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Long-time extinction of solutions of some semilinear parabolic equations

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Abstract

We study the long-time behavior of solutions of semilinear parabolic equation of the following type $\partial_t u - \Delta u + a_0(x)u^q = 0$ where $a_0(x) \geqslant d_0 \exp(-\frac{\omega(|x|)}{|x|^2})$, $d_0 > 0$, $1 > q > 0$, and ω is a positive continuous radial function. We give a Dini-like condition on the function ω by two different methods which implies that any solution of the above equation vanishes in a finite time. The first one is a variant of a local energy method and the second one is derived from semi-classical limits of some Schrödinger operators.
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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, be a bounded domain with C^1-boundary, $0 \in \Omega$. The aim of this paper is to investigate the time vanishing properties of generalized (energy) solutions of initial–boundary problem to a wide class of quasilinear parabolic equations with the model representative:

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$$\begin{cases} u_t - \Delta u + a_0(x)|u|^{q-1}u = 0 & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{1.1}$$

where $0 < q < 1$, $a_0(x) \geqslant 0$ and $u_0 \in L_2(\Omega)$. It is easy to see that if $a_0(x) \geqslant \varepsilon > 0$, then the comparison with the solution of corresponding ordinary equation $\varphi_t + \varepsilon|\varphi|^{q-1}\varphi = 0$ implies that the solution $u(x,t)$ of (1.1) vanishes for $t \geqslant T_0 = \varepsilon^{-1}(1-q)^{-1}\|u_0\|_{L_\infty}^{1-q}$. The property that any solution of problem (1.1) becomes identically zero for t large enough is called the time compact support property (TCS-property). On the opposite, if $a_0(x) \equiv 0$ for any x from some connected open subset $\omega \subset \Omega$, then any solution $u(x,t)$ of problem (1.1) is bounded from below by $\sigma \exp(-t\lambda_\omega)\varphi_\omega(x)$ on $\omega \times (0,\infty)$, where $\sigma = \operatorname{ess\,inf}_\omega u_0 > 0$, λ_ω and φ_ω are first eigenvalue and corresponding eigenfunction of $-\Delta$ in $W_0^{1,2}(\omega)$. It was Kondratiev and Véron [1] who first proposed a method of investigation of conditions of appearance of TCS-property in the case of general potential $a_0 \geqslant 0$. They introduced the fundamental states of the associated Schrödinger operator

$$\mu_n = \inf\left\{ \int_\Omega \left(|\nabla\psi|^2 + 2^n a_0(x)\psi^2\right) dx\colon\ \psi \in W^{1,2}(\Omega),\ \int_\Omega \psi^2\, dx = 1 \right\}, \quad n \in \mathbb{N}, \tag{1.2}$$

and proved that, if

$$\sum_{n=0}^{\infty} \mu_n^{-1} \ln(\mu_n) < \infty, \tag{1.3}$$

then (1.1) possesses the TCS-property. Starting from condition (1.3) in [2] an explicit conditions of appearance of TCS-property in terms of potential $a_0(x)$ was obtained. The analysis in [2] was based on the so-called semi-classical analysis [9], which uses sharp estimates of the spectrum of the Schrödinger operator [6,10,11]. Particularly, in the case of existence of the radially symmetric minorant

$$a_0(x) \geqslant d_0 \exp\left(-\frac{\omega(|x|)}{|x|^2}\right) := a\big(|x|\big) \quad \forall x \in \Omega,\ d_0 > 0, \tag{1.4}$$

the following statements was obtained in [2]:

Proposition 1.1. *(See [2, Theorem 4.5].) In Eq.* (1.1) *let* $a_0(x) = a(|x|)$, *where* $a(r)$ *is defined by* (1.4)*. Let* $u_0(x) \geqslant \nu > 0\ \forall x \subseteq \overline{\Omega}$ *and* $\omega(r) \to \infty$ *as* $r \to 0$*. Then arbitrary solution* u *of problem* (1.1) *never vanishes on* Ω*.*

Proposition 1.2. *(See [2, Corollary of Theorem 3.1].) If in assumption* (1.4) $a_0(x) = a(|x|)$ *and* $\omega(r) = r^\alpha$ *with* $0 < \alpha < 2$ *then an arbitrary solution of* (1.1) *enjoys the TCS-property.*

Thus, an open problem is to find sharp border which distinguish two different decay properties of solutions, described in Propositions 1.1 and 1.2. Moreover, the method of investigations used in [1,2] exploits essentially some regularity properties of solutions under consideration, particularly, sharp upper estimates of $\|u(x,t)\|_{L_\infty(\Omega)}$ with respect to t. Such an estimate is difficult to obtain or is unknown for solutions of equations of more general structure than (1.1). Particularly, it is absolutely impossible to have any information about such a behavior for higher order parabolic equations. We propose here some new energy method of investigations, which deals with energy norms of solutions $u(x,t)$ only and, therefore, may be applied, particularly, for higher order equations, too.

We suppose that function $\omega(s)$ from condition (1.4) satisfies the conditions:

(A_1) $\omega(r)$ is continuous and nondecreasing function $\forall r \geqslant 0$,
(A_2) $\omega(0)=0$, $\omega(r)>0$ $\forall r>0$,
(A_3) $\omega(s) \leqslant \omega_0 < \infty$ $\forall s \in \mathbb{R}^1_+$.

Our main result reads as follows:

Theorem 1.1. *Let $u_0(x)$ be an arbitrary function from $L_2(B_1)$, let function $\omega(r)$ from* (1.4) *satisfy assumptions (A_1)–(A_3) and the following main condition*:

$$\int_0^c \frac{\omega(s)}{s}\,ds < \infty \quad (\textit{Dini-like condition}). \tag{1.5}$$

Suppose also that $\omega(r)$ satisfies the following technical condition:

$$\frac{s\omega'(s)}{\omega(s)} \leqslant 2-\delta \quad \forall s \in (0,s_0),\ s_0>0,\ 2>\delta>0. \tag{1.6}$$

Then an arbitrary energy solution $u(x,t)$ of the problem (1.1) *vanishes on Ω in some finite time $T<\infty$.*

In the sequel of the paper we show that the sufficiency of the Dini condition (1.5) for the validity of TCS-property can be proved also by the methods from [1,2] if one uses L_∞ estimates of solution $u(x,t)$ of problem (1.1). This leads to the following result.

Proposition 1.3. *The assertion of Theorem* 1.1 *holds if the function $\omega(s)$ satisfies conditions (A_1)–(A_3), the Dini condition* (1.5) *and the following similar to* (1.6) *technical conditions*:

$$\omega(s) \geqslant s^{2-\delta} \quad \forall s \in (0,s_0),\ s_0>0,\ 2>\delta>0, \tag{1.7}$$

$$\textit{the function } \frac{\omega(s)}{s^2} \textit{ is decreasing on } (0,s_0). \tag{1.8}$$

Remark 1.1. It is easy to check that the function $\omega(s)=(\ln s^{-1})^{-\beta}$ satisfies all the conditions of Theorem 1.1 and Proposition 1.3 for arbitrary $\beta>1$.

2. The proof of main result

The proof of Theorem 1.1 is based on some variant of the local energy method, which was developed, particularly in [3,4]. First, we introduce the following families of subdomains:

$$\Omega(\tau)=\Omega\cap\{|x|>\tau\}, \qquad Q_s^{(T)}(\tau)=\Omega(\tau)\times(s,T), \quad T<\infty.$$

Definition 2.1. An energy solution of problem (1.1) is the function $u(x,t) \in L_2(0,T; W_2^1(\Omega))$: $\frac{\partial u}{\partial t} \in L_2(0,T; (W_2^1(\Omega))^*)$, $u(x,0) = u_0$, satisfying the following integral identity:

$$\int_0^T \langle u_t, \varphi\rangle\, dt + \int_{\Omega\times(0,T)} (\nabla_x u, \nabla_x \varphi)\, dx\, dt + \int_{\Omega\times(0,T)} a_0(x)|u|^{q-1} u\varphi\, dx\, dt = 0 \tag{2.1}$$

for arbitrary $\varphi \in L_2(0,T; W_2^1(\Omega))$ $\forall T < \infty$.

