Cardinal Interpolation and Spline Functions V. The B-Splines for Cardinal Hermite Interpolation*

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ABSTRACT

In the third paper of this series on cardinal spline interpolation [4] Lipow and Schoenberg study the problem of Hermite interpolation

$$S(\nu) = y_\nu, \qquad S'(\nu) = y_\nu', \ldots, S^{(r-1)}(\nu) = y_\nu^{(r-1)} \quad \text{for all } \nu.$$

The B-splines are there conspicuous by their absence, although they were found very useful for the case $r = 1$ of ordinary (or Lagrange) interpolation (see [5–10]). The purpose of the present paper is to investigate the B-splines for the case of Hermite interpolation ($r > 1$). In this sense the present paper is a supplement to [4] and is based on its results. This is done in Part I. Part II is devoted to the special case when we want to solve the problem

$$S(\nu) = y_\nu, \qquad S'(\nu) = y_\nu' \quad \text{for all } \nu \tag{1}$$

by quintic spline functions of the class $C'''(-\infty, \infty)$. This is the simplest nontrivial example for the general theory. In Part II we derive an explicit solution for the problem (1), where $\nu = 0, 1, \ldots, n$.

INTRODUCTION

Let m and r be natural numbers and let

$$n = 2m - 1, \qquad r \leqslant m. \tag{1}$$

Let, furthermore, $S_{n,r}$ denote the class of cardinal spline functions of the odd degree $n = 2m - 1$ and having all the integers as knots of multiplicity r. This last requirement means that

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$$S_{n,r} \subset C^{2m-1-r}(-\infty, \infty). \tag{2}$$

The problem of cardinal Hermite interpolation by elements of $S_{n,r}$ is as follows. A set of r sequences of numbers

$$y = (y_\nu), \qquad y' = (y_\nu'), \ldots, y^{(r-1)} = (y_\nu^{(r-1)}) \tag{3}$$

being prescribed, we are to find $S(x) \in S_{n,r}$ such that

$$S(\nu) = y_\nu, \qquad S'(\nu) = y_\nu', \ldots, S^{(r-1)}(\nu) = y_\nu^{(r-1)} \quad \text{for all } \nu. \tag{4}$$

The main result of [4] is the following

THEOREM 1. (i) *If the sequences* (3) *satisfy the growth conditions*

$$y_\nu^{(s)} = O(|\nu|^\gamma) \quad as \quad \nu \to \pm\infty, \quad (s = 0, 1, \ldots, r-1), \quad for\ some \quad \gamma \geqslant 0 \tag{5}$$

then there is a unique $S(x) \in S_{n,r}$ *satisfying the growth condition*

$$S(x) = O(|x|^\gamma) \quad as \quad x \to \pm\infty, \tag{6}$$

and the interpolatory conditions (4).

(ii) *Moreover, if the fundamental functions are*

$$L_0(x), L_1(x), \ldots, L_{r-1}(x), \tag{7}$$

i.e., they are the unique bounded elements of $S_{n,r}$ *satisfying the conditions*

$$L_s^{(\rho)}(0) = 0 \quad if \quad \rho \neq s, \quad L_s^{(s)}(0) = 1, \tag{8}$$

$$L_s(\nu) = L_s'(\nu) = \cdots = L_s^{(r-1)}(\nu) = 0 \quad for\ all \quad \nu \neq 0, \quad (s = 0, \ldots, r-1), \tag{9}$$

then the unique solution $S(x)$ *in* (i) *is given by the Lagrange-Hermite formula*

$$S(x) = \sum_{-\infty}^{\infty} f(\nu) L_0(x-\nu) + \sum_{-\infty}^{\infty} f'(\nu) L_1(x-\nu) + \cdots + \sum_{-\infty}^{\infty} f^{(r-1)}(\nu) L_{r-1}(x-\nu). \tag{10}$$

In [4] it was shown that if we define

$$S_{n,r}^0 = \{S(x);\, S(x) \in S_{n,r},\, S^{(s)}(\nu) = 0 \text{ for all } \nu \text{ and } s = 0, \ldots, r-1\} \tag{11}$$

then this is a subspace of $S_{n,r}$ of dimension

$$d = 2m - 2r. \tag{12}$$

Also that this subspace is spanned by d elements

$$S_1(x),\ S_2(x), \ldots, S_{2m-2r}(x), \tag{13}$$

that are solutions of the functional equations

$$S_j(x+1) = \lambda_j S_j(x) \quad (j = 1, \ldots, 2m-2r). \tag{14}$$

The functions (13) are called the *eigensplines* of $S_{n,r}$ and are determined only up to a multiplicative factor. The corresponding λ_ν are called *eigenvalues*. They are the simple zeros of a reciprocal monic polynomial

$$\begin{aligned} \Pi_{n,r}(x) = {} & c_{-(m-r)} + c_{-(m-r)+1}x + \cdots + c_0 x^{m-r} \\ & + c_1 x^{m-r+1} + \cdots + c_{m-r} x^{2m-2r} \end{aligned} \tag{15}$$

called the *characteristic polynomial* of $S_{n,r}$. Its coefficients are integers and

$$c_0 > 0, \quad c_{-\nu} = c_\nu, \quad c_{m-r} = \pm 1. \tag{16}$$

We also know [4] that (15) can be expressed in terms of the Pascal triangle by the determinant

$$\Pi_{n,r}(x) = (-1)^{m(r-1)} \times$$

$$\begin{vmatrix} 1 & \binom{r}{1} & \binom{r}{2} & \cdots & \binom{r}{r-1} & 1-x & 0 & \cdots & 0 \\ 1 & \binom{r+1}{1} & \binom{r+1}{2} & \cdots & \binom{r+1}{r-1} & \binom{r+1}{r} & 1-x & \cdots & 0 \\ \vdots & & & & & & & & \\ 1 & \binom{n-r}{1} & & \cdots & & \binom{n-r}{r} & \cdots & \binom{n-r}{n-r-1} & 1-x \\ 1 & \binom{n-r+1}{1} & & \cdots & & & & & \binom{n-r+1}{n-r} \\ \vdots & & & & & & & & \\ 1 & \binom{n}{1} & & \cdots & & & & & \binom{n}{n-r} \end{vmatrix}. \tag{17}$$

Its zeros

$$\lambda_1, \lambda_2, \ldots, \lambda_{2m-2r}, \tag{18}$$

are all simple, real, and have the sign of $(-1)^r$. We arrange them so that

$$0 < |\lambda_1| < |\lambda_2| < \cdots < |\lambda_{2m-2r}|. \tag{19}$$

In [4, Sec. 7] it was also shown how the fundamental functions (7) can be constructed in terms of one half of the set of eigensplines (13). The zeros (18) being reciprocal in pairs, and $d = 2m - 2r$ being an even number, we may make (19) more precise by writing

$$0 < |\lambda_1| < \cdots < |\lambda_{m-r}| < 1 < |\lambda_{m-r+1}| < \cdots < |\lambda_{2m-2r}|. \tag{20}$$

The eigensplines

$$S_1(x), S_2(x), \ldots, S_{m-r}(x), \tag{21}$$

correspond to the eigenvalues that are in absolute value below 1, and the relations (14) show that they have the property

$$\lim_{x \to +\infty} S_\nu(x) = 0 \quad (\nu = 1, \ldots, m - r). \tag{22}$$

We may describe them as the "decreasing" eigensplines.

From the construction of the $L_s(x)$ as given in [4, Sec. 7] we retain only the following result to be used below.

LEMMA 1. *In the interval* $[1, +\infty)$ *the fundamental functions* (7) *may be expressed as*

$$L_s(x) = \sum_{j=1}^{m-r} d_{j,s} S_j(x) \quad (s = 0, \ldots, r - 1, 1 \leqslant x < \infty), \tag{23}$$

with appropriate constant coefficient $d_{j,s}$.

The case of Lagrange interpolation, when $r = 1$, was discussed independently by several authors (see [12], [6], [1], [8], listing the papers in the order of their appearance in print). In [6] and [7] extensive use is made of the so-called B-splines. They seem particularly well suited for utilizing fully the fact that the concept of a cardinal spline function is invariant with respect to the group of translations by integers. Having bounded supports, the B-splines lead to *finite* recurrence relations. They also made possible the explicit solutions of the corresponding *finite* interpolation problems (see [9–11]).

The purpose of the present paper is to investigate the B-splines for the case of Hermite interpolation ($r > 1$). In this sense the present paper is a supplement to [4] on which it is based. The main result is Theorem 3 of Sec. 5. Unfortunately, the property of the B-splines there described is made to depend on a deep number-theoretic problem: The irreducibility in the rational field of the characteristic polynomial (17) for odd values of $n = 2m - 1$. This, however, is surely not in the nature of the problem. Rather, Assumption (5.1) of Theorem 3 should be removed altogether and the matter should be settled by showing that a certain determinant does not vanish. Part I contains the general discussion. Part II deals with the special case when

$$m = 3, \qquad r = 2.$$

For this case of Hermite interpolation by quintic splines we show how to obtain explicit solutions for the interpolation problems in a finite set of equidistant nodes. Since $2m - 2r = 2$ we have only two eigensplines and the problem is about as easy as for the case of interpolation by cubic splines (see [9], [10]).

The Hermite interpolation problems here considered can be described (in a notation due to Paul Turán) as problems of type $(0, 1, \ldots, r - 1)$. Using the same Turán notation, one could consider other interpolation problems such as (0, 2), which are not of the Hermite type. The method here developed can be adapted to furnish the solution of such problems, for the cardinal as well as the finite case. However, such problems require a corresponding set of B-splines, appropriate to the problem.

I. THE GENERAL THEORY

1. The B-Splines for Cardinal Hermite Spline Interpolation

We retain the notations of the Introduction.

DEFINITION 1. *We define the r elements of $S_{n,r}$*

$$N_0(x), N_1(x), \ldots, N_{r-1}(x) \tag{1.1}$$

by means of the relations

$$N_s(x) = \sum_{\nu=-(m-r)}^{m-r} c_\nu L_s(x-\nu), \quad (s = 0,\ldots, r-1). \tag{1.2}$$

We call them the B-splines of the class $S_{n,r}$.

That they deserve this name will follow from their properties described here and in Sec. 3 (Corollary 1) and Sec. 5 (Theorem 3).

