On multi-index assignment polytopes

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Abstract

We investigate an integer programming model for multi-dimensional assignment problems. This model enables us to establish the dimension for entire families of assignment polytopes, thus unifying and generalising previous results. In particular, we establish the dimension of the linear assignment polytope as well as that of every axial and planar assignment polytope. Further, for the axial polytopes, we identify a family of clique facets. We also give a necessary condition for the existence of a solution for assignment problems.

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1. Introduction

Assignment structures are embedded in numerous combinatorial optimisation problems. An assignment occurs whenever a member of an entity must be allocated/mapped to a member of another entity. The simplest case of an assignment problem is the well-known 2-index assignment, which is equivalent to weighted bipartite matching. Various applications can be found in [9,23].

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Extensions of the assignment structure to \( k \) entities give rise to multi-index (or multi-dimensional) assignment problems, formally introduced in \[25,26\]. These problems essentially ask for a minimum weight clique partition in the complete \( k \)-partite hypergraph (see also \[7\]). The objective function is defined as the weighted sum of the variables, a fact that justifies the alternative term linear sum assignment problems \[9\]. A \( k \)-index assignment problem is defined on \( k \) sets, usually (but not always) assumed to be of the same cardinality \( n \). The goal is to identify a minimum weight collection of \( n \) disjoint \( k \)-tuples, each including a single element from each set. This is the class of axial assignment problems \[4,34\], hereafter referred to as \((k, 1)AP_n\).

A different structure appears, if the aim is, instead, to identify a collection of \( n^2 \) \( k \)-tuples, partitioned into \( n \) disjoint sets of \( n \) disjoint tuples. By way of illustration, consider the problem of allocating \( n \) teachers to \( n \) student groups for sessions in one of \( n \) classrooms and using one of \( n \) laboratory facilities (i.e. \( n^2 \) quadruples), in such a way that all teachers teach all groups, using each time a different classroom and/or a different facility (for a relevant case, see \[15\]). These assignment problems are called planar and are directly linked to Mutually Orthogonal Latin Squares (MOLS) \[20\]. We denote the \( k \)-index planar assignment problems as \((k, 2)AP_n\). Generalising this concept, we could ask for a minimum weight set of \( n^s \) \( k \)-tuples \((s \leq k)\) thus defining the \((k, s)\) assignment problem of order \( n \), denoted as \((k, s)AP_n\). Thorough reviews on assignment problems appear in \[9,34\], where complexity and approximability issues are covered (see also \[10,17\]). Algorithms for certain cases of \((k, s)AP_n\) can be found in \[5,7,21,28\].

Apart from its theoretical significance, the \((k, s)AP_n\) possesses interesting applications. Multi-dimensional (axial) assignment structures have recently received substantial attention, because of their applicability in problems of data association. Such problems arise naturally in multi-target/multi-sensor tracking in satellite surveillance systems \[27\]. An application to the problem of tracking elementary particles at the large electron-positron collider of CERN is reported in \[30\]. The planar problems share the diverse application fields of MOLS \[20\], e.g. affine designs, multivariate design, \((t, m, s)\)-nets, graph factorisation, etc.

In this paper, we provide an integer programming (IP) formulation of the \((k, s)AP_n\), thus proposing a framework which enables the polyhedral study of entire families of assignment problems. The paper is organised as follows. In Section 2, we introduce the formulation of the \((k, s)AP_n\) and discuss related structures. The dimension of the linear \((k, s)\) assignment polytope (see Section 2 for definitions) is established in Section 3. In the same section, we also examine a number of properties of the convex hull of integer vectors and provide a necessary condition for the existence of a solution to the \((k, s)AP_n\). The dimension of the axial assignment polytopes is established in Section 4, where a family of clique facets is also exhibited. Finally, Section 5 establishes the dimension of the planar assignment polytopes.

2. The \((k, s)\) assignment problem

2.1. Mathematical formulation

The \((k, s)AP_n\) is a proper generalisation of the (two-index) assignment problem. In the two-index case, the assignment involves the elements of two disjoint \(n\)-sets in a way that each element appears exactly once at a given solution. In the \((k, s)AP_n\), the assignment involves the elements of \(k\) disjoint \(n\)-sets such that each \(s\)-tuple of elements, each from a different set, appears exactly once at any given solution. Hence, the problem consists of \(k\)-indexed \(0\)–\(1\) variables and equality constraints each of which has \(s\) indices fixed to specific values and \(k-s\) indices to be summed over
all the values of their domains. The right-hand side for each constraint is equal to one. Next, we introduce some formal definitions.

Let $K$ denote a set of $k$ indices defined as $K = \{1, \ldots, k\}$. Let $S \subseteq K$ denote a subset of $s$ indices (out of $k$). If $s = 0$ or $s = k$ then $S = \emptyset$, $S = K$, respectively. Let $Q_{k,s}$ denote the collection of all such distinct $S$ (i.e. $Q_{k,s} = \{S \subseteq K : |S| = s\}$). Clearly, $|Q_{k,s}| = \binom{k}{s}$. Consider $k$ disjoint $n$-sets $M_1, M_2, \ldots, M_k$ and let $m^i \in M_i$, for $i \in K$. For $S \subseteq K$, assume $S = \{i_1, i_2, \ldots, i_s\}$ such that $i_1 < i_2 < \cdots < i_s$. Let $M^S = M_{i_1} \times M_{i_2} \times \cdots \times M_{i_s}$ and $m^S \in M^S$, where $m^S = (m^{i_1}, m^{i_2}, \ldots, m^{i_s})$. Observe that, for $S' \subseteq S \subseteq K$, $m^{S'}$ defines a subset of indices of the tuple $m^S$, i.e. $m^{S'} \subset m^S$. Further, we follow the convention that $m^0_1, m^1_1, \ldots$ denote the elements of the set $M_i$, i.e. the values of the index $m^i$. In an analogous manner, $m^S_0$ denotes the specific $s$-tuple $(m^0_1, \ldots, m^0_s)$. For the mathematical programming formulation of $(k, s)AP_n$, consider binary variables $x_{m^K}$ and the mapping $w : M^K \rightarrow \mathbb{R}$. The problem is formulated as follows:

$$\min \sum \{w_{m^K} \cdot x_{m^K} : m^K \in M^K\},$$

s.t. \hspace{1cm} $$\sum \{x_{m^K} : m^K \in M^K \cap S\} = 1 \quad \forall m^S \in M^S, \ S \in Q_{k,s},$$

$$x_{m^K} \in \{0, 1\}^{n^K}, \quad \forall m^K \in M^K.$$ (1) (2) (3)

Note that there are exactly $s$ “fixed” indices in each constraint. $M^K \setminus S$ is the set of indices appearing in the sum, whereas $M^S$ is the set of indices common to all variables in an equality constraint. Let $A^{(k,s)}_n$ denote the $(0, 1)$ matrix of the constraints (2). The matrix $A^{(k,s)}_n$ has $n^k$ columns and $\binom{k}{s} \cdot n^s$ rows, i.e. $n^s$ constraints for each of the $\binom{k}{s}$ distinct $S \in Q_{k,s}$. Each constraint involves $n^{k-s}$ variables. Analogous formulations appear in [32–34].

Under these definitions, it is obvious that $(2,1)AP_n$ refers to the 2-index assignment problem, $(3,1)AP_n$ to the 3-index axial assignment problem [4,13], $(3,2)AP_n$ to the 3-index planar assignment problem [14,21], and $(4,2)AP_n$ to the 4-index planar assignment problem [2]. Note that parameter $s$ is central to the type of assignment required for each problem, i.e. the axial problems imply $s = 1$ and the planar problems imply $s = 2$.

