the K-theory of special groups developed in [5] and [8] to associate to a presheaf basis of special groups, $\mathfrak{G}$, a graded ring of presheaf bases of groups of exponent two

$$k_*\mathfrak{G} = \langle k_0\mathfrak{G}, k_1\mathfrak{G}, \ldots, k_n\mathfrak{G}, \ldots\rangle$$

together with a sequence $\omega = \langle \omega_1, \ldots, \omega_n, \ldots\rangle$ of morphisms of presheaf bases of groups,

$$\omega_n : k_n\mathfrak{G} \to k_{n+1}\mathfrak{G}, \quad n \geqslant 1,$$

corresponding to multiplication by $\lambda(-1)$. K-theoretic notation is as in 1(1).

Theorem 7.1. *Let X be a Boolean space and let $\mathcal{B}$ be the Boolean algebra of clopens in X. Let $\mathfrak{G}$ be a finitely complete presheaf basis of special groups over $\mathcal{B}$, with restriction morphisms $\{\rho_{VU}$: $U \subseteq V$ in $\mathcal{B}\}$.*

(a) *For each $n \geqslant 0$, the assignments*

$$\begin{cases} \;\; U \in \mathcal{B} \longmapsto k_n\mathfrak{G}(U); \\ U \subseteq_o V \longmapsto (\rho_{UV})_n : k_n\mathfrak{G}(V) \to k_n\mathfrak{G}(U), \end{cases}$$

constitute a finitely complete presheaf basis of groups, $k_n\mathfrak{G}$, such that

(1) *For all $n, m \geqslant 0$ and $U \in \mathcal{B}$, $\eta \in k_n\mathfrak{G}(U)$ and $\xi \in k_m\mathfrak{G}(U) \Rightarrow \eta\xi \in k_{n+m}\mathfrak{G}(U)$;*

(2) *For all $x \in X$, the map defined on generators by*

$$\big(\lambda(a_1)\cdots\lambda(a_n)\big)_x \in (k_n\mathfrak{G})_x \longmapsto \lambda(a_{1x})\cdots\lambda(a_{nx}) \in k_n\mathfrak{G}_x$$

extends to a (*natural*) ***isomorphism*** *from $(k_n\mathfrak{G})_x$ to $k_n\mathfrak{G}_x$, by which these groups will be identified.*

(b) *For $n \geqslant 1$, define $\omega_n = \{\omega_{nU}\colon U \in \mathcal{B}\}: k_n\mathfrak{G} \to k_{n+1}\mathfrak{G}$ by*

$$\text{For each } U \in \mathcal{B} \text{ and } \eta \in k_n\mathfrak{G}(U),\ \omega_{nU}(\eta) = \lambda(-1_{|U})\eta.$$

Then, ω_n is a morphism of presheaf bases of groups and for each $x \in X$, $\omega_{nx}: k_n\mathfrak{G}_x \to k_{n+1}\mathfrak{G}_x$ is precisely multiplication by $\lambda(-1_x)$, where $-1_x \in G_x$.

(c) *For $U \in \mathcal{B}$, if $\mathfrak{G}_x$ is* [SMC] *for all $x \in U$, then $\mathfrak{G}(U)$ is* [SMC]. *In particular, if every stalk of $\mathfrak{G}$ is* [SMC], *then $\mathfrak{G}(X)$, the SG of global sections of $\mathfrak{G}$, is* [SMC].

Proof. (a) By item (1) in Proposition 2.7, the K-theory functor from **SG** to **2-Grp** is geometrical, connecting the geometrical theories of special groups and groups of exponent 2 (see 2.2). Hence, Theorem 3.3 applies to yield the desired conclusions.

(b) It is clear that for $U \in \mathcal{B}$, ω_{nU} is a group morphism and that, for $U \subseteq V$ in $\mathcal{B}$ and $\eta \in k_n\mathfrak{G}(V)$, $\omega_{nV}(\eta)_{|U} = \omega_{nU}(\eta_{|U})$; hence, ω_n is a morphism of presheaf bases, as in 8.2(g). For $x \in X$, let $\omega_{nx} = \varinjlim_{U \in \mathcal{B}_x} \omega_{nU}$; by (a.2), given $\xi \in k_n\mathfrak{G}_x$, there is $U \in \mathcal{B}_x$ and $\eta \in k_n\mathfrak{G}(U)$ such that $\eta_x = \xi$. Then, Fact 2.4(c) and another application of (a.2) yield

$$\omega_{nx}(\xi) = \omega_{nx}(\eta_x) = \big(\omega_{nU}(\eta)\big)_x = \big(\lambda(-1_{|U})\eta\big)_x = \lambda(-1)_x\eta_x = \lambda(-1_x)\xi,$$

showing that ω_{nx} is multiplication by $\lambda(-1_x)$, as claimed.