Lemma 2.1. *An arbitrary energy solution u of the problem* (1.1) *satisfies the following global a priori estimate*

$$\int_\Omega \left|u(x,\hat t)\right|^2 dx + \int_{Q_0^{(\hat t)}(0)} \left(|\nabla_x u|^2 + a\left(|x|\right)|u|^{q+1}\right) dx\, dt \leqslant \int_\Omega |u_0|^2\, dx := y_0, \quad \forall \hat t > 0. \tag{2.2}$$

Testing integral identity (2.1) by $\varphi(x,t) = u(x,t)\xi(x)$, where $\xi(x)$ is arbitrary C^1-function, due to formula of integration by parts [8], we derive the following equality:

$$2^{-1}\int_\Omega \left|u(x,\hat t)\right|^2 \xi\, dx + \int_{\Omega\times(s,\hat t)} \left(|\nabla_x u|^2\xi + (\nabla_x u, \nabla_x \xi)u\right) dx\, dt + \int_{\Omega\times(s,\hat t)} a_0 \xi |u|^{q+1}\, dx\, dt$$

$$= 2^{-1}\int_\Omega \left|u(x,s)\right|^2 \xi\, dx, \quad 0 \leqslant s < \hat t < \infty. \tag{2.3}$$

Let $\eta(r) \in C^1(\mathbb{R}^1)$ be such that $0 \leqslant \eta(r) \leqslant 1$ $\forall r \in \mathbb{R}^1$, $\eta(r) = 0$ if $r \leqslant 0$, $\eta(r) = 1$ if $r > 1$. Fix arbitrary numbers $\tau > 0$, $\nu > 0$ and test (2.3) by

$$\xi(x) = \xi_{\tau,\nu}\left(|x|\right) := \eta\left(\frac{|x| - \tau}{\nu}\right).$$

Then passing to the limit $\nu \to 0$ we obtain

$$2^{-1}\int_{\Omega(\tau)} \left|u(x,\hat t)\right|^2 dx + \int_s^{\hat t}\int_{\Omega(\tau)} \left(|\nabla_x u|^2 + a_0(x)|u|^{q+1}\right) dx\, dt$$

$$= 2^{-1}\int_{\Omega(\tau)} \left|u(x,s)\right|^2 dx + \int_s^{\hat t}\int_{|x|=\tau} u\frac{\partial u}{\partial n}\, d\sigma\, dt \quad \forall \hat t\colon s < \hat t < \infty. \tag{2.4}$$

From (2.4) with $\tau = 0$, $s = 0$ the necessary global estimate (2.2) follows. Further we will denote by c, c_i different positive constants which depend on known parameters of the problem (1.1) only. Let us introduce the energy functions related to a fixed energy solution u of problem (1.1):

$$H(t,\tau)=\int\limits_{\Omega(\tau)}\big|u(x,t)\big|^2\,dx,\qquad I_s^{(v)}(\tau)=\int\limits_{Q_s^{(v)}(\tau)}\big(|\nabla_x u|^2+a\big(|x|\big)|u|^{q+1}\big)\,dx\,dt,$$

$$E(t,\tau)=\int\limits_{\Omega(\tau)}\big(\big|\nabla_x u(x,t)\big|^2+a\big(|x|\big)\big|u(x,t)\big|^{q+1}\big)\,dx,$$

$$J_s^{(v)}(\tau)=\int\limits_s^v\int\limits_{|x|=\tau}|\nabla_x u|^2\,dx\,dt. \tag{2.5}$$

Lemma 2.2. *Energy functions* (2.5) *related to arbitrary solution u of problem* (1.1) *satisfy the following relationship*:

$$\begin{aligned}H(T,\tau)+I_s^{(T)}(\tau)\leqslant{}& ca(\tau)^{-\frac{2(1-\theta_2)}{2-(1-\theta_2)(1-q)}}E(s,\tau)^{\frac{2}{2-(1-\theta_2)(1-q)}}\\&+c_1a(\tau)^{-\frac{2}{q+1}}E(s,\tau)^{\frac{2}{q+1}}+ca(\tau)^{-\frac{2}{q+1}}J_s^{(T)}(\tau)^{\frac{2}{q+1}}\\&+ca(\tau)^{-\frac{2(1-\theta_1)}{2-(1-\theta_1)(1-q)}}J_s^{(T)}(\tau)^{\frac{2}{2-(1-\theta_1)(1-q)}},\end{aligned}$$

$$0<\theta_1=\frac{(q+1)+n(1-q)}{2(q+1)+n(1-q)}<1,\quad \theta_2=\frac{n(1-q)}{2(q+1)+n(1-q)}. \tag{2.6}$$

Let us estimate the second term in right-hand side of (2.4). By interpolation (see, for example, [7]) we have

$$\begin{aligned}\int\limits_{|x|=\tau}|u|^2\,d\sigma\leqslant{}& d_1\Big(\int\limits_{\Omega(\tau)}|\nabla_x u|^2\,dx\Big)^{\theta_1}\Big(\int\limits_{\Omega(\tau)}|u|^{q+1}\,dx\Big)^{\frac{2(1-\theta_1)}{q+1}}\\&+d_2\Big(\int\limits_{\Omega(\tau)}|u|^{q+1}\,dx\Big)^{\frac{2}{q+1}}\quad\forall\tau>0,\ \theta_1\text{ is from (2.6)}.\end{aligned}\tag{2.7}$$

Using (2.7) we easily arrive at

$$\begin{aligned}&\int\limits_{|x|=\tau}|u||\nabla_x u|\,d\sigma\\&\leqslant c\Big(\int\limits_{|x|=\tau}|\nabla_x u|^2\,d\sigma\Big)^{1/2}\Big[\Big(\int\limits_{\Omega(\tau)}|\nabla_x u|^2\,dx\Big)^{\frac{\theta_1}{2}}\Big(\int\limits_{\Omega(\tau)}|u|^{q+1}\Big)^{\frac{1-\theta_1}{q+1}}+\Big(\int\limits_{\Omega(\tau)}|u|^{q+1}\,dx\Big)^{\frac{1}{q+1}}\Big]\\&=c\Big(\int\limits_{|x|=\tau}|\nabla_x u|^2\,d\sigma\Big)^{1/2}\Big(\int\limits_{\Omega(\tau)}|\nabla_x u|^2\,dx\Big)^{\frac{\theta_1}{2}}\end{aligned}$$

$$\times\Big(\int_{\Omega(\tau)}|u|^{q+1}dx\Big)^{\frac{1-\theta_1}{2}}\Big(\int_{\Omega(\tau)}|u|^{q+1}\,dx\Big)^{\frac{(1-q)(1-\theta_1)}{2(q+1)}}$$

$$+\,c\Big(\int_{|x|=\tau}|\nabla_x u|^2\,d\sigma\Big)^{1/2}\Big(\int_{\Omega(\tau)}|u|^{q+1}\,dx\Big)^{1/2}\Big(\int_{\Omega(\tau)}|u|^{q+1}\,dx\Big)^{\frac{1-q}{2(q+1)}}. \tag{2.8}$$

From condition (1.6) the monotonicity of function $a(s)$ from (1.4) follows easily. Therefore we can continue estimating (2.8) as follows:

$$\int_{|x|=\tau}|u||\nabla_x u|\,d\sigma$$

$$\leqslant c_1\Big(\int_{|x|=\tau}|\nabla_x u|^2\,d\sigma\Big)^{1/2}\Big(\int_{\Omega(\tau)}|u|^2\,dx\Big)^{\frac{(1-q)(1-\theta_1)}{4}}$$

$$\times\, a(\tau)^{-\frac{1-\theta_1}{2}}\Big(\int_{\Omega(\tau)}\big(|\nabla_x u|^2+a\big(|x|\big)|u^{q+1}|\big)\,dx\Big)^{1/2}$$

$$+\,c_1 a(\tau)^{-1/2}\Big(\int_{\Omega(\tau)}|u|^2\,dx\Big)^{\frac{1-q}{4}}\Big(\int_{|x|=\tau}|\nabla_x u|^2\,d\sigma\Big)^{1/2}\Big(\int_{\Omega(\tau)}a\big(|x|\big)|u|^{q+1}\,dx\Big)^{1/2}. \tag{2.9}$$

Integrating (2.9) in t and using the Young inequality with “ε” we obtain

$$\int_s^v\int_{|x|=\tau}|u||\nabla u|\,d\sigma\,dt$$

$$\leqslant \varepsilon\int_{Q_s^{(v)}(\tau)}\big(|\nabla_x u|^2+a\big(|x|\big)|u|^{q+1}\big)\,dx\,dt$$

$$+\,c(\varepsilon)a(\tau)^{-(1-\theta_1)}\sup_{s<t<v}\Big(\int_{\Omega(\tau)}\big|u(x,t)\big|^2\,dx\Big)^{\frac{(1-q)(1-\theta_1)}{2}}\int_s^v\int_{|x|=\tau}|\nabla_x u|^2\,d\sigma\,dt$$

$$+\,c(\varepsilon)a(\tau)^{-1}\sup_{s<t<v}\Big(\int_{\Omega(\tau)}\big|u(x,t)\big|^2\,dx\Big)^{\frac{1-q}{2}}\int_s^v\int_{|x|=\tau}|\nabla_x u|^2\,d\sigma\,dt \tag{2.10}$$

with arbitrary v: $s<v\leqslant T$. Let us fix now $v=\bar v=\bar v(\tau,s)$ such that the following inequality holds:

$$\int_{\Omega(\tau)}\big|u(x,\bar v)\big|^2\,dx\geqslant 2^{-1}\sup_{s\leqslant t\leqslant T}\int_{\Omega(\tau)}\big|u(x,t)\big|^2\,dx. \tag{2.11}$$