It is easy to see that multi-index variants of problems closely related to the (two-index) assignment problem can be modelled in a similar manner. Two such problems are the generalised assignment problem (GAP) [23] and the transportation problem. The multi-index version of GAP is defined for $s = 1$ (see [16,18] for an application having $k = 3$). For the latter, we refer to the solid (multi-index) transportation problem, introduced in [19].

Next, we take a closer look to the polytope defined by constraints (2) and (3).

2.2. Assignment polytopes and related structures

For definitions of polyhedral theory see [29]. The convex hull of the integer points satisfying the constraints (2) is the $(k, s)$ assignment polytope, denoted as $P^{(k,s)}_{n; I}$. Formally, $P^{(k,s)}_{n; I} = \text{conv}\{x \in [0, 1]^n : A^{(k,s)}_n x = e\}$, where $e$ is a column vector of ones. The linear relaxation of $P^{(k,s)}_{n; I}$, also called the linear $(k, s)$ assignment polytope, is the polytope $P^{(k,s)}_n = \{x \in \mathbb{R}^n : A^{(k,s)}_n x = e, x \geq 0\}$. Obviously, $P^{(k,s)}_{n; I} \subseteq P^{(k,s)}_n$. A generalisation of $P^{(k,s)}_n$ is obtained if we consider a real vector, namely $b$, as the right-hand side of the equality system (2) and $|M_i| = n_i, i \in K$, where all $n_i$ are
not necessarily of the same size. The resulting polytope is the multi-index transportation polytope defined as \( P(k, s) = \{ x \in \mathbb{R}^{\prod_{i \in K} n_i} : A_{n_1, \ldots, n_k} x = b, x \geq 0 \} \), where \( A_{n_1, \ldots, n_k} \) is the constraint matrix and \( b \) is a column vector with \( \sum \{ \prod n_i : i \in S \} : S \in Q_{k, s} \) entries \([36]\). The interest on the transportation polytope has been increasing, in the light of the recent discovery that any polytope is equivalent to some transportation polytope defined for \( k = 3, s = 2 \) \([12]\).

Clearly, the assignment polytope is a special case of the set-partitioning polytope defined as \( P_{SP} = \{ y \in \{0, 1\}^q : By = e \} \), where \( B \) is a 0–1 matrix. A close relative of \( P_{SP} \) is the set-packing polytope \( \tilde{P}_{SP} \), defined exactly as \( P_{SP} \) but with \( '=' \) replaced by \( '\leq' \). In our case, \( \tilde{P}_{n; I}^{(k, s)} = \text{conv}\{ x \in \{0, 1\}^{n_k} : A_{n_k}^{(k, s)} x \leq e \} \). A relation, inherited from the general case, is that \( P_{n; I}^{(k, s)} \) is a face of \( \tilde{P}_{n; I}^{(k, s)} \), implying \( \dim P_{n; I}^{(k, s)} \leq \dim \tilde{P}_{n; I}^{(k, s)} \). Polytope \( \tilde{P}_{n; I}^{(k, s)} \) is full dimensional, hence \( \dim \tilde{P}_{n; I}^{(k, s)} = n^k \), i.e. its dimension is independent of \( s \).

Two special cases of \( P_{n; I}^{(k, s)} \) arise for \( s = k \) and \( s = 0 \). For \( s = k \), constraints (2) reduce to the system of trivial equalities

\[
x_{m_1^1 m_2^2 \ldots m_k^k} = 1 \quad \forall (m_1^1, m_2^2, \ldots, m_k^k) \in M^K,
\]

whereas, for \( s = 0 \), constraints (2) result in the single equality constraint

\[
\sum\{ x_{m_1^1 m_2^2 \ldots m_k^k} : (m_1^1, m_2^2, \ldots, m_k^k) \in M^K \} = 1.
\]

The proofs of the following Lemmas are easy to devise.

**Lemma 1.** \( \text{rank} A_{n}^{(k, k)} = n^k \) and \( \dim P_{n; I}^{(k, k)} = \dim P_{n; I}^{(k, k)} = 0. \)

**Lemma 2.** \( \text{rank} A_{n}^{(k, 0)} = 1 \) and \( \dim P_{n; I}^{(k, 0)} = \dim P_{n; I}^{(k, 0)} = n^k - 1. \)

**Corollary 3.** \( P_{n; I}^{(k, 0)} \) is a facet of \( \tilde{P}_{n; I}^{(k, 0)}. \)

3. The \((k, s)\) assignment polytope

In this section, we study polytope \( P_{n; I}^{(k, s)} \), i.e. the linear relaxation of \( P_{n; I}^{(k, s)} \), and obtain its dimension. We subsequently examine polytope \( P_{n; I}^{(k, s)} \) and provide a necessary condition for \( P_{n; I}^{(k, s)} \neq \emptyset. \)

Apart from properly generalising previously known results, establishing the dimension of \( P_{n; I}^{(k, s)} \) provides an upper bound on the dimension of \( P_{n; I}^{(k, s)} \). It is well known that \( \dim P_{n; I}^{(k, s)} = n^k - \text{rank} A_{n}^{(k, s)}. \) Thus, we first establish the rank of matrix \( A_{n}^{(k, s)}. \)

**Theorem 4.** \( \text{rank} A_{n}^{(k, s)} = \sum_{r=0}^{s} \binom{k}{r} (n - 1)^r. \)

**Proof.** Assume for simplicity that \( M_I = \{0, 1, \ldots, n - 1\} \), for all \( i \in K. \) For \( x, y \in M^K, \langle x, y \rangle \) denotes their inner product. Let \( \varphi \) be an \( n \)th root of unity and \( U \) be the \( |M^K| \times |M^K| \) matrix with entries \( U_{x, y} = \varphi^{\langle x, y \rangle}, \) i.e., the rows and columns of \( U \) are indexed by the tuples of \( M^K. \) We denote by \( U^* \) the conjugate transpose of \( U. \) Let \( D_{n}^{(k, s)} = U^* \cdot A_{n}^{(k, s)} \cdot A_{n}^{(k, s)} \cdot U. \) It is easy to see that \( U^* \cdot U = n^k \cdot I, \) where \( I \) denotes the \( |M^K| \times |M^K| \) identity matrix, and \( \text{rank} A_{n}^{(k, s)} = \text{rank} D_{n}^{(k, s)} \).
The entry lying at an arbitrary row \( \alpha \) and column \( \beta \) of \( D^{(k,s)}_n (\alpha, \beta \in M^K) \) can be written as
\[
(U^* \cdot A^{(k,s)}_n \cdot A^{(k,s)}_n \cdot U)_{\alpha,\beta} = \sum_{S \in Q_{k,s}} \sum_{x,y \in M^K, S \cap \text{support}(y-x) = \emptyset} \varphi^{-\langle \alpha,x \rangle} \cdot \varphi \langle \beta,y \rangle
\]
where \( \delta_{\alpha,\beta} = 1 \) if \( \alpha = \beta \) and 0 otherwise.

The last term is non-zero if and only if \( \alpha = \beta \) and \( \text{support}(\alpha) \) is contained in at least one set \( S \in Q_{k,s} \). Thus, \( D^{(k,s)}_n \) has precisely
\[
\sum_{r=0}^{s} \binom{k}{r} \cdot (n-1)^{k-r}
\]
non-zero entries in the main diagonal and zero entries elsewhere. The result follows. □

**Corollary 5.** \( \dim P^{(k,s)}_{n} = \sum_{r=0}^{k-s-1} \binom{k}{r} \cdot (n-1)^{k-r}. \)

In the more general setting of the multi-index transportation polytope \( P^{(k,s)}_{n_1,...,n_k} \) (see Section 2.2), Theorem 4 becomes
\[
\text{rank} \ A^{(k,s)}_{n_1;...;n_k} = \sum_{r=0}^{s} \sum_{R \in Q_{k,r}} \prod_{i \in R} (n_i - 1),
\]
where the product over the empty set is taken to be 1. As a result, \( \dim P^{(k,s)}_{n_1,...,n_k} \leq \prod_{i \in K} n_i - \text{rank} \ A^{(k,s)}_{n_1;...;n_k} \), equality holding if \( P^{(k,s)}_{n_1,...,n_k} \neq \emptyset \) and all the entries of \( b \) are different than 0. For the special case where \( s = 1 \), this result is also reported in [36, p. 364].

