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Related Self-Adjoint Differential and Integro-Differential Systems*

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1. INTRODUCTION

The fundamental connections between the calculus of variations and the Sturmian theory for a real self-adjoint linear homogeneous differential equation of the second order rest upon the fact that equations of this latter type appear as the "Jacobi" or "accessory" differential equation for a simple integral variational problem. In turn, the involved variational principles are at the basis of the extension of the classical Sturmian theory to self-adjoint differential systems, as emanated from the basic work of Marston Morse [8–10]. At a relatively early stage (see [1, and references to other literature there cited; 2]), it was realized that for certain variational problems the "accessory system" was a boundary problem involving a self-adjoint integro-differential equation. In particular, Lichtenstein [6] treated a boundary problem involving a single integro-differential equation of the second order and a special set of two-point boundary conditions by means of the theory of quadratic forms in infinitely many variables. Under certain conditions, he established the existence of infinitely many eigenvalues, together with an expansion theorem for functions in terms of the corresponding eigenfunctions. Subsequently, Lichtenstein [7] used the results of his earlier paper to establish sufficient conditions for a weak relative minimum for a simple integral isoperimetric problem of the calculus of variations by expansion methods. Courant [1, Sects. 5, 13] treated an integro-differential boundary problem similar to that considered by Lichtenstein [6] by means of difference equations. A few years after Lichtenstein's paper [7], the author [11] considered a self-adjoint boundary problem involving a system of integro-differential equations and two-point boundary conditions, and in addition to the proof of the existence of infinitely many eigenvalues, he established comparison and oscillation theorems which are generalizations of such theorems of the

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classical Sturmian theory and contain as special instances the comparison and oscillation theorems of the Morse generalization to self-adjoint differential systems; indeed, [11] presented for the first time such theorems not involving any assumption of normality on subintervals. The method of proof of [11] may be described as functional in nature, employing in particular the "Green's matrix" for an integro-differential system as introduced by Tamarkin [18] and Jonah [4].

The present paper returns to the area of the earlier paper [11], and for a problem formulated in the general context of a Hamiltonian system with two-point boundary conditions there is a more detailed consideration of the interrelations that exist between such integro-differential systems and ordinary differential systems, although some of the basic techniques that are commonly employed for differential systems are no longer available for the study of integro-differential systems.

Section 2 is devoted to the formulation of the self-adjoint integro-differential system and basic properties of such system, while Section 3 presents some preliminary comparative results for integro-differential and differential boundary problems. Section 4 establishes the existence of a partial Green's matrix for the integro-differential boundary problem, together with a brief discussion of fundamental properties of this matrix. Section 5 is concerned with existence and properties of the set of eigenvalues and eigenfunctions of integro-differential boundary problems, and the area of comparison and oscillation theorems is surveyed in Section 6. Finally, Section 7 is devoted to remarks on interrelations between the methods employed herein and other possible methods of treatment, together with comments on the relationship between problems of the sort considered and generalized differential systems of the type previously treated by the author [13, 14, 16].

Matrix notation is used throughout; in particular, matrices of one column are called vectors, and for a vector (y_α), $(\alpha = 1,..., m)$, the norm $|y|$ is given by $[|y_1|^2 + \cdots + |y_m|^2]^{1/2}$. The $m \times m$ identity matrix is denoted by E_m, or merely by E when there is no ambiguity, and 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix M is denoted by M^*. If M is an $n \times m$ matrix the symbol $\|M\|$ is used for the supremum of $|My|$ on the unit closed ball $\{y: |y| \leqslant 1\}$ of complex m-space, $\mathbf{C}_m$. The relations $M \geqslant N$ $(M > N)$, are used to signify that M and N are hermitian matrices of the same dimensions and $M - N$ is a nonnegative (positive) definite matrix. For typographical simplicity, if $M = [M_{\alpha\beta}]$ and $N = [N_{\alpha\beta}]$, $(\alpha = 1,..., m; \beta = 1,..., r)$ are $m \times r$ matrices, then the $2m \times r$ matrix $P = [P_{\sigma\beta}]$, $(\sigma = 1,..., 2m; \beta = 1,..., r)$, with $P_{\alpha\beta} = M_{\alpha\beta}$, $P_{m+\alpha,\beta} = N_{\alpha\beta}$ is denoted by $(M; N)$.