Inserting inequality (2.10) with $v=\bar v$ into (2.4) with $\hat t=\bar v$ and fixing “ε” small enough we have

$$H(\bar{v},\tau)+I_s^{(\bar{v})}(\tau)\leqslant H(s,\tau)+ca(\tau)^{-(1-\theta_1)}H(\bar{v},\tau)^{\frac{(1-q)(1-\theta_1)}{2}}J_s^{(\bar{v})}(\tau)$$
$$+ca(\tau)^{-1}H(\bar{v},\tau)^{\frac{1-q}{2}}J_s^{(\bar{v})}(\tau), \tag{2.12}$$

where $J_s^{(v)}(\tau)$ is from (2.5). Using the Young inequality again we deduce from (2.12)

$$H(\bar{v},\tau)+I_s^{(\bar{v})}(\tau)\leqslant 2H(s,\tau)+ca(\tau)^{-\frac{2(1-\theta_1)}{2-(1-\theta_1)(1-q)}}\left(J_s^{(\bar{v})}(\tau)\right)^{\frac{2}{2-(1-\theta_1)(1-q)}}$$
$$+ca(\tau)^{-\frac{2}{1+q}}\left(J_s^{(\bar{v})}(\tau)\right)^{\frac{2}{1+q}}. \tag{2.13}$$

Fixing now $v=T$ in (2.10) and using property (2.11) we obtain the inequality

$$\int_s^T\int_{|x|=\tau}|u||\nabla_x u|\,d\sigma\,dt\leqslant \varepsilon I_s^{(T)}(\tau)+c(\varepsilon)a(\tau)^{-(1-\theta_1)}H(\bar{v},\tau)^{\frac{(1-q)(1-\theta_1)}{2}}J_s^{(T)}(\tau)$$
$$+c(\varepsilon)a(\tau)^{-1}H(\bar{v},\tau)^{\frac{1-q}{2}}J_s^{(T)}(\tau). \tag{2.14}$$

By $\hat{t}=T$ it follows from (2.4) due to (2.14) with $\varepsilon=\frac{1}{2}$

$$H(T,\tau)+I_s^{(T)}(\tau)\leqslant H(s,\tau)+ca(\tau)^{-(1-\theta_1)}H(\bar{v},\tau)^{\frac{(1-q)(1-\theta)}{2}}J_s^{(T)}(\tau)$$
$$+ca(\tau)^{-1}H(\bar{v},\tau)^{\frac{1-q}{2}}J_s^{(T)}(\tau). \tag{2.15}$$

From (2.13) we have

$$H(\bar{v},\tau)^{\nu}\leqslant cH(s,\tau)^{\nu}+ca(\tau)^{-\frac{2(1-\theta_1)\nu}{2-(1-\theta_1)(1-q)}}\left(J_s^{(\bar{v})}(\tau)\right)^{\frac{2\nu}{2-(1-\theta_1)(1-q)}}$$
$$+ca(\tau)^{-\frac{2\nu}{1+q}}\left(J_s^{(\bar{v})}(\tau)\right)^{\frac{2\nu}{1+q}}\quad\forall\nu>0. \tag{2.16}$$

Using this estimate with $\nu_1=\frac{(1-q)(1-\theta_1)}{2}$ and $\nu_2=\frac{1-q}{2}$ from (2.15) we deduce that

$$H(T,\tau)+I_s^{(T)}(\tau)\leqslant H(s,\tau)+ca(\tau)^{-(1-\theta_1)}H(s,\tau)^{\nu_1}J_s^{(T)}(\tau)$$
$$+ca(\tau)^{-1}H(s,\tau)^{\nu_2}J_s^{(T)}(\tau)+ca(\tau)^{-(1-\theta_1)(1+\frac{2\nu_1}{2-(1-\theta_1)(1-q)})}$$
$$\times\left(J_s^{(T)}(\tau)\right)^{1+\frac{2\nu_1}{2-(1-\theta_1)(1-q)}}+ca(\tau)^{-(1-\theta_1)-\frac{2\nu_1}{1+q}}\left(J_s^{(T)}(\tau)\right)^{1+\frac{2\nu_1}{1+q}}$$
$$+ca(\tau)^{-1-\frac{2(1-\theta_1)\nu_2}{2-(1-\theta_1)(1-q)}}\left(J_s^{(T)}(\tau)\right)^{1+\frac{2\nu_2}{2-(1-\theta_1)(1-q)}}$$
$$+ca(\tau)^{-1-\frac{2\nu_2}{1+q}}\left(J_s^{(T)}(\tau)\right)^{1+\frac{2\nu_2}{1+q}}. \tag{2.17}$$

Using the Young inequality we infer from (2.17)

$$H(T,\tau)+I_s^{(T)}(\tau)\leqslant 2H(s,\tau)+ca(\tau)^{-\frac{2}{1+q}}\left(J_s^{(T)}(\tau)\right)^{\frac{2}{1+q}}$$
$$+ca(\tau)^{-\frac{2(1-\theta_1)}{2-(1-\theta_1)(1-q)}}\left(J_s^{(T)}(\tau)\right)^{\frac{2}{2-(1-\theta_1)(1-q)}}. \tag{2.18}$$

Now we have to estimate from above the term $H(s,\tau)$ in right-hand side of (2.18). Due to the Gagliardo–Nirenberg interpolation inequality we have

$$\int_{\Omega(\tau)} |u(x,s)|^2\,dx \leqslant d_3\Big(\int_{\Omega(\tau)} |\nabla_x u(x,s)|^2\,dx\Big)^{\theta_2}\Big(\int_{\Omega(\tau)} |u(x,s)|^{q+1}\Big)^{\frac{2(1-\theta_2)}{q+1}} + d_4\Big(\int_{\Omega(\tau)} |u(x,s)|^{q+1}dx\Big)^{\frac{2}{q+1}}, \quad \theta_2 \text{ is from (2.6)}, \tag{2.19}$$

and constants $d_3>0$, $d_4>0$ do not depend on τ as $\tau\to 0$. Taking into account the monotonicity of function $a(\tau)$ we deduce from (2.19)

$$\begin{aligned}
&\int_{\Omega(\tau)} |u(x,s)|^2\,dx \\
&\quad\leqslant d_3\Big(\int_{\Omega(\tau)} |\nabla_x u(x,s)|^2\,dx\Big)^{\theta_2}\Big(\int_{\Omega(\tau)} a(|x|)|u(x,s)|^{q+1}\,dx\Big)^{1-\theta_2} \\
&\qquad\times a(\tau)^{-(1-\theta_2)}\Big(\int_{\Omega(\tau)} |u(x,s)|^{q+1}\,dx\Big)^{\frac{(1-\theta_2)(1-q)}{1+q}} \\
&\qquad+ d_4 a(\tau)^{-\frac{2}{q+1}}\Big(\int_{\Omega(\tau)} a(|x|)|u(x,s)|^{q+1}\,dx\Big)^{\frac{2}{q+1}} \\
&\quad\leqslant c a(\tau)^{-(1-\theta_2)}\int_{\Omega(\tau)} \big(|\nabla_x u|^2 + a(|x|)|u(x,s)|^{q+1}\big)\,dx\Big(\int_{\Omega(\tau)} |u(x,s)|^2\,dx\Big)^{\frac{(1-\theta_2)(1-q)}{2}} \\
&\qquad+ d_4 a(\tau)^{-\frac{2}{q+1}}\Big(\int_{\Omega(\tau)} a(|x|)|u(x,s)|^{q+1}\,dx\Big)^{\frac{2}{q+1}}.
\end{aligned}$$

Estimating the first term in the right-hand side by the Young inequality with "ε," we have

$$\begin{aligned}
\int_{\Omega(\tau)} u^2(x,s)\,dx &\leqslant c_1 a(\tau)^{-\frac{2}{q+1}}\Big(\int_{\Omega(\tau)} a(|x|)|u(x,s)|^{q+1}\,dx\Big)^{\frac{2}{q+1}} \\
&\quad+ c a(\tau)^{-\frac{2(1-\theta_2)}{2-(1-\theta_2)(1-q)}}\Big(\int_{\Omega(\tau)} \big(|\nabla_x u|^2 + a(|x|)|u(x,s)|^{q+1}\big)\,dx\Big)^{\frac{2}{2-(1-\theta_2)(1-q)}}.
\end{aligned} \tag{2.20}$$

Using (2.20) in (2.18) we obtain the required (2.6).