Let us for the moment consider the case $r = 1$ and show that

$$N_0(x) = (2m-1)!M_{2m}(x), \tag{1.3}$$

where

$$M_{2m}(x) = \frac{1}{(2m-1)!}\sum_{i=0}^{2m}(-1)^i\binom{2m}{i}(x-i+m)_+^{2m-1} \tag{1.4}$$

is the ordinary (central) B-spline of degree $2m-1$ (see 5, Sec. 3.13). To see this we observe that (10) reduces to the Lagrange formula

$$S(x) = \sum_{-\infty}^{\infty} f(\nu)L_0(x-\nu)$$

which is exact for bounded elements of $S_{2m-1,1}$. But then we have the identity

$$M_{2m}(x) = \sum_{\nu=-(m-1)}^{m-1} M_{2m}(\nu)L_0(x-\nu), \tag{1.5}$$

while

$$c_\nu = (2m-1)!M_{2m}(\nu) \qquad (\nu = -(m-1),\ldots, m-1)$$

are precisely the coefficients of the Euler-Frobenius polynomial $\Pi_{2m-1,1}(x) = \Pi_{2m-1}(x)$ of [8]. Now (1.5) is seen to be identical to (1.2), except for the factor $(2m-1)!$, if $r = 1$.

We return to the general case and begin with

LEMMA 2. *The B-splines* $N_s(x)$ $(s = 0,\ldots, r-1)$ *have their support in the interval*

$$I = (-(m-r+1), m-r+1), \tag{1.6}$$

hence

$$N_s(x) = 0 \quad \textit{if} \quad x \leqslant -m+r-1, \quad \textit{or} \quad x \geqslant m-r+1. \tag{1.7}$$

Proof. Let

$$x \geqslant m - r + 1, \tag{1.8}$$

hence

$$x - \nu \geqslant 1 \quad \text{if} \quad \nu = -(m-r), -(m-r)+1, \ldots, m-r.$$

This shows that we can apply the representation (23) of Lemma 1 to all terms of the right side of (1.2) to obtain

$$N_s(x) = \sum_{\nu=-(m-r)}^{m-r} c_\nu L_s(x-\nu) = \sum_\nu c_\nu \sum_{j=1}^{m-r} d_{j,s} S_j(x-\nu)$$

$$= \sum_{j=1}^{m-r} d_{j,s} \sum_\nu c_\nu S_j(x-\nu).$$

The vanishing of $N_s(x)$, for x satisfying (1.8), will follow as soon as we establish *that*

$$\sum_{-(m-r)}^{m-r} c_\nu S_j(x-\nu) = 0 \quad \textit{for all real } x. \tag{1.9}$$

From the functional equations (14) we find that

$$S_j(x-\nu) = \lambda_j^{-\nu} S_j(x)$$

and substituting this into the left side of (1.9) we find that

$$\sum_{-(m-r)}^{m-r} c_\nu S_j(x-\nu) = \sum_\nu c_\nu \lambda_j^{-\nu} S_j(x) = S_j(x) \sum_{-(m-r)}^{m-r} c_\nu \lambda_j^{-\nu},$$

and the last sum vanishes, because λ_j is a zero of the *reciprocal* polynomial (15). This establishes the second half of (1.7).

The proof of Lemma 2 will be complete as soon as we establish the *symmetry relations*

$$N_s(-x) = (-1)^s N_s(x) \qquad (s = 0, \ldots, r-1). \tag{1.10}$$

These are shown as follows. From (1.2)

$$N_s(-x) = \sum_\nu c_\nu L_s(-x-\nu) = (-1)^s \sum_\nu c_\nu L_s(x+\nu)$$

$$= (-1)^s \sum_\nu c_{-\nu} L_s(x-\nu) = (-1)^s \sum_\nu c_\nu L_s(x-\nu)$$

$$= (-1)^s N_s(x). \quad \blacksquare$$

2. A Construction of the B-Splines $N_s(x)$

From their definition (1.2) we see that the $N_s(x)$ have the properties

$$N_s^{(s)}(\nu) = c_\nu \text{ if } -(m-r) \leqslant \nu \leqslant m-r, \tag{2.1}$$

$$N_s^{(\rho)}(\nu) = 0 \text{ if } -(m-r) \leqslant \nu \leqslant m-r \text{ and } \rho \neq s, \text{ and } 0 \leqslant \rho \leqslant r-1. \tag{2.2}$$

However, by using the relations (1.10), it is clear that it suffices to construct $N_s(x)$ for $x \geqslant 0$. Cutting down the relations (2.1) and (2.2) we obtain

$$N_s^{(s)}(\nu) = c_\nu \quad \text{for} \quad \nu = 0, 1, \ldots, m-r, \tag{2.3}$$

$$N_s^{(\rho)}(\nu) = 0 \quad \text{for} \quad \nu = 0, 1, \ldots, m-r, \quad \rho \neq s. \tag{2.4}$$

It is interesting that these last relations allow us to obtain the $N_s(x)$, explicitly. Indeed, if we realize that $N_s(x)$ is a spline function of degree $2m-1$ and has r-fold knots we see that it must have the form

$$\begin{aligned} N_s(x) & \\ = a_{1,1}(1-x)^{2m-1} &+ a_{2,1}(2-x)_+^{2m-1} + \cdots + a_{r-m+1,1}(m-r+1-x)_+^{2m-1} \\ + a_{1,2}(1-x)^{2m-2} &+ a_{2,2}(2-x)_+^{2m-2} + \cdots + a_{r-m+1,2}(m-r+1-x)_+^{2m-2} \\ \vdots \qquad & \qquad \vdots \qquad\qquad\qquad \vdots \\ + a_{1,r}(1-x)^{2m-r} &+ a_{2,r}(2-x)_+^{2m-r} + \cdots + a_{r-m+1,r}(m-r+1-x)_+^{2m-r}, \\ & \text{if} \quad x \geqslant 0. \end{aligned} \tag{2.5}$$

The computation of the coefficients $a_{i,j}$, which all turn out to be rationals, is extremely simple, if we go about the right way. Using the relations (2.3) and (2.4) for $\nu = m-r$ and substituting into (2.5) the value $x = m-r$ we see that all terms vanish, except the r terms in the last column. This gives

$$a_{r-m+1,\nu} \quad (\nu = 1, \ldots, r).$$

These being known, we now substitute $x = m - r - 1$ and obtain the column

$$a_{r-m,\nu} \quad (\nu = 1, \ldots, r).$$

Finally substituting $x = 0$ gives us the first column

$$a_{1,\nu} \quad (\nu = 1, \ldots, r).$$

As examples we work out the B-splines for the following two cases:

$$m = 3, \qquad r = 2, \tag{2.6}$$

and

$$m = 4, \qquad r = 2. \tag{2.7}$$

To begin with, we need the coefficients of the corresponding characteristic polynomials. These are

$$\Pi_{5,2}(x) = -\begin{vmatrix} 1 & 2 & 1-x & 0 \\ 1 & 3 & 3 & 1-x \\ 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \end{vmatrix} = -x^2 + 6x - 1, \tag{2.8}$$

and

$$\Pi_{7,2}(x) = \begin{vmatrix} 1 & 2 & 1-x & 0 & 0 & 0 \\ 1 & 3 & 3 & 1-x & 0 & 0 \\ 1 & 4 & 6 & 4 & 1-x & 0 \\ 1 & 5 & 10 & 10 & 5 & 1-x \\ 1 & 6 & 15 & 20 & 15 & 6 \\ 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix} \tag{2.9}$$

$$= x^4 - 72x^3 + 262x^2 - 72x + 1,$$

respectively. Applying the general method just described we easily find the following:

If $m = 3$, $r = 2$ then

$$N_0(x) = \begin{cases} 8(1-x)_+^{\,5} + 4(2-x)_+^{\,5} \\ -50(1-x)_+^{\,4} - 5(2-x)_+^{\,4} & \textit{if} \quad x \geqslant 0, \end{cases}$$

$$N_0(x) = N_0(-x) \qquad \textit{if} \quad x \leqslant 0, \tag{2.10}$$

and

$$N_1(x) = \begin{cases} 10(1-x)_+^5 + (2-x)_+^5 \\ -26(1-x)_+^4 - (2-x)_+^4 & \text{if} \quad x \geqslant 0, \end{cases}$$

$$N_1(x) = -N_1(-x) \qquad \text{if} \quad x \leqslant 0. \tag{2.11}$$

If $m = 4$, $r = 2$ *then*

$$N_0(x) = \begin{cases} -90(1-x)_+^7 - 144(2-x)_+^7 - 6(3-x)_+^7 \\ +1715(1-x)_+^6 + 392(2-x)_+^6 + 7(3-x)_+^6 & \text{if} \quad x \geqslant 0, \end{cases}$$

$$N_0(x) = N_0(-x) \qquad \text{if} \quad x \leqslant 0; \tag{2.12}$$

and

$$N_1(x) = \begin{cases} -245(1-x)_+^7 - 56(2-x)_+^7 - (3-x)_+^7 \\ +1191(1-x)_+^6 + 120(2-x)_+^6 + (3-x)_+^6 & \text{if} \quad x \geqslant 0, \end{cases}$$

$$N_1(x) = -N_1(-x) \qquad \text{if} \quad x \leqslant 0. \tag{2.13}$$

As a check on our computation we mention the following.

1. For the case $m = 3$, $r = 2$, the functions (2.10) and (2.11) must satisfy the condition

$$N_0'''(0) = 0$$

because $N_0(x) \in C'''(-\infty, \infty)$, and

$$N_1''(0) = 0$$

for the same reason.

2. For the case $m = 4$, $r = 2$, the functions (2.12) and (2.13) must satisfy

$$N_0'''(0) = 0, \qquad N_0^{(5)}(0) = 0,$$

because $N_0(x) \in C^5(-\infty, \infty)$, and

$$N_1''(0) = 0, \qquad N_1^{(4)}(0) = 0,$$

for the same reason. These six conditions are actually satisfied.