(c) If $n \geqslant 1$, since $k_n\mathfrak{G}$ is a presheaf basis over $\mathcal{B}$, (a.2) and Proposition 3.2(c) imply that the map

$$\gamma_n^U : k_n\mathfrak{G}(U) \to \Gamma_n(U) = \prod_{x \in U} k_n\mathfrak{G}_x$$

is a group embedding, where $\Gamma_n(U)$ has the product structure, defined coordinatewise. By item (b), the following diagram commutes:

$$\begin{tikzcd} k_n\mathfrak{G}(U) \arrow[r, "\gamma_n^U"] \arrow[d, "\omega_{nU}"'] & \Gamma_n(U) \arrow[d, "\Pi_{x\in U}\omega_{nx}"] \\ k_{n+1}\mathfrak{G}(U) \arrow[r, "\gamma_{n+1}^U"'] & \Gamma_{n+1}(U) \end{tikzcd}$$

Now let $\eta \in k_n\mathfrak{G}(U)$ be such that $\omega_{nU}(\eta) = \lambda(-1_{|U})\eta = 0$ in $k_{n+1}\mathfrak{G}(U)$. By commutativity of the diagram above, we get that for all $x \in U$, $\omega_{nx}(\eta_x) = \lambda(-1_x)\eta_x = 0$ in $k_{n+1}\mathfrak{G}_x$; since $\mathfrak{G}_x$ is [SMC], we conclude that $\eta_x = 0$ in $k_n\mathfrak{G}_x$, for all $x \in U$. But then, the extensionality of $k_n\mathfrak{G}$ entails $\eta = 0$ in $k_n\mathfrak{G}(U)$, as needed to verify that $\mathfrak{G}(U)$ is [SMC]. $\square$

We now have

Theorem 7.2. *If R is a vN-ring in which 2 is a unit and T is a proper preorder of R, then $G_T(R)$ is* [SMC]. *In particular, if R is a formally real vN-ring, $G_{red}(R)$ is* [SMC].

Proof. By Proposition 6.8(d) it suffices to show that the result holds for a **strict** preorder on R. Indeed, with notation as in 6.8, since k_* is a functor, the map $(q_I^\pi)_* : k_*G_T(R) \to k_*G_{T/I}(R/I)$ is an isomorphism, and one of these groups will be [SMC] iff the same is true of the other.

Assume that T is a strict preorder on R. By Corollary 6.9(b), the stalk at each $P \in \mathrm{Spec}(R)$ of the presheaf basis, $\mathfrak{G}$, of RSGs associated to $\langle R, T\rangle$, is the RSG corresponding to the proper preorder T/P on the field R/P, i.e., $G_{T/P}(R/P)$. Since R/P is a formally real field, it follows from Theorem 6.4 and (the proof of) Theorem 6.9 in [5] that $G_{T/P}(R/P)$ is [SMC]. Hence, for all $P \in \mathrm{Spec}(R)$, $\mathfrak{G}_P$ is [SMC] and the desired conclusion follows from item (c) of Theorem 7.1. □

8. Basic results and constructions

A. Presheaves of first-order structures

8.1. Notation. Let X be a topological space.

(a) $\Omega(X)$ denotes the collection of opens of X, while $B(X)$ is the Boolean algebra (BA) of clopens of X.
(b) A subset $\mathcal{B}$ of $\Omega(X)$ is a **basis** for X if it is closed under finite intersections and all opens in X are the union of elements of $\mathcal{B}$. Whenever convenient, we assume that $\emptyset$, $X \in \mathcal{B}$.
(c) Write $U \subseteq_o V$ to mean that $V \in \Omega(X)$ and U is an *open* subset of V.

Definition 8.2. (Essentially in [9].) Let X be a topological space and let L be a first-order language with equality. Let $\mathcal{B}$ be a basis for X.

