Weierstrass Gap Sets for Quadruples of Points on Compact Riemann Surfaces

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Let $M$ be a compact Riemann surface of genus $g$, and let $P_1,\ldots,P_n$ be distinct points on $M$. We study the Weierstrass gap set $G(P_1,\ldots,P_n)$ and prove the conjecture of Ballico and Kim on the upper bound of $\#G(P_1,\ldots,P_n)$ affirmatively in case $M$ is $d$-gonal curve of genus $g \geq 5$ with $d = 2$ or $d \geq 5$.

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0. INTRODUCTION

0.1. Let $M$ be a compact Riemann surface of genus $g \geq 2$, and let $P_1,\ldots,P_n$ be distinct points on $M$. We define the Weierstrass gap set $G(P_1,\ldots,P_n)$ by

$$G(P_1,\ldots,P_n) = \{(\gamma_1,\ldots,\gamma_n) \in \mathbb{N}_0^n \mid \exists \text{ meromorphic function } f \text{ on } M$$

whose pole divisor $(f)_\pi$ is $\gamma_1 P_1 + \cdots + \gamma_n P_n\},$$

where $\mathbb{N}_0$ is the set of non-negative integers.

In case $n = 1$, $G(P_1)$ is the set of Weierstrass gaps at $P_1$ and the cardinality $\#G(P_1)$ is equal to the genus $g$ of $M$.

But in case $n \geq 2$, $\#G(P_1,\ldots,P_n)$ depends on the choice of $M$ and the set of points $(P_1,\ldots,P_n)$ on $M$. 
0.2. The set $G(P_1, \ldots, P_n)$ appeared in a book by Arbarello et al. [1]. They gave the inequality

$$\#G(P_1, P_2) \geq \binom{g+2}{2} - 1$$

as an exercise (the proof can be seen in [3, 4, 5]). It was also mentioned that the equality holds when $P_1$ and $P_2$ are general on an arbitrary $M$ (we can refer to [3] for the proof). This inequality also can be generalized to an arbitrary integer $n \geq 2$ [4]. That is,

$$\#G(P_1, \ldots, P_n) \geq \binom{n+g}{g} - 1,$$

and it can be seen that the equality holds when $P_1, \ldots, P_n$ are general points on $M$ by the same way as in [3].

On the upper bound of $\#G(P_1, \ldots, P_n)$, first Kim proved that

$$\#G(P_1, P_2) \leq (3g^2 + g)/3,$$

and the equality holds if and only if $M$ is hyperelliptic and $|2P_1| = |2P_2| = g^2$ [5].

Then the author proved the inequality

$$\#G(P_1, P_2, P_3) \leq \sum_{0 \leq m \leq 3} \left( \frac{3}{m} \binom{g}{m} 2^m - \binom{g+3}{3} \right) = g(7g^2 + 6g + 5)/6,$$

and the equality holds if and only if $M$ is hyperelliptic and $|2P_1| = |2P_2| = |2P_3| = g^2$ [4].

Recently Ballico and Kim have computed $\#G(P_1, \ldots, P_n)$ in case $P_1, \ldots, P_n$ are (distinct) arbitrary points on a hyperelliptic curve $M$, and they presented the following conjecture [2].

**Conjecture.** Assume $g$ is very large with respect to $n(2n \leq g^2)$. Then the inequality

$$\#G(P_1, \ldots, P_n) \leq \sum_{0 \leq m \leq n} \binom{n}{m} \binom{g}{m} 2^m - \binom{g+n}{n}$$

is satisfied, and the equality holds if and only if $M$ is hyperelliptic and $|2P_i| = g^2$ ($i = 1, \ldots, n$).
0.3. In this paper we will prove that the conjecture is true in case
\( n = 4 \) and \( M \) is a \( d \)-gonal curve of genus \( g \geq 5 \) with \( d = 2 \) or \( d \geq 5 \). That is:

**Theorem A.** Assume \( M \) is a \( d \)-gonal curve of genus \( g \geq 5 \) with \( d = 2 \) or \( d \geq 5 \). Then

\[
\#G(P_1, \ldots, P_4) \leq \sum_{0 \leq m \leq 4} \binom{4}{m} \binom{g}{m} 2^m - \binom{g + 4}{4} \\
= g(14 + 45g + 22g^2 + 15g^3)/24,
\]

and the equality holds if and only if \( M \) is hyperelliptic (i.e., \( d = 2 \)) and
\( |2P_1| = |2P_2| = |2P_3| = |2P_4| = g_2^1 \).

0.4. Outline of the proof. Let \( g \geq 2 \) be fixed. From now on \( M_h \)
denotes a hyperelliptic curve of genus \( g \), and \( Q_i \) \( (i = 1, 2, 3, 4) \) denote distinct points on \( M_h \) satisfying \( |2Q_i| = g_2^1 \) \( (i = 1, \ldots, 4) \). For an arbitrary curve \( M \) of genus \( g \) and distinct points \( P_i \) \( (i = 1, 2, 3, 4) \) on \( M \), we denote by \( K \) the set of effective canonical divisors whose support is contained in the set \( \{P_1, P_2, P_3, P_4\} \). That is,

\[
K = \{ \Gamma = \gamma_1P_1 + \gamma_2P_2 + \gamma_3P_3 + \gamma_4P_4 \mid \gamma_i \geq 0 \ (i = 1, 2, 3, 4) \}.
\]

In particular if \( M = M_h \) and \( P_i = Q_i \), we write \( K_h \) for the \( K \) above.

Since \( g_2^1 \) is unique and the canonical linear system on \( M_h \) is \((g - 1)g_2^1\), we can see

\[
\#K_h = \binom{g + 2}{3}.
\]

And we already know the following facts by the calculation of Ballico and Kim [2]:

\[
\#G(Q_1, Q_2, Q_3, Q_4) = g(14 + 45g + 22g^2 + 15g^3)/24 \\
= 9\#K_h + \frac{15}{24}g^4 - \frac{14}{24}g^3 - \frac{63}{24}g^2 - \frac{58}{24}g. \tag{0.4.1}
\]

More precisely \( G(Q_1, Q_2, Q_3, Q_4) \) can be obtained as

\[
V^{(4)} = \left\{ \left( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \right) \mid 0 \leq \sum \gamma_i \leq 2g - 2, \text{ and all } \gamma_i \text{ are even} \right\} \\
\cup \left\{ \left( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \right) \mid \sum \gamma_i = 2g - 2, \text{ and all } \gamma_i \text{ are odd} \right\} \\
\cup \left\{ \left( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \right) \mid \sum \gamma_i = 2g - 1, \text{ and exactly one of } \gamma_i \text{ is even} \right\}.
\]
Let $R_1, \ldots, R_4$ be distinct points on $M_h$, and assume that at least one of $|2R_i|$ ($i = 1, \ldots, 4$) is not equal to $g^1$. Then

$$
\#G(R_1, R_2, R_3, R_4) < \#G(Q_1, Q_2, Q_3, Q_4). \tag{0.4.2}
$$

For the proof of Theorem A, first we will prove the following inequality:

**Proposition B.** Let $M$ be a non-hyperelliptic curve of genus $g \geq 3$. Then

$$
\#G(P_1, P_2, P_3, P_4)
\leq 9\#K + 3\#K_h + \frac{15}{24} g^4 - \frac{14}{24} g^3 - \frac{3}{24} g^2 + \frac{2}{24} g. \tag{0.4.3}
$$

From (0.4.1) and (0.4.3) we have

$$
\#G(Q_1, Q_2, Q_3, Q_4) - \#G(P_1, P_2, P_3, P_4)
\geq 6\#K_h - 9\#K - \frac{5}{2} g^2 - \frac{5}{2} g, \tag{0.4.4}
$$

where $P_1, P_2, P_3, P_4$ are distinct points on a non-hyperelliptic curve $M$.

Next we show the right hand side of (0.4.4) is positive when $M$ is a $d$-gonal curve of genus $g \geq 5$ with $d \geq 5$ by proving the following theorem.

**Theorem C.** Assume $M$ is a $d$-gonal curve of genus $g$. Then

$$
\#K \leq \left( \left\lfloor \frac{2g - 2}{3} \right\rfloor + 3 \right)
$$

provided $g \geq 3$ and $d \geq 5$, where $[r]$ is the largest integer $n$ satisfying $n \leq r$ for a real number $r$.