3. Expressing the $L_s(x)$ in Terms of the $N_s(x)$

This is a simple matter that is useful. Let

$$F(x) = 1 \Big/ \sum_{-(m-r)}^{m-r} c_\nu x^\nu, \tag{3.1}$$

and observe that in view of the inequalities (20), $F(x)$ is regular in an open ring containing the unit circumference $|x| = 1$. We expand it there in a Laurent series

$$F(x) = \sum_{-\infty}^{\infty} \omega_\nu x^\nu \quad \text{if} \quad |x| = 1. \tag{3.2}$$

From the symmetry relations (16) we conclude that

$$\omega_{-\nu} = \omega_\nu \quad \text{for all } \nu. \tag{3.3}$$

It is also clear that

$$\omega_\nu \to 0 \quad \text{exponentially, as} \quad \nu \to \pm\infty. \tag{3.4}$$

The relations (3.1) and (3.2) imply that

$$\sum_{\nu=-\infty}^{\infty} c_{j-\nu}\omega_\nu = \delta_j \quad (c_j = 0 \text{ if } |j| > m - r). \tag{3.5}$$

THEOREM 2. *We can invert the relations* (1.2) *to obtain*

$$L_s(x) = \sum_{-\infty}^{\infty} \omega_\nu N_s(x - \nu). \tag{3.6}$$

Proof. For using (1.2) we have

$$\begin{aligned}\sum_\nu \omega_\nu N_s(x-\nu) &= \sum_\nu \omega_\nu \sum_j c_j L_s(x-\nu-j)\\ &= \sum_\nu \omega_\nu \sum_j c_{j-\nu} L_s(x-j)\\ &= \sum_j L_s(x-j) \sum_\nu c_{j-\nu}\omega_\nu\\ &= \sum_j L_s(x-j)\delta_j = L_s(x).\end{aligned}$$

The functions $L_s(x)$ and $N_s(x)$, for a particular value of s, are elements of a subclass of $S_{n,r}$ which we now define by

$$S_{n,r}^{(s)} = \{S(x); S(x) \in S_{n,r}, S^{(\rho)}(\nu) = 0 \text{ for all } \nu \text{ if } \rho \neq s\} \quad (s = 0, 1, \ldots, r-1). \tag{3.7}$$

COROLLARY 1. *If*

$$S(x) \in S_{2m-1,r}^{(s)} \tag{3.8}$$

and

$$S(x) = O(|x|^\gamma) \quad \textit{as} \quad x \to \pm\infty, \quad \textit{for some} \quad \gamma \geqslant 0, \tag{3.9}$$

then $S(x)$ *admits a unique representation of the form*

$$S(x) = \sum_{-\infty}^{\infty} a_j N_s(x-j). \tag{3.10}$$

Proof. The assumption (3.9) implies that also

$$S^{(\rho)}(x) = O(|x|^\gamma) \quad \text{as} \quad x \to \pm\infty \quad (\rho = 0, 1, \ldots, r-1). \tag{3.11}$$

We may therefore apply to $S(x)$ the Lagrange-Hermite expansion (10) (Theorem 1) and obtain the expansion

$$S(x) = \sum_{-\infty}^{\infty} S^{(s)}(j) L_s(x-j). \tag{3.12}$$

Indeed, by (3.7) we see that all other terms of (10) vanish. Using the expansion (3.6), (3.12) gives

$$\begin{aligned} S(x) &= \sum_j S^{(s)}(j) \sum_\nu \omega_\nu N_s(x-j-\nu) \\ &= \sum_j S^{(s)}(j) \sum_\nu \omega_{\nu-j} N_s(x-\nu) \\ &= \sum_\nu N_s(x-\nu) \sum_j S^{(s)}(j)\omega_{\nu-j}, \end{aligned}$$

and finally

$$S(x) = \sum_{-\infty}^{\infty} a_\nu N_s(x-\nu), \tag{3.13}$$

where

$$a_\nu = \sum_{j=-\infty}^{\infty} \omega_{\nu-j} S^{(s)}(j). \tag{3.14}$$

Observe that these series converge by (3.4) and (3.11).

The *existence* of the expansion (3.10) is thereby established. However, we have also established its *unicity*, for if $S(x)$ were $= 0$ for all x, then (3.11) surely holds, hence (3.13) and (3.14) are valid, while (3.14) shows that all a_ν vanish.

REMARK. Corollary 1 already justifies the name of B-splines for the $N_s(x)$. However, it is clear that $S(x)$, defined by (3.10), is an element of $S_{n,r}^{(s)}$, no matter what the values of the coefficients a_j may be. *We would very much like to remove the assumption* (3.9) *and show that every*

$$S(x) \in S_{n,r}^{(s)} \tag{3.15}$$

may be represented in the form (3.10).

This we shall actually establish below under the following additional

ASSUMPTION 1. *The polynomial*

$$\Pi_{2m-1,r}(x) \tag{3.16}$$

defined by (17) *is irreducible in the rational field.*

REMARK. The statement that the polynomial (3.16) is irreducible over the rationals is most certainly true and also certainly a deep theorem. The polynomials (2.8), (2.9), and

$$\Pi_{7,3}(x)$$

are easily shown to be irreducible.

4. A Few Lemmas

Let $N(x) = N_s(x)$ be any one of the r B-splines having support in

$$I = (-(m - r + 1), m - r + 1). \tag{4.1}$$

It has $2m - 2r + 2$ *components*, by which we mean the polynomials representing $N(x)$ in the $2m - 2r + 2$ consecutive unit intervals that make up I. If we confine x to

$$-(m-r+1) \leqslant x \leqslant -(m-r), \tag{4.2}$$

then these component polynomials may be represented by

$$N_s(x), N_s(x+1), \ldots, N_s(x+2m-2r+1). \tag{4.3}$$

LEMMA 3. *The* $2m-2r+2$ *polynomials* (4.3), *for* x *satisfying* (4.2) *are linearly independent if and only if there is no nontrivial element of* $S_{n,r}^{(s)}$ *having its support composed of only* $2m-2r+1$ *consecutive unit intervals.*

1. The condition is sufficient. Assume that it holds, i.e., that the support of $N_0(x)$ is the shortest possible and let us show that the polynomials (4.3) are independent. For suppose that they are not and that we have a linear relation

$$a_0 N_s(x) + a_1 N_s(x+1) + \cdots + a_{2m-2r+1} N_s(x+2m-2r+1) = 0 \quad \text{for } x \text{ in (4.2)} \tag{4.4}$$

with coefficients a_ν that do not all vanish.

We consider the spline function

$$S(x) = \sum_{\nu=0}^{2m-2r+1} a_\nu N_s(x+\nu) \quad \text{for all real } x. \tag{4.5}$$

It is surely an element of $S_{n,r}^{(s)}$. Notice that

$$N_s(x) = 0 \quad \text{if} \quad x < -(m-r+1) \quad \text{or} \quad x > m-r+1.$$

It follows that

$$S(x) = 0 \tag{4.6}$$

if x is such that

$$x \geqslant m-r+1.$$

Similarly

$$x \leqslant -3m+3r-2$$

implies that, if $0 \leqslant \nu \leqslant 2m-2r+1$, then

$$x + v \leqslant - m + r - 1$$

of that again $S(x) = 0$ by Eq. (4.5) and Lemma 2.

This means that the support of $S(x)$ is confined to the interval

$$- 3m + 3r - 2 < x < m - r + 1. \tag{4.7}$$

On the other hand, the identity (4.4) shows that

$$S(x) = 0 \quad \text{if} \quad - (m - r + 1) < x < - (m - r + 1) + 1. \tag{4.8}$$

Now the interval of (4.8) is exactly in the middle of (4.7). Hence, (4.8) shows that the support of $S(x)$ is confined to the nonabutting intervals

$$(- 3m + 3r - 2, - (m - r + 1)) \quad \text{and} \quad (- m + r, m - r + 1) \tag{4.9}$$

both of which are of the same length $2m - 2r + 1$.

In one of these intervals $S(x)$ is not identically zero. Reason: $S(x)$ *satisfies assumptions of Corollary* 1.

Let I_1 be that interval (or one of them) among (4.9) where $S(x) \not\equiv 0$. Let

$$\tilde{S}(x) = \begin{cases} S(x) & \text{if} \quad x \in I_1, \\ 0 & \text{elsewhere.} \end{cases}$$

This, however, is an element of $S_{n,r}^{(s)}$ of support $2m - 2r + 1$, or less.

2. *The condition is necessary.* For suppose that it is violated and that $S(x) \in S_{n,r}^{(s)}$ is nontrivial and has its support in

$$I_1 = (0, 2m - 2r + 1).$$

To fix the ideas and notations, let

$$m = 3, \quad r = 2,$$

when the support of $N_s(x)$ is in $(- 2, 2)$ and the support of $S(x)$ is in $I_1 = (0, 3)$.

To $S(x)$ we can surely apply Corollary 1, since all assumptions of Corollary 1 are satisfied and obtain the unique representation

$$S(x) = \sum_{-\infty}^{\infty} a_j N_s(x - j) \quad \text{for all } x, \quad (\text{not } all\ a_j = 0). \tag{4.10}$$

Notice that

$$\text{in } (-1, 0) \text{ (4.10) reduces to } \sum_{-2}^{1} a_j N_s(x - j) = 0; \tag{4.11}$$

$$\text{in } (3, 4) \text{ it reduces to } \sum_{2}^{5} a_j N(x - j) = 0. \tag{4.12}$$

Moreover in $(\nu, \nu + 1)$ (4.10) reduces to

$$\sum_{\nu-1}^{\nu+2} a_j N(x - j) \quad \text{in} \quad (\nu, \nu + 1). \tag{4.13}$$

By assumption we know that $S(x) \not\equiv 0$ in some one of the three intervals

$$(\nu, \nu + 1) \quad (\nu = 0, 1, 2).$$

This means that among the three sets of coefficients

$$(a_{-1}, a_0, a_1, a_2), \quad (a_0, a_1, a_2, a_3), \quad \text{and} \quad (a_1, a_2, a_3, a_4),$$

at least *one* set is not entirely composed of zero elements. This implies that at least one of the coefficients

$$a_{-1}, a_0, a_1, a_2, a_3, a_4, \quad \text{is} \quad \neq 0.$$

But this surely implies that at least *one* of the sums (4.11) and (4.12) is not trivial. Suppose it is the first, hence

$$a_{-2}N_s(x + 2) + a_{-1}N(x + 1) + a_0N(x) + a_1N(x - 1) = 0, \quad \text{in} \quad (-1, 0).$$

Equivalently, setting $x = t + 1$ we find that

$$a_1N(t) + a_0N(t + 1) + a_{-1}N(t + 2) + a_{-2}N(t + 3) = 0 \quad \text{in} \quad 0 \leqslant t \leqslant 1.$$

This is what we wanted to prove. ■

The proof obviously generalizes.

LEMMA 4. *If m and r are such that Assumption* 1 *is satisfied, then, for every $s = 0, 1, \ldots, r - 1$, the $2m - 2r + 2$ components* (4.3) *are linearly independent.*

Proof. For suppose they were not. It follows from Lemma 3 that $S_{2m-1,r}^{(s)}$ contains a *nontrivial* element $S(x)$ with support in

$$I_1 = (0, 2m - 2r + 1).$$

We claim: *At least one of the quantities*

$$S^{(s)}(1),\ S^{(s)}(2), \ldots, S^{(s)}(2m - 2r), \tag{4.14}$$

is $\neq 0$.