(a) A **presheaf basis of L-structure over $\mathcal{B}$**, is a contravariant functor, $\mathfrak{A} : \mathcal{B} \to \boldsymbol{L}$**-mod** (2.1(c)),

$$U \longmapsto \mathfrak{A}(U) \quad \text{and} \quad U \subseteq_o V \longmapsto \alpha_{VU} : \mathfrak{A}(V) \to \mathfrak{A}(U),$$

satisfying the following *separation or extensionality* condition

$$\begin{array}{l}\text{If } \bar{s} \in \mathfrak{A}(U)^n,\ R \text{ is a } n\text{-ary relation in } L,\ U \in \mathcal{B}, \text{ and} \\ \{U_i \subseteq_o U\colon\ i \in I\} \subseteq \mathcal{B} \text{ is a covering of } U, \\ \text{then, } \forall i \in I,\ \mathfrak{A}(U_i) \models R[\alpha_{UU_i}(s_1), \ldots, \alpha_{UU_i}(s_n)] \Rightarrow \mathfrak{A}(U) \models R[\bar{s}].\end{array} \tag{ext}$$

For $U \in \mathcal{B}$, $\mathfrak{A}(U)$ is called the **L-structure of sections of $\mathfrak{A}$ over U** and the **L**-morphism $\alpha_{VU} : \mathfrak{A}(V) \to \mathfrak{A}(U)$, $U \subseteq_o V$ in $\mathcal{B}$, is the **restriction morphism**; when no confusion is extant, this morphism is written as $\cdot_{|U}$. In this notation, condition [ext] may be expressed as

$$\forall i \in I,\ \mathfrak{A}(U_i) \models R[\bar{s}_{|U_i}] \ \Rightarrow\ \mathfrak{A}(U) \models R[\bar{s}] \quad \big(\bar{s}_{|U_i} = \langle s_{1|U_i}, \ldots, s_{n|U_i}\rangle\big).$$

We shall assume, without loss of generality, that the L-structures $\mathfrak{A}(U)$ are *pairwise disjoint* and that $\mathfrak{A}(\emptyset) = \{0\}$, the singleton L-structure, with trivially defined function, relation and constant symbols from L.

The set $|\mathfrak{A}| = \bigcup_{U \in \mathcal{B}} \mathfrak{A}(U)$ is the **domain of $\mathfrak{A}$** and an element of $|\mathfrak{A}|$ is called a **section of $\mathfrak{A}$**. For each $s \in |\mathfrak{A}|$, let

$$Es = \text{the unique } U \in \mathcal{B} \text{ such that } s \in \mathfrak{A}(U),$$

called the **extent of s**. A section whose extent is X is a **global section of $\mathfrak{A}$**. We say that $s, t \in |\mathfrak{A}|$ are **compatible**, if $s_{|Es \cap Et} = t_{|Es \cap Et}$. Clearly, sections with disjoint extents are compatible.

(b) If Σ is a set of sentences in L, a presheaf basis of **models of Σ** over $\mathcal{B}$ is a presheaf basis of L-structures over $\mathcal{B}$, such that **for all $U \neq \emptyset$** in $\mathcal{B}$, $\mathfrak{A}(U)$ is a model of Σ.

(c) A **presheaf of L-structures over X** is a presheaf basis such that $\mathcal{B} = \Omega(X)$.

(d) Let $\mathfrak{A}$ be a presheaf basis of L-structures over $\mathcal{B}$ and let $U \in \mathcal{B}$.

(1) $\mathfrak{A}$ is **finitely complete (fc)** over U if for all *finite* $S \subseteq |\mathfrak{A}|$ such that $U = \bigcup_{s \in S} Es$, if the elements of S are pairwise compatible, then there is $t \in \mathfrak{A}(U)$ such that $s = t_{|Es}$, for all $s \in S$; since the extensionality condition [ext] applies to equality, this t is *unique* and is called the **gluing of S** in $\mathfrak{A}$;
(2) $\mathfrak{A}$ is **complete** over U, if the condition in (1) holds for *arbitrary* subsets S of $|\mathfrak{A}|$ satisfying $U = \bigcup_{s \in S} Es$.

(e) $\mathfrak{A}$ is **complete or finitely complete (fc) over $\mathcal{B}$** if it is complete or fc over every $U \in \mathcal{B}$, respectively.

(f) A **sheaf of L-structures over X** is a presheaf over X that is complete over all $U \in \Omega(X)$.

(g) If $\mathfrak{A}$, $\mathfrak{B}$ are presheaf bases over $\mathcal{B}$, a **morphism**, $f : \mathfrak{A} \to \mathfrak{B}$, is a natural transformation of contravariant functors, that is, a family of L-morphisms, $f = \{\mathfrak{A}(U) \xrightarrow{f_U} \mathfrak{B}(U)\colon U \in \mathcal{B}\}$, such that for $U \subseteq_o V$ in $\mathcal{B}$ and $x \in \mathfrak{A}(V)$, $f_V(x)_{|U} = f_U(x_{|U})$.

Remark 8.3.

(a) It is straightforward that the extensionality condition [ext] in Definition 8.2(a) holds for *conjunctions of atomic formulas*. If $U \in \mathcal{B}$ is **compact**, it suffices to consider finite coverings of U.
(b) If equality is the only relation symbol in the language, then extensionality is standard *separation*, i.e., two sections with the same extent which are locally identical, are equal.
(c) Presheaf bases are important because frequently the data for a sheaf are given only on a basis for the topology of X, as in the case of the affine scheme of a commutative ring, as in Section 5.