For polytope \( P^{(k,s)}_{n;I} \), the known values for \( \dim P^{(k,s)}_{n;I} \), together with the appropriate references, are illustrated in Table 1. It is easy to see that all these values coincide with the corresponding values of \( \dim P^{(k,s)}_n \) obtained from Corollary 5. For general \( (k, s) \), establishing the exact value of \( \dim P^{(k,s)}_n \) is not straightforward, one reason being that \( P^{(k,s)}_n = \emptyset \) for certain values of \( k, s, n \) (for example, consider the polytope for \( k = 4, s = 2, n = 2 \)). An obvious bound on \( \dim P^{(k,s)}_n \) is provided by Corollary 5 and the observation that \( P^{(k,s)}_{n;I} \subseteq P^{(k,s)}_n \) (i.e. \( \dim P^{(k,s)}_{n;I} \leq \sum_{r=0}^{k-s-1} \binom{k}{r} \cdot (n-1)^{k-r} \)). Next, we take a closer look at the integer points of \( P^{(k,s)}_{n;I} \).

**Remark 6.** At an integer point of \( P^{(k,s)}_{n;I} \), each element of any of the sets \( M_1, \ldots, M_k \) appears in exactly \( n^{s-1} \) variables set to one.

The above remark is useful in terms of providing a mechanism of transition from one integer point to another. Consider an arbitrary integer point \( x \in P^{(k,s)}_{n;I} \), and a pair of values \( m_0^1, m_1^1 \in \).
is \( k \leq n + s - 1. \)
Proof. First observe that, according to (2), variables appearing at the same row have at least \( s \) indices in common (common values for indices belonging to \( m^s \)).

Given that \( P_{n;I}^{(k,s)} \neq \emptyset \) and \( s \geq 2 \), consider the integer point \( x' \in P_{n;I}^{(k,s)} \) having \( x'_1 \cdots x'_{m^s_1 - m^s_2} = 1 \) such that \( x'_1 \cdots x'_m = \cdots = x'_1 \cdots x'_m = 1 \). Such a point always exists, since all rows \( 1, \ldots, 1, m^s_j \) \( j = 1, \ldots, n \), must have exactly one variable equal to 1. Also note that all indices \( m^s_j, j = 1, \ldots, n \), must be pairwise different since a constraint would otherwise have its left-hand side equal to 2. Consider the following sequence of conditional interchanges, whose aim is to set the first \( s - 1 \) indices equal to 1 and the indices \( s, \ldots, k \) to identical values, at each position of vector \( x' \).

\[
x^s = x'(m^s_1 \neq 1?m^s_1 \leftrightarrow 1)_s \cdots (m^s_n \neq n?m^s_n \leftrightarrow n)_s,
\]
\[
x^{s+1} = x^s(m^{s+1}_1 \neq 1?m^{s+1}_1 \leftrightarrow 1)_{s+1} \cdots (m^{s+1}_n \neq n?m^{s+1}_n \leftrightarrow n)_{s+1},
\]
\[
\vdots
\]
\[
x^k = x^{k-1}(m^k_1 \neq 1?m^k_1 \leftrightarrow 1)_k \cdots (m^k_n \neq n?m^k_n \leftrightarrow n)_k.
\]

Let \( x = x^k \). Vector \( x \) has indeed \( x_1 \cdots x_{s-1} = x_{s+1} = \cdots = x_{m^n} = 1 \). Point \( x \) must satisfy all constraints, including row \( 2, 1, \ldots, 1 \) \( I \). Consequently, there must be exactly one variable of the form \( x_{21} \cdots x_{m^{s+1}} \cdots x_k \) with value 1. The approach adopted is to show that the affine hull of \( P_{n;I}^{(k,s)} \) coincides with that of \( P_{n;I}^{(k,s)} \). Consider the inequality \( \dim P_{n;I}^{(k,s)} \leq \dim P_{n;I}^{(k,s)} \) and suppose that \( ax = a_0 \) for all \( x \in P_{n;I}^{(k,s)} \), where \( a \in \mathbb{R}^{nk}, a_0 \in \mathbb{R} \). Then, the above inequality is satisfied as equality if and only if \( ax = a_0 \) can be derived as a linear combination of the system \( A_{n;I}^{(k,s)} x = e \). More concisely, if any equation \( ax = a_0 \) is satisfied for all \( x \in P_{n;I}^{(k,s)} \), is a linear combination of the equations of the system \( A_{n;I}^{(k,s)} x = e \), then this is equivalent to \( \dim P_{n;I}^{(k,s)} = \dim P_{n;I}^{(k,s)} \) (see [29]). This must be proved by using exclusively points of \( P_{n;I}^{(k,s)} \). In the proofs that follow, we often make use of the equation \( ay = az \), for \( y, z \in P_{n;I}^{(k,s)} \). This is valid for any pair of points of \( P_{n;I}^{(k,s)} \), since, by assumption, they both satisfy \( ax = a_0 \).

4. The axial assignment polytopes

The two most prominent representatives of this class are the polytopes \( P_{n;I}^{(2,1)} \) and \( P_{n;I}^{(3,1)} \). The basic properties of \( P_{n;I}^{(2,1)} \) can be found in [6,8,9]. The facial structure of \( P_{n;I}^{(3,1)} \) has also been studied substantially. As mentioned earlier, the dimension of this polytope is established, independently,
in [4,13]. Several classes of facets are identified in [4,31]. Separation algorithms for some of these classes are given in [3].

To the best of our knowledge, (2,1) $AP_n$ and (3,1) $AP_n$ are the only axial assignment problems whose underlying polyhedral structure has been studied. However, several applications of axial assignment problems for $k > 3$, have been reported (see [9,26]). This suggests that the study of $P_{n;1}^{(k,1)}$, for general $k$, is of both practical and theoretical interest. We have already mentioned that $P_{n;1}^{(k,1)} \neq \emptyset$, for $k, n \in \mathbb{Z}_+$. Furthermore, it can be proved, by induction on $n$, that the number of integer points of $P_{n;1}^{(k,1)}$ is equal to $(n!)^{k-1}$ (see also [36, Corollary 3.6]). Next, we establish the dimension of $P_{n;1}^{(k,1)}$, thus unifying and generalising the corresponding results for $P_{n;1}^{(2,1)}$ and $P_{n;1}^{(3,1)}$.

**Theorem 9.** For $n \geq 3$, $\dim P_{n;1}^{(k,1)} = \dim P_{n;1}^{(k,1)} = \sum_{r=0}^{k-2} \binom{k}{r} \cdot (n-1)^{k-r}$.