Proof. We realize that $S(x)$, besides having support in I_1, has the properties

$$S^{(\rho)}(1) = S^{(\rho)}(2) = \cdots = S^{(\rho)}(2m - 2r) = 0, \tag{4.15}$$

$$\rho = 0, 1, \ldots, s - 1, s + 1, \ldots, r - 1.$$

Suppose that $S^{(s)}(1) = 0$. It follows that

$$S(0) = S'(0) = \cdots = S^{(r-1)}(0) = \cdots = S^{2m-r-1}(0) = 0$$

$$\text{(because } S \in C^{2m-r-1}\text{)}$$

and

$$S(1) = S'(1) = \cdots = S^{(r-1)}(1) = 0.$$

But then the polynomial component $P(x)$ of $S(x)$ in $(0, 1)$ has at $x = 0$ a zero of order $2m - r$, and at $x = 1$ a zero of order r, thus a total of

$$(2m - r) + r = 2m.$$

This is one zero too many for its degree $2m - 1$ and we conclude that $P(x) \equiv 0$. Now if $S^{(s)}(2) = 0$, we similarly conclude that $S(x) \equiv 0$ in $(1, 2)$, and so on.

We claim: *We may assume the quantities* (4.14) *to be rationals or even integers.*

Proof. We may assume $S(x)$ to be of the form

$$\begin{aligned} S(x) &= a_{1,0}x_+^{2m-1} + a_{1,1}(x-1)_+^{2m-1} + \cdots + a_{1,2m-2r+1}(x - 2m + 2r - 1)_+^{2m-1} \\ &\quad + a_{2,0}x_+^{2m-2} + a_{2,1}(x-1)_+^{2m-2} + \cdots \\ &\qquad \vdots \qquad\qquad \vdots \\ &\quad + a_{r,0}x_+^{2m-r} + a_{r,1}(x-1)_+^{2m-r} + \cdots + a_{r,2m-2r+1}(x - 2m + 2r - 1)_+^{2m-r}, \end{aligned}$$

for all real x. This linear form in the a_{ij} satisfies numerous linear homogeneous equations, namely Eqs. (4.15) and also

$$S^{(\nu)}(2m - 2r + 1) = 0 \quad \text{for} \quad \nu = 0, 1, \ldots, 2m - r - 1.$$

This homogeneous system does have a nontrivial solution because the quantities (4.14) do not all vanish. The existence of a nontrivial solution means that the rank of the system is less than the number of unknowns. Moreover, all coefficients of the system are integers. It follows that the system has also a *rational* solution that is as close to the nontrivial one as we wish. Multiplying it by an appropriate integer we obtain a solution such that *the quantities* (4.14) *are all integers, not all* $= 0$.

Let us now consider the function

$$\Phi(x; t) = \sum_{-\infty}^{\infty} t^j S(x - j), \quad (t \neq 0), \tag{4.16}$$

which satisfies

$$\Phi(x + 1; t) = t\Phi(x, t). \tag{4.17}$$

This means that if

$$\Phi^{(s)}(0; t_0) = 0, \tag{4.18}$$

then

$$\Phi^{(s)}(\nu; t_0) = 0 \quad \text{for all integers } \nu.$$

Thus $\Phi(x; t_0)$ is an eigenspline of $S_{2m-1,r}$ for the eigenvalue t_0 and this because it satisfies

$$\Phi(x + 1; t_0) = t_0\Phi(x; t_0).$$

However (4.18) is equivalent to

$$\sum_{-\infty}^{\infty} t_0{}^j S^{(s)}(-j) = S^{(s)}(1)t_0{}^{-1} + S^{(s)}(2)t_0{}^{-2} + \cdots$$
$$+ S^{(s)}(2m - 2r)t_0{}^{-(2m-2r)} = 0,$$

or

$$S^{(s)}(1)t_0{}^{2m-2r-1} + S^{(s)}(2)t_0{}^{2m-2r-2} + \cdots + S^{(s)}(2m - 2r) = 0. \tag{4.19}$$

Therefore

$$P(x) = S^{(s)}(1)x^{2m-2r-1} + \cdots + S^{(s)}(2m - 2r) \tag{4.20}$$

is a *nontrivial* polynomial of degree $< 2m - 2r$, all of whose zeros are also zeros of

$$\Pi_{n,r}(x).$$

Therefore $P(x)$ is a factor of $\Pi_{n,s}(x)$ with integer coefficients. This contradicts Assumption 1 and proves the lemma. ∎

5. The Main Result

We can now prove

THEOREM 3. *If m and r are such that*

$$\Pi_{2m-1,r}(x) \quad \textit{is irreducible over the rationals,} \tag{5.1}$$

then every $S(x) \in S^{(s)}_{2m-1,r}$ admits a unique representation of the form

$$S(x) = \sum_{-\infty}^{\infty} a_j N_s(x - j). \tag{5.2}$$

Proof. By Lemma 4 we know that the components of (4.3), in (4.2), are linearly independent. But then the proof runs along familiar lines that were used before:

1. *Let $S(x) = 0$ if $x < 0$.* Observe that if we set

$$q = m - r + 1$$

then $N_s(x - q) = Q(x)$ also vanishes for $x < 0$ since

$$Q(x) \text{ has support in } (0, 2m - 2r + 2)$$

and its components are linearly independent. But then

$$\varphi_1(x) = S(x) - \frac{S^{(s)}(1)}{Q^{(s)}(1)} Q(x)$$

has property $\varphi_1{}^{(s)}(1) = 0$ and therefore $\varphi_1(x) = 0$ in $(0, 1)$ by a previous argument. Continuing in like manner we see that

$$S(x) = \sum_{j=0}^{\infty} a_j Q(x - j) \tag{5.3}$$

with coefficients a_j that are uniquely determined.

2. *Let* $S(x) \in S_{n,r}^{(s)}$. Let $0 \leqslant x \leqslant 1$. Since

$$Q(x), Q(x+1), \ldots, Q(x+2m-2r+1) \quad (0 \leqslant x \leqslant 1) \tag{5.4}$$

are linearly independent in (0, 1) we can determine a_j so that

$$S(x) = \sum_{-(2m-2r+1)}^{0} a_j Q(x-j) \quad \text{in} \quad 0 \leqslant x \leqslant 1. \tag{5.5}$$

This requires some explanation: The component of $S(x)$ in (0, 1) is a polynomial π_{2m-1} satisfying

$$P^{(\rho)}(0) = P^{(\rho)}(1) = 0 \quad \text{if} \quad \rho = 0, \ldots, s-1, s+1, \ldots, r-1,$$

this is a set of $2r-2$ homogeneous conditions. Such polynomials *form a linear space of dimension* $2m-2r+2$. Since the (5.4) are linearly independent, they *must span* this space and (5.5) follows.

Now

$$S_0(x) = S(x) - \sum_{-(2m-2r+1)}^{0} a_j Q(x-j) \tag{5.6}$$

vanishes in $0 \leqslant x \leqslant 1$. Let

$$S_1(x) = \begin{cases} S_0(x) & \text{if} \quad x \geqslant 1 \\ 0 & \text{if} \quad x < 1 \end{cases}$$

$$S_2(x) = \begin{cases} 0 & \text{if} \quad x > 0 \\ S_0(x) & \text{if} \quad x \leqslant 0. \end{cases}$$

By the first case 1 of the proof we know that

$$S_1(x) = \sum_{1}^{\infty} a_j Q(x-j)$$

and

$$S_2(x) = \sum_{-\infty}^{-(2m-2r+2)} a_j Q(x-j).$$

Since

$$S_0(x) = S_1(x) + S_2(x) \quad \text{for all } x,$$

we conclude from (5.6) that

$$S(x) = S_2(x) + \sum_{-(2m-2r+1)}^{0} a_j Q(x-j) + S_1(x) = \sum_{-\infty}^{\infty} a_j Q(x-j). \quad \blacksquare$$

REMARK. It would be interesting to establish Theorem 3 without the Assumption (5.1), as undoubtedly is true. The matter hinges on showing the nonvanishing of a certain determinant.

6. *The Use of B-Splines in Cardinal Spline Interpolation*

We know that the interpolation problem (4) is solved by the expansion

$$S(x) = \sum_{\nu} y_\nu L_0(x-\nu) + \sum_{\nu} y_\nu' L_1(x-\nu) + \cdots + \sum_{\nu} y_\nu^{(r-1)} L_{r-1}(x-\nu). \tag{6.1}$$

This can be given a more advantageous form if we use the expansions (3.6) of Theorem 2. Indeed we find that

$$\sum_{\nu} y_\nu^{(s)} L_s(x-\nu) = \sum_{j} c_j^{(s)} N_s(\nu - j),$$

where

$$c_j^{(s)} = \sum_{\nu=-\infty}^{\infty} y_\nu^{(s)} \omega_{j-\nu}, \quad (s = 0, \ldots, r-1). \tag{6.2}$$

We record this as

COROLLARY 2. *The solution $S(x)$ of the problem* (4) *can be expressed in the form*

$$S(x) = \sum_{-\infty}^{\infty} c_j^{(0)} N_0(x-j) + \sum_{-\infty}^{\infty} c_j^{(1)} N_1(x-j) + \cdots + \sum_{-\infty}^{\infty} c_j^{(r-1)} N_{r-1}(x-j), \tag{6.3}$$

where the $c_j^{(s)}$ are computed from (6.2).

Of course, we have assumed here that the conditions (5) of Theorem 1 are satisfied. The computational advantages of the representation (6.3) were already pointed out in [5, Appendix, p. 89] for the special case when $r = 1$.

II. INTERPOLATION BY QUINTIC SPLINE FUNCTIONS

Let $m = 3$, so that we deal with 5th degree spline functions. We may now choose the multiplicity r such that $1 \leqslant r \leqslant m = 3$. If we choose $r = 3$ then the problem of interpolation (4), as well as the problem of interpolation in a finite number of nodes, become trivial because it reduces to a succession of separate two-point Hermite interpolation problems (see [4, p. 32]). The choice $r = 1$ is the subject of the note [11]. In that case $2m - 2r = 4$ and the problems depended on the *four* eigenvalues which are the zeros of the Euler-Frobenius polynomial

$$\Pi_{5,1}(x) = x^4 + 26x^3 + 66x^2 + 26x + 1.$$

For the remainder of this paper we assume that $r = 2$. While not trivial, this case is simple because it depends on the *two* zeros of the polynomial (2.8).