The following result gives, among other things, a useful criterion for a contravariant functor from the BA of clopens of a Boolean space to **L-mod** to be a finitely complete and extensional presheaf basis.

Proposition 8.4. *Let X be a Boolean space and let $\mathcal{B}$ be the Boolean algebra of clopens in X. Let $\mathfrak{A} : \mathcal{B} \to$ **L-mod** be a contravariant functor.*

(a) *The following are equivalent*:
  (1) *$\mathfrak{A}$ is an extensional, finitely complete presheaf basis over $\mathcal{B}$*;
  (2) *For all $U \in \mathcal{B}$, if $\overline{V} = \langle V_1, \ldots, V_n\rangle$ is a clopen partition of U, then, the L-morphism*

$$\alpha_{\mathfrak{A}}\big(U; \overline{V}\big) : \mathfrak{A}(U) \to \prod_{j=1}^{n} \mathfrak{A}(V_j) = \mathfrak{A}\big(\overline{V}\big),$$

given by $s \longmapsto \langle s_{|V_1}, \ldots, s_{|V_n} \rangle$, *where* $\mathfrak{A}(\overline{V})$ *has the product structure, is an isomorphism, making the following diagram commutative, where* $p_j : \mathfrak{A}(\overline{V}) \to \mathfrak{A}(V_j)$ *is the canonical coordinate projection*:

$$\begin{tikzcd} \mathfrak{A}(U) \arrow[rr, "\alpha_{\mathfrak{A}}(U;\overline{V})"] \arrow[dr, "\cdot_{|V_j}"'] & & \mathfrak{A}(\overline{V}) \arrow[dl, "p_j"] \\ & \mathfrak{A}(V_j) & \end{tikzcd} \qquad (1 \leqslant j \leqslant n)$$

(b) *If* $\mathfrak{A}$ *is a fc presheaf basis over* $\mathcal{B}$, *then*:
   (1) *For all* $U \subseteq V$ *in* $\mathcal{B}$, *the restriction* L*-morphism from* $\mathfrak{A}(V)$ *to* $\mathfrak{A}(U)$ *is surjective*;
   (2) *For all* $x \in X$ *and all* $U \in \mathcal{B}_x$, *the stalk* L*-morphism*, $s \in \mathfrak{A}(U) \longmapsto s_x \in \mathfrak{A}_x$, *is surjective* (*see Definition* 8.6 *below*).

Proof. (a) Write $\alpha(U; \overline{V})$ for $\alpha_{\mathfrak{A}}(U; \overline{V})$.

(1) $\Rightarrow$ (2): Since $\mathfrak{A}$ is fc and extensional with respect to equality, the L-morphism $\alpha(U, \overline{V})$ is bijective; hence, to show it is an isomorphism it suffices to check that α reflects atomic formulas, that is, if $\varphi(v_1, \ldots, v_m)$ is an atomic formula in L and $\bar{s} = \langle s_1, \ldots, s_m \rangle \in \mathfrak{A}(U)^m$, then

$$\mathfrak{A}(\overline{V}) \models \varphi\big[\langle s_{1|V_1}, \ldots, s_{m|V_1} \rangle, \ldots, \langle s_{1|V_n}, \ldots, s_{m|V_n} \rangle\big] \Rightarrow \mathfrak{A}(U) \models \varphi[\bar{s}]. \tag{I}$$

The antecedent in (I) means $\mathfrak{A}(V_j) \models \varphi[s_{1|V_j}, \ldots, s_{m|V_j}]$, $1 \leqslant j \leqslant n$, and hence Remark 8.3(a) entails $\mathfrak{A}(U) \models \varphi[\bar{s}]$, as needed. It is clear that the displayed diagram in (2) is commutative for all $1 \leqslant j \leqslant n$.