In fact if this theorem is true, then the right hand side of (0.4.4) is not less than

$$
6\left( \frac{g + 2}{3} \right) - 9\left( \left\lfloor \frac{2g - 2}{3} \right\rfloor + 3 \right) - \frac{5}{2} g^2 - \frac{5}{2} g,
$$

and this value is positive when $g \geq 5$. Thus, by (0.4.2) above, we get Theorem A.
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In [4] the author defined a “Riemann–Roch graph” to draw the distribution of $G(P_1, \ldots, P_n)$ in $\mathbb{N}^n$. Then we start this section by giving a brief summary of the Riemann–Roch graph.

**Definition 1.1.** Fix positive integers $g$ and $n$. Let $e_i$ be the $n$-tuple $(0, \ldots, 0, 1, 0, \ldots, 0)$ in $\mathbb{N}_0^n$. We usually express a point $\Gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}_0^n$ as an element of the free module $\mathbb{Z}^n$ over $\mathbb{Z}$ generated by $e_i$ ($i = 1, \ldots, n$). For example $\Gamma = \sum_{i=1}^n \gamma_i e_i$, $\Gamma - e_i = (\gamma_1, \ldots, \gamma_i - 1, \ldots, \gamma_n)$, and so on. Let $V(n)$ denote the subset

$$\{ \Gamma = \sum \gamma_i e_i \mid \gamma_i \in \mathbb{N}_0, 0 \leq \gamma_1 + \cdots + \gamma_n \leq 2g - 1 \}$$

in $\mathbb{N}_0^n$. We write $\deg \Gamma$ for $\gamma_1 + \cdots + \gamma_n$.

Let $E(n)$ denote the subset

$$\bigcup_{i=1}^n \{ (\Gamma - e_i) \Gamma \mid \Gamma \in V(n) \text{ whose coefficient at } e_i \text{ is positive} \}$$

in $V(n) \times V(n)$. Let $D(n) = \{ V(n), E(n) \}$ be the graph consisting of $V(n)$ and $E(n)$ as a set of vertices and a set of edges, respectively. For two elements $\Gamma = \sum \gamma_i e_i$ and $\Gamma' = \sum \gamma_i' e_i$ in $V(n)$, we write $\Gamma' \leq \Gamma$ if $\gamma_i' \leq \gamma_i$ for every $i$. When $\Gamma' \leq \Gamma$, we call any chain of successive $(\deg \Gamma - \deg \Gamma')$ edges

$$\Gamma \Gamma', \Gamma' \Gamma^2, \ldots, \Gamma^{\deg \Gamma - \deg \Gamma - 1} \Gamma$$

a path from $\Gamma'$ to $\Gamma$, and write them $\Gamma' \Gamma$ abusively. Moreover, assume each edge to be labeled “0” or “1,” which is called the weight of the edge, and the labeling has the following properties

(*)$_n - 1$ Let $\Gamma = \sum \gamma_i e_i$ and $\tilde{\Gamma} = \sum \tilde{\gamma}_i e_i$ be in $V(n)$. Assume $\tilde{\Gamma} \geq \Gamma$ and $\gamma_i = \tilde{\gamma}_i > 0$ with some $i$. If the edge $(\Gamma - e_i) \Gamma$ is of weight 1, then so is the edge $(\tilde{\Gamma} - e_i) \tilde{\Gamma}$.

(*)$_n - 2$ Let $O = \sum 0 e_i$ and $\Gamma = \sum \gamma_i e_i$ be in $V(n)$ with $\deg \Gamma = 2g - 1$. The number of edges of weight 1 (resp. 0) on every path $O \Gamma$ is $g - 1$ (resp. $g$).

From now on, we will call the above type of graph $(D(n), *_n)$ a Riemann–Roch graph.

**Definition 1.2.** Define the gap set $G(n)$ of $(D(n), *_n)$ by

$$G(n) := \{ \Gamma \in V(n) \mid \exists i \text{ such that the edge } (\Gamma - e_i) \Gamma \in E(n) \text{ is of weight 0} \}.$$ 

$H(n)$ denotes the complement $V(n) - G(n)$ of $G(n)$ in $V(n)$.
Remark. \( O = (0, \ldots, 0) \in H^{(n)}. \)

Example 1.3. Let \( M \) and \( \{P_1, \ldots, P_n\} \) be as above. Then the following facts on an effective divisor \( E = \gamma_1 P_1 + \gamma_2 P_2 + \cdots + \gamma_n P_n \) are known:

1. if \( \deg E = \gamma_1 + \cdots + \gamma_n = 2g - 1 \), then \( l(E) = h^0(\mathcal{O}(E)) = g; \)
2. if \( P_i \) is not a base point of the linear system \( |E| \) for some \( i \), then \( P_i \) is not a base point of any linear system

\[ |\tilde{\gamma}_1 P_1 + \tilde{\gamma}_2 P_2 + \cdots + \tilde{\gamma}_i P_i + \cdots + \tilde{\gamma}_n P_n|, \]

where \( \tilde{\gamma}_k \geq \gamma_k \) for \( k = 1, \ldots, n \) and \( \tilde{\gamma}_i = \gamma_i. \)

Identify each effective divisor \( E = \sum_{i=1}^{n} \gamma_i P_i \) of degree \( \leq 2g - 1 \) with the vertex \( \Gamma = \sum_{i=1}^{n} \gamma_i e_i \), and give 1 to the edge \( (\Gamma - e_i )\Gamma \) if and only if \( P_i \) is not a base point of \( |\gamma_i P_i| \). Then we get a Riemann–Roch graph. \( D_M(P_1, \ldots, P_n) = D(P_1, \ldots, P_n) \) denotes this graph. The gap set \( G^{(n)} \) obtained from \( D_M(P_1, \ldots, P_n) \) coincides with the Weierstrass gap set \( G(P_1, \ldots, P_n) \) in 0.1.

Fix a Riemann–Roch graph \( (D^{(n)}, *_n) \). Here \( (D^{(n-1)}, *_{n-1}) \) is a subgraph of \( (D^{(n)}, *_n) \) obtained by identifying \( (\gamma_1, \ldots, \gamma_{n-1}) \in V^{(n-1)} \) with \( (\gamma_1, \ldots, \gamma_{n-1}, 0) \in V^{(n)} \) and restricting \( *_n \) to \( V^{(n-1)} \). Let \( \Gamma_n \) denote an element of \( V^{(n-1)} \). Then \( G^{(n-1)} \) (resp. \( H^{(n-1)} \)) of \( (D^{(n-1)}, *_{n-1}) \) is embedded in \( G^{(n)} \) (resp. \( H^{(n)} \)) of \( (D^{(n)}, *_n) \).

Definition 1.4. For \( \Gamma_n = (\gamma_1, \ldots, \gamma_{n-1}) \in G^{(n-1)} \), define a positive integer \( \delta^{(n)}_\Gamma \) by

\[ \delta^{(n)}_\Gamma = \delta^{(n)} = \min \left\{ \delta \mid \delta \text{ is a positive integer satisfying} \right. \]

\[ (\Gamma_n, \delta) = \sum_{i=1}^{n-1} \gamma_i e_i + \delta e_n \notin G^{(n)} \left. \right\}. \]

Definition 1.5. For \( \Gamma \) and \( \Gamma' \) in \( V^{(n)} \) with \( \Gamma \geq \Gamma' \), every path \( \Gamma' \) to \( \Gamma \) has the same number of edges of weight 1, and we write \( [\Gamma', \Gamma] \) for this number. Define non-negative integers \( \ell(\Gamma) \) and \( i(\Gamma) \) by

\[ \ell(\Gamma) := \lfloor O \Gamma \rfloor + 1 \quad \text{and} \quad i(\Gamma) := \ell(\Gamma) - 1 + g - \deg \Gamma. \]

We can easily see that \( \ell(\Gamma) \) is the number of edges of weight 0 on a path \( \Gamma \Gamma \) with \( \Gamma \leq \Gamma \) and \( \deg \Gamma = 2g - 1. \)

If \( (D^{(n)}, *_n) \) is the graph of Example 1.3 and \( \Gamma = \sum \gamma_i e_i \), then \( \ell(\Gamma) = \dim_c H^0(\mathcal{O}(\sum \gamma_i P_i), M). \)
Let $\Gamma = (\Gamma_n, \gamma) \in V^{(n)}$. In case $\Gamma_n \in H^{(n-1)}$, we can see from the property $\ast_n$ that $\Gamma \in G^{(n)}$ if and only if $[\Gamma - e_n, \Gamma] = 0$. In case $\Gamma_n \in G^{(n-1)}$, we can see that $\Gamma \in G^{(n)}$ if and only if $\gamma < \delta^{\Gamma_n}$ or $\gamma > \delta^{\Gamma_n}$ and $[\Gamma - e_n, \Gamma] = 0$. From these facts we have the following theorem.