7. *Statement of Results*

We assume throughout that

$$m = 3, \qquad r = 2.$$

We have determined explicitly the corresponding B-splines (2.10) and (2.11). They have the properties

$$\begin{aligned} &N_0(-1) = -1, \quad &&N_0(0) = +6, \quad &&N_0(1) = -1, \\ &N_0'(-1) = 0, &&N_0'(0) = 0, &&N_0'(1) = 0; \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} &N_1(-1) = 0, \quad &&N_1(0) = 0, \quad &&N_1(1) = 0, \\ &N_1'(-1) = -1, &&N_1'(0) = +6, &&N_1'(1) = -1; \end{aligned} \tag{7.1'}$$

and sketches of their graphs are as shown in Fig. 1.

Unlike our previous use of the letter n (when we wrote $n = 2m - 1$), let now n denote any natural number. Let $S_{5,2}[0, n]$ denote the class of quintic splines defined in $[0, n]$ and having double knots at the points $x = 1, 2, \ldots, n - 1$. Likewise let

$$S^{(0)}[0, n] = \{S(x); S(x) \in S_{5,2}[0, n], \quad S'(\nu) = 0 \quad (\nu = 0, 1, \ldots, n)\}, \tag{7.2}$$

$$S^{(1)}[0, n] = \{S(x); S(x) \in S_{5,2}[0, n], \quad S(\nu) = 0 \quad (\nu = 0, 1, \ldots, n)\}. \tag{7.3}$$

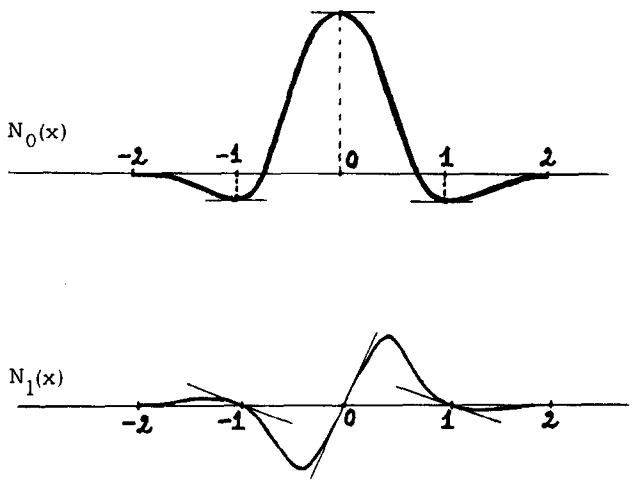

FIG. 1

We wish to solve the following

PROBLEM 1. The function $f(x)$ being in $C^1[0, n]$ we are to find an element of $S_{5,2}[0, n]$ such that

$$S(\nu) = f(\nu), \qquad S'(\nu) = f'(\nu) \quad (\nu = 0, 1, \ldots, n), \tag{7.4}$$

and satisfying the boundary conditions

$$S'''(0) = 0, \qquad S'''(n) = 0. \tag{7.5}$$

The boundary conditions (7.5) are sometimes referred to as "natural," because the solution $S(x)$ which they produce has the property of minimizing

$$\int_0^n [S'''(x)]^2 \, dx$$

among all functions having a square-integrable third derivatives and satisfying the interpolation conditions (7.4).

In describing our solution of Problem 1 we depart from our previous notation (7) and will use the symbol $L_\nu(x)$ for another purpose. In fact, let

$$S(x) = \sum_0^n f(\nu)L_\nu(x) + \sum_0^n f'(\nu)\Lambda_\nu(x) \tag{7.6}$$

be the Lagrange-Hermite formula which describes the solution of our interpolation problems (7.4) and (7.5). In other words, the $L_\nu(x)$ and $\Lambda_\nu(x)$ are the *fundamental functions* of our process. They have the property that *all quantities* $L_\nu(j)$, $L_\nu'(j)$, $\Lambda_\nu(j)$ *and* $\Lambda_\nu'(j)$ *vanish, except that*

$$L_\nu(\nu) = 1 \quad and \quad \Lambda_\nu'(\nu) = 1. \tag{7.7}$$

From these it follows that

$$L_\nu(x) \in S_{5,2}^{(0)}[0, n], \quad \Lambda_\nu(x) \in S_{5,2}^{(2)}[0, n] \quad (\nu = 0, \ldots, n). \tag{7.8}$$

In this connection the following finite analogue of Theorem 3 is of relevance.

LEMMA 5. 1. *Every* $S(x) \in S_{5,2}^{(0)}[0, n]$ *admits in* $[0, n]$ *a unique representation*

$$S(x) = \sum_{-1}^{n+1} c_j N_0(x - j). \tag{7.9}$$

2. *Every* $S(x) \in S_{5,2}^{(1)}[0, n]$ *admits in* $[0, n]$ *a unique representation*

$$S(x) = \sum_{-1}^{n+1} \gamma_j N_1(x - j). \tag{7.10}$$

In view of (7.8) and Lemma 5, we may represent the fundamental functions in the form

$$L_\nu(x) = \sum_{-1}^{n+1} c_{j,\nu} N_0(x - j), \tag{7.11}$$

$$\Lambda_\nu(x) = \sum_{-1}^{n+1} \gamma_{j,\nu} N_1(x - j), \tag{7.12}$$

with coefficients $c_{j,\nu}$ and $\gamma_{j,\nu}$ that have appropriate rational values. The explicit values of these coefficients are given by

THEOREM 4. *We define*

$$\lambda = 3 - 2\sqrt{2} = 0.17158 \tag{7.13}$$

as the smaller root of the quadratic equation

$$-\Pi_{5,2}(x) = x^2 - 6x + 1 = 0. \tag{7.14}$$

In terms of λ *we define for every integer* k *two sequences of integers* (a_k) *and* (b_k) *by*

$$\lambda^k = a_k + b_k\sqrt{2}. \tag{7.15}$$

We may also define them as sequences satisfying the recurrence relation

$$x_{k+2} - 6x_{k+1} + x_k = 0 \quad \text{for all integer } k, \tag{7.16}$$

with the initial values

$$a_0 = 1, \quad a_1 = 3 \quad \text{and} \quad b_0 = 0, \quad b_1 = -2,$$

respectively. We easily find the table of values

k	-2	-1	0	1	2	3	4	5	6	7
a_k	17	3	1	3	17	99	577	3363	19601	114243
b_k	12	2	0	-2	-12	-70	-408	-2378	-13860	-80782

Observe that (a_k) *is an even sequence, while* (b_k) *is odd.*

In terms of these integers, the coefficients in (7.11) *and* (7.12) *are expressed as follows:*

$$c_{j,0} = -\frac{1}{8}\frac{a_{n-|j|}}{b_n}, \tag{7.17}$$

$$c_{j,\nu} = \begin{cases} -\dfrac{1}{4b_n} a_j a_{n-\nu} & \text{if } j \leqslant \nu, \quad 0 < \nu < n, \\ -\dfrac{1}{4b_n} a_\nu a_{n-j} & \text{if } j \geqslant \nu, \quad 0 < \nu < n. \end{cases} \tag{7.18}$$

Moreover, we have the symmetry relations

$$c_{j,\nu} = c_{n-j,n-\nu} \tag{7.19}$$

and therefore, in particular,

$$c_{j,n} = c_{n-j,0} \tag{7.20}$$

which are given by (7.17).

Also

$$\gamma_{j,0} = \frac{3}{32}\frac{b_{n-j}}{b_n} \quad \text{if} \quad j \geqslant 0, \tag{7.21}$$

$$\gamma_{-1,0} = 1 + \frac{3}{32}\frac{b_{n+1}}{b_n}, \tag{7.22}$$

$$\gamma_{j,\nu} = \begin{cases} -\dfrac{1}{2b_n} b_j b_{n-\nu} & \text{if} \quad j \leqslant \nu, \quad 0 < \nu < n, \\ -\dfrac{1}{2b_n} b_\nu b_{n-j} & \text{if} \quad j \geqslant \nu, \quad 0 < \nu < n. \end{cases} \tag{7.23}$$

Finally, we have

$$\gamma_{j,\nu} = \gamma_{n-j,n-\nu}, \tag{7.24}$$

and, in particular

$$\gamma_{j,n} = \gamma_{n-j,0}, \tag{7.25}$$

which are given by (7.21) *and* (7.22).

Writing

$$C_n = ||c_{j,\nu}|| \quad \text{and} \quad \Gamma_n = ||\gamma_{j,\nu}||, \tag{7.26}$$

where j indicates the row and ν the column, for these $(n+3) \times (n+1)$ matrices of coefficients, we find as an example that

$$C_4 = \frac{1}{3264}\left\|\begin{matrix} 99 & 594 & 102 & 18 & 3 \\ 577 & 198 & 34 & 6 & 1 \\ 99 & 594 & 102 & 18 & 3 \\ 17 & 102 & 578 & 102 & 17 \\ 3 & 18 & 102 & 594 & 99 \\ 1 & 6 & 34 & 198 & 577 \\ 3 & 18 & 102 & 594 & 99 \end{matrix}\right\|, \tag{7.27}$$

and

$$\Gamma_4 = \frac{1}{816}\left\|\begin{matrix} -987(3/8) & -140 & -24 & -4 & -1(3/8) \\ 204(3/8) & 0 & 0 & 0 & 0 \\ 35(3/8) & 140 & 24 & 4 & 1(3/8) \\ 6(3/8) & 24 & 144 & 24 & 6(3/8) \\ 1(3/8) & 4 & 24 & 140 & 35(3/8) \\ 0 & 0 & 0 & 0 & 204(3/8) \\ -1(3/8) & -4 & -24 & -140 & -987(3/8) \end{matrix}\right\|. \tag{7.28}$$

8. Semicardinal Interpolation

From (7.15) we obtain that

$$a_k = \tfrac{1}{2}(\lambda^k + \lambda^{-k}), \qquad b_k = \frac{1}{2\sqrt{2}}(\lambda^k - \lambda^{-k}). \tag{8.1}$$

From these we obtain the relations

$$\lim_{n\to\infty} \frac{a_{n-j}}{b_n} = -\sqrt{2}\,\lambda^j, \quad \lim_{n\to\infty} \frac{b_{n-j}}{b_n} = \lambda^j \quad \text{for all } j. \tag{8.2}$$