A matrix function $M(t) = [M_{\alpha\beta}(t)]$ is called continuous, integrable, etc., if each element $M_{\alpha\beta}(t)$ possesses the specified property. If a hermitian matrix function $M(t)$, $t \in [a, b]$ is such that $M(t_2) - M(t_1) \geqslant 0$, $(\leqslant 0)$, for $a \leqslant t_1 < t_2 \leqslant b$, then $M(t)$ is called nondecreasing (nonincreasing) on $[a, b]$.

If a matrix function $M(t)$ is a.c. (absolutely continuous) on $[a, b]$, then $M'(t)$ signifies the matrix of derivatives at values where these derivatives exist, and zero elsewhere. Similarly, if $M(t)$ is (Lebesgue) integrable on $[a, b]$ then $\int_a^b M(t)\,dt$ denotes the matrix of integrals of respective elements of $M(t)$. For a given compact interval $[a, b]$ the symbols $\mathfrak{C}_{hk}[a, b]$, $\mathfrak{L}_{hk}[a, b]$, $\mathfrak{L}^{\infty}_{hk}[a, b]$, $\mathfrak{L}^2_{hk}[a, b]$, $\mathfrak{BV}_{hk}[a, b]$, and $\mathfrak{A}_{hk}[a, b]$ are used to denote the class of $h \times k$ matrix functions $M(t)$ on $[a, b]$ which are respectively continuous, (Lebesgue) integrable, (Lebesgue) measurable, and essentially bounded, measurable, and $\| M(t)\|^2$ integrable, of bounded variation, and absolutely continuous. For brevity, in the designation of one of the above classes whenever $k = 1$ the double subscript "$h1$" is reduced to merely "h." Also, whenever a matrix function $M(t)$ defined on a general interval I of the real line is such that $M \in \mathfrak{L}_{hk}[a, b]$ for arbitrary compact intervals $[a, b]$ of I, then M is said to be "locally of class $\mathfrak{L}_{hk}$ on I," with similar meanings for "locally of class $\mathfrak{L}^2_{hk}$ on I," etc.

2. Formulation and Basic Properties of the Problem

Corresponding to the manner in which the general self-adjoint Hamiltonian system of ordinary differential equations may be written (see, for example, [15; Chap. VII]), the integro-differential system to be considered is

$$L_1[u, v](t) = -v'(t) + C(t)\,u(t) - A^*(t)\,v(t) + \int_a^b N(t, s)\,u(s)\,ds = 0,$$
$$L_2[u, v](t) = u'(t) - A(t)\,u(t) - B(t)\,v(t) = 0, \tag{2.1}$$

in n-dimensional vector functions $u(t)$, $v(t)$. Moreover, for the subsequent discussion it is supposed that $A(t)$, $B(t)$, $C(t)$ are $n \times n$ matrix functions on a given interval I on the real line, and $N(t, s)$ is an $n \times n$ matrix function on $I \times I$ satisfying on arbitrary compact subintervals $[a, b]$ of I the following hypothesis.

($\mathfrak{H}$) (i) *$A(t)$, $B(t)$, $C(t)$ are of class $\mathfrak{L}^{\infty}_{nn}[a, b]$, and $B(t)$ and $C(t)$ are hermitian*;

(ii) *$B(t) \geqslant 0$ for t a.e. (almost everywhere), on $[a, b]$*;

(iii) *$N(t, s)$ is of class $\mathfrak{L}^{\infty}$ on $[a, b] \times [a, b]$, and*

$$N(t, s) = [N(s, t)]^* \qquad \textit{for } t, s \in [a, b] \times [a, b].$$

For $\mathscr{S}$ a linear subspace of $\mathbf{C}_{2n}$ and $[a, b]$ a compact subinterval of I there will be associated with (2.1) two-point boundary conditions of the form

$$\hat{u} \in \mathscr{S}, \qquad T[u, v] \equiv Q\hat{u} + D\hat{v} \in \mathscr{S}^{\perp} \tag{2.2}$$