(2) $\Rightarrow$ (1): We fix $U \in \mathcal{B}$ and a clopen covering, $\mathcal{C}$, of U, whose elements are all contained in U. Let $\varphi(v_1, \ldots, v_m)$ be an atomic formula in L, let $\bar{s} = \langle s_1, \ldots, s_m \rangle \in \mathfrak{A}(U)^m$ and assume that for $O \in \mathcal{C}$, $\mathfrak{A}(O) \models \varphi[\bar{s}_{|O}]$, where $\bar{s}_{|O} = \langle s_{1|O}, \ldots, s_{m|O} \rangle$; since U is compact, there is $\{U_1, \ldots, U_n\} \subseteq \mathcal{C}$ that is also a covering of U. Now consider

$$V_1 = U_1 \quad \text{and,} \quad \text{for } 2 \leqslant j \leqslant n, \quad V_j = U_j \setminus \Big(\bigcup_{i<j} V_i\Big).$$

Then, $\{V_1, \ldots, V_n\}$ is a pairwise disjoint clopen covering of U, subordinate to $\{U_1, \ldots, U_n\}$. Since $\mathfrak{A}(U_j) \models \varphi[\bar{s}_{|U_j}]$ and restriction is an L-morphism, we get that

$$\mathfrak{A}(V_j) \models \varphi[\bar{s}_{|V_j}], \quad 1 \leqslant j \leqslant n.$$

Therefore, $\alpha(U; \overline{V}) : \mathfrak{A}(U) \to \mathfrak{A}(\overline{V})$ being a L-isomorphism, we conclude that $\mathfrak{A}(U) \models \varphi[\bar{s}]$, establishing the extensionality of $\mathfrak{A}$.

For finite completeness, let $\{s_1, \ldots, s_n\}$ be a set of pairwise compatible sections in $|\mathfrak{A}|$, with $U = \bigcup_{j=1}^n E s_j$. The disjointing procedure above yields a disjoint clopen covering $\{W_1, \ldots, W_n\}$ of U, subordinate to the covering $\{Es_1, \ldots, Es_n\}$. Since $\alpha(U; \overline{W})$, where $\overline{W} = \langle W_1, \ldots, W_n \rangle$, is an L-isomorphism, there is $t \in \mathfrak{A}(U)$ such that

$$\text{For all } 1 \leqslant j \leqslant n, \quad t_{|W_j} = s_{j|W_j}. \tag{II}$$

Fix j between 1 and n; then, if $A_{ji} = Es_j \cap W_i$, $\{A_{j1}, \ldots, A_{jn}\}$ is a disjoint clopen covering of Es_j; moreover, since $A_{ji} = Es_j \cap W_i \cap Es_i$ and the collection $\{s_1, \ldots, s_n\}$ is compatible, (II) yields, for $1 \leqslant i \leqslant n$,

$$s_{j|A_{ji}} = (s_{j|Es_i \cap Es_j})_{|W_i} = (s_{i|Es_i \cap Es_j})_{|W_i} = s_{i|A_{ji}} = (s_{i|W_i})_{|A_{ji}} = (t_{|W_i})_{|A_{ji}} = t_{|A_{ji}}.$$

Thus, since $U = \bigcup_{j,i} A_{ji}$, extensionality of $\mathfrak{A}$ with respect to equality entails $t_{|Es_j} = s_j$, completing the proof of (a).

(b) Item (1) follows from (a) because the map $s \in \mathfrak{A}(V) \longmapsto \langle s_{|U}, s_{|V \setminus U} \rangle$ is a L-isomorphism. For (2), fix $x \in U$ and let $\xi \in \mathfrak{A}_x$. Then, for some $W \in \mathcal{B}_x$, with $W \subseteq U$, there is $s \in \mathfrak{A}(W)$ such that $\xi = s_x$. By (1), there is $t \in \mathfrak{A}(U)$ such that $t_{|W} = s$, and so $t_x = s_x = \xi$, as needed. $\square$

8.5. Completions. (a) A presheaf basis of L-structures on X, $\mathfrak{A}$, can be embedded in a sheaf over X, $c\mathfrak{A}$, called its completion, that is unique up to isomorphism. Usually this construction involves taking projective limits (see [18], Lemma 4.2.6, pp. 83–84), although there are better methods (e.g., *singletons*, a notion due to D. Scott, that appears in [10]).

(b) The completion process preserves the extent of the sections originally given over a basis of X, and each new section in the completion is a gluing of original ones; the stalks of the completion (cf. 8.6) are L-isomorphic to those of the given presheaf basis; it does not add new sections over an open U of the given basis if and only if the original presheaf is complete over U; it preserves and reflects positive quantifier-free formulas. Moreover, morphisms of presheaf bases extend uniquely to their respective completions.

We now define the *stalk* of a presheaf at a point of X.

Let $\mathcal{B}$ be a basis for the topological space X. Let $\mathfrak{A}$ be a presheaf basis of L-structures over $\mathcal{B}$. Write ν_x for the filter of *open* neighborhoods of $x \in X$ and define $\mathcal{B}_x = \nu_x \cap \mathcal{B}$. Note that:

* ν_x and $\mathcal{B}_x$, are rd-posets (cf. 2.2) under the opposite of the partial order of inclusion, $\subseteq^{op}$, because they are closed under finite intersections;
* Since $\mathfrak{A}$ is a *contravariant* functor from $\langle \mathcal{B}, \subseteq \rangle$, it yields, by restriction to $\langle \mathcal{B}_x, \subseteq^{op} \rangle$, a *covariant* functor from this rd-poset to L-mod, that is, an inductive system of L-structures over $\langle \mathcal{B}_x, \subseteq^{op} \rangle$.