Let us introduce the positive nondecreasing function

$$s(\tau) = \tau^4 \omega(\tau)^{-1}, \tag{2.21}$$

where $\omega(\tau) > 0$ is from (1.2). Define the energy function

$$y(\tau) = I^{(T)}_{s(\tau)}(\tau), \quad \text{where } I^{(T)}_s(\tau) \text{ is from (2.5).} \tag{2.22}$$

Lemma 2.3. *The energy function $y(\tau)$ from* (2.22) *is the solution of the following Cauchy problem for the ordinary differential inequality*:

$$y(\tau) \leqslant c_0 \sum_{i=0}^{2} \left(-\frac{y'(\tau)}{\psi_i(\tau)} \right)^{1+\lambda_i} \quad \forall \tau > 0, \tag{2.23}$$

$$y(0) \leqslant y_0, \quad y_0 \textit{ is from } (2.2), \tag{2.24}$$

where

$$\psi_0(\tau) = a(\tau)s'(\tau), \qquad \psi_1(\tau) = a(\tau)^{1-\theta_1}, \qquad \psi_2(\tau) = a(\tau)^{1-\theta_2}s'(\tau),$$

$$\lambda_0 = \frac{1-q}{1+q} > \lambda_2 = \frac{(1-\theta_2)(1-q)}{2-(1-\theta_2)(1-q)} > \lambda_1 = \frac{(1-\theta_1)(1-q)}{2-(1-\theta_1)(1-q)} > 0.$$

It is easy to verify the following equality

$$\begin{aligned} \frac{d}{d\tau} I^{(T)}_{s(\tau)}(\tau) = {} & - \int_{s(\tau)}^{T} \int_{\{|x|=\tau\}} \left(|\nabla_x u|^2 + a(|x|) \left| u(x, s(\tau)) \right|^{q+1} \right) d\sigma \, dt \\ & - s'(\tau) \int_{\Omega(\tau)} \left(\left| \nabla_x u(x, s(\tau)) \right|^2 + a(|x|) \left| u(x, s(\tau)) \right|^{q+1} \right) dx. \end{aligned} \tag{2.25}$$

Since $s'(\tau) \geqslant 0$, from (2.25) it follows that

$$\int_{s(\tau)}^{T} \int_{\{|x|=\tau\}} |\nabla_x u|^2 \, d\sigma \, dt = J^{(T)}_{s(\tau)}(\tau) \leqslant -\frac{d}{d\tau} I^{(T)}_{s(\tau)}(\tau), \tag{2.26}$$

$$\begin{aligned} & \int_{\Omega(\tau)} \left(\left| \nabla_x u(x, s(\tau)) \right|^2 + a(|x|) \left| u(x, s(\tau)) \right|^{q+1} \right) dx \\ & \quad = E\big(s(\tau), \tau\big) \leqslant -\big(s'(\tau)\big)^{-1} \frac{d}{d\tau} I^{(T)}_{s(\tau)}(\tau). \end{aligned} \tag{2.27}$$

Inserting these estimates in (2.6) and using additionally that $s'(\tau) \to 0$ as $\tau \to 0$ after simple calculations we obtain ODI (2.23) and the initial condition (2.24).

Now we will study the asymptotic behavior of an arbitrary solution $y(\tau)$ of system (2.23), (2.24). We have to prove the existence of a continuous function $\bar\tau = \bar\tau(y_0)$ such that $y(\tau) \leqslant 0$ for arbitrary $\tau \geqslant \bar\tau(y_0)$. Moreover, we have to find the sharp upper estimate for the function $\bar\tau(y)$ as $y \to 0$. It is related to the optimal choice of the function $s(\tau)$, defined by (2.21). Consider the following auxiliary Cauchy problem:

$$Y(\tau) = 3c_0 \max_{0\leqslant i\leqslant 2}\left\{\left(-\frac{Y'(\tau)}{\psi_i(\tau)}\right)^{1+\lambda_i}\right\}, \qquad Y(0) = y_0 > 0, \tag{2.28}$$

where $c_0 > 0$ is from (2.23). It is easy to check the following comparison property:

$$y(\tau) \leqslant Y(\tau) \quad \forall\tau > 0, \tag{2.29}$$

where $y(\tau)$ is arbitrary solution of the Cauchy problem (2.23), (2.24).

Lemma 2.4. *Let $Y(\tau)$ be an arbitrary solution of the Cauchy problem* (2.28). *Then there exists a function $\bar\tau(r) < \infty$ $\forall r > 0$ such that $Y(\tau) \leqslant 0$ $\forall\tau > \bar\tau(y_0)$.*

Let us consider the following additional ordinary differential equations (ODE):

$$Y_i(\tau) = 3c_0\left(-\frac{Y_i'(\tau)}{\psi_i(\tau)}\right)^{1+\lambda_i}, \quad i = 0, 1, 2, \tag{2.30}$$

or equivalently,

$$Y_i'(\tau) = -\psi_i(\tau)\left(\frac{Y_i(\tau)}{3c_0}\right)^{\frac{1}{1+\lambda_i}} := -F_i\big(\tau, Y_i(\tau)\big). \tag{2.31}$$

Let us define the following subdomains Ω_i, $i = 0, 1, 2$,

$$\Omega_0 = \big\{(\tau, y) \in \mathbb{R}^2_+ := \{\tau > 0, y > 0\}\colon\ F_0(\tau, y) = \min_{0\leqslant i\leqslant 2}\big\{F_i(\tau, y)\big\}\big\},$$

$$\Omega_1 = \big\{(\tau, y) \in \mathbb{R}^2_+\colon\ F_1(\tau, y) = \min_{0\leqslant i\leqslant 2}\big\{F_i(\tau, y)\big\}\big\},$$

$$\Omega_2 = \big\{(\tau, y) \in \mathbb{R}^2_+\colon\ F_2(\tau, y) = \min_{0\leqslant i\leqslant 2}\big\{F_i(\tau, y)\big\}\big\}.$$

It is easy to see that

$$\Omega_0 \cup \Omega_1 \cup \Omega_2 = \mathbb{R}^2_+.$$

Due to (2.28), (2.30), (2.31) it is easy to see that arbitrary solution $Y(\tau)$ of the problem (2.28) has the following structure:

$$Y(\tau) = \big\{Y_i(\tau)\ \forall(\tau, Y) \in \Omega_i,\ i = 0, 1, 2\big\}, \tag{2.32}$$

where $Y_i(\tau)$ is solution of Eq. (2.30) (or (2.31)). It is easy to check that

$$\Omega_0 = \left\{(\tau, y):\ y \geqslant 3c_0 a(\tau)^{\frac{2}{1-q}}\right\},$$

$$\Omega_1 = \left\{(\tau, y):\ y \leqslant 3c_0 a(\tau)^{\frac{2}{1-q}} s'(\tau)^{\frac{2}{(1-q)(\theta_1-\theta_2)}}\right\}, \quad s'(\tau) = \frac{ds(\tau)}{d\tau},$$

$$\Omega_2 = \left\{(\tau, y):\ 3c_0 a(\tau)^{\frac{2}{1-q}} s'(\tau)^{\frac{2}{(1-q)(\theta_1-\theta_2)}} \leqslant y \leqslant 3c_0 a(\tau)^{\frac{2}{1-q}}\right\}.$$

Therefore the solution $Y(\tau)$ of the Cauchy problem (2.28) is dominated by the following curve:

$$\widetilde{Y}(\tau) = \begin{cases} y_0, & \text{if } 0 \leqslant \tau \leqslant \tau', \\ \widetilde{Y}_2(\tau), & \text{if } \tau' \leqslant \tau \leqslant \tau'', \\ \widetilde{Y}_1(\tau), & \text{if } \tau'' \leqslant \tau \leqslant \tau''', \end{cases} \tag{2.33}$$

where τ' is defined by equality

$$y_0 = 3c_0 a(\tau')^{\frac{2}{1-q}} \quad \Longrightarrow \quad \frac{\tau'^2}{\omega(\tau')} = \frac{2}{1-q}\big(\ln(3c_0) - \ln y_0\big)^{-1}, \tag{2.34}$$

$\widetilde{Y}_2(\tau)$ is the solution of the Cauchy problem

$$Y_2'(\tau) = -\psi_2(\tau)\left(\frac{Y_2(\tau)}{3c_0}\right)^{\frac{1}{1+\lambda_2}}, \qquad Y_2(\tau') = y_0, \tag{2.35}$$

τ'' is defined by the equality

$$\widetilde{Y}_2(\tau'') = 3c_0 a(\tau'')^{\frac{2}{1-q}} s'(\tau'')^{\frac{2}{(1-q)(\theta_1-\theta_2)}}. \tag{2.36}$$