in the $2n$-dimensional boundary vectors $\hat{u} = (u(a); u(b))$, $\hat{v} = (v(a); v(b))$, where Q is an hermitian $2n \times 2n$ matrix and $D = \operatorname{diag}\{-E_n; E_n\}$. The boundary problem involving (2.1) and the boundary condition (2.2) is denoted by $\mathscr{B}$. Let $\mathscr{D}[a, b]$ denote the class of n-dimensional vector functions $\eta \in \mathfrak{A}_n[a, b]$ such that there exists a $\zeta \in \mathfrak{L}_n^2[a, b]$ satisfying with η the differential equation $L_2[\eta, \zeta] = 0$ on $[a, b]$. The subclass of $\mathscr{D}[a, b]$ on which $\eta(a) = 0 = \eta(b)$ is designated by $\mathscr{D}_0[a, b]$. Also, the symbol $\mathscr{D}[\mathscr{B}]$ is used to denote the class $\{\eta : \eta \in \mathscr{D}[a, b], \hat{\eta} \in \mathscr{S}\}$.

Note that for the problems treated herein an alternate set of hypotheses would merely require the coefficient matrix functions $A(t)$, $B(t)$, $C(t)$ to be of class $\mathfrak{L}_{nn}[a, b]$ on arbitrary compact subintervals $[a, b]$ of I, and that $N(t, s)$ be of class $\mathfrak{L}$ on $[a, b] \times [a, b]$. With such modification of the above hypothesis ($\mathfrak{H}$), the definition of the class $\mathscr{D}[a, b]$ would be altered te require the involved ζ to be of class $\mathfrak{L}_n^{\infty}[a, b]$. For a treatment of differential systems and generalized differential systems in the context of such modified hypotheses the reader is referred to [13]. Hypothesis ($\mathfrak{H}$) follows the procedure of [15, Chap. VII], and under such conditions the Dirichlet functional $\hat{J}$ defined below is in a Hilbert space setting.

Of basic importance for the present discussion is the fact that if $(\eta_\alpha, \zeta_\alpha) \in \mathfrak{L}_n^2[a, b] \times \mathfrak{L}_n^2[a, b]$, then the functionals

$$J[\eta_1; \zeta_1, \eta_2; \zeta_2 : a, b] = \int_a^b \{\zeta_2^* B \zeta_1 + \eta_2^* C \eta_1\}\, dt + \int_a^b \int_a^b \eta_2^*(t)\, N(t, s)\, \eta_1(s)\, dt\, ds \tag{2.3}$$

$$\hat{J}[\eta_1; \zeta_1, \eta_2; \zeta_2 : a, b] = \hat{\eta}_2^* Q \hat{\eta}_1 + J[\eta_1; \zeta_1, \eta_2; \zeta_2 : a, b] \tag{2.3'}$$

are hermitian. Moreover, if $\eta_\alpha \in \mathscr{D}[a, b] : \zeta_\alpha$, $(\alpha = 1, 2)$, then although in general the ζ_α are not determined uniquely, the values of the functionals (2.3), (2.3′) are independent of the particular ζ_α associated with the η_α, and consequently, in this case the symbols for these functionals are reduced to $J[\eta_1, \eta_2 : a, b]$ and $\hat{J}[\eta_1, \eta_2 : a, b]$. Also, in accord with common terminology, if $\eta \in \mathscr{D}[a, b] : \zeta$, the symbols $J[\eta, \eta : a, b]$ and $\hat{J}[\eta, \eta : a, b]$ are further reduced to $J[\eta; a, b]$ and $\hat{J}[\eta; a, b]$.

LEMMA 2.1. *If* $(u, v) \in \mathfrak{C}_n[a, b] \times \mathfrak{A}_n[a, b]$, *and* $\eta \in \mathscr{D}[a, b] : \zeta$, *then*

$$\hat{J}[u; v, \eta; \zeta : a, b] = \hat{\eta}^* T[u, v] + \int_a^b \eta^*(t) L_1[u, v](t)\, dt.$$

LEMMA 2.2. *If* $(u, v) \in \mathfrak{C}_n[a, b] \times \mathfrak{L}_n^2[a, b]$ *the following conditions are equivalent:*

(a) $\hat{J}[u; v, \eta; \zeta: a, b] = 0$ *for* $\eta \in \mathscr{D}_0[a, b]: \zeta$;

(b) *there exists a* v_0 *such that* $v_0 \in \mathfrak{A}_n[a, b]$, $B[v - v_0] = 0$ *a.e. on* $[a, b]$, *and* $L_1[u, v_0](t) = 0$ *on* $[a, b]$.