**Proof.** We must show that there exist scalars $\lambda_{m^1}^1, \lambda_{m^{k-1}}^2, \ldots, \lambda_{m^1}^k$, for all $m^1 \in M_1, \ldots, m^k \in M_k$, satisfying

$$a_{m^1m^2\ldots m^k} = \lambda_{m^1}^1 + \lambda_{m^{k-1}}^2 + \ldots + \lambda_{m^1}^k,$$  

(6)

$$a_0 = \sum_{i=1}^{k} \sum \{ \lambda_{m^{k-i+1}}^i : m^{k-i+1} \in M_{k-i+1} \}. $$

(7)

We define: $\lambda_{m^1}^1 = a_{1\ldots 1m^k}, \lambda_{m^{k-1}}^2 = a_{1\ldots 1m^{k-1}} - a_{1\ldots 1}, \ldots, \lambda_{m^1}^k = a_{m^1\ldots 1} - a_{1\ldots 1}$. By substituting $\lambda$s in (6) we obtain

$$a_{m^1m^2\ldots m^k} = a_{1\ldots 1m^k} + \ldots + a_{m^1\ldots 1} - (k-1)a_{1\ldots 1}. $$

(8)

For $v \in M^K$, let $T(v)$ denote the set of indices of $v$ with value different than one. Obviously $T(v) \subseteq K$thus $|T(v)| = \ell \leq k$. Observe that for $t \leq 1$, $v$ has at least $k-1$ indices equal to one, in which case (8) is true by definition. It remains to show (8) for $2 \leq t \leq k$. The proof is based on points derived from $x_{\text{diag}}$ by applying a series of interchanges. By Remark 7, all these points belong to $P_{n;1}^{(k,1)}$.

Assume that $T(v) = \{1, \ldots, t\}$, all other cases being symmetrical. For $n \geq 3$, let $1, m_0^1, m_1^1$ be three distinct elements of $M_i$, for all $i \in K$. Let $w = (m_w^1, \ldots, m_w^k) \neq (1, \ldots, 1)$, such that $x_{\text{diag}} = 1$. Then for $2 \leq t \leq k$ we consider the point

$$x^t = x_{\text{diag}} \quad (m_w \neq m_1^1 \leftrightarrow m_1^1) \ldots (m_w \neq m_t^{-1} \leftrightarrow m_t^{-1}) \leftrightarrow m_1^{t-1} \leftrightarrow m_t^{-1}).$$

Also let $\hat{x}^t = x^t(1 \leftrightarrow m_0^1)$. The points $x^t, \hat{x}^t$ differ in exactly two variables set to one (Remark 7). Hence, after cancelling out identical terms, $ax^t = a\hat{x}^t$ becomes

$$a_{1\ldots 1} + a_{m_1^1\ldots m_t^{-1}m_0^0\ldots m_0^0} = a_{1\ldots 1m_0^1\ldots 1} + a_{m_1^1\ldots m_t^{-1}m_0^0\ldots m_0^0}. $$

(9)

Consider points $\tilde{x}^t = \hat{x}^t(1 \leftrightarrow m_0^1) \ldots (1 \leftrightarrow m_0^t) \leftrightarrow m_1^{t-1})$ and $\tilde{x}^t = \hat{x}^t(1 \leftrightarrow m_0^t)$. Hence, after cancelling out identical terms, $ax_t = a\tilde{x}^t$ yields

$$a_{m_0^1\ldots m_0^t-1m_1^1\ldots 1} + a_{m_1^1\ldots m_t^{-1}m_0^0\ldots m_0^0} = a_{m_0^1\ldots m_0^t-1\ldots 1} + a_{m_1^1\ldots m_t^{-1}m_0^0\ldots m_0^0}. $$

(10)
Adding Eqs. (9) and (10) results in
\[ a_{m_1 \cdots m_n} = a_{m_1 \cdots m_{i-1} 1} + a_{1 \cdots m_n} - a_{1 \cdots 1}. \]  
\[ (11) \]

Observe that (11) defines a recurrence relation with respect to \( t \). Thus, starting for \( t = k \) and recursively substituting term \( a_{m_1 \cdots m_n} \) until \( t = 2 \), we obtain Eq. (8) for \( m_i = m_0 \in M_i \setminus \{1\} \), \( i \in K \). To prove (8) for a \( k \)-tuple with \( q \) \( (1 \leq q < k \leq k - 2) \) indices equal to one, we perform the recursive step starting from \( t = k - q \). This proves Eq. (6).

To prove Eq. (7), consider \( ax\text{diag} = a_0 \) implying \( a_0 = a_{1 \cdots 1} + a_{2 \cdots 2} + \cdots + a_{n \cdots n} \) and observe that
\[ a_{1 \cdots 1} + a_{2 \cdots 2} + \cdots + a_{n \cdots n} = \sum_{i=1}^{k} \sum_{t=1}^{k} \{ \lambda_{m_{i-t+1}}^i : m_{i-t+1}^k \in M_{k-t+1} \}. \]
This completes the proof. \( \Box \)

Let us examine which of the faces induced by the constraints of \( P_{n;1}^{(k,1)} \) are facets of \( P_{n;1}^{(k,1)} \). Constraints (2) are satisfied by all points of \( P_{n;1}^{(k,1)} \), therefore defining improper faces of \( P_{n;1}^{(k,1)} \). For the trivial inequalities \( 0 \leq x_{m^k} \leq 1 \), we provide the following propositions.

**Proposition 10.** For \( m^k \in M_k \), \( x_{m^k} \geq 0 \) defines a facet of \( P_{n;1}^{(k,1)} \).

**Proposition 11.** For \( m^k \in M_k \), \( x_{m^k} \leq 1 \) does not define a facet of \( P_{n;1}^{(k,1)} \).

Next, we establish a class of facets for \( P_{n;1}^{(k,1)} \) induced by cliques of the intersection graph of \( A_{n}^{(k,1)} \). We briefly introduce some definitions. Let \( C \) denote the index set of columns of the \( 0 \)-\( 1 \) matrix \( A \). We refer to a column of the \( A \) matrix as \( \alpha^c \) for \( c \in C \). The intersection graph \( G_A(C, E) \) has a node \( c \in C \) for every \( \alpha^c \in A \) and an edge \( (c, d) \in E \) for every pair of nodes with \( \alpha^c \cdot \alpha^d \geq 1 \). Let \( G_A(C^{(k,1)}, E^{(k,1)}) \) denote the intersection graph of \( A_{n}^{(k,1)} \). Then, \( C^{(k,1)} = M_K \) and \( (c, d) \in E^{(k,1)} \) for all \( c, d \in C^{(k,1)} \) with \( |c \cap d| \geq 1 \). By definition, \( c \in C^{(k,1)} \) refers to the \( k \)-tuple \( (m_1^c, \ldots, m_k^c) \in M^K \). Hence, the variable \( x_c \) can be denoted as \( x_{m_1^c \cdots m_k^c} \).

**Proposition 12.** \( G_A(C^{(k,1)}, E^{(k,1)}) \) is regular of degree \( \sum_{i=1}^{k-1} \binom{k}{i} (n - 1)^{k-t} \).

For the rest of the section, assume \( k \geq 3 \). For \( c \in C^{(k,1)} \), we define \( Q^*(c) = \{ d \in C^{(k,1)} : k - 1 \geq |c \cap d| \geq \left\lfloor \frac{k}{2} \right\rfloor + 1 \} \) and \( Q^1(c) = Q^*(c) \cup \{ c \} \). Further, for \( c \in C^{(k,1)} \) and \( S \subset K \), let \( C^{(k,1)}_S(c) = \{ d \in C^{(k,1)} : \text{support}(c - d) \cap S = \emptyset \} \). It is easy to verify that \( |d \cap h| = 0 \), for all \( d \in C^{(k,1)}_S(c), h \in C^{(k,1)}_K \setminus S(c) \). The following definitions apply to the case of \( k \) being even. Let \( G = \{ S \subset K : |S| = \left\lfloor \frac{k}{2} \right\rfloor \} \), \( |G| = \binom{k}{\left\lfloor \frac{k}{2} \right\rfloor} \). Observe that, for \( k \) even, \( S \in G \) if and only if \( K \setminus S \in G \).

Therefore, the set \( G \) can be partitioned into sets \( G^+ \), \( G^- \), such that \( G = G^+ \cup G^- \) and \( S \in G^+ \) if and only if \( K \setminus S \in G^- \). There are \( 2^{\left\lfloor \frac{k}{2} \right\rfloor} \) such partitions. Finally, let \( Q^2_{G^+}(c) = \bigcup_{S \in G^+} C^{(k,1)}_S(c) \).