Finally, $\widetilde{Y}_1(\tau)$ is the solution of the Cauchy problem

$$Y_1'(\tau) = -\psi_1(\tau)\left(\frac{Y_1(\tau)}{3c_0}\right)^{\frac{1}{1+\lambda_1}}, \qquad Y_1(\tau'') = \widetilde{Y}_2(\tau''), \tag{2.37}$$

and τ''' is such that $\widetilde{Y}_1(\tau) \leqslant 0\ \forall \tau \geqslant \tau'''$. It is easy to check that the solution of (2.35) is

$$\begin{aligned} \widetilde{Y}_2(\tau) &= \left[y_0^{\frac{\lambda_2}{1+\lambda_2}} - \frac{\lambda_2}{(1+\lambda_2)(3c_0)^{\frac{1}{1+\lambda_0}}} \int_{\tau'}^{\tau} \psi_2(r)\,dr \right]^{\frac{1+\lambda_2}{\lambda_2}} \\ &= \left[y_0^{\frac{(1-\theta_2)(1-q)}{2}} - \frac{(1-\theta_2)(1-q)}{2(3c_0)^{\frac{1}{1+\lambda_0}}} \int_{\tau'}^{\tau} a(r)^{1-\theta_2} s'(r)\,dr \right]^{\frac{2}{(1-\theta_2)(1-q)}}. \end{aligned} \tag{2.38}$$

Equation (2.36) for τ'' then yields

$$y_0^{\frac{(1-\theta_2)(1-q)}{2}} - \frac{(1-\theta_1)(1-q)}{2(3c_0)^{\frac{1}{1+\lambda_2}}} \int_{\tau'}^{\tau''} a(r)^{1-\theta_2} s'(r)\,dr$$
$$= (3c_0)^{\frac{(1-\theta_2)(1-q)}{2}} a(\tau'')^{1-\theta_2} s'(\tau'')^2 \quad \left(\text{since } \frac{1-\theta_2}{\theta_1-\theta_2} = 2\right). \tag{2.39}$$

We will say that $a(\tau) \approx b(\tau)$, if there exists constant C, which does not depend on τ, such that

$$0 < C^{-1} a(\tau) \leqslant b(\tau) \leqslant C a(\tau) \quad \forall \tau\colon\ 0 < \tau < \tau_0.$$

Due to condition (1.6) it follows easily too:

$$(2+\delta)\frac{\tau^3}{\omega(\tau)} \leqslant s'(\tau) \leqslant \frac{4\tau^3}{\omega(\tau)} \quad \forall \tau > 0. \tag{2.40}$$

From definition (2.21) of $s(r)$ by virtue of (2.40) and Lemma A.1 we deduce

$$\int_0^{\tau} a(r)^{1-\theta_2} s'(r)\,dr \approx \int_0^{\tau} \exp\left(-\frac{(1-\theta_2)\omega(r)}{r^2}\right) r^3 \omega(r)^{-1}\,dr$$
$$\approx \tau^6 \omega(\tau)^{-2} \exp\left(-\frac{(1-\theta_2)\omega(\tau)}{\tau^2}\right)$$
$$\approx a(\tau)^{1-\theta_2} \big(s'(\tau)\big)^2 \quad \forall \tau\colon\ 0 < \tau < \tau_0 < \infty. \tag{2.41}$$

Thus, from (2.39) due to (2.41) one obtains the following estimate for τ'':

$$c_1 y_0^{\frac{(1-\theta_2)(1-q)}{2}} \leqslant a(\tau'')^{1-\theta_2} s'(\tau'')^2 \leqslant c_2 y_0^{\frac{(1-\theta_2)(1-q)}{2}}, \tag{2.42}$$

where positive constants c_1, c_2 does not depend on y_0. Now, the solution of the Cauchy problem (2.37) is

$$\widetilde{Y}_1(\tau) = \left[\widetilde{Y}_2(\tau'')^{\frac{(1-\theta_1)(1-q)}{2}} - \frac{(1-\theta_1)(1-q)}{2(3c_0)^{\frac{1}{1+\lambda_1}}} \int_{\tau''}^{\tau} a(r)^{1-\theta_1}\,dr\right]^{\frac{2}{(1-\theta_1)(1-q)}}. \tag{2.43}$$

Thus, τ''' is defined by the equation

$$\widetilde{Y}_2(\tau'')^{\frac{(1-\theta_1)(1-q)}{2}} - \frac{(1-\theta_1)(1-q)}{2(3c_0)^{\frac{1}{1+\lambda_1}}} \int_{\tau''}^{\tau'''} a(r)^{1-\theta_1}\,dr. \tag{2.44}$$

Due to Lemma A.1 we have

$$\left(\int_0^\tau a(r)^{1-\theta_1}\,dr\right)^2 \approx \left(\int_0^\tau \exp\left(-\frac{\beta\omega(r)}{r^2}\right)dr\right)^2$$
$$\approx \left(\frac{\tau^3}{\omega(\tau)}\exp\left(-\frac{\beta\omega(\tau)}{\tau^2}\right)\right)^2$$
$$\approx s'(\tau)^2 a(\tau)^{1-\theta_2} \quad \forall\tau > 0, \tag{2.45}$$

where $1-\theta_1 = \beta = \frac{q+1}{2(q+1)+u(1-q)} = \frac{1-\theta_2}{2}$. It is easy to see that

$$\int_0^\tau a(r)^{1-\theta_1}\,dr \approx \int_{\tau/2}^\tau a(r)^{1-\theta_1}\,dr \quad \text{if } \tau \to 0.$$

Therefore due to (2.44) the following inequalities are sufficient conditions for τ''':

$$a(\tau''')^{1-\theta_2} s'(\tau''')^2 \leqslant c_3 \widetilde{Y}_2(\tau'')^{\frac{(1-\theta_2)(1-q)}{2}}, \qquad \tau''' > 2\tau''.$$

Finally, by virtue of (2.38) we obtain the following unique sufficient condition which defines τ''':

$$a(\tau''')^{1-\theta_2} s'(\tau''')^2 \leqslant c_4 y_0^{\frac{(1-\theta_2)(1-q)}{2}}. \tag{2.46}$$

Condition (2.46) can be rewritten in the form

$$\frac{\exp(-\frac{(1-\theta_2)(1-\nu)\omega(\tau''')}{(\tau''')^2})\cdot\exp(-(1-\theta_2)\nu\frac{\omega(\tau''')}{(\tau''')^2})\omega(\tau''')}{(\frac{\omega(\tau''')}{(\tau''')^2})^3} \leqslant c_5 y_0^{\frac{(1-\theta_2)(1-q)}{2}} \tag{2.47}$$

with arbitrary $1 > \nu > 0$. It is obviously, that the following is a sufficient condition for (2.47)

$$\exp\left(-\frac{(1-\theta_2)(1-\nu)\omega(\tau''')}{(\tau''')^2}\right) \leqslant c_6 y_0^{\frac{(1-\theta_2)(1-q)}{2}}, \quad c_6 = c_6(\nu, \omega_0, c_5),$$

or,

$$\frac{(\tau''')^2}{\omega(\tau''')} \leqslant c_7\left(\ln y_0^{-1}\right)^{-1}, \quad c_7 = c_7(c_6, \nu, \omega_0), \quad \omega_0 \text{ is from } (A_3). \tag{2.48}$$

Thus, the assertion of Lemma 2.4 holds with $\bar\tau(r)$ defined by

$$\frac{\bar\tau(r)^2}{\omega(\bar\tau(r))} = c_7\left(\ln r^{-1}\right)^{-1} \quad \forall r > 0. \tag{2.49}$$

Proof of Theorem 1.1. Due to Lemma A.3 from Appendix A we can suppose that

$$y_0 \ll 1 \quad \text{and} \quad \bar\tau(y_0) < 1. \tag{2.50}$$

From definition (2.23) of function $y(\tau)$ due to Lemma 2.4 and property (2.29) it follows that

$$I^{(T)}_{s(\bar{\tau}(y_0))}\big(\bar{\tau}(y_0)\big) = 0 \quad \text{for arbitrary } T < \infty.$$

Therefore our solution $u(x,t)$ has the following property:

$$u(x,t) \equiv 0 \quad \forall (x,t) \in \big\{|x| \geqslant \tau_1, t \geqslant s(\tau_1)\big\}, \quad \tau_1 = \bar{\tau}(y_0). \tag{2.51}$$

From identity (2.4) with $\tau = 0$ we deduce that

$$\frac{d}{dt}\int_{\Omega} \big|u(x,t)\big|^2\,dx + \int_{\Omega} \big(\big|\nabla_x u(x,t)\big|^2 + a_0(x)|u|^{q+1}\big)\,dx \leqslant 0 \quad \forall t \in \big(s(\tau_1), T\big). \tag{2.52}$$