LEMMA 2.3. *If* $u \in \mathfrak{A}_n[a, b]$ *there exists a* v_0 *such that* $(u; v_0)$ *is a solution of* (2.1), (2.2) *if and only if there exists a* v *such that* $u \in \mathscr{D}[a, b]: v$ *and*

$$\hat{J}[u; v, \eta; \zeta: a, b] = 0 \qquad \text{for } \eta \in \mathscr{D}_0[a, b]: \zeta.$$

LEMMA 2.4. *If* $(u, v) \in \mathfrak{C}_n[a, b] \times \mathfrak{L}_n^2[a, b]$, *the following conditions are equivalent:*

(a) $\hat{J}[u; v, \eta; \zeta: a, b] = 0$ *for* $\eta \in \mathscr{D}[\mathscr{B}]$;

(b) *there exists a* v_0 *such that* $v_0 \in \mathfrak{A}_n[a, b]$, $B[v - v_0] = 0$ *a.e. on* $[a, b]$, *and* $T[u, v_0] \in S^{\perp}$, $L_1[u, v_0](t) = 0$ *on* $[a, b]$.

In particular, as for differential systems, one has the following result.

COROLLARY. *If* $[a, b]$ *is a compact subinterval of* I, $\hat{J}[\eta: a, b]$ *is nonnegative definite on* $\mathscr{D}[\mathscr{B}]$ *and there exists an element* $u \in \mathscr{D}[\mathscr{B}]$ *satisfying* $\hat{J}[u: a, b] = 0$, *then there exists a* $v \in A_n[a, b]$, *such that* $(u; v)$ *is a solution of* (2.1), (2.2).

The results of Lemmas 2.2 and 2.3 and those of Lemma 2.4 and its Corollary may be established by steps analogous to those used in the proofs of [13, Theorems 2.1 and 2.2].

For $[a, b]$ a compact subinterval of I, let $(\mathfrak{H}_K: a, b)$ denote the following hypothesis.

$(\mathfrak{H}_K: a, b)$. *$K(t)$ is a nonnegative hermitian $n \times n$ matrix function of class* $\mathfrak{L}_{nn}^{\infty}[a, b]$, *and such that the set* $\{t: t \in [a, b], K(t) \neq 0\}$ *is of positive measure.*

For brevity, we introduce the notations

$$K[\eta_1, \eta_2: a, b] = \int_a^b \eta_2^*(t)\, K(t)\, \eta_1(t)\, dt, \qquad K[\eta_1: a, b] = K[\eta_1, \eta_1: a, b]; \tag{2.4}$$

clearly $K[\eta_1, \eta_2: a, b]$ is an hermitian functional on $\mathfrak{L}_n^2[a, b] \times \mathfrak{L}_n^2[a, b]$. In particular, for $K(t) \equiv E_n$ we write

$$E[\eta_1, \eta_2: a, b] = \int_a^b \eta_2^*(t)\, \eta_1(t)\, dt, \qquad E[\eta_1: a, b] = E[\eta_1, \eta_1: a, b]. \tag{2.4'}$$

Now we present some readily established properties of solutions of the boundary problem

$$\begin{aligned} &\text{(i)}\quad L_1[u, v; \lambda](t) \equiv L_1[u, v](t) - \lambda K(t)\, u(t) = 0, \qquad t \in [a, b],\\ &\text{(ii)}\quad L_2[u, v](t) = 0,\\ &\text{(iii)}\quad u \in \mathscr{S}, \qquad T[u, v] \in \mathscr{S}^{\perp}, \end{aligned} \tag{2.5}$$