Definition 8.6. With notation as above, for $x \in X$, the **stalk of $\mathfrak{A}$ at x** is defined as

$$\mathfrak{A}_x = \varinjlim \mathfrak{A}_{|\mathcal{B}_x}.$$

For $U \in \mathcal{B}_x$, let $\alpha_{Ux} : \mathfrak{A}(U) \to \mathfrak{A}_x$ be the L-morphism given by the inductive limit construction. If $U \subseteq_o V$ are in $\mathcal{B}_x$, then the diagram below left is commutative:

$$\begin{tikzcd}
\mathfrak{A}(V) \arrow[rr, "\alpha_{VU}"] \arrow[dr, "\alpha_{Vx}"'] & & \mathfrak{A}(U) \arrow[dl, "\alpha_{Ux}"] \\
& \mathfrak{A}_x &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{A}(V) \arrow[rr, "\alpha_{VW}"] \arrow[dr, "\alpha_{Vx}"'] & & \mathfrak{A}(W) \arrow[dl, "\alpha_{Wx}"] & & \mathfrak{A}(U) \arrow[ll, "\alpha_{UW}"'] \arrow[dlll, "\alpha_{Ux}"] \\
& \mathfrak{A}_x & & &
\end{tikzcd}$$

If $s \in |\mathfrak{A}|$, $x \in Es$ and $U \in \mathcal{B}_x$ is such that $U \subseteq_o Es$, define **the germ of *s* at *x*** to be the value

$$s_x = \alpha_{Ux}(s_{|U}). \tag{$*$}$$

Remark 8.7. Given any other $V \in \mathcal{B}_x$ such that $V \subseteq Es$, let $W = U \cap V$. Commutativity of the diagram above right shows: $\alpha_{Ux}(s_{|U}) = \alpha_{Wx}(\alpha_{UW}(s_{|U})) = \alpha_{Wx}(s_{|W}) = \alpha_{Vx}(s_{|V})$, i.e., $(*)$ is independent of the choice of $U \in \mathcal{B}_x$ contained in Es. In this notation, the commutativity of the diagram above left is expressed as

$$\text{For all } U \subseteq_o V \text{ in } \mathcal{B}_x \text{ and all } s \in \mathfrak{A}(V),\ s_x = (s_{|U})_x. \tag{germ}$$

Lemma 8.8. *Let $\mathcal{B}$ be a basis for a topological space X and let $\mathfrak{A}$ be a presheaf basis over $\mathcal{B}$.*

(a) *If $\langle s_1, \ldots, s_n\rangle \in |\mathfrak{A}|^n$, $\varphi(v_1, \ldots, v_n)$ is a positive quantifier-free formula in L, and $x \in \bigcap_{i=1}^n Es_i$, then*

$$\mathfrak{A}_x \models \varphi[s_{1x}, \ldots, s_{nx}] \quad \Leftrightarrow \quad \begin{cases} \exists V \in \mathcal{B}_x \textit{ such that } V \subseteq \bigcap_{i=1}^n Es_i \\ \textit{and } \mathfrak{A}(V) \models \varphi[s_{1|V}, \ldots, s_{n|V}]. \end{cases}$$

(b) *A morphism of presheaf bases over $\mathcal{B}$, $f : \mathfrak{A} \to \mathfrak{B} = \langle \mathfrak{B}(U);\ \beta_{VU}\colon U \subseteq V \text{ in } \mathcal{B}\rangle$, induces, for each $x \in X$, a L-morphism, $f_x : \mathfrak{A}_x \to \mathfrak{B}_x$, such that for all $U \in \mathcal{B}$, $\beta_{Ux} \circ f_U = f_x \circ \alpha_{Ux}$.*

Proof. (a) Since a positive quantifier-free formula is constructed from atomic formulas using the connectives $\wedge$, $\vee$, and $\mathcal{B}$ is closed under finite intersections, it is enough to verify the stated equivalence for *atomic formulas*. But this follows readily from items (a.2) and (b) of Fact 2.4. Item (b) holds exactly as in the classical case of algebraic structures. □

B. Special groups and p-rings

8.9. Construction. If $\langle A, T\rangle$ is a p-ring (see Definition 6.1), $T^* = T \cap A^*$ is a subgroup of the multiplicative group A^*. Indeed, if $t \in T^*$, then $1/t = t \cdot (1/t)^2 \in T$ because T is closed under products and contains A^2.