**Theorem 1.6** [4].

$$
\# G^{(n)} = \sum_{\Gamma_n \in H^{(n-1)}} l(\Gamma_n) + \sum_{\Gamma_n \in G^{(n-1)}} l(\Gamma_n, \delta^{\Gamma_n}) - \sum_{k=0}^{2g-1} k\left(n + k - 2\right) + (g - 1)\left(\frac{2g - 2 + n}{2g - 1}\right).
$$

In this section we study $\# G^{(4)}$. Let $(D^{(4)}, \ast_4)$ be a Riemann–Roch graph, and let

$$
(D^{(4)}, \ast_4) \supset (D^{(3)}, \ast_3) \supset (D^{(2)}, \ast_2) \supset (D^{(1)}, \ast_1)
$$

be its subgraphs as in Section 1. According to these subgraphs we have successive gap subsets

$$
G^{(4)} \supset G^{(3)} \supset G^{(2)} \supset G^{(1)}.
$$

Let $\Gamma_k = (\gamma_1, \ldots, \gamma_k)$ be an element of $V^{(k-1)}$ $(k = 2, 3, 4)$ and let $\Gamma = (\gamma_1, \ldots, \gamma_4)$ be an element of $V^{(4)}$. For $\Gamma_k \in G^{(k-1)}$, $\delta^{\Gamma_k}$ is the number defined in 1.4, i.e.,

$$
\delta^{\Gamma_k} = \min\{\delta \mid (\Gamma_k, \delta) \notin G^{(k)} (k = 2, 3, 4)\}.
$$

**Definition 2.1.** Define subsets $W_l (l = 1, \ldots, g + 1)$ of $G^{(3)}$ by

$$
W_l = \{\Gamma_k \in G^{(3)} \mid l(\Gamma_k, \delta^{\Gamma_k}) = l\}.
$$

We will prove the following inequality on $\# G^{(4)}$.

**Proposition 2.2.** For an arbitrary $(D^{(4)}, \ast_4)$ with $g \geq 2$, we have

$$
\# G^{(4)} \leq 2 \times \# W_{g+1} + \# W_g + \frac{15}{24}g^4 - \frac{14}{24}g^3 - \frac{3}{24}g^2 + \frac{2}{24}g.
$$

**Proof.** Start the proof from the equation of Theorem 1.6 with $n = 4$, divide $G^{(3)}$ into three subsets, $W_{g+1}$, $W_g$, and $G^{(3)} - (W_{g+1} \cup W_g)$, and use
Theorem 1.6 with \( n = 3 \). Then

\[
\# G^{(4)} = \sum_{\Gamma_i \in G^{(3)}} l(\Gamma_i, \delta \Gamma_i) + \sum_{\Gamma_i \in H^{(3)}} l(\Gamma_i)
\]

\[
- 2^{g-1} \sum_{k=0}^{k+2} \binom{k+2}{k} + (g-1) \binom{2g+2}{2g-1}
\]

\[
= \sum_{k=1}^{g+1} k \# W_k + \sum_{\Gamma_i \in H^{(3)}} l(\Gamma_i) - \sum_{k=0}^{2^{g-1}} k \binom{k+2}{k} + (g-1) \binom{2g+2}{2g-1}
\]

\[
\leq (g+1) \# W_{g+1} + g \# W_g + (g-1)(\# G^{(3)} - \# W_{g+1} - \# W_g)
\]

\[
+ \sum_{\Gamma_i \in H^{(3)}} l(\Gamma_i) - \sum_{k=0}^{2^{g-1}} k \binom{k+2}{k} + (g-1) \binom{2g+2}{2g-1}
\]

\[
= 2 \# W_{g+1} + \# W_g + \sum_{\Gamma_i \in H^{(3)}} l(\Gamma_i)
\]

\[
+ (g-1) \left( \sum_{\Gamma_3 \in G^{(2)}} l(\Gamma_3, \delta \Gamma_3) + \sum_{\Gamma_i \in H^{(2)}} l(\Gamma_i) \right)
\]

\[
- \sum_{k=0}^{2^{g-1}} \binom{k+1}{k} + (g-1) \binom{2g+1}{2g-1}
\]

\[
- \sum_{k=0}^{2^{g-1}} \binom{k+2}{k} + (g-1) \binom{2g+2}{2g-1}
\]

\[
= 2 \# W_{g+1} + \# W_g
\]

\[
+ \sum_{\Gamma_i \in H^{(3)}} l(\Gamma_i) + (g-1) \left( \sum_{\Gamma_i \in G^{(2)}} l(\Gamma_i, \delta \Gamma_i) + \sum_{\Gamma_i \in H^{(2)}} l(\Gamma_i) \right)
\]

\[
- \frac{1}{6} (2g+1) g (g+1) (4g-1)
\]

\[
= *_1.
\]
The set \( H^{(3)} \) can be divided into two subsets as
\[
H^{(3)} = \{ (\Gamma_3, \gamma) \mid \Gamma_3 \in H^{(2)}, 0 \leq \gamma \leq 2g - 1 - \deg \Gamma_3, \\
((\Gamma_3, \gamma - 1), (\Gamma_3, \gamma)] = 1 \}
\cup \{ (\Gamma_3, \gamma) \mid \Gamma_3 \in G^{(2)}, \delta^{\Gamma_3} \leq \gamma \leq 2g - 1 - \deg \Gamma_3, \\
((\Gamma_3, \gamma - 1), (\Gamma_3, \gamma)] = 1 \}.
\]

Here the second set is considered to be empty if \( \delta^{\Gamma_3} = 2g - \deg \Gamma_3 \) (and \( l(\Gamma_3, \delta^{\Gamma_3}) = g + 1 \)).

Then
\[
*_{1} = 2\#W_{g+1} + \#W_{g} + (g - 1) \left\{ \sum_{\Gamma_3 \in G^{(2)}} l(\Gamma_3, \delta^{\Gamma_3}) + \sum_{\Gamma_3 \in H^{(2)}} l(\Gamma_3) \right\}
\]
\[
+ \sum_{\Gamma_3 \in G^{(2)}} \{ l(\Gamma_3, \delta^{\Gamma_3}) + \cdots + g \} + \sum_{\Gamma_3 \in H^{(2)}} \{ l(\Gamma_3) + \cdots + g \}
\]
\[
- \frac{1}{6} (2g + 1) g(g + 1)(4g - 1)
\]

(the sum \( \{ l(\Gamma_3, \delta^{\Gamma_3}) + \cdots + g \} \) is considered to be 0 if \( l(\Gamma_3, \delta^{\Gamma_3}) = g + 1 \))
\[
= 2\#W_{g+1} + \#W_{g}
\]
\[
+ \sum_{\Gamma_3 \in G^{(2)}} \left\{ (g - 1) l(\Gamma_3, \delta^{\Gamma_3}) + l(\Gamma_3, \delta^{\Gamma_3}) + \cdots + g \right\}
\]
\[
+ \sum_{\Gamma_3 \in H^{(2)}} \left\{ (g - 1) l(\Gamma_3) + \frac{(g + l(\Gamma_3))(g - l(\Gamma_3) + 1)}{2} \right\}
\]
\[
- \frac{1}{6} (2g + 1) g(g + 1)(4g - 1)
\]
\[
= \ast_{2}.
\]

Since
\[
(g - 1) l(\Gamma_3, \delta^{\Gamma_3}) + \{ l(\Gamma_3, \delta^{\Gamma_3}) + \cdots + g \}
\]
\[
= \begin{cases} 
(g - 1) l(\Gamma_3, \delta^{\Gamma_3}) + \frac{(g + l(\Gamma_3, \delta^{\Gamma_3}))(g - l(\Gamma_3) + 1)}{2} & \text{if } l(\Gamma_3, \delta^{\Gamma_3}) \leq g \\
(g - 1)(g + 1) & \text{if } l(\Gamma_3, \delta^{\Gamma_3}) = g + 1,
\end{cases}
\]
\[
(n_{1} - 1) / \left( 2 + \frac{1}{6} (2g + 1) g(g + 1)(4g - 1) \right)
\]
\[
= \ast_{2}.
\]