These relations show that the computation of the $c_{j,\nu}$ and $\gamma_{j,\nu}$ as given by Theorem 4 never presents any difficulty or loss of accuracy no matter how large n may be. In fact the situation is as follows. If we write $c_{j,\nu} = c_{j,\nu}^{(n)}$ and $\gamma_{j,\nu} = \gamma_{j,\nu}^{(n)}$ to indicate the dependence of these coefficients on n, and use the formulas (7.17), (7.18), (7.21)–(7.23), and (8.2) we find that if $n \to \infty$

$$\lim c_{j,0}^{(n)} = c_{j,0}^{+} = \frac{\sqrt{2}}{8}\lambda^{|j|}, \tag{8.3}$$

$$\lim c_{j,\nu}^{(n)} = c_{j,\nu}^{+} = \begin{cases} \sqrt{2}\,4^{-1}a_j\lambda^\nu & \text{if } j \leqslant \nu, \quad \nu > 0, \\ \sqrt{2}\,4^{-1}a_\nu\lambda^j & \text{if } j \geqslant \nu, \quad \nu > 0, \end{cases} \tag{8.4}$$

$$\lim \gamma_{-1,0}^{(n)} = \gamma_{-1,0}^{+} = -1 + (3/32)\lambda^{-1}, \tag{8.5}$$

$$\lim \gamma_{j,0}^{(n)} = \gamma_{j,0}^{+} = (3/32)\lambda^j \quad \text{if } j \geqslant 0, \tag{8.6}$$

$$\lim \gamma_{j,\nu}^{(n)} = \gamma_{j,\nu}^{+} = \begin{cases} -\tfrac{1}{2}b_j\lambda^\nu & \text{if } j \leqslant \nu, \\ -\tfrac{1}{2}b_\nu\lambda^j & \text{if } j \geqslant \nu. \end{cases} \tag{8.7}$$

It follows that the fundamental functions (7.11) and (7.12) converge to the spline functions

$$L_\nu^+(x) = \sum_{-1}^{\infty} c_{j,\nu}^+ N_0(x-j), \tag{8.8}$$

and

$$\Lambda_\nu^+(x) = \sum_{-1}^{\infty} \gamma_{j,\nu}^+ N_1(x-j), \tag{8.9}$$

respectively, where the coefficients have the values as given by Eqs. (8.3)–(8.7).

We may state without proving it here the following

THEOREM 5. *If the data $f(\nu)$, $f'(\nu)$ $(\nu = 0, 1, \ldots)$ are such that*

$$f(\nu) = O(\nu^\gamma),\ f'(\nu) = O(\nu^\gamma) \quad \text{as} \quad \nu \to \infty, \quad \text{for some} \quad \gamma \geqslant 0 \tag{8.10}$$

then there is a unique $S(x) \in S_{5,2}[0, \infty)$ satisfying the conditions

$$S(x) = O(x^\gamma) \quad \text{as} \quad x \to \infty, \tag{8.11}$$

$$S(\nu) = f(\nu), \quad S'(\nu) = f'(\nu) \quad (\nu = 0, 1, 2, \ldots), \tag{8.12}$$

$$S'''(0) = 0. \tag{8.13}$$

This unique solution is given by the expansion

$$S(k) = \sum_{0}^{\infty} f(\nu) L_\nu^+(x) + \sum_{0}^{\infty} f'(\nu) \Lambda_\nu^+(x). \tag{8.14}$$

9. Interpretation of Results of Sec. 7 in Terms of Matrix Inversions

Let us consider the $(n+3) \times (n+1)$ matrices C_n and Γ_n defined by (7.26) and define two new $(n+1) \times (n+1)$ matrices $C_n{}^*$ and $\Gamma_n{}^*$ obtained from C_n and Γ_n, respectively, and obtained from them by deleting their first and last rows. Thus

$$C_n{}^* = \|c_{j,\nu}\|, \quad \Gamma_n{}^* = \|\gamma_{j,\nu}\| \quad (j, \nu = 0, 1, \ldots, n). \tag{9.1}$$

We define the following $(n+1) \times (n+1)$ square matrices

$$P_n = \begin{Vmatrix} +6 & -2 & 0 & 0 & \cdots & & & 0 \\ -1 & +6 & -1 & 0 & & & & \\ 0 & -1 & +6 & -1 & & & & \\ & & & \ddots & & & & \\ \vdots & & & -1 & +6 & -1 & & \\ & & & & -1 & +6 & -1 & \\ 0 & & \cdots & & & 0 & -2 & +6 \end{Vmatrix} \tag{9.2}$$

and

$$Q_n = \begin{Vmatrix} 32/3 & 0 & & \cdots & & & 0 \\ -1 & 6 & -1 & 0 & & & \\ 0 & -1 & 6 & -1 & & & \\ & & & \ddots & & & \\ & & & & -1 & 6 & -1 \\ 0 & & & & 0 & 0 & 32/3 \end{Vmatrix}, \tag{9.3}$$

Both matrices are what they seem to look like: Toeplitz matrices with rows $0, 0, -1, 6, -1, 0, \ldots$, except that the first and last rows are modified.

THEOREM 6. *The following matrix relations hold*

$$C_n{}^* = P_n{}^{-1}, \tag{9.4}$$

and

$$\Gamma_n{}^* = Q_n{}^{-1}. \tag{9.5}$$

Proof. 1. Let us attack directly the interpolation problems (7.4) and (7.5) by means of our B-splines $N_0(x)$ and $N_1(x)$ of Fig. 1. The problem as it stands depends on solving a linear system of (roughly) $2n$ unknowns. The peculiar properties of the B-splines allow to reduce this problem to the solution of *two* systems each in n unknowns. Indeed, consider the spline function

$$U(x) = \sum_{-1}^{n+1} A_j N_0(x - j) \tag{9.6}$$

which if restricted to $[0, n]$ becomes an element of $S_{5,2}[0, n]$. Automatically it has the property

$$U'(\nu) = 0 \quad (\nu = 0, 1, \ldots, n), \tag{9.7}$$

whatever the constant coefficients A_j may be. Let us determine these such that

$$U(\nu) = f(\nu) \quad (\nu = 0, \ldots, n) \tag{9.8}$$

and

$$U'''(0) = 0, \qquad U'''(n) = 0. \tag{9.9}$$

If we substitute (9.6) into (9.8) we get the equations

$$\begin{aligned}
&-A_{-1} + 6A_0 - A_1 &&= f(0)\\
&\quad -A_0 + 6A_1 - A_2 &&= f(1)\\
&\qquad \cdots\\
&\qquad -A_{n-2} + 6A_{n-1} - A_n &&= f(n-1)\\
&\qquad\quad -A_{n-1} + 6A_n - A_{n+1} &&= f(n).
\end{aligned} \tag{9.10}$$

$N_0(x)$ being an *even* function, we conclude that $N_0'''(0) = 0$ *while* $N_0'''(-1)$ *and* $N_0'''(1)$ *are equal and of opposite sign.* In fact they have the values ± 120, but that is unimportant, because (9.9) leads to the two equations

$$A_{-1} = A_1, \qquad A_{n-1} = A_{n+1}. \tag{9.11}$$

Eliminating A_{-1} and A_{n+1} between Eqs. (9.10) and (9.11) we obtain the square system

$$\begin{aligned}
&+6A_0 - 2A_1 &&= f(0)\\
&-A_0 + 6A_1 - A_2 &&= f(1)\\
&\qquad \cdots\\
&\qquad -A_{n-2} + 6A_{n-1} - A_n &&= f(n-1)\\
&\qquad\quad -2A_{n-1} + 6A_n &&= f(n)
\end{aligned}$$

whose matrix is P_n. Solving the system, we obtain

$$||A_0, A_1, \ldots, A_n||^T = P_n^{-1} \cdot ||f(0), \ldots, f(n)||^T. \tag{9.12}$$

On the other hand, from Eqs. (7.6) and (7.11) we find that

$$U(x) = \sum_0^n f(\nu)L_\nu(x) = \sum_{\nu=0}^n f(\nu) \sum_{j=-1}^{n+1} c_{j,\nu} N_0(x-j)$$

$$U(x) = \sum_{j=-1}^{n+1} N_0(x-j) \sum_{\nu=0}^n c_{j,\nu} f(\nu).$$

Comparing with Eq. (9.6) and using the *unicity* of the representation (9.6), we conclude that

$$A_j = \sum_{\nu=0}^n c_{j,\nu} f(\nu) \quad (j = -1, 0, \ldots, n+1). \tag{9.13}$$

Leaving out the first and the last equation, we obtain

$$A_j = \sum_{\nu=0}^n c_{j,\nu} f(\nu) \quad \text{for} \quad j = 0, 1, \ldots, n, \tag{9.14}$$

or

$$||A_0, A_1, \ldots, A_n||^T = C_n^* ||f(0), \ldots, f(n)||^T. \tag{9.15}$$

A comparison of this with (9.12) establishes (9.4). ■

2. We establish (9.5) similarly. We consider

$$V(x) = \sum_{-1}^{n+1} B_j N_1(x-j), \tag{9.16}$$

which is to satisfy

$$V'(\nu) = f'(\nu) \quad (\nu = 0, \ldots, n), \tag{9.17}$$

and

$$V'''(0) = 0, \qquad V'''(n) = 0. \tag{9.18}$$

Now we need the values of $N_1'''(-1)$, $N_1'''(0)$, $N_1'''(1)$. Actually they are found to be 36, 168, 36, but the numbers 3, 14, 3 proportional to them will do, for now we have from (9.17) the system

$$
\begin{aligned}
-B_{-1}+6B_0-B_1 &= f'(0)\\
-B_0+6B_1-B_2 &= f'(1)\\
&\;\;\cdots\\
-B_{n-2}+6B_{n-1}-B_n &= f'(n-1)\\
-B_{n-1}+6B_n-B_{n+1} &= f'(n)
\end{aligned}
\tag{9.19}
$$

while (9.18) and (9.16) lead to the two equations

$$3B_{-1}+14B_0+3B_1=0, \qquad 3B_{n-1}+14B_n+3B_{n+1}=0. \tag{9.20}$$

Substituting

$$B_{-1}+B_1=-\frac{14}{3}B_0, \qquad B_{n-1}+B_{n+1}=-\frac{14}{3}B_n$$

into Eqs. (9.19) we obtain the system

$$
\begin{aligned}
\frac{32}{3}B_0 &= f'(0)\\
-B_0+6B_1-B_2 &= f'(1)\\
&\;\;\cdots\\
-B_{n-2}+6B_{n-1}-B_n &= f'(n-1)\\
\frac{32}{3}B_n &= f'(n)
\end{aligned}
\tag{9.21}
$$

having the matrix Q_n. Solving we find

$$||B_0, B_1, \ldots, B_n||^T = Q_n^{-1}||f'(0), \ldots, f'(n)||. \tag{9.22}$$

On the other hand

$$V(x)=\sum_0^n f'(\nu)A_\nu(x)=\sum_{\nu=0}^n f'(\nu)\sum_{j=-1}^{n+1}\gamma_{j,\nu}N_1(x-j),$$

$$V(x)=\sum_{j=-1}^{n+1}N_1(x-j)\sum_{\nu=1}^n \gamma_{j,\nu}f'(\nu).$$

Comparing with (9.16) we find that

$$B_j = \sum_{\nu=0}^{n} \gamma_{j,\nu} f'(\nu), \qquad j = -1, 0, \ldots, n+1. \tag{9.23}$$