and the related nonhomogeneous system

$$\begin{aligned} &\text{(i)}\quad L_1[u, v; \lambda](t) = f(t), \qquad L_2[u, v](t) = 0, \qquad t \in [a, b],\\ &\text{(ii)}\quad u \in \mathscr{S}, \qquad T[u, v] \in \mathscr{S}^{\perp}, \end{aligned} \tag{2.6}$$

where hypotheses ($\mathfrak{H}$i, ii, iii: a, b) and ($\mathfrak{H}_K$: a, b) are supposed to be satisfied, and in (2.6) it is supposed that $f \in \mathfrak{L}_n[a, b]$. Also, we set

$$J[\eta_1; \zeta_1, \eta_2; \zeta_2; \lambda: a, b] = \hat{J}[\eta_1; \zeta_1, \eta_2; \zeta_2: a, b] - \lambda K[\eta_1, \eta_2: a, b], \tag{2.7}$$

with similar meanings for $\hat{J}[\eta_1, \eta_2; \lambda: a, b]$ and $\hat{J}[\eta_1; \lambda: a, b]$.

LEMMA 2.5. *If (u, v) is a solution of (2.6) then*

$$\hat{J}[u, \eta; \lambda: a, b] = E[f, \eta: a, b];$$

in particular,

$$\hat{J}[u; \lambda: a, b] = E[f, u: a, b].$$

COROLLARY 1. *If (u_1, v_1) is a solution of (2.6) for $\lambda = \lambda_1$ and $f = f_1$, and (u_1, v_2) is a solution of (2.6) for $\lambda = \lambda_2$ and $f = f_2$, then*

$$(\lambda_1 - \bar{\lambda}_2)\, K[u_1, u_2: a, b] + E[f_1, u_2: a, b] - E[u_1, f_2: a, b] = 0. \tag{2.8}$$

COROLLARY 2. *If (u, v) is a solution of (2.5) for a value λ, then*

$$\hat{J}[u; \lambda: a, b] = 0.$$

COROLLARY 3. *If (u_1, v_1) is a solution of (2.5) for a value λ_1 and (u_2, v_2) is a solution of (2.5) for a value λ_2, then $(\lambda_1 - \bar{\lambda}_2)\, K[u_1, u_2: a, b] = 0$. In particular, if (u, v) is a solution of (2.5) for a value λ such that $K[u: a, b] \neq 0$, then λ is real.*

For a compact subinterval $[a_0, b_0]$ of I, in view of hypothesis ($\mathfrak{H}$iii) there exists a constant $k[a_0, b_0]$ such that $\| N(t, s)\| \leqslant k[a_0, b_0]$ for

$$(t, s) \in [a_0, b_0] \times [a_0, b_0],$$

and with the aid of Schwarz' inequality it follows that if $\eta \in \mathfrak{L}_n^2[a_0, b_0]$, then

$$\left| \int_{a_0}^{b_0} \int_{a_0}^{b_0} \eta^*(t) N(t, s) \eta(s) \, dt \, ds \right| \leqslant k[a_0, b_0] \int_{a_0}^{b_0} |\eta(t)|^2 \, dt. \tag{2.9}$$

Moreover, since by hypothesis ($\mathfrak{H}$i) there is a constant $c = c[a_0, b_0]$ such that $\| B(t) \| \leqslant c$ a.e. on $[a_0, b_0]$, and as $B(t) \geqslant 0$ a.e. on $[a_0, b_0]$, it then follows that $B(t) - (1/c) B^2(t) \geqslant 0$ a.e. on $[a_0, b_0]$. In view of these inequalities, and with the aid of [15, Problem VII.4.4] and a method similar to that employed in the proofs of [15, Lemma VII.11.1 and its Corollary], one may establish the following result.

LEMMA 2.6. *For $I_0 = [a_0, b_0]$ a compact subinterval of I and a constant d satisfying $0 < d < b_0 - a_0$, there exist corresponding constants*

$$l_0 = l_0[I_0, d] > 0, \qquad l_1 = l_1[I_0, d] \geqslant 0$$

such that if $[a, b]$ is a compact subinterval of I_0 with $b - a \geqslant d$, then for arbitrary $\eta \in \mathscr{D}[a, b]$ and $s \in [a, b]$ we have

$$\begin{aligned} J[\eta: a, b] \geqslant l_0[I_0, d] \left\{ |\eta(a)|^2 + |\eta(b)|^2 + |\eta(s)|^2 + \int_a^b |\eta'(t)|^2 \, dt \right\} \\ - l_1[I_0, d] \int_a^b |\eta(t)|^2 \, dt. \end{aligned} \tag{2.10}$$

As in the case of differential systems, an important class of systems (2.5) involves a nonnegative hermitian matrix function $B(t)$ which satisfies the following condition.