Since
\[
\ast_{2} = \ast_{1} - \frac{1}{6} (2g + 1) g(g + 1)(4g - 1)
\]
this term attains the maximal value \( g^2 \) when \( l(\Gamma_3, \delta_{\Gamma_3}) = g - 1 \) or \( g \). Then the term \( \sum_{\Gamma_3 \in G(\delta)} \binom{\ast}{\ast} \) is not bigger than \( g^2 \# G^{(2)} \). Divide \( H^{(2)} \) into two subsets

\[
H^{(2)} = \{(\Gamma_2, \gamma) \mid 0 \leq \gamma \leq 2g - 1 - \Gamma_2, [(\Gamma_2, \gamma - 1), (\Gamma_2, \gamma)] = 1\}
\cup \{(\Gamma_2, \gamma) \mid \delta_{\Gamma_2} \leq \gamma \leq 2g - 1 - \Gamma_2, [(\Gamma_2, \gamma - 1), (\Gamma_2, \gamma)] = 1\}.
\]

The second set is considered to be empty if \( \delta_{\Gamma_2} = 2g - \Gamma_2 \) and \( l(\Gamma_2, \delta_{\Gamma_2}) = g + 1 \), and keep on computing further. Then

\[
\begin{align*}
*_2 & \leq 2\#W_{g+1} + \#W_g + g^2 \# G^{(2)} \\
& + \sum_{\Gamma_2 \in G^{(1)}} \sum_{k=0}^{g} \left( (g - 1)k + \frac{(g + k)(g - k + 1)}{2} \right) \\
& + \frac{1}{6}(2g + 1)g(g + 1)(4g - 1)
\end{align*}
\]

where \( \sum_{k=l(\Gamma_2, \delta_{\Gamma_2})}^{g} \binom{\ast}{\ast} \) is 0 if \( l(\Gamma_2, \delta_{\Gamma_2}) = g + 1 \).

Use Theorem 1.6 with \( n = 2 \); then

\[
\begin{align*}
*_3 &= 2\#W_{g+1} + \#W_g + g^2 \left[ \sum_{\Gamma_2 \in G^{(1)}} l(\Gamma_2, \delta_{\Gamma_2}) \right] + \frac{g(g - 1)}{2} \\
& + \sum_{\Gamma_2 \in G^{(1)}} \sum_{k=0}^{g} \left( (g - 1)k + \frac{(g + k)(g - k + 1)}{2} \right) \\
& + \sum_{\Gamma_2 \in H^{(1)}} \sum_{k=l(\Gamma_2)}^{g} \left( (g - 1)k + \frac{(g + k)(g - k + 1)}{2} \right) \\
& - \frac{1}{6}(2g + 1)g(g + 1)(4g - 1)
\end{align*}
\]

where the sum \( \sum_{k=l(\Gamma_2, \delta_{\Gamma_2})}^{g} \binom{\ast}{\ast} \) is 0 if \( l(\Gamma_2, \delta_{\Gamma_2}) = g + 1 \).
Since \( \{ l(\Gamma_2) \mid \Gamma_2 \in H^{(1)} \} = \{1, 2, \ldots, g\} \), the term (B) is equal to

\[
\sum_{l=1}^{g} \sum_{k=l}^{g} \left( (g - 1)k + \frac{(g + k)(g - k + 1)}{2} \right)
= \frac{1}{24}g(g + 1)(11g^2 + 3g - 2),
\]

and we have

\[
\begin{align*}
\sigma_3 &= 2\#W_{g+1} + \#W_g + g^2 \left\{ \sum_{\Gamma_2 \in G^{(1)}} l(\Gamma_2, \delta_{\Gamma_2}) \right\} \\
&+ \sum_{\Gamma_2 \in G^{(1)}} \sum_{k=l(\Gamma_2, \delta_{\Gamma_2})}^{g} \left( (g - 1)k + \frac{(g + k)(g - k + 1)}{2} \right) \\
&+ \frac{1}{2}g^3(g - 1) + \frac{1}{24}g(g + 1)(11g^2 + 3g - 2) \\
&- \frac{1}{6}(2g + 1)g(g + 1)(4g - 1) \\
&= 2\#W_{g+1} + \#W_g \\
&+ \sum_{\Gamma_2 \in G^{(1)}} \left[ g^2 l(\Gamma_2, \delta_{\Gamma_2}) \\
&+ \sum_{k=l(\Gamma_2, \delta_{\Gamma_2})}^{g} \left( (g - 1)k + \frac{(g + k)(g - k + 1)}{2} \right) \right] \\
&+ \frac{1}{2}g^3(g - 1) + \frac{1}{24}g(g + 1)(11g^2 + 3g - 2) \\
&- \frac{1}{6}(2g + 1)g(g + 1)(4g - 1).
\end{align*}
\]

Since

\[
g^2 l(\Gamma_2, \delta_{\Gamma_2}) + \sum_{k=l(\Gamma_2, \delta_{\Gamma_2})}^{g} \left( (g - 1)k + \frac{(g + k)(g - k + 1)}{2} \right)
= \left\{ \begin{array}{ll}
\frac{1}{6}l(\Gamma_2, \delta_{\Gamma_2})^3 - \frac{1}{2}gl(\Gamma_2, \delta_{\Gamma_2})^2 + \left( \frac{1}{2}g^2 - \frac{1}{6} \right)l(\Gamma_2, \delta_{\Gamma_2}) \\
+ \frac{5}{6}g^3 + g^2 + \frac{1}{6}g & \text{if } l(\Gamma_2, \delta_{\Gamma_2}) \leq g \\
g^2(g + 1) & \text{if } l(\Gamma_2, \delta_{\Gamma_2}) = g + 1,
\end{array} \right.
\]

and we have
this term attains the maximal value $g^3 + g^2$ if and only if $l(\Gamma_2, \delta^{\Gamma}) = g - 1$ or $g$ or $g + 1$

As $\#G^{(1)} = g$, we have

$$
* \leq 2\#W_{g+1} + \#W_g \\
+ g(g^3 + g^2) \\
+ \frac{1}{2}g^3(g - 1) + \frac{1}{2}g(g + 1)(11g^2 + 3g - 2) \\
- \frac{1}{6}(2g + 1)g(g + 1)(4g - 1).
$$

Thus we get our assertion,

$$
\#G^{(4)} \leq 2\#W_{g+1} + \#W_g + \frac{15g^4 - 14g^3 - 3g^2 + 2g}{24}.
$$

3

Let $M$ be a compact Riemann surface of genus $g$, and through this section and the next we assume that $(D^{(4)}, \ast)$ is the graph of Example 1.3. In this section we study the inequality of Proposition 2.2 in case $G^{(4)} = G_P(P_1, P_2, P_3, P_4)$ and prove Proposition B in Section 0. We use the following facts on Riemann surface $M$ of genus $g \geq 2$.

(3.1) A canonical divisor is base point free.

(3.2) A canonical divisor is very ample if $M$ is non-hyperelliptic.

**Lemma 3.1.** Let $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in V^{(4)}$ with $\deg \Gamma \leq 2g - 2$. Assume that precisely $s$ ($0 \leq s \leq 3$) edges of four edges $\Gamma(\Gamma + e_i)$ ($i = 1, 2, 3, 4$) have weight 1. Then

$$
\deg \Gamma \leq 2g - 2 - s \quad \text{and} \quad l(\Gamma) \leq g - s.
$$

Moreover, if $\deg \Gamma = 2g - 2 - s$, then $l(\Gamma) = g - s$.

Proof. Assume $\deg \Gamma > 2g - 2 - s$.