Due to (2.51) and the Poincaré inequality it follows from (2.52)

$$H'(t) + \frac{\bar{c}}{\tau_1^2} H(t) \leqslant 0 \quad \forall t > s(\tau_1),\ \bar{c} = \text{const} > 0, \tag{2.53}$$

where $H(t) := H(t,0)$, $H(t,\tau)$ is defined by (2.9), constant $\bar{c} > 0$ does not depend on t. Integrating ODI (2.53) we deduce the following relationship easily:

$$H\big(t + s(\tau_1)\big) \leqslant H\big(s(\tau_1)\big)\exp\bigg(-\frac{\bar{c}t}{\tau_1^2}\bigg) \quad \forall t > 0.$$

Using additionally estimate (2.2) with $\hat{t} = s(\tau_1)$ we deduce

$$H\big(t + s(\tau_1)\big) \leqslant y_0 \exp\bigg(-\frac{\bar{c}t}{\tau_1^2}\bigg) \quad \forall t > 0. \tag{2.54}$$

Define $t_1 > 0$ by

$$y_0 \exp\bigg(-\frac{\bar{c}t_1}{\tau_1^2}\bigg) = y_0^{1+\gamma} \quad \Longleftrightarrow \quad t_1 = \frac{\gamma \ln y_0^{-1}}{\bar{c}}\tau_1^2, \quad \gamma = \text{const} > 0. \tag{2.55}$$

Due to (2.49) from (2.55) it follows that

$$t_1 = \frac{\gamma c_7}{\bar{c}}\omega(\tau_1). \tag{2.56}$$

Thus, we have

$$H\big(t_1 + s(\tau_1)\big) = \int_{\Omega} \big|u\big(x, t_1 + s(\tau_1)\big)\big|^2\,dx \leqslant y_0^{1+\gamma}, \quad \gamma > 0. \tag{2.57}$$

So, we finished first round of computations. For the second round we will consider our initial–boundary problem (1.1) in the domain $\Omega \times (t_1 + s(\tau_1), \infty)$ with initial data (2.57) instead of (2.2). Repeating all previous computations we deduce the following analogue of estimate (2.57)

$$H\big(t_2 + s(\tau_2) + t_1 + s(\tau_1)\big) \leqslant y_0^{(1+\gamma)^2}, \tag{2.58}$$

where as in (2.49) and (2.55)

$$\tau_2^2 = c_7\omega(\tau_2)\big(\ln y_0^{-(1+\gamma)}\big)^{-1} = \frac{c_7}{1+\gamma}\omega(\tau_2)\big(\ln y_0^{-1}\big)^{-1}, \quad \tau_2 = \bar{\tau}\big(y_0^{1+\gamma}\big). \tag{2.59}$$

Analogously to (2.56) we have also

$$t_2 = \frac{\gamma \ln y_0^{-(1+\gamma)}}{\bar{c}}\tau_2^2 = \frac{\gamma c_7}{\bar{c}}\omega(\tau_2). \tag{2.60}$$

Now using estimate (2.58) as a starting point for next round of computations we find τ_3, t_3 and so on. As result, after j rounds we get

$$H\left(\sum_{i=1}^{j} t_i + \sum_{i=1}^{j} s(\tau_i)\right) \leqslant y_0^{(1+\gamma)^j} \to 0 \quad \text{as } j \to 0, \tag{2.61}$$

where

$$\tau_i^2 \leqslant \frac{c_7\omega(\tau_i)}{(1+\gamma)^{i-1}}\big(\ln y_0^{-1}\big)^{-1}. \tag{2.62}$$

Due to condition (A_3) it follows from (2.62):

$$\tau_i^2 \leqslant \frac{c_7\omega_0(\ln y_0^{-1})^{-1}}{(1+\gamma)^{i-1}}.$$

From definition (2.21) of function $s(\tau)$ due to condition (1.6) it follows the estimate

$$s(\tau) \leqslant \tau_0^2\omega(\tau_0)^{-1}\tau^2 \quad \forall\tau_0 > 0,\ \forall\tau > 0.$$

Therefore inequality (2.62) yields

$$\sum_{i=1}^{\infty} s(\tau_i) < \tilde{c} < \infty. \tag{2.63}$$

Obviously, we have also: $t_i = \frac{\gamma c_7}{\bar{c}}\omega(\tau_i)$. Therefore, due to (2.62) we have

$$\sum_{i=1}^{j} t_i = \frac{\gamma c_7}{\bar{c}}\sum_{i=1}^{j}\omega(\tau_i) \leqslant C\sum_{i=1}^{j}\omega\big(C_1\lambda^i\big), \tag{2.64}$$

where $C = \frac{\gamma c_7}{\bar{c}}$, $C_1 = (\frac{c_7 \omega_0}{\ln y_0^{-1}(1+\gamma)})^{1/2}$, $\lambda = (1+\gamma)^{-1/2} < 1$. In virtue of condition (1.5) it is easy to check that

$$\sum_{i=1}^{j} \omega\big(C_1 \lambda^i\big) \approx \ln \lambda^{-1} \int_{C_1 \lambda^j}^{C_1} \frac{\omega(s)}{s}\, ds < c < \infty \quad \forall j \in \mathbb{N}. \tag{2.65}$$

From (2.61) due to (2.63), (2.65) and condition (1.5) it follows that

$$H(R) = 0, \quad R = \sum_{i=1}^{\infty} t_i + \sum_{i=1}^{\infty} s(\tau_i) < \infty,$$

which completes the proof of Theorem 1.1. □

3. Dini condition (1.5) of extinction in finite time via semi-classical limit of Schrödinger operator

Here we prove Proposition 1.3. We recall the definition of $\lambda_1(h)$ and $\mu(\alpha)$ for $h > 0$ and $\alpha > 0$:

$$\lambda_1(h) = \inf\left\{ \int_{B_1} |\nabla v|^2 + h^{-2} a\big(|x|\big)|v|^2\, dx\colon\ v \in W^{1,2}(B_1),\ \|v\|_{L_2(B_1)} = 1 \right\}$$

and

$$\mu(\alpha) = \lambda_1\big(\alpha^{\frac{1-q}{2}}\big).$$

We define $r(z) = a^{-1}(z)$ or equivalently $z = a(r(z))$ and $\rho(z) = z(r(z))^2$ for z small enough. We will use the following technical statement.

Lemma 3.1. *(See [5, Corollaries 2.23, 2.31].) Under assumptions* (A_1)–(A_3) *and* (1.8)*, there exist four positives constants C_1, C_2, C_3 and C_4 such that for h small enough,*

$$C_1 h^{-2} \rho^{-1}\big(C_2 h^2\big) \leqslant \lambda_1(h) \leqslant C_3 h^{-2} \rho^{-1}\big(C_4 h^2\big).$$

Our main starting point in the proof of Proposition 1.3 is the following

Theorem A. *(See [2, Theorem 2.2].) Under assumptions* (A_1)–(A_3)*, if there exists a decreasing sequence* (α_n) *of positive real numbers such that*

$$\sum_{n=0}^{+\infty} \frac{1}{\mu(\alpha_n)} \left(\ln\big(\mu(\alpha_n)\big) + \ln\left(\frac{\alpha_n}{\alpha_{n+1}}\right) + 1 \right) < +\infty,$$

then problem (1.1) *satisfies the TCS-property.*

The first step in the proof of Proposition 1.3 is the estimation of ρ^{-1} in a neighborhood of zero.