Given a p-ring $\langle A, T\rangle$, let $G_T(A) = A^*/T^*$ and $q_T : A^* \to G_T(A)$ be the quotient group and canonical projection, respectively; to ease notation, write a^T for $q_T(a)$. Thus, for $a, b \in A^*$,

$$a^T = b^T \ \Leftrightarrow\ ab \in T^* \ \Leftrightarrow\ \exists t \in T^* \text{ such that } b = at \tag{$*$}$$

and $G_T(A) = \{a^T\colon a \in A^*\}$. We also abuse notation, denoting by 1 and -1 both the elements of A^*, and 1^T, $(-1)^T$, respectively. Because $A^2 \subseteq T$, $G_T(A)$ is a group of exponent 2; moreover,

$$\begin{cases} G_T(A) = \{1\} \ \Leftrightarrow\ \langle A, T\rangle \text{ is the trivial p-ring;} \\ 1 \neq -1 \text{ in } G_T(A) \ \Leftrightarrow\ \langle A, T\rangle \text{ is a proper p-ring.} \end{cases} \tag{pp}$$

For $x, y \in A^*$, define

$$D_T(x, y) = \{z \in A^*\colon \exists t_1, t_2 \in T \text{ such that } z = t_1x + t_2y\}, \tag{D_T}$$

called the **set of elements represented by x and y** in A^*. Since $0, 1 \in T$, it is clear that $\{x, y\} \subseteq D_T(x, y)$. The basic properties of these sets are contained in the following fact; the proofs of Lemma 1.30 and Proposition 1.31 of [4] (pp. 22–23), done for fields of characteristic $\neq 2$, transfer *verbatim* to the case of p-rings.

Fact 8.10. *With notation as above, let $x, y, u, v \in A^*$ and $t \in T^*$.*

(a) $u D_T(x, y) = D_T(ux, uy)$ *and* $D_T(x, y) = D_T(tx, ty)$.
(b) $u \in D_T(x, y)$ *and* $u^T = v^T \Rightarrow v \in D_T(x, y)$.
(c) $x^T = u^T$ *and* $y^T = v^T \Rightarrow D_T(x, y) = D_T(u, v)$.
(d) $D_T(1, x)$ *is a subgroup of* A^*.
(e) $x \in D_T(1, y) \Rightarrow D_T(x, xy) = x D_T(1, y) = D_T(1, y)$.
(f) $u \in D_T(x, y) \Leftrightarrow D_T(u, uxy) = D_T(x, y)$.
(g) *The following are equivalent*:
  (1) $(xy)^T = (uv)^T$ *and* $D_T(x, y) = D_T(u, v)$;
  (2) $(xy)^T = (uv)^T$ *and* $D_T(x, y) \cap D_T(u, v) \neq \emptyset$.

Define a binary relation, $\equiv_T$, on $G_T(A) \times G_T(A)$, called **binary isometry mod T**, as follows: for $a, b, c, d \in A^*$

$$\langle a^T, b^T \rangle \equiv_T \langle c^T, d^T \rangle \Leftrightarrow a^T b^T = c^T d^T \text{ and } D_T(a, b) = D_T(c, d). \qquad (\equiv_T)$$

Fact 8.10 yields

Fact 8.11. (Cf. [4, Definition 1.2, p. 2].)

(a) *The relation $\equiv_T$ satisfies the following properties, for all $a, b, c, d, x \in A^*$*:
  [SG 0]: $\equiv_T$ *is an equivalence relation on* $G_T(A) \times G_T(A)$;
  [SG 1]: $\langle a^T, b^T \rangle \equiv_T \langle b^T, a^T \rangle$;
  [SG 2]: $\langle a^T, -a^T \rangle \equiv_T \langle 1, -1 \rangle$;
  [SG 3]: $\langle a^T, b^T \rangle \equiv_T \langle c^T, d^T \rangle \Rightarrow a^T b^T = c^T d^T$;
  [SG 5]: $\langle a^T, b^T \rangle \equiv_T \langle c^T, d^T \rangle \Rightarrow \langle x^T a^T, x^T b^T \rangle \equiv_T \langle x^T c^T, x^T d^T \rangle$.
(b) *(Reduction)* $\langle a^T, a^T \rangle \equiv_T \langle 1, 1 \rangle \Leftrightarrow a^T = 1 \Leftrightarrow a \in T^*$.