Case $s = 3$. We may assume that

$$
[\Gamma, \Gamma + e_1] = 0 \quad \text{and} \quad [\Gamma, \Gamma + e_2] = [\Gamma, \Gamma + e_3] = [\Gamma, \Gamma + e_4] = 1.
$$

Put $k = 2g - 2 - \deg \Gamma$. Then $s = 3 > k \geq 0$. If $k = 2$, then

$$
\deg(\Gamma + e_2 + e_3 + e_4) = 2g - 1.
$$
By \((\ast_4 - 1)\) (Definition 1.1) 
\[
[\Gamma, \Gamma + e_2] = [\Gamma + e_2, \Gamma + e_2] + e_3 = [\Gamma + e_2 + e_3, \Gamma + e_2 + e_3 + e_4] = 1. 
\] (3.1.1)

Take a path from \(\Gamma\) to \(\Gamma + e_2 + e_3 + e_4\). Then every edge on this path is of weight 1 by (3.1.1). This contradicts to our assumption \([\Gamma, \Gamma + e_1] = 0\). In case \(k = 0\) or 1, we also get a contradiction in the same way as above. Therefore we have \(\deg \Gamma \leq 2g - 2 - s\) in case \(s = 3\).

Moreover, when \(s = 3\), the equations (3.1.1) hold again. Then we have \(l(\Gamma) \leq g - s\), and \(l(\Gamma) = g - s\) if \(\deg \Gamma = 2g - 2 - s\).

**Case** \(s = 0\) or 1 or 2. In these cases we can also prove the assertion by the same manner as in case \(s = 3\).

Let \(p_4\) be the projection from \(V^{(4)}\) to \(V^{(3)}\) with \(\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)\) corresponding to \((\gamma_1, \gamma_2, \gamma_3)\). We write \(p_4\) for an element of \(V^{(3)}\).

**Definition 3.2.** Define the subset \(C_0\) and \(U_0\) of \(V^{(3)}\) by
\[
C_0 = \{\Gamma_4 \in V^{(3)} \mid \deg \Gamma_4 \leq 2g - 2, [\Gamma_4, \Gamma_4 + \hat{e}_i] = 0 \text{ for every } i = 1, 2, 3\},
\]
\[
U_0 = (C_0 + \hat{e}_1) \cup (C_0 + \hat{e}_2) \cup (C_0 + \hat{e}_3),
\]
where \(\hat{e}_1 = (1, 0, 0) \in V^{(3)}\) and so on.

The symbols \(K\) and \(K_h\) are as in 0.4. By the condition \((\ast_4\text{-ii})\), we can see
\[
K = \{\Gamma \in V^{(4)} \mid \deg \Gamma = 2g - 2, [\Gamma, \Gamma + e_i] = 0 \text{ for some } i\}
\]
\[
= \{\Gamma \in V^{(4)} \mid \deg \Gamma = 2g - 2, [\Gamma, \Gamma + e_i] = 0 \text{ for all } i\}.
\]

**Remark 3.3.** Let \(P = \{\Gamma \in V^{(4)} \mid \deg \Gamma = 2g - 2\}\). Restricting the projection \(p_4\) to \(P\) causes an injection from \(P\) onto \(V^{(3)}\). In particular, \(p_4\) induces an injection from \(K\) to \(C_0\), and we have
\[
\#K \leq \#C_0.
\]

We also remark the following which comes from the property of curves and the definition of \(\delta^{(4)}\).

**Remark 3.4.** For \(\Gamma_4 = (\gamma_1, \gamma_2, \gamma_3) \in V^{(3)}\),

1. \(l(\Gamma_4, \delta^{(4)}) = g + 1\) if and only if \(\deg \Gamma_4 + \delta^{(4)} = 2g\).
2. If \(l(\Gamma_4, \delta^{(4)}) \geq g\), then \(\deg \Gamma_4 + \delta^{(4)} \geq 2g - 2\).
In the rest of this section we will show the following inequalities.

(A) \( \#W_{g+1} = 3\#K \).

If \( M \) is not hyperelliptic,

(B) \( \#W_g \leq 3\#K + 3\#K_h \).

Combining these inequalities and Proposition 2.2, we get Proposition B.

We start from proving the following lemma.

**Lemma 3.5.** Let \( \Gamma_4 = (\gamma_1, \gamma_2, \gamma_3) \) be in \( V^{(3)} \).

1. \( l(\Gamma_4, \delta^{\Gamma_4}) = g + 1 \) if and only if there exists a unique \( i \) (1 \( \leq i \leq 3 \)) such that the vertex \( (\Gamma_4, \delta^{\Gamma_4}) - \varepsilon_i - \varepsilon_4 \) is in \( K \). In this case \( \Gamma_4 - \hat{\varepsilon}_i \in C_0 \).

2. Assume \( M \) is not hyperelliptic. Then
   \[ l(\Gamma_4, \delta^{\Gamma_4}) = g \quad \text{if and only if} \quad \deg(\Gamma_4, \delta^{\Gamma_4}) = 2g - 1. \]

Moreover, the following two cases can occur if \( l(\Gamma_4, \delta^{\Gamma_4}) = g \).

Case (i). There is a unique pair \((i, j)\) satisfying
   \[ (\Gamma_4, \delta^{\Gamma_4}) + \varepsilon_i - \varepsilon_j - \varepsilon_4 \in K. \]

Conversely, if \((\Gamma_4, \delta^{\Gamma_4}) + \varepsilon_i - \varepsilon_j - \varepsilon_4 \in K\), then \( l(\Gamma_4, \delta^{\Gamma_4}) = g \).

Case (ii). There exists \( k \) (1 \( \leq k \leq 3 \)) such that
   \[ \left[ (\Gamma_4, \delta^{\Gamma_4}) - \varepsilon_k - \varepsilon_4, (\Gamma_4, \delta^{\Gamma_4}) + \varepsilon_j - \varepsilon_k - \varepsilon_4 \right] = 0 \]

for any 1 \( \leq j \leq 3 \). In this case \( \Gamma_4 - \hat{\varepsilon}_k \in C_0 \) and \( \Gamma_4 \in U_0 \).

**Proof.**

1. Assume \( l(\Gamma_4, \delta^{\Gamma_4}) = g + 1 \). Then \( \deg \Gamma_4 + \delta^{\Gamma_4} = 2g \). By the definition of \( \delta^{\Gamma_4} \), we have \((\Gamma_4, \delta^{\Gamma_4}) - \varepsilon_4 \in G^{(d)} \). If
   \[ \left[ (\Gamma_4, \delta^{\Gamma_4}) - 2\varepsilon_4, (\Gamma_4, \delta^{\Gamma_4}) - \varepsilon_4 \right] = 0, \]

then \((\Gamma_4, \delta^{\Gamma_4}) - 2\varepsilon_4 \) is in \( K \). From (3.1), \((\Gamma_4, \delta^{\Gamma_4}) - 2\varepsilon_4 \) must be in \( H^{(3)} \).

This contradicts the definition of \( \delta^{\Gamma_4} \). Thus we may assume that
   \[ \left[ (\Gamma_4, \delta^{\Gamma_4}) - \varepsilon_i - \varepsilon_4, (\Gamma_4, \delta^{\Gamma_4}) - \varepsilon_4 \right] = 0 \quad \text{for some} \quad i \quad (1 \leq i \leq 3), \]

and then \((\Gamma_4, \delta^{\Gamma_4}) - \varepsilon_i - \varepsilon_4 \) is in \( K \) for this \( i \).

Conversely, if \((\Gamma_4, \delta^{\Gamma_4}) - \varepsilon_i - \varepsilon_4 \in K\), then \( \deg \Gamma_4 + \delta^{\Gamma_4} = 2g \) and, by Remark 3.4, we have \( l(\Gamma_4, \delta^{\Gamma_4}) = g + 1 \).

If \((\Gamma_4, \delta^{\Gamma_4}) - \varepsilon_i - \varepsilon_4 \) and \((\Gamma_4, \delta^{\Gamma_4}) - \varepsilon_j - \varepsilon_4 \) are in \( K \) with 1 \( \leq i, j \leq 3 \), then \( P_i - P_j \) is a principal divisor. Since \( g > 0 \), we have \( i = j \).