($\mathfrak{H}_B$: a, b). *If $[a, b]$ is a compact subinterval of I there exists a positive constant $k_0[a, b]$ such that $B^2(t) - k_0[a, b] B(t) \geqslant 0$ a.e. on $[a, b]$.*

With the aid of elementary integral inequalities one establishes readily the following result, which is somewhat complementary to that of Lemma 2.6. In this regard the reader is referred to [15, Problem VII.4.6] and a corresponding inequality in [15, p. 388].

LEMMA 2.7. *Suppose that $I_0 = [a_0, b_0]$ is a compact subinterval of I for which hypotheses ($\mathfrak{H}$(i, ii, iii): a_0, b_0) and ($\mathfrak{H}_B$: a_0, b_0) hold. Then there exists a constant $l_2[I_0]$ such that if $[a, b]$ is a nondegenerate subinterval of I_0 then for arbitrary $\eta \in \mathscr{D}[a, b]$ we have*

$$J[\eta: a, b] \leqslant l_2[I_0] \left\{ |\eta(a)|^2 + |\eta(b)|^2 + \int_a^b [|\eta'(t)|^2 + |\eta(t)|^2] \, dt \right\}. \tag{2.11}$$

Also, if $0 < d < b_0 - a_0$ *then there exists a constant* $l_3[I_0]$ *such that if* $[a, b]$ *is a compact subinterval of* I_0 *with* $b - a \geqslant d$ *then for arbitrary* $\eta \in \mathscr{D}[a, b]$ *we have*

$$\hat{J}[\eta: a, b] \leqslant l_3[I_0] \int_a^b \{|\eta'(t)|^2 + |\eta(t)|^2\}\, dt. \tag{2.11'}$$

As for the differential system to which (2.1), (2.2) reduces for $N(t, s) \equiv 0$, (see, for example, [15, Sects. VII.3, VII.9), for $[a, b]$ a compact subinterval of I let $\Lambda[a, b]$ denote the vector space of n-dimensional vector functions $v(t)$ which on $[a, b]$ are solutions of the differential equation

$$v'(t) + A^*(t)\, v(t) = 0$$

and satisfy $B(t)\, v(t) = 0$ a.e. on $[a, b]$. That is, $v \in \Lambda[a, b]$ if and only if $u(t) \equiv 0$, $v(t)$ is a solution of the integro-differential system (2.1). If $v \in \Lambda[a, b]$ and $\eta \in \mathscr{D}[a, b]: \zeta$, it follows readily that $v^*(t)\, \eta(t)$ is constant on $[a, b]$. Also, for a given integro-differential problem $(\mathscr{B})$ involving a subspace $\mathscr{S}$ of $\mathbf{C}_{2n}$, the subspace of $\Lambda[a, b]$ on which the $2n$-dimensional vector Dv belongs to $\mathscr{S}^\perp$ will be denoted by $\Lambda\{\mathscr{S}\}$. Clearly, $v \in \Lambda\{\mathscr{S}\}$ if and only if $u(t) \equiv 0$, $v(t)$ is a solution of $(\mathscr{B})$. If $\Lambda\{\mathscr{S}\}$ is zero-dimensional, the boundary problem $(\mathscr{B})$ is said to be *normal* or to have *order of abnormality equal to zero*, whereas if $\Lambda\{\mathscr{S}\}$ has dimension $\delta > 0$, the problem $(\mathscr{B})$ is said to be abnormal, with order of abnormality equal to δ. If w_ν, $(\nu = 1,..., 2n - d)$, is a basis for $\mathscr{S}^\perp$ and $v_\beta(t)$, $(\beta = 1,..., d_a)$, is a basis for $\Lambda[a, b]$, then $(\mathscr{B})$ is normal if and only if the $2n \times (2n - d + d_a)$ matrix $[w_\nu\ Dv_\beta]$ has rank $2n - d + d_a$. If $(\mathscr{B})$ has order of normality equal to $\delta > 0$, then this matrix has rank $2n - d + d_a - \delta$, and upon deleting a suitable set $w_\nu{}^*\eta = 0$, $(\nu = \nu_1, ..., \nu_\delta)$, of the conditions defining $\mathscr{S}$ the remaining conditions $w_\sigma{}^*\eta = 0$, $(\sigma \neq \nu_j, j = 1,..., \delta)$, defines a subspace $\mathscr{S}_\mu$ of $2n$-dimensional space that is of dimension $d + \delta$, is such that $\mathscr{S} \subset \mathscr{S}_\mu$, and the corresponding integro-differential problem