Lemma 3.2. *Under assumptions* (A_1)–(A_3) *with* (1.7) *there holds*

$$\frac{s}{(1+\alpha)}\ln\left(\frac{1}{s}\right)\frac{1}{\omega\big(\big(\frac{\omega_0(1+\alpha)}{\ln(\frac{1}{s})}\big)^{1/2}\big)}\leqslant\rho^{-1}(s)\leqslant s\ln\left(\frac{1}{s}\right)\frac{1}{\omega\big(\big(\frac{1}{\ln(\frac{1}{s})}\big)^{1/\delta}\big)},\tag{3.1}$$

for arbitrary $\alpha>0$, *for all* $s>0$ *small enough.*

First of all, we prove the following estimate for $\rho(z)$:

$$\rho(z)\ln\left(\frac{1}{z}\right)\left[\omega\left(\left(\frac{\omega_0}{\ln(\frac{1}{z})}\right)^{\frac{1}{2}}\right)\right]^{-1}\leqslant z\leqslant\rho(z)\ln\left(\frac{1}{z}\right)\left[\omega\left(\left(\frac{1}{\ln(\frac{1}{z})}\right)^{\frac{1}{\delta}}\right)\right]^{-1}.\tag{3.2}$$

Starting with $r>0$ small enough, we have from (1.7) the relationship $r^{2-\delta}\leqslant\omega(r)\leqslant\omega_0$ and since for $z>0$ small enough,

$$\big(r(z)\big)^2\ln\left(\frac{1}{z}\right)=\omega\big(r(z)\big)\quad\Longrightarrow\quad r(z)^{2-\delta}\leqslant\big(r(z)\big)^2\ln\left(\frac{1}{z}\right)\leqslant\omega_0.$$

Therefore, we obtain

$$\left(\frac{1}{\ln(\frac{1}{z})}\right)^{\frac{1}{\delta}}\leqslant r(z)\leqslant\left(\frac{\omega_0}{\ln(\frac{1}{z})}\right)^{\frac{1}{2}}.\tag{3.3}$$

Since ω is a nondecreasing function,

$$\omega\left(\left(\frac{1}{\ln(\frac{1}{z})}\right)^{\frac{1}{\delta}}\right)\leqslant\omega\big(r(z)\big)\leqslant\omega\left(\left(\frac{\omega_0}{\ln(\frac{1}{z})}\right)^{\frac{1}{2}}\right).$$

Substituting the definition of $\omega(r)$:

$$\omega\left(\left(\frac{1}{\ln(\frac{1}{z})}\right)^{\frac{1}{\delta}}\right)\leqslant\big(r(z)\big)^2\ln\left(\frac{1}{z}\right)\leqslant\omega\left(\left(\frac{\omega_0}{\ln(\frac{1}{z})}\right)^{\frac{1}{2}}\right).$$

It follows the estimate for $\rho(z)$:

$$z\frac{1}{\ln(\frac{1}{z})}\omega\left(\left(\frac{1}{\ln(\frac{1}{z})}\right)^{\frac{1}{\delta}}\right)\leqslant\rho(z)\leqslant z\frac{1}{\ln(\frac{1}{z})}\omega\left(\left(\frac{\omega_0}{\ln(\frac{1}{z})}\right)^{\frac{1}{2}}\right).\tag{3.4}$$

By an easy calculation, we have (3.2).

Here and further, $z=\rho^{-1}(s)$. By using (3.3) and $\rho(z)=z(r(z))^2$,

$$\rho(z)\geqslant z\left(\frac{1}{\ln(\frac{1}{z})}\right)^{\frac{2}{\delta}} \iff \frac{1}{\rho(z)}\leqslant \frac{1}{z}\left(\ln\left(\frac{1}{z}\right)\right)^{\frac{2}{\delta}},$$

or equivalently,

$$\ln\left(\frac{1}{\rho(z)}\right)\leqslant \ln\left(\frac{1}{z}\right)+\frac{2}{\delta}\ln\left(\ln\left(\frac{1}{z}\right)\right).$$

Let $\alpha>0$. Then for z small enough, since $\ln(\ln(z^{-1}))\ll \ln z^{-1}$,

$$\ln\left(\frac{1}{\rho(z)}\right)\leqslant (1+\alpha)\ln\left(\frac{1}{z}\right) \iff \rho(z)\geqslant z^{1+\alpha} \implies \rho^{-1}(s)\leqslant s^{\frac{1}{1+\alpha}}. \tag{3.5}$$

Substituting $z=\rho^{-1}(s)$ in (3.2) yields

$$s\ln\left(\frac{1}{\rho^{-1}(s)}\right)\left[\omega\left(\left(\frac{\omega_0}{\ln(\frac{1}{\rho^{-1}(s)})}\right)^{\frac{1}{2}}\right)\right]^{-1}\leqslant \rho^{-1}(s),$$

and due to (3.5),

$$\frac{s}{(1+\alpha)}\ln\left(\frac{1}{s}\right)\left[\omega\left(\left(\frac{\omega_0(1+\alpha)}{\ln(\frac{1}{s})}\right)^{\frac{1}{2}}\right)\right]^{-1}\leqslant \rho^{-1}(s),$$

since ω is a nondecreasing function.

For the right-hand side of (3.1), we substitute $z=\rho^{-1}(s)$ in (3.2):

$$\rho^{-1}(s)\leqslant s\ln\left(\frac{1}{\rho^{-1}(s)}\right)\left[\omega\left(\left(\frac{1}{\ln(\frac{1}{\rho^{-1}(s)})}\right)^{\frac{1}{\delta}}\right)\right]^{-1}.$$

But from (3.3), $r(z)\to z$ so we have for z small enough, $\rho(z)\leqslant z$, which gives $\rho^{-1}(s)\geqslant s$. Consequently,

$$\rho^{-1}(s)\leqslant s\ln\left(\frac{1}{s}\right)\left[\omega\left(\left(\frac{1}{\ln(\frac{1}{s})}\right)^{\frac{1}{\delta}}\right)\right]^{-1},$$

which completes the proof.

Lemma 3.3. *Under* (A_1)–(A_3) *with* (1.7) *and* (1.8), *if*

$$\sum_{n=n_0}^{+\infty}\frac{\omega(\frac{1}{(n\ln n)^{1/2}})}{n}<+\infty, \tag{3.6}$$

then all solutions of (1.1) *vanish in a finite time. Moreover,*

$$\sum_{n=n_0}^{+\infty} \frac{\omega(\frac{1}{(n\ln n)^{1/2}})}{n} < +\infty \iff \int_0^c \frac{\omega(x)}{x}\,dx < +\infty. \tag{3.7}$$

From Lemmas 3.3 and 3.1 we get

$$K_1 \ln\left(\frac{1}{h}\right)\omega\left(\frac{K_2}{(\ln(\frac{1}{h}))^{1/2}}\right) \leqslant \lambda_1(h) \leqslant K_3 \ln\left(\frac{1}{h}\right)\left[\omega\left(\frac{K_4}{(\ln(\frac{1}{h}))^{1/\delta}}\right)\right]^{-1},$$

and since $\omega(r) \geqslant r^\theta$ for r small enough, we have

$$K_1 \ln\left(\frac{1}{h}\right)\left[\omega\left(\frac{K_2}{(\ln(\frac{1}{h}))^{1/2}}\right)\right]^{-1} \leqslant \lambda_1(h) \leqslant K_3' \ln\left(\frac{1}{h}\right)^{1+\frac{2-\delta}{\delta}},$$

which leads to

$$C_1' \ln\left(\frac{1}{h}\right)\left[\omega\left(\frac{C_2'}{(\ln(\frac{1}{h}))^{1/2}}\right)\right]^{-1} \leqslant \lambda_1(h) \leqslant C_3' \ln\left(\frac{1}{h}\right)^{\frac{2}{\delta}}. \tag{3.8}$$

The real number α is defined by $h = \alpha^{\frac{1-q}{2}}$ and thus,

$$C_1'' \ln\left(\frac{1}{\alpha}\right)\left[\omega\left(\frac{C_2''}{(\ln(\frac{1}{\alpha}))^{1/2}}\right)\right]^{-1} \leqslant \mu(\alpha) \leqslant C_3'' \ln\left(\frac{1}{\alpha}\right)^{\frac{2}{\delta}}.$$

From Theorem A, if (α_n) is a decreasing sequence of positive real numbers and

$$\sum_{n=n_0}^{+\infty} \frac{\omega\left(\frac{C_2''}{(\ln(\frac{1}{\alpha_n}))^{1/2}}\right)}{\ln(\frac{1}{\alpha_n})}\left[\ln\left(\ln\left(\frac{1}{\alpha_n}\right)\right) + \ln\left(\frac{\alpha_n}{\alpha_{n+1}}\right) + 1\right] < +\infty,$$

then all the solutions of (1.1) vanish in a finite time.

The main point is the sequence (α_n). In [2], they set $\alpha_n = 2^{-n}$. A better choice is $\alpha_n = n^{-Kn}$ for some $K > 0$ since $\ln(\ln(\frac{1}{\alpha_n})) \sim \ln(\frac{\alpha_n}{\alpha_{n+1}})$ which leads to (3.6).

Now, we have to show that

$$\sum_{n=n_0}^{+\infty} \frac{\omega(\frac{1}{(n\ln n)^{1/2}})}{n} < +\infty \iff \int_0^c \frac{\omega(x)}{x}\,dx < +\infty.$$

The series is finite if and only if

$$\int_{n_0}^{+\infty} \frac{\omega(\frac{1}{(x\ln x)^{1/2}})}{x}\,dx = \int_0^{1/n_0} \frac{\omega((\frac{x}{-\ln x})^{1/2})}{x}\,dx$$

is finite. The following inequalities hold for $c > 0$ small enough:

$$\int_0^c \frac{\omega(x)}{x}\,dx \leqslant \int_0^c \frac{\omega((\frac{x}{-\ln x})^{1/2})}{x}\,dx \leqslant \int_0^c \frac{\omega(x^{1/2})}{x}\,dx = 2\int_0^{\sqrt{c}} \frac{\omega(x)}{x}\,dx,$$

which completes the proof of Proposition 1.3.