2. Assume that \( l(\Gamma_4, \delta^{\Gamma_4}) = g \). Then \( \deg(\Gamma, \delta^{\Gamma_4}) \geq 2g - 2 \) from Remark 3.4. If \( \deg(\Gamma, \delta^{\Gamma_4}) = 2g - 2 \), then \((\Gamma_4, \delta^{\Gamma_4}) \in K \) and \((\Gamma_4, \delta^{\Gamma_4}) - \varepsilon_4 \)
\( \in H^{(d)} \) by (3.2). This contradicts the definition of \( \delta^{\Gamma_1} \). Thus we have 
\[ \text{deg}(\Gamma_4, \delta^{\Gamma_1}) = 2g - 1. \]
By the minimality of \( \delta^{\Gamma_1} \), the three equations below do not happen at the same time:
\[
[(\Gamma_4, \delta^{\Gamma_1}) - e_i - e_4, (\Gamma_4, \delta^{\Gamma_1}) - e_4] = 1 \quad (i = 1, 2, 3).
\]
Thus we may assume that
\[
[(\Gamma_4, \delta^{\Gamma_1}) - e_i - e_4, (\Gamma_4, \delta^{\Gamma_1}) - e_4] = 0. \quad (3.5.1)
\]
Since \( (\Gamma_4, \delta^{\Gamma_1}) \in H^{(d)} \), we have
\[
[(\Gamma_4, \delta^{\Gamma_1}) - e_4, (\Gamma_4, \delta^{\Gamma_1})] = [(\Gamma_4, \delta^{\Gamma_1}) - e_4, (\Gamma_4, \delta^{\Gamma_1})] = 1.
\]
By these equations and (3.5.1), we also have
\[
[(\Gamma_4, \delta^{\Gamma_1}) - e_i - e_4, (\Gamma_4, \delta^{\Gamma_1}) - e_i] = 0. \quad (3.5.2)
\]
Regard \( (\Gamma_4, \delta^{\Gamma_1}) - e_i - e_4 \) as \( \Gamma \) in Lemma 3.1. Since \( \Gamma \) in Lemma 3.1 is equal to 0 or 1 since \( \text{deg} \Gamma = 2g - 3. \) By (3.5.1) and (3.5.2), we can assume that

(I)
\[
[(\Gamma_4, \delta^{\Gamma_1}) - e_4, (\Gamma, \delta^{\Gamma_1}) - e_1 + e_2 - e_4] = 1
\]
and
\[
[(\Gamma_4, \delta^{\Gamma_1}) - e_4, (\Gamma, \delta^{\Gamma_1}) - e_1 + e_3 - e_4] = 0
\]
or

(II)
\[
[(\Gamma_4, \delta^{\Gamma_1}) - e_4, (\Gamma, \delta^{\Gamma_1}) - e_1 + e_2 - e_4] = 0
\]
and
\[
[(\Gamma_4, \delta^{\Gamma_1}) - e_4, (\Gamma, \delta^{\Gamma_1}) - e_1 + e_3 - e_4] = 0.
\]

Case (I). Since \( l((\Gamma_4, \delta^{\Gamma_1}) - e_1, \delta^{\Gamma_1}) = g, \) \( l((\Gamma_4, \delta^{\Gamma_1}) - e_1) = g - 1 \) by the definition of \( \delta^{\Gamma_1} \). By (3.5.2), \( l((\Gamma_4, \delta^{\Gamma_1}) - e_1 - e_4) = g - 1. \) And, by the first equation of (I), we have \( l((\Gamma_4, \delta^{\Gamma_1}) - e_1 + e_2 - e_4) = g. \) Since \( \text{deg}(\Gamma_4, \delta^{\Gamma_1}) - e_1 + e_2 - e_4) = 2g - 2. \) \((\Gamma_4, \delta^{\Gamma_1}) - e_1 + e_2 - e_4 \) is in \( K. \) Thus the equations (I) imply the case (ii).
Conversely, assume \((\Gamma_4, \delta^{\Gamma_4}) - e_1 + e_2 - e_4\) is in \(K\). Then
\[
\left[ (\Gamma_4, \delta^{\Gamma_4}) - e_1 - e_4, \left(\Gamma_4, \delta^{\Gamma_4}\right) - e_1 + e_2 - e_4 \right] = 1
\]
and
\[
\left[ (\Gamma_4, \delta^{\Gamma_4}) - e_1 - e_4, \left(\Gamma_4, \delta^{\Gamma_4}\right) - e_4 \right]
\]
\[
= \left[ \left(\Gamma_4, \delta^{\Gamma_4}\right) - e_1 + e_2 - e_4, \left(\Gamma_4, \delta^{\Gamma_4}\right) + e_2 - e_4 \right] \]
\[
= 0.
\]

From these equations, we have
\[
l(\left(\Gamma_4, \delta^{\Gamma_4}\right) - e_4)
\]
\[
= l\left(\left(\Gamma_4, \delta^{\Gamma_4}\right) - e_1 - e_4\right)
\]
\[
= l\left(\left(\Gamma_4, \delta^{\Gamma_4}\right) - e_1 - e_4 + e_2\right) - 1
\]
\[
= g - 1.
\]

Then, by the definition of \(\delta^{\Gamma_4}\), we have \(l(\Gamma_4, \delta^{\Gamma_4}) = g\).

If \((\Gamma_4, \delta^{\Gamma_4}) + e_i - e_j - e_4\) and \((\Gamma_4, \delta^{\Gamma_4}) + e_j - e_i - e_4\) are in \(K\), then \(P_i - P_j \sim P_i - P_j\). Since \(M\) is not hyperelliptic, \(i = i'\) and \(j = j'\).

**Case (II).** We can easily see that the equations (3.5.1) and (3.5.2) and (II) imply the case (i).

**Remark.** We do not deny the possibility that two cases (i) and (ii) of (2) occur at the same time for one \(\Gamma_4\).

**Definition 3.6.** We define the subsets \(U_0(M)\) and \(U_1(M)\) of \(V^{(3)}\) by
\[
U_0(M) = \tilde{U}_0 = \left\{ \Gamma_4 \mid (\Gamma_4, 2g - \deg \Gamma_4) - e_i - e_4 \in K \right. \\
\text{for some } i \left(1 \leq i \leq 3\right) \}
\]
and
\[
U_1(M) = \tilde{U}_1 = \left\{ \Gamma_4 \mid (\Gamma_4, 2g - 1 - \deg \Gamma_4) + e_i - e_j - e_4 \in K \right. \\
\text{for some pair } (i, j) \left(i \neq j, 1 \leq i, j \leq 3\right) \}
\]

**Remark 3.7.** (1) In the same way as in the proof of Lemma 3.5, we have
\[
U_0(M) = \left\{ \Gamma_4 \mid \exists i \text{ such that } (\Gamma_4, 2g - \deg \Gamma_4) - e_i - e_4 \in K \right\}.
\]
If $M$ is non-hyperelliptic,
\[
\tilde{U}_1(M) = \{ \Gamma_4 \mid \exists \{i,j\} \ (i \neq j, 1 \leq i,j \leq 3) \text{ such that} \\
(\Gamma_4, 2g - 1 - \deg \Gamma_4 + e_i - e_j - e_4 \in K) \}.
\]

(2) By Remark 3.3, $\tilde{U}_0 \subset U_0$ and $\# \tilde{U}_0 \leq U_0$.

Here we give some properties of $\tilde{U}_0(M)$ and $\tilde{U}_1(M)$.

**Lemma 3.8.** (1) The three conditions

(i) $\Gamma_4 \in \tilde{U}_0(M)$,

(ii) $\delta^{\Gamma_4} = 2g - \deg \Gamma_4$, and

(iii) $l(\Gamma_4, \delta^{\Gamma_4}) = g + 1$

are equivalent, and then $\tilde{U}_0 = W_{g+1}$.

(2) If $\Gamma_4 \in \tilde{U}_1$, then $\delta^{\Gamma_4} = 2g - 1 - \deg \Gamma_4$ and $l(\Gamma_4, \delta^{\Gamma_4}) = g$. Therefore we have $U_1(M) \subset W_g$.

*Proof.* (1) The equivalence between (ii) and (iii) is Remark 3.4 (1).

Assume $\Gamma_4 \in \tilde{U}_0(M)$ and $(\Gamma_4, 2g - \deg \Gamma_4) - e_i - e_4 \in K$ for a unique $i$ $(1 \leq i \leq 3)$. Then

\[
[(\Gamma_4, 2g - \deg \Gamma_4) - e_i - e_4, (\Gamma_4, 2g - \deg \Gamma_4) - e_4] = 0.
\]

So, by $(\ast_4)$, $\delta^{\Gamma_4} = 2g - \deg \Gamma_4$.