$$\begin{aligned} &\text{(i)}\quad L_1[u, v; \lambda](t) = 0, \qquad L_2[u, v] = 0, \qquad t \in [a, b] \\ &\text{(ii)}\quad \hat{u} \in \mathscr{S}_\mu, \qquad T[u, v] \in \mathscr{S}_\mu{}^\perp \end{aligned} \tag{2.11}$$

is normal. Moreover, since $\hat{v}_\beta{}^* D\hat{\eta} = 0$ for arbitrary $\eta \in \mathscr{D}[a, b]$ (see, [15, Lemma VII.3.2]), an n-dimensional vector function η belongs to $\mathscr{D}[\mathscr{B}]$ if and only if η belongs to $\mathscr{D}[\mathscr{B}_\mu]$. Also, $(\mathscr{B}_\mu)$ is a normal problem equivalent to the original problem $(\mathscr{B})$ in the following sense: If $(u(t); v(t))$ is a nonidentically vanishing solution of $(\mathscr{B}_\mu)$ then $u(t) \not\equiv 0$ on $[a, b]$, and $(u(t); v(t))$ is a solution of $(\mathscr{B})$, whereas if $(u(t); v(t))$ is a solution of $(\mathscr{B})$ there exist unique constants c_β, $(\beta = 1,..., d_a)$ such that $(u(t); v(t) + \Sigma_\beta c_\beta v_\beta(t))$ is a solution of $(\mathscr{B}_\mu)$.

Let $\mathscr{S}_a$ denote the $(2n - d_a)$-dimensional subspace of $\mathbf{C}_{2n}$ defined as

$$\mathscr{S}_a = \{\hat{\eta}: \hat{v}^* D\hat{\eta} = 0, \text{ for } v \in \Lambda[a, b]\}. \tag{2.12}$$

In particular, if $(\mathscr{B}^0)$ denotes the problem $(\mathscr{B})$ with $\mathscr{S}$ the zero-dimensional subspace of $\mathbf{C}_{2n}$, then the associated normal problem $(\mathscr{B}_\mu{}^0)$ determined by the above-described process is $(\mathscr{B})$ with $\mathscr{S} = \mathscr{S}_a{}^\perp$; that is, the system involving the integro-differential equation (2.1) and the boundary condition

$$\hat{u} \in \mathscr{S}_a{}^\perp, \qquad T[u, v] \in \mathscr{S}_a. \tag{2.13}$$

For a normal problem $(\mathscr{B})$, the condition that the matrix $[w_\gamma \;\; Dv_\beta]$ be of rank $2n - d + d_a$ is equivalent to the condition that

$$\dim\{\mathscr{S}[\mathscr{B}] \cap \mathscr{S}_a\}^\perp = \dim \mathscr{S}[\mathscr{B}]^\perp + \dim \mathscr{S}_a{}^\perp = (2n - d) + d_a,$$

so that $\dim\{\mathscr{S}[\mathscr{B}] \cap \mathscr{S}_a\} = d - d_a$. Now consider with a given problem $(\mathscr{B})$ a second problem $(\mathscr{B}_\#)$ involving the same integro-differential system (2.1) and the boundary conditions

$$\hat{u} \in \mathscr{S}_\#, \qquad T[u, v] \in \mathscr{S}_\#{}^\perp, \tag{2.14}$$