Retaining all of these except the first and last we have

$$B_j = \sum_{\nu=0}^{n} \gamma_{j,\nu} f'(\nu) \quad \text{for} \quad j = 0, \ldots, n$$

or

$$\|B_0, \ldots, B_n\|^T = \Gamma_n^* \|f'(0), \ldots, f'(n)\|.$$

A comparison with Eq. (9.22) establishes Eq. (9.5). ∎

REMARKS. 1. Theorem 6 gives all elements of C_n in terms of P_n^{-1} except the first row $\gamma_{-1,\nu}$ and the last row $\gamma_{n+1,\nu}$. However (9.13) shows that

$$A_{-1} = \sum_0^n c_{-1,\nu} f(\nu), \quad A_{n+1} = \sum_0^n c_{n+1,\nu} f(\nu),$$

and now the relations (9.11) show that

$$c_{-1,\nu} = c_{1,\nu}, \quad c_{n+1,\nu} = c_{n-1,\nu} \quad (\nu = 0, \ldots, n). \tag{9.23}$$

Similarly, Eqs. (9.23) and (9.20) show that

$$\gamma_{-1,\nu} = -\frac{14}{3}\gamma_{0,\nu} - \gamma_{1,\nu}, \qquad \gamma_{n+1,\nu} = -\frac{14}{3}\gamma_{n,\nu} - \gamma_{n-1,\nu},$$
$$(\nu = 0, \ldots, n). \tag{9.24}$$

The relations (9.23) and (9.24) are readily seen to hold on the numerical examples (7.27) and (7.28).

2. Explicit expressions for the inverses of the matrices P_n and Q_n, defined by (9.2) and (9.3), would therefore furnish our results of Theorem 4. Such explicit inverses can be constructed by means of some results of Gantmacher and Krein [2, p. 95]. See also [3]. This, however, is not the way in which we obtained Theorem 4. The remainder of this paper describes the way by which we derived Theorem 4.

10. Preparations for Deriving Theorem 4

These fall into several parts.

A. Proof of Lemma 5 of Sec. 7. If $S(x)$, defined in $[0, n]$, is an element of $S_{5,2}^{(0)}[0, n]$, then it is fairly clear that we can extend the definition $S(x)$ to all real x so as to become an element of $S_{5,2}^{(0)}$. This means that there exists an

$$\tilde{S}(x) \in S_{5,2}^{(0)} \tag{10.1}$$

such that

$$\tilde{S}(x) = S(x) \quad \text{if} \quad 0 \leqslant x \leqslant n. \tag{10.2}$$

Indeed, we can first extend $S(x)$ to all real x by removing the knots at $x = 0$ and $x = n$ and obtain $S_1(x)$ $(-\infty < x < \infty)$ such that

$$S_1(x) = S(x) \quad \text{if} \quad 0 \leqslant x \leqslant n,$$

while

$$S_1(x) \in \pi_5 \quad \text{in} \quad (-\infty, 1] \quad \text{and also} \quad S_1(x) \in \pi_5 \quad \text{in} \quad [n-1, \infty).$$

Next we define

$$\begin{aligned} \tilde{S}(x) = S_1(x) &+ a_n(x-n)_+^5 + a_{n+1}(x-n-1)_+^5 + \cdots \\ &+ b_n(x-n)_+^4 + b_{n+1}(x-n-1)_+^4 + \cdots \\ &+ a_0(-x)_+^5 + a_{-1}(-x-1)_+^5 + \cdots \\ &+ b_0(-x)_+^4 + b_{-1}(-x-1)_+^4 + \cdots, \end{aligned}$$

and determine a_n, b_n, (this can be done in many ways) such that $\tilde{S}'(n+1) = 0$. Next a_{n+1}, b_{n+1}, so that $\tilde{S}'(n+2) = 0$, a.s.o. Similarly we can achieve that $\tilde{S}'(-1) = \tilde{S}'(-2) = \cdots = 0$. But then (10.1) surely holds. To $\tilde{S}(x)$ we apply Theorem 3 to obtain the expansion

$$\tilde{S}(x) = \sum_{-\infty}^{\infty} c_j N_0(x-j).$$

Restricting x to $[0, n]$ and using (10.2) we obtain the desired representation (7.9). The two lemmas of Sec. 4 imply its unicity. A similar proof establishes the second part of Lemma 5.

B. Determining the Fundamental Functions $\mathscr{L}_0(x)$, $\mathscr{L}_1(x)$, and the Eigensplines $S_1(x)$, $S_2(x)$, of the Class $S_{5,2}$. We change notation from that used in Theorem 1 and let us denote by

$$\mathscr{L}_0(x) \quad \text{and} \quad \mathscr{L}_1(x) \tag{10.3}$$

the fundamental functions of $S_{5,2}$. Furthermore, let

$$S_1(x) \quad \text{and} \quad S_2(x) \tag{10.4}$$

denote the eigensplines of the class $S_{5,2}$ (see [4, Sec. 2]). These span the class

$$S_{5,2}^0 = \{S(x);\, S(x) \in S_{5,2},\, S(\nu) = S'(\nu) = 0 \text{ for all } \nu\}. \tag{10.5}$$

We begin by finding the functions (10.3). By Theorem 2 we have

$$\mathscr{L}_0(x) = \sum_{-\infty}^{\infty} \omega_\nu N_0(x - \nu), \tag{10.6}$$

$$\mathscr{L}_1(x) = \sum_{-\infty}^{\infty} \omega_\nu N_1(x - \nu), \tag{10.7}$$

where the ω_ν are the coefficients of the Laurent expansion

$$\frac{1}{-x + 6 - x^{-1}} = \sum_{-\infty}^{\infty} \omega_\nu x^\nu. \tag{10.8}$$

Using λ, defined by (7.13), as the smaller root of (7.14), we find

$$\begin{aligned}\sum_{-\infty}^{\infty} \lambda^{|j|} x^j &= 1 + \lambda x + \lambda^2 x^2 + \cdots + \lambda x^{-1} + \lambda^2 x^{-2} + \cdots \\ &= \frac{1}{1 - \lambda x} + \frac{\lambda x^{-1}}{1 - \lambda x^{-1}} = \frac{1 - \lambda^2}{(1 - \lambda x)(1 - \lambda x^{-1})} = \frac{1 - \lambda^2}{1 - \lambda x - \lambda x^{-1} + \lambda^2}.\end{aligned}$$

Since $\lambda^2 - 6\lambda + 1 = 0$ we obtain

$$\sum_{-\infty}^{\infty} \lambda^{|j|} x^j = \frac{2 - 6\lambda}{6\lambda - \lambda x - \lambda x^{-1}} = \frac{2\lambda^{-1} - 6}{-x + 6 - x^{-1}} = \frac{4\sqrt{2}}{-x + 6 - x^{-1}}$$

because $2\lambda^{-1} - 6 = 2(3 + 2\sqrt{2}) - 6 = 4\sqrt{2}$. Thus

$$\frac{1}{-x + 6 - x^{-1}} = \frac{1}{4\sqrt{2}} \sum_{-\infty}^{\infty} \lambda^{|j|} x^j \tag{10.9}$$

which shows that

$$\omega_\nu = \frac{1}{4\sqrt{2}} \lambda^{|\nu|} \quad \text{for all } \nu. \tag{10.10}$$

Finally (10.6) and (10.7) become

$$\mathscr{L}_0(x) = \frac{1}{4\sqrt{2}} \sum_{-\infty}^{\infty} \lambda^{|j|} N_0(x - j) \tag{10.11}$$

and

$$\mathscr{L}_1(x) = \frac{1}{4\sqrt{2}} \sum_{-\infty}^{\infty} \lambda^{|j|} N_1(x - j). \tag{10.12}$$

That these functions do have the properties

$$\mathscr{L}_0(0) = 1, \quad \mathscr{L}_0(\nu) = 0 \quad \text{if} \quad \nu \neq 0, \quad \mathscr{L}_0'(\nu) = 0 \quad \text{for all } \nu$$

$$\mathscr{L}_1'(0) = 1, \quad \mathscr{L}_1'(\nu) = 0 \quad \text{if} \quad \nu \neq 0, \quad \mathscr{L}_1(\nu) = 0 \quad \text{for all } \nu \tag{10.13}$$

is easily verified if we use the relations (7.1) and (7.1′).

We now turn to the eigensplines (10.4). As mentioned before, these are determined only up to a constant factor. We claim that, with $\lambda = 3 - 2\sqrt{2}$, we may set

$$S_1(x) = \sum_{-\infty}^{\infty} \lambda^j N_0(x - j), \quad S_2(x) = \sum_{-\infty}^{\infty} \lambda^{-j} N_0(x - j). \tag{10.14}$$

This is clear, because

$$S_1(x + 1) = \lambda S_1(x) \quad \text{for all } x, \tag{10.15}$$

$$S_1'(\nu) = 0 \quad \text{for all } \nu, \tag{10.16}$$

while (10.14) implies also

$$S_1(0) = \lambda - 6 + \lambda^{-1} = \lambda^{-1}(\lambda^2 - 6\lambda + 1) = 0$$

and therefore

$$S_1(\nu) = 0 \quad \text{for all } \nu, \tag{10.17}$$

in view of the functional relation.

Finally

$$S_2(x) = \sum_j \lambda^{-j} N_0(x - j) = \sum_j \lambda^j N_0(x + j) = \sum_j \lambda^j N_0(-x - j),$$

so that

$$S_2(x) = S_1(-x). \tag{10.18}$$

But then

$$S_2(x + 1) = S_1(-x - 1) = \lambda^{-1} S_1(-x) = \lambda^{-1} S_2(x),$$

showing that $S_2(x)$ is indeed an eigenspline corresponding to the eigenvalue $\lambda^{-1} = 3 + 2\sqrt{2}$.