Conversely if $\delta^{\Gamma_4} = 2g - \deg \Gamma_4$, then $(\Gamma_4, 2g - \deg \Gamma_4) - e_4 \in G^{(4)}$. By the minimality of $\delta^{\Gamma_4}$, there exists $i$ $(1 \leq i \leq 3)$ such that

\[
[(\Gamma_4, 2g - \deg \Gamma_4) - e_i - e_4, (\Gamma_4, 2g - \deg \Gamma_4) - e_4] = 0
\]

(see the proof of Lemma 3.5 (2)). Then

\[
l((\Gamma_4, 2g - \deg \Gamma_4) - e_i - e_4) = l((\Gamma_4, 2g - \deg \Gamma_4) - e_4) = g,
\]

and we have $(\Gamma_4, 2g - \deg \Gamma_4) - e_i - e_4 \in K$. Hence we get the equivalence between (i) and (ii).

(2) Assume $\Gamma_4 \in \tilde{U}_1$. Then there exists a pair $(i,j) (i \neq j, 1 \leq i,j \leq 3)$ satisfying $(\Gamma_4, 2g - 1 - \deg \Gamma_4) + e_i - e_j - e_4 \in K$. Put $\Gamma = (\Gamma_4, 2g - 1 - \deg \Gamma_4) + e_i - e_j - e_4$. Then we have $l(\Gamma - e_i) = g - 1$,

\[
[\Gamma - e_i, \Gamma - e_j + e_i] = [\Gamma - e_i, \Gamma - e_j + e_4] = 0 \quad (3.8.1)
\]
and

\[ l(\Gamma - e_i + e_j) = l(\Gamma - e_i + e_4) = g - 1. \]

By these equations and \( \text{deg}(\Gamma - e_i + e_j + e_4) = 2g - 1 \), we have

\[ l(\Gamma - e_i + e_j + e_4) = g \]

and

\[ [\Gamma - e_i + e_4, \Gamma - e_i + e_j + e_4] = [\Gamma - e_i + e_j, \Gamma - e_i + e_j + e_4] = 1. \]  
(3.8.2)

Since \( \Gamma \) is very ample,

\[ [\Gamma - e_i - e_k, \Gamma - e_i] = 1 \]

for an arbitrary \( k \) with \( 1 \leq k \leq 4 \). If \( k \neq j \) and \( k \neq 4 \), then, by \((*4) - 1\), we have

\[ [\Gamma - e_i + e_j + e_4 - e_k, \Gamma - e_i + e_j + e_4] = 1. \]  
(3.8.3)

Equations (3.8.2) and (3.8.3) mean \((\Gamma_4, 2g - 1 - \text{deg} \Gamma_4) \in H^{(4)}\). Moreover, by (3.8.1), we have \( \delta^{\Gamma_4} = 2g - 1 - \text{deg} \Gamma_4 \).

From these lemmas, we have

**Proposition 3.9.**  
(1) For an arbitrary curve \( M \) of genus \( g \), we have

\[ \#W_{g+1} = \#\tilde{U}_0 = 3\#K. \]

(2) Assume \( M \) is non-hyperelliptic. Then

\[ \#W_{g} \leq 3\#K + \#U_0. \]

**Proof.**  
(1) By Lemma 3.8 (1), \( W_{g+1} = \tilde{U}_0 \). By the definition of \( \tilde{U}_0 \) and Remark 3.7 (1), we have \( \#\tilde{U}_0 = 3\#K \).

(2) By the definition of \( \tilde{U}_1 \) and Remark 3.7 (1) again, \( \Gamma_4 \) is in \( \tilde{U}_1 \) if and only if there exist a unique \( \Gamma = \sum_{i=1}^{4} \gamma_i e_i \in K \) and a unique pair \((i, j)\) satisfying \( 1 \leq i, j \leq 3, i \neq j, \gamma_j > 0 \), and \( \Gamma_4 = p_4(\Gamma + e_i - e_j) \). Therefore \( \#\tilde{U}_1 \leq 6\#K \).
By Lemma 3.8 (2), \( \#W' = \#U_1 + \#(W'_g - \hat{U}_1) \). If \( l(\Gamma_4, \delta_4') = g \) and \( \Gamma_4 \not\equiv \hat{U}_1 \), then \( \Gamma_4 \) satisfies the condition of Lemma 3.5 (2) (ii) and \( \Gamma_4 \in (U_0 - \hat{U}_0) \). Using \( \#U_1 \leq 6K \) and \( \#U_0 = 3K \), we have

\[
\#W'_g \leq \#U_1 + \#U_0 - \#\hat{U}_0 \leq 3K + \#U_0.
\]

At last we reach the inequalities (A) and (B) by showing the following lemma.

**Lemma 3.10.** For an arbitrary curve \( M \), we have

\[
\#C_0 \leq \#K_h \quad \text{and} \quad \#U_0 \leq 3\#K_h.
\]

**Proof.** Since \( \#U_0 \leq 3\#C_0 \) by Definition 3.2, it suffices to show the inequality \( \#C_0 \leq \#K_h = (g, 2) \).

Recall the notations at the beginning of Section 2. Let \( C^{(1)}_0 \) be the subset of \( V^{(1)} \) defined by

\[
C^{(1)}_0 = \{ \alpha \in V^{(1)} | \alpha \leq 2g - 2, [\alpha, \alpha + 1] = 0 \}.
\]

Put \( C^{(1)}_0 = \{ \alpha_i | 1 \leq t \leq g, \alpha_i < \alpha_i + 1 \} \). We also define the subset \( C^{(2)}_0 \) of \( V^{(2)} \) by \( C^{(2)}_0 = \{ (\alpha, \beta) \in V^{(2)} | \alpha + \beta \leq 2g - 2, [\alpha, \beta, \alpha + 1, \beta] = 0 \} \). We write \( C^{(j)}_0 \) for the set \( C_0 \). For each \( C^{(j)}_0 \) \((j = 1, 2, 3)\) and for an integer \( k \) with \( 1 \leq k \leq g \), define \( C^{(j)}_0(k) \) by \( \{ \Gamma_{j+1} \in C^{(j)}_0 | i(\Gamma_{j+1}) = k \} \), which is a subset of \( C^{(j)}_0 \). In particular \( C^{(2)}_0(k) = \{ \alpha_{g-k+1} \} \).

Let \( p_{j+1} \) be the projection from \( V^{(j+1)} \) to \( V^{(j)} \) \((j = 1, 2, 3)\). Then \( p_{j+1} \) induces the map from \( C^{(j+1)}_0 \) into \( C^{(j)}_0 \). On the other hand, for \( \Gamma_{j+1} \in C^{(j)}_0(k) \),

\[
p_{j+1}^{-1}((\Gamma_{j+1}, \gamma_{j+1})) \cap C^{(j+1)}_0 \subset \{ (\Gamma_{j+1}, \gamma_{j+1}) | [(\Gamma_{j+1}, \gamma_{j+1}), (\Gamma_{j+1}, \gamma_{j+1} + 1)] = 0 \}.
\]

The cardinality of the last set is equal to \( k \). Then we have

\[
\#C^{(j+1)}_0 \leq \sum_{k=1}^g \#C^{(j)}_0(k).
\]

(3.10.1)

Let \( k \) be fixed, and let \( (\Gamma_{j+1}, \gamma_{j+1}) \) be in \( C^{(j+1)}_0(k) \). Then \( p_{j+1}((\Gamma_{j+1}, \gamma_{j+1})) = \Gamma_{j+1} \) must be in \( C^{(j)}_0(t) \) with \( t \geq k \). Conversely if \( \Gamma_{j+1} \in C^{(j)}_0(t) \), then

\[
p_{j+1}^{-1}(\Gamma_{j+1}) \cap C^{(j+1)}_0(k) = \emptyset \quad \text{if } t < k
\]

and

\[
\#\{p^{-1}_{j+1}(\Gamma_{j+1}) \cap C^{(j+1)}_0(k)\} = 0 \text{ or } 1 \quad \text{if } t \geq k.
\]
Therefore
\[ \#C_{0}^{(j+1)}(k) \leq \sum_{t=k}^{g} \#C_{0}^{(j)}(t). \] (3.10.2)

Starting from \( \#C_{0}^{(1)}(k) = 1 \), we get \( \#C_{0}^{(2)}(k) \leq g - k + 1 \) by (3.10.2) and then \( \#C_{0}^{(3)} \leq \sum_{k=1}^{3} k(g - k + 1) = (\frac{3g^2 - 3g}{2}) \) by (3.10.1).