The node set \( Q^2_{G^-}(c) \) is defined analogously. We define
Theorem 14. As an example, consider $k = 4$ and $c = (n, n, n, n)$. We have

$$Q(c) = \begin{cases} Q_1^1(c), & \text{if } k \text{ odd,} \\ Q_1^1(c) \cup Q_{G+}^2(c), & \text{if } k \text{ even.} \end{cases}$$

whereas there are $2^{\left\lfloor \frac{2k}{3} \right\rfloor}$ choices for $Q_{G+}^2(c)$. Consider, for example,

$$Q_{G+}^2(c) = \bigcup_{(m^3, m^4) \in M_3 \times M_4} \{(n, n, m^3, m^4)\} \cup \bigcup_{(m^1, m^4) \in M_1 \times M_4 \setminus n} \{(m^1, n, n, m^4)\} \cup \bigcup_{(m^2, m^4) \in M_2 \times M_4 \setminus n} \{(n, m^2, n, m^4)\}.$$

Using simple counting arguments, it is not difficult to show that

$$|Q(c)| = \sum_{t=\left\lceil \frac{k}{2} \right\rceil + 1}^{k} {\binom{k}{t}} \cdot (n - 1)^{k-t} + (1 - (k \mod 2)) \cdot \frac{1}{2} \cdot {\binom{k}{\frac{k}{2}}} \cdot (n - 1)^{\frac{k}{2}}.$$

A clique is defined as a maximal complete subgraph.

Proposition 13. For each $c \in C^{(k,1)}$, the node set $Q(c)$ induces a clique. There are

$$n^k \cdot \left((k \mod 2) + (1 - (k \mod 2)) \cdot 2^{\left\lfloor \frac{|c|}{2} \right\rfloor}\right)$$

cliques of this type.

Proof. Let $c_1, c_2 \in Q(c)$. If they both belong to $Q_1^1(c)$ then they both have at least $\left\lceil \frac{k}{2} \right\rceil + 1$ indices in common with $c$. Hence, $|c_1 \cap c_2| \geq 1$. If one of them, say $c_2$, belongs to $Q_{G+}^2(c)$ then this uniquely implies that $k$ is even. If $c_1 \in Q_1^1(c)$ then again $|c_1 \cap c_2| \geq 1$, since $|c \cap c_1| \geq \frac{k}{2} + 1$, $|c \cap c_2| = \frac{k}{2}$. Otherwise, $|c_1 \cap c_2| \geq 1$ by definition, since $c_1$ belongs to $Q_{G+}^2(c)$ also. In all cases $(c_1, c_2) \in E^{(k,1)}$.

To show that $Q(c)$ is maximal, consider $c_3 \in C^{(k,1)} \setminus Q(c)$. This implies $|c \cap c_3| \leq \left\lceil \frac{k}{2} \right\rceil - 1$ or $|c \cap c_3| = \left\lceil \frac{k}{2} \right\rceil$. The first case can occur for all values of $k$, whereas the second only if $k$ is even. In the first case, there exists an element $c_4 \in Q_1^1(c)$ with $|c_3 \cap c_4| = 0$. In the second case, $c_3 \notin Q_{G+}^2(c)$, therefore there exists at least one element $c_4 \in Q_{G+}^2(c)$ such that $|c_3 \cap c_4| = 0$. In both cases, $(c_3, c_4) \notin E^{(k,1)}$ implying that the subgraph $Q(c) \cup \{c_3\}$ is not complete.

For each $c \in C^{(k,1)}$ the set $Q_1^1(c)$ is uniquely defined, i.e., there are $n^k$ distinct such sets. For $k$ even, there are $2^{\left\lceil \frac{|c|}{2} \right\rceil}$ sets $Q_{G+}^2(c)$ for each $c$. Hence, the number of cliques of this type is $n^k$ for $k$ odd and $n^k \cdot 2^{\left\lceil \frac{|c|}{2} \right\rceil}$ for $k$ even. □

Theorem 14. For $k \geq 3$, $n \geq 5$, the inequality

$$\sum_{q \in Q(c)} |x_q| \leq 1$$

defines a facet of $P_{n,1}^{(k,1)}$, for every $c \in C^{(k,1)}$. 

(13)
Proof. Let $F$ denote the face of $P_{n:1}^{(k,1)}$ induced by (13). To show that $F$ is a facet we use an analogous approach to that used in the proof of Theorem 9. Hence, we must prove that if $ax = a_0$ for all $x \in F$, then $ax = a_0$ can be derived as a linear combination of $\Lambda_{n}^{(k,1)} \lambda = e$ and (13). Additionally, we must show that (i) (13) is valid for every point of $P_{n:1}^{(k,1)}$, and (ii) $F \neq \emptyset$, $F \subset P_{n:1}^{(k,1)}$ (see [29, Theorem 3.16]).

To see that (13) is valid, observe that it is induced by a clique of $G_A(C^{(k,1)}, E^{(k,1)})$. Thus, it defines a facet of $\tilde{P}_{n:1}^{(k,1)}$ [24]. Since $P_{n:1}^{(k,1)} \subset \tilde{P}_{n:1}^{(k,1)}$, it follows that (13) is also valid for all $x \in P_{n:1}^{(k,1)}$.

We define $F = P_{n:1}^{(k,1)}(Q(c)) = \{x \in P_{n:1}^{(k,1)} : \sum \{x_q : q \in Q(c)\} = 1\}$. To show that $P_{n:1}^{(k,1)}(Q(c)) \neq \emptyset$, consider the point

$$x = x_{\text{diag}}(m^1_c \neq 1?m^1_c \leftrightarrow 1) \cdot \ldots \cdot (m^k_c \neq 1?m^k_c \leftrightarrow 1)k.$$ 

Because $x_{1 \ldots 1} = 1$, also $x_c = 1$ implying $x \in P_{n:1}^{(k,1)}(Q(c))$. On the opposite side now, let $x$ be an integer point of $P_{n:1}^{(k,1)}(Q(c))$ with $x_c = 1$. We can safely assume that if $k$ is even $(m^1_c, \ldots, m^k_c, m^{k+1}_2, \ldots, m^k) \notin Q^2_{G+}(c)$, the opposite case being symmetrical. At point $x$, let $x_u, x_w$ be two variables set to one other than $x_c$. Let

$$\hat{x} = x(m^1_u \leftrightarrow m^1_u) \cdot \ldots \cdot (m^k_u \leftrightarrow m^k_u)(m^1_w \leftrightarrow m^1_w) \cdot \ldots \cdot (m^k_w \leftrightarrow m^k_w).$$ 

Obviously $\hat{x} \in P_{n:1}^{(k,1)} \setminus P_{n:1}^{(k,1)}(Q(c))$; if $k$ is odd then there are two variables set to one, each with a distinct set of $\lfloor k/2 \rfloor$ indices from $c$ and one variable with the remaining index from $c$. If $k$ is even then the variables set to one having indices from the tuple $c$ are $\hat{x}_{m^1_u \ldots m^k_u/m^k_w \ldots m^k_w}, \hat{x}_{m^1_u \ldots m^k_w/m^k_u \ldots m^k_u}, \hat{x}_{m^1_u \ldots m^k_u/m^k_u \ldots m^k_w}, \hat{x}_{m^1_u \ldots m^k_w/m^k_w \ldots m^k_u}$. Observe that no variable set to one is indexed by a tuple from $Q(c)$. Therefore, $P_{n:1}^{(k,1)} \setminus P_{n:1}^{(k,1)}(Q(c)) \neq \emptyset$.