However, we obtain a second set of eigensplines by proceding similarly by using the second B-spline $N_1(x)$. Arguments similar to the above show that

$$\tilde{S}_1(x) = \sum_{-\infty}^{\infty} \lambda^j N_1(x - j), \qquad \tilde{S}_2(x) = \sum_{-\infty}^{\infty} \lambda^{-j} N_1(x - j), \tag{10.19}$$

is also an acceptable set of eigensplines for $S_{5,2}$ also corresponding to the eigenvalues λ and λ^{-1}, respectively. It follows that we must have

$$\tilde{S}_1(x) = C_1 S_1(x), \quad \tilde{S}_2(x) = C_2 S_2(x). \tag{10.20}$$

The values of the constants C_1 and C_2 are readily determined as follows. Observe first that the relation between the functions (10.19) is a little different from the relation (10.18) because

$$\tilde{S}_2(-x) = \sum_j \lambda^{-j} N_1(-x - j) = \sum \lambda^j N_1(-x + j)$$
$$= -\sum_j \lambda^j N_1(x - j),$$

and therefore

$$\tilde{S}_2(-x) = -\tilde{S}_1(x). \tag{10.21}$$

Next we record the values

$$N_0''(-1) = 20, \qquad N_0''(0) = -40, \quad N_0''(1) = 20, \tag{10.22}$$

$$N_1''(-1) = -8, \quad N_1''(0) = 0, \qquad N_1''(1) = 8, \tag{10.23}$$

that are easily obtained from the explicit expressions (2.10) and (2.11). From (10.14) and (10.19) we obtain

$$S_1''(0) = 80, \qquad \tilde{S}_1''(0) = 32\sqrt{2} \tag{10.24}$$

and the first relation (10.20) becomes

$$\tilde{S}_1(x) = \frac{2\sqrt{2}}{5} S_1(x). \tag{10.25}$$

But then, by (10.18) and (10.21) we find

$$\tilde{S}_2(x) = -\tilde{S}_1(-x) = -\frac{2\sqrt{2}}{5} S_2(x)$$

hence

$$\tilde{S}_2(x) = -\frac{2\sqrt{2}}{5} S_2(x). \tag{10.26}$$

We note in particular the relations (10.11), (10.12), (10.14), (10.19), (10.25), and (10.26) to be used below.

11. Determining the Fundamental Functions $L_\nu(x)$ in (7.6)

Using the formulae (10.11) and (10.14) we try to determine $L_\nu(x)$ $(0 \leqslant \nu \leqslant n)$ from a representation

$$L_\nu(x) = \mathscr{L}_0(x-\nu) + c_1 S_1(x) + c_2 S_2(x), \tag{11.1}$$

with coefficients c_1 and c_2 to be appropriately determined. This looks promising because the $L_\nu(x)$ defined by Eq. (11.1) for all real x, already satisfy the conditions that $L_\nu(j)$, $L_\nu'(j)$, vanish for all integer j, except that

$$L_\nu(\nu) = 1.$$

The only remaining conditions to be satisfied are

$$L_\nu'''(0) = 0, \qquad L_\nu'''(n) = 0, \tag{11.2}$$

and these will yield the values of the constants c_1 and c_2.

Requiring (11.1) to satisfy (11.2) leads to the equations

$$\begin{aligned} c_1 S_1'''(0) + c_2 S_2'''(0) &= -\mathscr{L}_0'''(-\nu), \\ c_1 S_1'''(n) + c_2 S_2'''(n) &= -\mathscr{L}_0'''(n-\nu). \end{aligned} \tag{11.3}$$

By direct calculation we obtain from (2.10) that

$$N_0'''(-1) = 120, \qquad N_0'''(0) = 0, \qquad N_0'''(1) = -120 \tag{11.4}$$

and the first expansion (10.14) shows that

$$S_1'''(0) = -480\sqrt{2}. \tag{11.5}$$

The functional equation (10.15) yields $S_1(x+n) = \lambda^n S_1(x)$ whence

$$S_1'''(n) = \lambda^n S_1'''(0) = -480\sqrt{2}\,\lambda^n. \tag{11.6}$$

Since $\mathscr{L}_0(x)$ is an even function in C''' we have

$$\mathscr{L}_0'''(0) = 0. \tag{11.7}$$

For $k > 0$, however, we find from (10.11) and (11.4) that

$$\mathscr{L}_0'''(k) = -120\lambda^k, \quad (k > 0). \tag{11.8}$$

Therefore

$$\mathscr{L}_0'''(n - \nu) = -120\lambda^{n-\nu}, \quad (\nu < n), \tag{11.9}$$

and finally, again from (10.11),

$$\mathscr{L}_0'''(-\nu) = -\mathscr{L}_0'''(\nu) = 120\lambda^\nu. \tag{11.10}$$

Finally (10.18), (11.5) and (11.6) show that

$$S_2'''(0) = -S_1'''(0) = 480\sqrt{2}, \tag{11.11}$$

$$S_3'''(n) = -S_1'''(-n) = 480\sqrt{2}\,\lambda^{-n}. \tag{11.12}$$

We now distinguish two cases.

CASE 1: $\nu = 0$. Using (11.5), (11.6), (11.11), (11.12), (11.7), and (11.8) for the coefficients of (11.3), we find that

$$c_1 = c_2 = -\frac{\lambda^n}{4\sqrt{2}\,(\lambda^n - \lambda^{-n})}.$$

Substituting these into (11.1) and using the expansions (10.11) and (10.14) we find that

$$L_0(x) = \sum_{-\infty}^{\infty} c_{j,0} N_0(x - j), \tag{11.13}$$

where

$$4\sqrt{2}\, c_{j,0} = \lambda^{|j|} - \frac{\lambda^{n+j} + \lambda^{n-j}}{\lambda^n - \lambda^{-n}}.$$

If $j \geqslant 0$, this yields

$$4\sqrt{2}\, c_{j,0} = -\frac{\lambda^{n-j} + \lambda^{-n+j}}{\lambda^n - \lambda^{-n}},$$

while if $j \leqslant 0$ we get

$$4\sqrt{2}\, c_{j,0} = -\frac{\lambda^{n+j} + \lambda^{-n-j}}{\lambda^n - \lambda^{-n}}.$$

In either of these two cases we can write

$$4\sqrt{2}\, c_{j,0} = -\frac{\lambda^{n-|j|} + \lambda^{-n+|j|}}{\lambda^n - \lambda^{-n}}.$$

However, using the integers defined by (8.1), we may write

$$c_{j,0} = -\frac{1}{8}\frac{a_{n-|j|}}{b_n},$$

which is the desired result (7.17).

CASE 2: $0 < \nu < n$. In this case we obtain from (11.3) the values

$$c_1 = -\frac{1}{4\sqrt{2}}\frac{\lambda^{n-\nu} + \lambda^{-n+\nu}}{\lambda^n - \lambda^{-n}}, \qquad c_2 = -\frac{1}{4\sqrt{2}}\frac{(\lambda^\nu + \lambda^{-\nu})\lambda^n}{\lambda^n - \lambda^{-n}}.$$

Substituting these into (11.1) we obtain

$$L_\nu(x) = \sum_{-\infty}^{\infty} c_{j,\nu} N_0(x - j),$$

where

$$4\sqrt{2}\, c_{j,\nu} = \lambda^{|j-\nu|} - \frac{\lambda^{n-\nu} + \lambda^{-n+\nu}}{\lambda^n - \lambda^{-n}}\lambda^j - \frac{\lambda^\nu + \lambda^{-\nu}}{\lambda^n - \lambda^{-n}}\lambda^{n-j}. \tag{11.14}$$

We must again distinguish two cases.

CASE 1: $j \leqslant \nu$. In this case we find that

$$c_{j,\nu} = -\frac{1}{4b_n} a_j a_{n-\nu}.$$

CASE 2: $j \geqslant \nu$. Then

$$c_{j,\nu} = -\frac{1}{4b_n} a_\nu a_{n-j}.$$

These results establish (7.18). For our purpose we need them only for $j = -1, 0, 1, \ldots, n+1$. The symmetry relations (7.19) follow from the identity

$$L_\nu(x) = L_{n-\nu}(n-x)$$

in view of the fact that

$$L_{n-\nu}(n-x) = \sum_{j=-1}^{n+1} c_{j,n-\nu} N_0(n-x-j) = \sum_{j=-1}^{n+1} c_{n-j,n-\nu} N_0(-x+j)$$
$$= \sum_{-1}^{n+1} c_{n-j,n-\nu} N_0(x-j).$$

12. Determining the Fundamental Functions $\Lambda_\nu(x)$ in (7.6)

The main difference from Sec. 11 is that we now try to determine $\Lambda_\nu(x)$ from a representation

$$\Lambda_\nu(x) = \mathscr{L}_1(x-\nu) + c_1\tilde{S}_1(x) + c_2\tilde{S}_2(x) \tag{12.1}$$

where all three terms on the right side by (10.12) and (10.19) are expressed in terms of the second B-splines $N_1(x-j)$. We are hereby assured that the right side of (12.1) already vanishes for all integer values of x. Thus

$$\Lambda_\nu(j) = \Lambda_\nu'(j) = 0 \quad \text{for all } j, \text{ except that } \quad \Lambda_\nu'(\nu) = 1.$$

The only conditions that still have to be satisfied are

$$\Lambda_\nu'''(0) = 0, \qquad \Lambda_\nu'''(n) = 0, \tag{12.2}$$

leading to the system

$$\begin{aligned} c_1\tilde{S}_1'''(0) + c_2\tilde{S}_2'''(0) &= -\mathscr{L}_1'''(-\nu) \\ c_1\tilde{S}_1'''(n) + c_2\tilde{S}_2'''(n) &= -\mathscr{L}_1'''(n-\nu). \end{aligned} \tag{12.3}$$

Now we need the values

$$N_1'''(-1) = -36, \qquad N_1'''(0) = -168, \qquad N_1'''(1) = -36,$$

to find the values

$$\mathscr{L}_1'''(0) = 36 - 48\sqrt{2}, \quad \mathscr{L}_1'''(k) = -48\sqrt{2}\,\lambda^k, \quad (k > 0).$$

The quantities $\tilde{S}_1'''(0)$, $\tilde{S}_2'''(0)$ are immediately obtained from (11.5), (11.11), and the relations (10.25) and (10.26).

The remaining calculations are so very much similar to those of Sec. 11 that they may be safely omitted.

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