In this section we compute the cardinality of \( K \) for a \( d \)-gonal curve \( M \) with \( d \geq 5 \) and prove Theorem C. The notation is same as in the last section.

**Definition 4.1.** Let \( P \) be the subset of \( V^{(4)} \) defined by
\[ P = \{ \Gamma \in V^{(4)} \mid \text{deg } \Gamma = 2g - 2 \}, \]
and let \( v_i \) be the vector \( \mathbf{e}_i - \mathbf{e}_4 \) (\( i = 1, 2, 3 \)).

For \( \Gamma, \Gamma' \in P \), \( \Gamma - \Gamma' \) can be uniquely expressed as a linear combination of \( v_i \) (\( i = 1, 2, 3 \)). Put \( \Gamma_0 = (0, 0, 0, 2g - 2) \). By placing each \( \Gamma \in P \) at the point \( (\delta_1, \delta_2, \delta_3) \) in \( \mathbb{R}^3 \) if \( \Gamma - \Gamma_0 = \sum_{i=1}^{3} \delta_i v_i \), and we can regard \( P \) as the set of lattice points in the tetrahedron \( \{(\delta_1, \delta_2, \delta_3) \mid 0 \leq \delta_1 + \delta_2 + \delta_3 \leq 2g - 2\} \subset \mathbb{Z}^3 \).

**Definition 4.2.** Let \( \Gamma = \sum_{i=1}^{4} \gamma_i \mathbf{e}_i \) and \( \Gamma' = \sum_{i=1}^{4} \gamma'_i \mathbf{e}_i \) be in \( P \). Then the non-negative integer \( \sum_{i=1}^{4} |\gamma_i - \gamma'_i| \) is even, and we define the distance \( d(\Gamma, \Gamma') \) between \( \Gamma \) and \( \Gamma' \) by
\[ d(\Gamma, \Gamma') := \frac{1}{2} \sum_{i=1}^{4} |\gamma_i - \gamma'_i|. \]

For a subset \( S \) of \( P \), define the diameter \( d(S) \) of \( S \) by
\[ d(S) := \max \{ d(\Gamma, \Gamma') \mid \Gamma \in S, \Gamma' \in S \}. \]

**Lemma 4.3.** For \( \Gamma \) and \( \Gamma' \) in \( P \), assume \( \Gamma - \Gamma' = \sum_{i=1}^{3} \delta_i v_i \) as in 4.1. Put \( \Delta = \{ \delta_i \neq 0 \} \). Divide \( \Delta \) into two sets \( \Delta^+ \) and \( \Delta^- \) according to the signature of \( \delta_i \). That is, \( \Delta^+ \) (resp. \( \Delta^- \)) consists of positive (resp. negative) \( \delta_i \).

Then
\[ d(\Gamma, \Gamma') = \max \left\{ \sum_{\Delta^+ \ni \delta} |\delta|, \sum_{\Delta^- \ni \delta} |\delta| \right\}. \]
Proof. Let $\Gamma = \sum_{i=1}^{4} \gamma_i e_i$ and let $\Gamma' = \sum_{i=1}^{4} \gamma'_i e_i$. Then $\gamma_i - \gamma'_i = \delta_i$ $(i = 1, 2, 3)$ and $\gamma_4 - \gamma'_4 = -\sum_{i=1}^{3} \delta_i$. Moreover we may assume that
\[
\sum_{\Delta^+} \delta \geq \sum_{\Delta^-} \delta > 0.
\]

We prove our assertion by checking up every possibilities of $\Delta^+$ and $\Delta^-$. For example, assume $\Delta^+ = \{\delta_1, \delta_2\}$ and $\Delta^- = \{\delta_3\}$. Then $\delta_1 + \delta_2 \geq -\delta_1 > 0$ and
\[
\frac{1}{2} \sum_{i=1}^{4} |\gamma_i - \gamma'_i| = \frac{1}{2} \{\delta_1 + \delta_2 + (\delta_1 + \delta_2 + \delta_3)\} = \delta_1 + \delta_2.
\]

Therefore in this case we proved our assertion.

Since $K$ is the subset of $P$, we can consider the distance of two points of $K$.

**Proposition 4.4.** Let $M$ be an arbitrary curve of genus $g \geq 2$, and let $P_1, \ldots, P_4$ be distinct points on $M$. Let $K$ be the set defined in 0.4. Assume $d(\Gamma, \Gamma') \geq 5$ for any $\Gamma, \Gamma' \in K$. Then we have
\[
\#K \leq \left\lceil \left(\frac{2g - 2}{3}\right) + 3 \right\rceil.
\]

**Proof.** For $(x, y, z) \in P$, define subsets $\alpha_{(x, y, z)}$ and $\beta_{(x, y, z)}$ of $\mathbb{N}_0^3$ by
\[
\alpha_{(x, y, z)} = \{(4x + a, 4y + b, 4z + c) | 0 \leq a + b + c \leq 4, 0 \leq a, b, c \leq 4\} \cap P
\]
and
\[
\beta_{(x, y, z)} = \{(4x + a, 4y + b, 4z + c) | 4 < a + b + c \leq 9, 0 \leq a, b, c < 4\} \cap P.
\]

We can check that the diameters of $\alpha_{(x, y, z)}$ and $\beta_{(x, y, z)}$ are less than 5 by Lemma 4.3. Then, by our assumption on $K$, we have
\[
\#(K \cap \alpha_{(x, y, z)}) \leq 1 \quad \text{and} \quad \#(K \cap \beta_{(x, y, z)}) \leq 1.
\]

Let
\[
\mathcal{A} = \left\{\alpha_{(x, y, z)} | 0 \leq x + y + z \leq \left\lfloor\frac{2g - 2}{4}\right\rfloor - 1\right\}
\]
and

$$\mathcal{B} = \{ \beta(x, y, z) \mid 0 \leq x + y + z \leq \left\lfloor \frac{2g - 2}{4} \right\rfloor - 2 \}.$$ 

where \([r]\) is the largest integer \(n\) satisfying \(n \leq r\) for a real number \(r\). Then it can easily be seen that \(\mathcal{A} \cup \mathcal{B}\) is a cover of \(P\), and we have

$$\#K \leq \#A + \#B.$$ 

By making a comparison between

$$\#A + \#B = \left( \left\lfloor \frac{2g - 2}{4} \right\rfloor - 1 + 3 \right) + \left( \left\lfloor \frac{2g - 2}{4} \right\rfloor - 2 + 3 \right)$$

and \(\left\lceil \frac{(2g - 2)/3 + 3}{3} \right\rceil\), we can get our assertion.

4.5. The proof of Theorem C. If \(M\) is a \(d\)-gonal curve with \(d \geq 5\), then the condition on the set \(K\) is satisfied. In fact, take distinct elements \(\Gamma\) and \(\Gamma'\) of \(K\); then \(\Gamma - \Gamma'\) is a principal divisor of degree not less than \(d\). This means \(d(\Gamma, \Gamma') \geq 5\). Thus, by the above proposition, we have Theorem C.

Remark 4.6. The assumption “\(d(\Gamma, \Gamma') \geq 5\)” of Proposition 4.4 is also satisfied in the cases below.

(a) \(M\) is a trigonal curve of genus \(g \geq 7\), and take \(P_i\) \((i = 1, \ldots, 4)\) such that there is no principal divisor \(\sum_{i=1}^{4} s_i P_i\) of degree 3.

(b) \(M\) is a 4-gonal curve of genus \(g \geq 13\), and take \(P_i\) \((i = 1, \ldots, 4)\) such that there is no principal divisor \(\sum_{i=1}^{4} s_i P_i\) of degree 4.

In fact, when \(M\) is trigonal (resp. 4-gonal) of genus \(g \geq 7\) (resp. 13), every principal divisor \(\sum_{i=1}^{4} s_i P_i\) is of degree bigger than 4 (resp. 5) [6, p. 87], for \(P_i\)'s taken as above. Then we can weaken the assumption of Theorem A more or less.

But the estimation in Proposition B and Theorem C may be too rough to determine the exact upper bound of \(\#K\) or \(\#G(P_1, \ldots, P_4)\) for \(d\)-gonal curves \(M\) with \(d \geq 3\). And in order to remove the assumption of Theorem A or C on gonality, we have to investigate the set \(K\) of arbitrary \(P_1, \ldots, P_4\) on trigonal and 4-gonal curves more closely.
REFERENCES