Without loss of generality, assume that $c = c_n = (n, n, \ldots, n)$. To show that $P_{n:1}^{(k,1)}(Q(c_n))$ is a facet of $P_{n:1}^{(k,1)}$, we consider the scalars $\lambda_k$ defined in the proof of Theorem 9 and an additional scalar $\pi$ for the clique inequality. If $ax = a_0$ for all $x \in P_{n:1}^{(k,1)}$, we must prove

(a) Eq. (6) for every $(m^1, m^2, \ldots, m^k) \in C^{(k,1)} \setminus Q(c_n)$,

(b) Eq. (6)$\pi$: $a_{m^1m^2 \ldots m^k} = \lambda_{m^1}^1 + \ldots + \lambda_{m^k}^k + \pi$, for every $(m^1, m^2, \ldots, m^k) \in Q(c_n)$,

(c) Eq. (7)$\pi$ which is defined from (7) by adding the scalar $\pi$ to its right-hand side.

For $v = (m^1_v, \ldots, m^k_v) \in C^{(k,1)}$, let $T(v)$ be defined as in the proof of Theorem 9. Let $R(v)$ denote the set of indices of $v$ with value equal to $n$, i.e. $|R(v)| = r = |v \cap c_n|$. By definition, $R(v) \subset T(v)$ implying $r \leq t$. For $v$ to belong to $C^{(k,1)} \setminus Q(c_n)$, we must have $0 \leq r \leq \lfloor k/2 \rfloor$. If $k$ is even, without loss of generality, we consider $(m^1, \ldots, m^k, n, \ldots, n) \in Q^2_{G+}(c_n)$, for all $m^i \in M_i \setminus \{n\}$, $i = 1, \ldots, k/2$. Now, we can safely assume that if $R(v) \neq \emptyset$ then $R(v) = \{1, \ldots, r\}$, all other cases being symmetrical. Under these assumptions, the tuple $v$ has (i) $m^i_v = n$, for $i \in R(v)$, (ii) $m^i_v \in M_i \setminus \{n\}$, for $i \in T(v) \setminus R(v)$ and (iii) $m^i_v = 1$, for $i \in K \setminus T(v)$. Then, by replacing, for every $i \in \{1, \ldots, t\}$, the index-value $m^0_i$ by $m^i_v(m^i_v \neq m^i_1)$ in the proof of Theorem 9, we show (6) for $v \in C^{(k,1)} \setminus Q(c_n)$. 
Observe that, at all points \( x', \tilde{x}', \tilde{x} \), there exists one variable set to one indexed by a tuple from the set \( Q(c_n) \); if \( k \) is odd then this variable has at least \( \lfloor \frac{k}{2} \rfloor + 1 \) indices equal to \( n \), whereas if \( k \) is even it has at least the last \( k/2 \) indices equal to \( n \). This is because \( x_{n...n} \) is even and the rest from (6), we obtain to derive these points affect only the first \( r \) indices of this variable (\( r \leq \lfloor \frac{k}{2} \rfloor \)). Hence \( x', \tilde{x}', \tilde{x} \in P^{(k,1)}_{n;\ell}(Q(c_n)) \). Specifically, for \( r = 0 \), \( x_{n...n} = 1 \), at all these points.

To prove (6)\( \pi \) we first consider \( v \in Q^{1}(c_n) \), i.e., \( \lfloor \frac{k}{2} \rfloor + 1 \leq r \leq k \). Assume again that the first \( r \) indices are equal to \( n \), all other cases being symmetrical. We define

\[
\pi_{n...n} = a_{n...n} + a_{n...n} = \pi
\]

for \( q \in \{ r + 1, \ldots, k - 1 \} \). Consider a \( k \)-tuple \( (m^1, \ldots, m^k) \in M^K \) such that \( m^i \in M_i \setminus \{ n \} \). Let \( y \) be an integer point of \( P^{(k,1)}_{n;\ell}(Q(c_n)) \) such that \( y_{n...n} = y_{m^1m^2...m^k} = 1 \). Then,

\[
y^q = (m^1 \leftrightarrow n)_1 \cdots (m^q \leftrightarrow n)_q \quad \text{for} \quad q \in \{ r + 1, \ldots, k - 1 \}.
\]

Observe that points \( y^q \), \( y^{q-1} \) have a variable set to one, which has at least \( \lfloor \frac{k}{2} \rfloor + 1 \) indices equal to \( n \). Equation \( ay^q = ay^{q-1} \), after cancelling out identical terms, yields

\[
a_{n...n} + a_{n...n} = a_{n...n} + a_{n...n} = \pi_{n...n} = \pi_{n...n} = \pi.
\]

The first terms of both sides are indexed by \( k \)-tuples belonging to \( Q(c_n) \), whereas the other two \( k \)-tuples belong to \( C^{(k,1)} \setminus Q(c_n) \). Therefore, substituting the first terms, of both sides, from (14) and the rest from (6), we obtain

\[
\sum_{t=1}^{k-q} \lambda^t_{m^{k+1-t}} + \sum_{t=k-q+1}^{k} \lambda^t_{n} + \pi_{n...n} - \pi_{n...n} = \sum_{t=1}^{k} \lambda^t_{n} + \sum_{t=k-q+1}^{k} \lambda^t_{m^{k+1-t}} + \sum_{t=k-q+1}^{k} \lambda^t_{m^{k+1-t}} + \sum_{t=1}^{k} \lambda^t_{n},
\]

Cancelling out identical terms yields (15). In a similar manner, equation \( ay = ay^{k-1} \) yields \( \pi_{n...n} = \pi_{n...n} = \pi \).

The proof of (b) with respect to \( v \in Q^{2}_{G^{+}}(c_n) \) (implying that \( k \) is even) is done in a similar manner. In particular, given our assumption that \( (m^1, \ldots, m^k, n, \ldots, n) \in Q^{2}_{G^{+}}(c_n) \), all other cases being symmetrical, we define

\[
\pi_{m^1...m^k/n...n} = a_{m^1...m^k/n...n} - \sum_{t=1}^{k/2} \lambda^t_{m^{k+1-t}} - \sum_{t=(k/2)+1}^{k} \lambda^t_{n}.
\]

We must show that \( \pi_{m^1...m^k/n...n} = \pi \).

Consider integer points \( y \), \( y^{q=k/2} \) defined as above. Equation \( ay = ay^{k/2} \) yields

\[
a_{n...n} + a_{m^1...m^k} = a_{m^1...m^k} + a_{n...n} = \pi_{n...n} + \pi_{m^1...m^k} = \pi_{n...n} + \pi_{m^1...m^k} = \pi.
\]
Substituting the first term of the right-hand side from (16), the first term of the left-hand side from (6) and the remaining two terms from (6) yields \( \pi_{m^1 \ldots m^k/2n \ldots n} = \pi_{n \ldots n} = \pi \), after cancelling out identical terms.

To show (c) consider that \( x^{\text{diag}} \in P^{(k,1)}_{n;1} \). Equation \( ax^{\text{diag}} = a_0 \), after substituting term \( a_{n \ldots n} \) from (6) and the remaining terms from (6), becomes (7). \( \square \)

For \( k = 3 \), this class of facets is also described in [4].

**Lemma 15.** For \( k \geq 3 \) and odd, the inequalities (13) belong to the elementary closure of \( P^{(k,1)}_{n;1} \).

**Proof.** Let \( c_0 \in C^{(k,1)} \), where \( c_0 = m_0^K = (m_0^1, m_0^2, \ldots, m_0^k), m_0^i \in M_i, i \in K \). There are \( k \) equality constraints, one for each of the \( \binom{k}{1} \) row sets of (2), containing variable \( x_{m_0^i} \), i.e.

\[
\sum_{m^i \backslash \{ i \}} x_{m^i} = 1 \quad \forall i \in K, \tag{17}
\]

where \( m^i = m_0^i \).