Proof. We comment only on [SG 2] and (b). For [SG 2], since $2 \in A^*$, any element in A is a difference of two squares. Hence, if $a \in A^*$, we have $a \in D_T(a, -a) \cap D_T(1, -1)$. Since $a^T(-a)^T = (-1)^T$, 8.10(g) guarantees that $\langle a^T, -a^T \rangle \equiv_T \langle 1, -1 \rangle$.

(b) Since $(a^T)^2 = 1$, the isometry in the antecedent is equivalent to $D_T(a, a) = D_T(1, 1)$; in particular, $a \in D_T(1, 1)$, which is clearly equivalent to $a \in T^*$. $\square$

Remark 8.12. Under the very general conditions of Construction 8.9, axiom [SG 4] in Definition 1.2 of [4] may fail. The point is that all known proofs of this axiom resort to an analogue, for preorders, of the transversality condition ($\natural$) in the proof of Lemma 4.11: if $\langle A, T \rangle$ is a p-ring and $u, v, w \in A^*$

$$w \in D_T(u, v) \;\Rightarrow\; \exists p, q \in T^* \text{ so that } w = up + vq. \qquad \text{(T)}$$

A large class of rings with many units satisfy [T], which follows, in fact, from a more general transversality principle (see [19], Propositions 3.6.1, p. 25, and 4.1.8, pp. 32–33). This is, in fact, the reason for considering the notion of proto special group, introduced in 6.3(a).

Lemma 8.13. *If $G = \langle G, \equiv_G, -1\rangle$ is a π-SG and a, b, c, $d \in G$, then*:

(a) *$D_G(1, a)$ is a subgroup of G.*
(b) *$\langle a, b\rangle \equiv_G \langle c, d\rangle \Leftrightarrow ab = cd$ and $ac \in D_G(1, cd)$.*
(c) *If $H = \langle H, \equiv_H, -1\rangle$ is a π-SG and $G \xrightarrow{h} H$ is a group morphism, such that $h(-1) = -1$, then h is a π-SG morphism iff for all $a, b \in G$, $a \in D_G(1, b) \Rightarrow f(a) \in D_H(1, h(b))$.*
(d) *If $\langle A, T\rangle$ is p-ring, then $G_T(A)$ is a π-SG, which is reduced iff $\langle A, T\rangle$ is a non-trivial p-ring. Moreover, for all a, b, c, $d \in A^*$, $\langle a^T, b^T\rangle \equiv_T \langle c^T, d^T\rangle \Leftrightarrow a^T b^T = c^T d^T$ and $ac \in D_T(1, cd)$.*

Proof. Item (a) is straightforward. The proof of Lemma 1.5(a) of [4] (p. 3) uses only [SG 3] and [SG 5] and yields (b). Item (c) is an immediate consequence of (b) and the definition of morphism in 6.3(b), while (d) follows easily from the definition of $\equiv_T$ and Fact 8.11. $\square$

Lemma 8.14. *A p-ring morphism, $h : \langle A_1, T_1\rangle \to \langle A_2, T_2\rangle$, induces a morphism of π-SGs,*

$$h^\pi : G_{T_1}(A_1) \to G_{T_2}(A_2), \text{ given by } h^\pi\big(a^{T_1}\big) = h(a)^{T_2}. \tag{$*$}$$

Furthermore, $Id^\pi_{A_1} = Id_{G_{T_1}(A_1)}$ and if $g : \langle A_2, T_2\rangle \to \langle A_3, T_3\rangle$ is a morphism of p-rings, then $(g \circ h)^\pi = g^\pi \circ h^\pi$.

Proof. Since h is a p-ring morphism, $h^* = h_{|A_1^*} : A_1^* \to A_2^*$ is a group morphism, with $h^*(-1) = -1$, and $h^*(T_1^*) \subseteq T_2^*$. Hence h^* induces a group morphism given by $(*)$, such that $h^\pi(-1) = -1$. By 8.13(c), h^π will be π-SG morphism if for $a, b \in A_1^*$,

$$a^{T_1} \in D_{T_1}\big(1, b^{T_1}\big) \Rightarrow h^\pi\big(a^{T_1}\big) = h(a)^{T_2} \in D_{T_2}\big(1, h\big(b^{T_1}\big)\big) = D_{T_2}\big(1, h(b)^{T_2}\big). \tag{I}$$

The antecedent in (I) means that there are $t_1, t_2 \in T_1$ such that $a = t_1 + t_2 b$; thus,

$$h(a) = h(t_1) + h(t_2)h(b). \tag{II}$$

Since $h(T_1) \subseteq T_2$, (II) implies $h(a) \in D_{T_2}(1, h(b))$, which in turn, because of condition (D_T) in 8.8, entails the consequent in (I). The preservation of identity and composition is clear. $\square$

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