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Appendix A

Lemma A.1. *Let the nonnegative nondecreasing function $\omega(s)$, $s \geqslant 0$, satisfy condition* (1.6). *Then for any $m \in \mathbb{R}^1$, $l \in \mathbb{R}^1$, $A > 0$, one has*

$$\int_0^\tau s^{m-2}\omega(s)^{l+1}\exp\left(-\frac{A\omega(s)}{s^2}\right)ds \approx \tau^{m+1}\omega(\tau)^l\exp\left(-\frac{A\omega(\tau)}{\tau^2}\right) \quad \textit{as } \tau \to 0. \tag{A.1}$$

It is easy to check the following equality

$$\begin{aligned}
&\frac{d}{ds}\left(s^{m+1}\omega(s)^l\exp\left(-\frac{A\omega(s)}{s^2}\right)\right)\\
&\quad = s^m\omega(s)^l\exp\left(-\frac{A\omega(s)}{s^2}\right)\left[(m+1) + l\frac{s\omega'(s)}{\omega(s)} + \frac{A\omega(s)}{s^2}\left(2 - \frac{s\omega'(s)}{\omega(s)}\right)\right]\\
&\quad \equiv s^m\omega(s)^l\exp\left(-\frac{A\omega(s)}{s^2}\right)[I_1 + I_2 + I_3].
\end{aligned} \tag{A.2}$$

Integrating condition (1.6) we get

$$\omega(s) \geqslant s^{2-\delta} \quad \forall s \in (0, s_0), \tag{A.3}$$

and, as a consequence, $\frac{\omega(s)}{s^2} \to \infty$ as $s \to 0$. Now due to (1.6) it follows that

$$I_3 \gg |I_1|, \qquad I_3 \gg |I_2| \quad \text{as } s \to 0.$$

Therefore, integrating (A.2) we obtain (A.1).

Lemma A.2. *Let Ω be a domain from problem* (1.1), *let Ω_0 be a subdomain of Ω: $\overline{\Omega}_0 \subset \Omega$. Then the following interpolation inequality holds*:

$$\left(\int_\Omega v^2(x)\,dx\right)^{1/2} \leqslant c_1\left(\int_\Omega |\nabla_x v|^2\,dx\right)^{1/2} + c_2\left(\int_{\Omega_0} |v|^\lambda\,dx\right)^{1/\lambda} \quad \forall v \in W_2^1(\Omega), \tag{A.4}$$

where λ: $1 < \lambda \leqslant 2$, positive constants c_1, c_2 does not depend on v.

We start from the standard interpolation inequality

$$\left(\int_\Omega v^2(x)\,dx\right)^{1/2} < c_1\left(\int_\Omega |\nabla_x v|^2\,dx\right)^{1/2} + c_2\left(\int_\Omega |v|^\lambda\,dx\right)^{1/\lambda} \quad \forall v \in W_2^1(\Omega). \tag{A.5}$$

It is clear that

$$\left(\int_\Omega |v|^\lambda\,dx\right)^{1/\lambda} \leqslant \left(\int_{\Omega_0} |v|^\lambda\,dx\right)^{1/\lambda} + \left(\int_{\Omega\setminus\Omega_0} |v|^\lambda\,dx\right)^{1/\lambda}. \tag{A.6}$$

Let Ω_0' be a subdomain of Ω_0 such that $\overline{\Omega}_0' \subset \Omega_0$. Let $\xi(x) \geqslant 0$ be C^1-smooth function such that

$$\xi(x) = 0 \quad \forall x \in \overline{\Omega}_0', \qquad \xi(x) = 1 \quad \forall x \in \Omega \setminus \Omega_0. \tag{A.7}$$

Then we have due to the Poincaré inequality

$$\begin{aligned}\left(\int_{\Omega\setminus\Omega_0} |v|^\lambda\,dx\right)^{1/\lambda} &\leqslant \left(\int_{\Omega\setminus\Omega_0'} |v\xi|^\lambda\,de\right)^{1/\lambda} \leqslant c\left(\int_{\Omega\setminus\Omega_0'} \left|\nabla_x(b\xi)\right|^\lambda\,dx\right)^{1/\lambda} \\ &\leqslant c\left(\int_{\Omega\setminus\Omega_0'} |\nabla v|^\lambda\,dx\right)^{1/\lambda} + c\left(\int_{\Omega_0\setminus\Omega_0'} |\nabla\xi|^\lambda |v|^\lambda\,dx\right)^{1/\lambda} \\ &\leqslant c_1\left(\int_{\Omega\setminus\Omega_0'} |\nabla v|^2\,dx\right)^{1/2} + c_2\left(\int_{\Omega_0\setminus\Omega_0'} |v|^\lambda\,dx\right)^{1/\lambda}. \end{aligned} \tag{A.8}$$

From (A.5) due to (A.6)–(A.8) one obtains (A.4). Lemma A.2 is proved.

Lemma A.3. *Let $u(x,t)$ be an arbitrary energy solution of problem* (1.1). *Then $H(t) = \int_\Omega |u(x,t)|^2\,dx \to 0$ as $t \to \infty$.*

It is clear that there exists a constant $a_0 > 0$ and a subdomain $\Omega_0 \subset \Omega$ such that $a(x) \geqslant a_0 > 0$ for all $x \in \overline{\Omega}_0$. From (2.54) it follows that

$$\frac{d}{dt}\int_\Omega \left|u(x,t)\right|^2 dx + \int_\Omega \left|\nabla_x u(x,t)\right|^2 dx + a_0\int_{\Omega_0} \left|u(x,t)\right|^{q+1} dx \leqslant 0. \tag{A.9}$$

Due to Lemma A.2 we have

$$\varepsilon\int_\Omega |u|^2\,dx \leqslant \varepsilon c_1\int_\Omega |\nabla_x u|^2\,dx + \varepsilon c_2\left(\int_{\Omega_0} |u|^{q+1}\,dx\right)^{\frac{2}{1+q}} \quad \forall \varepsilon > 0. \tag{A.10}$$

Adding (A.9) and (A.10) we get

$$\frac{d}{dt}\int_{\Omega}\left|u(x,t)\right|^2 dx+\varepsilon\int_{\Omega}\left|u(x,t)\right|^2 dx+(1-\varepsilon c_1)\int_{\Omega}|\nabla_x u|^2 dx$$
$$+a_0\int_{\Omega_0}|u|^{q+1}dx-c_2\varepsilon\left(\int_{\Omega_0}|u|^{q+1\,dx}\right)^{\frac{2}{q+1}}\leqslant 0. \tag{A.11}$$

From (A.9) it follows that

$$\int_{\Omega}\left|u(x,t)\right|^{1+q}dx\leqslant(\operatorname{mes}\Omega)^{\frac{1-q}{2}}\left(\int_{\Omega}\left|u(x,t)\right|^2dx\right)^{\frac{q+1}{2}}$$
$$\leqslant(\operatorname{mes}\Omega)^{\frac{1-q}{2}}\left(\int_{\Omega}\left|u_0(x)\right|^2dx\right)^{\frac{q+1}{2}}$$
$$=(\operatorname{mes}\Omega)^{\frac{1-q}{2}}y_0^{\frac{q+1}{2}}=\widetilde{C}=\text{const}\quad\forall t>0. \tag{A.12}$$

Now due to (A.12) we have

$$a_0\int_{\Omega_0}\left|u(x,t)\right|^{q+1}dx-c_2\varepsilon\left(\int_{\Omega_0}\left|u(x,t)\right|^{q+1}dx\right)^{\frac{2}{q+1}}$$
$$=\int_{\Omega}\left|u(x,t)\right|^{q+1}dx\left(a_0-c_2\varepsilon\left(\int_{\Omega_0}\left|u(x,t)\right|^{q+1}dx\right)^{\frac{1-q}{1+q}}\right)$$
$$\geqslant\int_{\Omega}\left|u(x,t)\right|^{q+1}dx\left(a_0-c_2\varepsilon\widetilde{C}^{\frac{1-q}{1+q}}\right)\geqslant 0 \tag{A.13}$$

if ε is small enough, namely,

$$\varepsilon\leqslant\frac{a_0}{c_2\widetilde{C}^{\frac{1-q}{1+q}}}. \tag{A.14}$$

Thus, if ε satisfies (A.14), then from (A.11) it follows that

$$\frac{d}{dt}\int_{\Omega}\left|u(x,t)\right|^2dx+\varepsilon\int_{\Omega}\left|u(x,t)\right|^2dx\leqslant 0\quad\forall t>0.$$

The last inequality implies the assertion of Lemma A.3.

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