For \( r \in \{-\left\lfloor \frac{k}{2} \right\rfloor, \ldots, \left\lfloor \frac{k}{2} \right\rfloor + 1 \} \), a tuple having \( \left\lfloor \frac{k}{2} \right\rfloor + r \) values from \( m_0^K \) appears in exactly \( \left\lfloor \frac{k}{2} \right\rfloor + r \) equality constraints (17). Hence, by adding up constraints (17), dividing the resulting equation by \( \left\lfloor \frac{k}{2} \right\rfloor + 1 \), replacing ‘=’ with ‘\( \leq \)’ and rounding down, we obtain (13), for \( Q(c_0) \). \( \square \)

5. The planar assignment polytopes

Each integer point of \( P^{(3,2)}_{n;1} \) corresponds to a Latin square of order \( n \) (see, for example, [14]). The polytope \( P^{(3,2)}_{n;1} \) is also referred to as the Latin square polytope. We briefly introduce some definitions (see [20]). A Latin square \( L \) of order \( n \) is an \( n \times n \) square array consisting of \( n^2 \) entries of \( n \) different elements, each occurring exactly once in each row and column. Two Latin squares \( L_1 = \begin{bmatrix} 1_{i,j}^1 \end{bmatrix} \) and \( L_2 = \begin{bmatrix} 1_{i,j}^2 \end{bmatrix} \) are called orthogonal, if every ordered pair of symbols occurs exactly once among the \( n^2 \) ordered pairs \( (1_{i,j}^1, 1_{i,j}^2), i, j = 1, \ldots, n \). This definition is extended to a set of more than two Latin squares, which are said to be mutually orthogonal if they are pairwise orthogonal. In [22], it is noted that an integer point of \( P^{(k,2)}_{n;1} \) corresponds to a set of \( k - 2 \) MOLS, with 1 MOLS being conventionally defined as a Latin Square. Therefore, \( P^{(k,2)}_{n;1} \) is called the \( (k - 2)\text{MOLS} \) polytope.

The connection between MOLS and \( P^{(k,2)}_{n;1} \) provides information on the non-emptiness of \( P^{(k,2)}_{n;1} \). An immediate result is that \( P^{(3,2)}_{n;1} \neq \emptyset \), for \( n \in \mathbb{Z}_+ \backslash \{1\} \), since there exists at least one Latin square for any \( n \geq 2 \). It is also known that \( P^{(4,2)}_{n;1} \neq \emptyset \), for \( n \in \mathbb{Z}_+ \backslash \{1, 2, 6\} \) ([20, Theorem 2.9]). The theory of MOLS provides us with further results. Let \( N(n) \) denote the maximum number of MOLS of order \( n \). First, \( N(n) \leq n - 1 \) ([20, Theorem 2.1]), the bound being attainable if \( n \) is a prime power, i.e., \( n = p^t \), where \( p \) is a prime and \( t \geq 1 \) is an integer ([20, Theorem 2.3]). Consequently, for \( n \) being a prime power, \( P^{(k,2)}_{n;1} \neq \emptyset \), for \( k - 2 \leq n - 1 \), implying \( k \leq n + 1 \).

For general \( n \), this is a necessary condition for \( P^{(k,2)}_{n;1} \neq \emptyset \), a result also obtained from Proposition 8 for \( s = 2 \). Moreover, if \( p_1 \times p_2 \times \cdots \times p_q \) is the factorization of \( n \) into distinct prime powers
with \( p_1 < p_2 < \cdots < p_q \), then there exist at least \( p_1 + 1 \) MOLS of order \( n \) ((20, Theorem 2.10)). This implies that a sufficient condition for \( P_{\n;I}^{(k,2)} \neq \emptyset \) is \( k \geq p_1 + 1 \). Finally, a more general result is that \( N(n) \) tends to infinity with \( n \) (see [11], where it is shown that \( N(n) > \frac{1}{3}n^{3/4} \)). Hence, \( P_{\n;I}^{(k,2)} \neq \emptyset \), for \( k \leq n + 1 \), for all sufficiently large \( n \).

The facial structure of \( P_{\n;I}^{(3,2)} \) and \( P_{\n;I}^{(4,2)} \) have been studied in [1,14] and [2], respectively. This section generalises these results with respect to the dimension. First, we introduce the related notation and an auxiliary proposition.

As mentioned above, an integer point of \( P_{\n;I}^{(k,2)} \) can be illustrated as a collection of \((k-2)\) MOLS. Each of the \((k-2)\) Latin squares is formed by the values of an index belonging to a set \( M_1, \ldots, M_k \), whereas the remaining two indices are used for the rows and columns of the Latin squares. Conventionally, the role of row (column) set is played by \( M_1 (M_2) \) and \( M_i \) the set of elements appearing at Latin square \( i-2 \) \((i=3, \ldots, k)\). For this Latin square, we denote as \( m^i(m^1,m^2) \) the element appearing at row \( m^1 \) and column \( m^2 \). The notation \( m^i(x;m^1,m^2) \) refers to the same element at a specific integer point \( x \in P_{\n;I}^{(k,2)} \). For conciseness, the tuple \((m^3(x; m^1, m^2), \ldots, m^k(x; m^1, m^2))\) is denoted as \((m^3 \cdots m^k)(x; m^1, m^2)\). For illustration purposes, the collection of the \((k-2)\) MOLS will be depicted as a single square containing a \((k-2)\)-tuple at each cell.

To establish the dimension of \( P_{\n;I}^{(k,2)} \), we have applied equations relating elements of the vector \( a \) (see Eqs. (9) and (10)). Analogous equations are necessary for showing the dimension of \( P_{\n;I}^{(k,2)} \). We are looking for a generic equation that is simple and includes a constant number of terms (i.e. not based on \( n \)). The following proposition establishes such an equation. The proof is based on a series of points obtained from one another by a single interchange. In particular, considering an integer point \( x \in P_{\n;I}^{(k,2)} \) as a MOLS configuration, we derive neighbouring points by interchanging the contents of the cells of two specific rows and columns.

**Proposition 16.** Let \( n \geq k - 1 \), \( P_{\n;I}^{(k,2)} \neq \emptyset \), and suppose there exist \( a \in \mathbb{R}^{n^k}, a_0 \in \mathbb{R} \) such that \( ay = a_0 \) for all \( y \in P_{\n;I}^{(k,2)} \). Then, for an integer point \( x \) of \( P_{\n;I}^{(k,2)} \) it holds that

\[
\begin{align*}
{am^1_1m_2^1m_3^1m_4^1}_{x; m^1_1, m^1_2} + {am^1_2m^2_2m_3^1m_4^1}_{x; m^1_2, m^1_2} + {am^1_2m^2_2m_3^1m_4^1}_{x; m^1_3, m^1_2} \\
+ {am^2_1m^2_1m_3^2m_4^1}_{x; m^1_1, m^1_2} + {am^1_2m^2_2m_3^2m_4^1}_{x; m^1_2, m^1_2} + {am^1_2m^2_2m_3^2m_4^1}_{x; m^1_3, m^1_2} \\
+ {am^2_1m^2_1m_3^2m_4^1}_{x; m^1_1, m^1_2} + {am^1_2m^2_2m_3^2m_4^1}_{x; m^1_2, m^1_2} + {am^1_2m^2_2m_3^2m_4^1}_{x; m^1_3, m^1_2} \\
= am^1_1m_2^1m_3^2m_4^2_{x; m^1_1, m^1_2} + am^1_2m^2_2m_3^2m_4^2_{x; m^1_2, m^1_2} + am^1_2m^2_2m_3^2m_4^2_{x; m^1_3, m^1_2} \\
+ am^1_1m_2^1m_3^2m_4^2_{x; m^1_1, m^1_2} + am^1_2m^2_2m_3^2m_4^2_{x; m^1_2, m^1_2} + am^1_2m^2_2m_3^2m_4^2_{x; m^1_3, m^1_2} \\
+ am^1_1m_2^1m_3^2m_4^2_{x; m^1_1, m^1_2} + am^1_2m^2_2m_3^2m_4^2_{x; m^1_2, m^1_2} + am^1_2m^2_2m_3^2m_4^2_{x; m^1_3, m^1_2}
\end{align*}
\]

for \( m^1_1, m^1_2 \in M_1 \) and \( m^2_1, m^2_2 \in M_2 \).

**Proof.** Consider an arbitrary integer point \( x \in P_{\n;I}^{(k,2)} \) in the \((k-2)\) MOLS format. Then \( m^1_1, m^1_2 \in M_1 (m^2_1, m^2_2 \in M_2) \) index two rows (columns) of the structure. Let \( x' = x(m^1_1 \leftrightarrow m^1_2) \), \( x'' = x(m^1_1 \leftrightarrow m^2_2) \). Observe that points \( x, x' \) have the same variables set to one except the ones indexed by \( m^1_1, m^1_2 \). Thus, \( ax = ax' \) consists only of terms indexed by the elements of rows \( m^1_1, m^1_2 \). The same is true for \( a\tilde{x} = a\tilde{x}' \). Also observe that in \( ax = a\tilde{x} \) all terms
not indexed by the elements of columns \( m_1^2, m_2^2 \) cancel out since they appear in both sides of the equation. The same is true for \( ax' = a\bar{x}' \). Hence, \( ax - ax' = a\bar{x} - a\bar{x}' \) yields the result. \( \square \)

When applied to an integer point \( x \in P_{n;I}^{(k,2)} \), the equation of Proposition 16 is denoted as \( a[x; (m_1^1, m_2^1)_1, (m_1^2, m_2^2)_2] \).

**Theorem 17.** For \( n \geq \max \{5, k\} \), if \( P_{n;I}^{(k,2)} \neq \emptyset \) then \( \dim P_{n;I}^{(k,2)} = \dim P_n^{(k,2)} = \sum_{r=0}^{k-3} \binom{k}{r} \cdot (n - 1)^{k-r} \).

**Proof.** We must show that there exist scalars \( \lambda_{m^1, m^2, \ldots, m^k}^1, \lambda_{m^1, m^2, \ldots, m^k}^2 \), for all \( m^1 \in M_1, \ldots, m^k \in M_k \), such that

\[
a_m = \sum_{m^k \in M_k} \lambda_{m^1, m^2, \ldots, m^k}^1 m^1 + \cdots + \sum_{m^1, m^2 \in M_1 \times M_2} \lambda_{m^1, m^2}^2 m^2.
\]

We define

\[
\lambda_{m^1, m^2}^1 = a_{m^1 m^2},
\]

\[
\lambda_{m^2}^2 = a_{1\ldots m^2} - a_{1\ldots 1m^2},
\]

\[
\lambda_{m^2}^3 = a_{1\ldots m^2 - 1\ldots m^2 - 1} - a_{1\ldots 1m^2 - 1\ldots m^2 - 1} + a_{1\ldots 1m^2} - a_{1\ldots m^2 - 1},
\]

\[
\vdots
\]

\[
\lambda_{m^2}^k = a_{m^2 - 1\ldots m^2 - 1 - 1} - a_{m^2 - 1\ldots 1 - 1} + a_{1\ldots 1m^2 - 1}.
\]

Notice that each multiplier indexed by \( m^k \), except \( \lambda_{m^1, m^2}^1 \), is defined in a manner analogous to \( \lambda_{m^2}^2 \), whereas all other multipliers are defined in a manner analogous to \( \lambda_{m^2}^3 \).

By substituting the values of the scalars in Eq. (18), we obtain

\[
a_m = a_{m^1 m^2} + a_{m^1 m^3 - 1\ldots m^3 - 1} + \cdots + a_{m^1 m^k - 1\ldots m^k - 1} + a_{1\ldots 1m^1} - (k - 2) \cdot (a_{1\ldots 1m^1} - a_{1\ldots m^1 - 1}) + (k - 1)(k - 2)/2 \cdot a_{1\ldots 1}.
\]

We consider \( v, T(v) \) exactly as in the proof of Theorem 9. Observe that for \( |T(v)| = t \leq 2 \), \( v \) has at least \( k - 2 \) indices equal to one, in which case (20) is true by definition. It remains to show (20) for \( 3 \leq t \leq k \). Consider the integer point \( x^2 \), illustrated in Table 2, where \( c_i, d_i, e_i \in M_k \setminus \{1\}, c_i, d_i \neq e_i \), for all \( i \in \{3, \ldots, k\} \). It is easy to see that such a point exists for \( n \geq k \).

Assume that \( T(v) = \{1, \ldots, t\} \), all other cases being symmetrical. For \( n \geq 5 \), there exist \( m_1^t \in M_t \setminus \{1, c_i, d_i, e_i\} \), \( t \in \{3, \ldots, k\} \). Let \( x^t = x^2 (1 \leftrightarrow m_1^t) \cdots (1 \leftrightarrow m_1^t) \). The fact that \( x^2 \in P_{n;I}^{(k,2)} \) ensures that \( x^t \in P_{n;I}^{(k,2)} \), for every \( t \in \{3, \ldots, k\} \) (Remark 7).

\[
a[x^t; (1, m_1^0)_1, (1, m_1^0)_2] \]
As in the case of Theorem 9, we have produced a recursive equation with respect to \( t \). Handling (21) as (11), we obtain (20) thus proving (18). Eq. (19) follows from the assumption that \( P^{(k,2)}_{n;1} \neq \emptyset \). □

As in the case of \( P^{(k,1)}_{n;1} \) (Propositions (10) and (11)), it can be shown that, for \( n \geq \max\{5, k\} \) and \( P^{(k,2)}_{n;1} \neq \emptyset \), the inequalities \( x_c \geq 0 \) define facets of \( P^{(k,2)}_{n;1} \), whereas inequalities \( x_c \leq 1 \) are redundant.

6. Concluding remarks

This work investigates an IP model for the class of multi-index assignment problems. This model establishes a framework for unifying the polyhedral analysis of all assignment polytopes belonging to this class. In particular, the dimension of the linear relaxation of all members of this class is derived. The properties of integer points are encompassed in the definition of the interchange operator, which exploits inherent isomorphisms. Focusing on the classes of axial and planar assignment polytopes, and via the derived recurrence relations, we prove that their dimension equals a sum of terms from Newton’s polynomial. It is noteworthy that, for both the axial and the planar case, the dimension of the convex hull of the integer points (polytope \( P^{(k,s)}_{n;1} \)) equals the dimension of its linear relaxation (polytope \( P^{(k,s)}_{n;1} \)), provided that \( P^{(k,s)}_{n;1} \neq \emptyset \).
This, together with the fact that \( \dim P_{n;I}^{(k,s)} = \dim P_{n}^{(k,s)} \), for \( s = 0, k \) (Lemmas 1, 2), offer strong evidence that the dimensions of these two polytopes coincide for general \( k, s \) (always under the condition \( P_{n;I}^{(k,s)} \neq \emptyset \)). Resolving this claim requires, among other things, (a) the definition of multipliers reducing, for \( s = 1, 2 \), to the multipliers exhibited in the proofs of Theorems 9, 17, respectively, (b) the derivation of an equation analogous to that of Proposition 16 and (c) establishing the existence of appropriate integer vectors of \( P_{n;I}^{(k,s)} \). Despite the general and complex facial structure of \( P_{n;I}^{(k,s)} \), the framework presented here has obvious advantages; results obtained for a specific value of \( s \) are applicable to all the polytopes of the specific class (defined by the value of \( s \)) rather than to an individual polytope. Along this line, we have identified a family of clique facets for all axial assignment polytopes, for \( k \geq 3 \).

References