where $\mathscr{S}_\#$ is a second subspace of $\mathbf{C}_{2n}$. If $\dim \mathscr{S} = d$, $\dim \mathscr{S}_\# = d_\#$, and each of the systems $(\mathscr{B})$, $(\mathscr{B}_\#)$, is normal, then $\dim[\mathscr{S} \cap \mathscr{S}_a] = d - d_a$ and $\dim[\mathscr{S}_\# \cap \mathscr{S}_a] = d_\# - d_a$. If $\mathscr{S}_\# \cap \mathscr{S}_a \subset \mathscr{S} \cap \mathscr{S}_a$, then $d \geq d_\#$ and $(\mathscr{B}_\#)$ is called a subproblem of $(\mathscr{B})$ of dimension $d - d_\#$. If $d > d_\#$ then there exist $d - d_\#$ linear forms $\theta_\tau[\hat{\eta}] = \theta_\tau{}^*\hat{\eta}$, $(\tau = 1,\ldots, d - d_\#)$ such that

$$\mathscr{S}_\# \cap \mathscr{S}_a = \{\hat{\eta} : \hat{\eta} \in \mathscr{S} \cap \mathscr{S}_a,\ \theta_\tau[\hat{\eta}] = 0,\ \tau = 1,\ldots, d - d_\#\}. \tag{2.15}$$

Now suppose that the matrix function $K(t)$ satisfies the following additional hypothesis, which clearly holds if there exists a positive constant c such that $K(t) \geq cE_n$ a.e. on I.

($\mathfrak{H}$iv: a, b). *$[a, b]$ is a compact subinterval of I, and there exists a constant $c = c[a, b]$ such that*

$$K[\eta : a, b] \geq cE[\eta : a, b], \qquad \textit{for } \eta \in \mathscr{D}[a, b]. \tag{2.16}$$

In view of the results of Lemmas 2.1–2.5 we have the following property of boundary problems (2.1), (2.2).

LEMMA 2.8. *If the boundary problem $(\mathscr{B})$ defined by* (2.1), (2.5) *satisfies hypotheses* ($\mathfrak{H}$(i, ii, iii): a, b), ($\mathfrak{H}_K$: a, b), ($\mathfrak{H}$iv: a, b) *and is normal, then*

(i) *there exists a value λ_0 such that*

$$J[\eta; \lambda_0 : a, b] > 0 \textit{ for arbitrary } \eta \neq 0 \textit{ belonging to } \mathscr{D}[a, b]; \tag{2.17}$$

(ii) *all eigenvalues λ of the boundary value problem $(\mathscr{B})$ specified by* (2.1), (2.2) *are real and satisfy $\lambda > \lambda_0$;*

(iii) *if $(u; v)$ is an eigenfunction of $(\mathscr{B})$ corresponding to an eigenvalue λ, then $K[u : a, b] > 0$.*

3. Some Comparative Results for Integro-differential and Differential Boundary Problems

As the results presented in the lemmas of Section 2 are of the type that feature prominently in the treatment of ordinary differential boundary problems, one might surmise that the overall theory of integro-differential boundary problems may be developed in a manner highly analogous to that commonly used for differential systems, as in [15, Chap. VII]. That there are fundamental differences, however, is pointed out in [11, Sect. 4]. For example, in general it is not true that for given initial values u^0, v^0 there exists a solution $(u(t), v(t))$ of (2.1) assuming the values $u(t_0) = u^0$, $v(t_0) = v^0$ at a given initial value $t = t_0$. This possibility is illustrated by the integro-differential system (2.1), with $n = 1$, $A \equiv 0$, $B \equiv 1$, $C \equiv 0$, $N(t, s) = t + s$, $a = 0$, and b the positive zero of the polynomial $p(b) = b^8 + 408b^4 - 2880$. In this example the system (2.1) is equivalent to the scalar linear homogeneous integro-differential equation of the second order

$$u''(t) - \int_0^b (t + s)\, u(s)\, ds = 0, \qquad t \in [0, b]. \tag{3.1}$$

Clearly any solution of this equation is of the form

$$u(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3,$$

and upon substitution it is found that for b a positive zero of $p(b)$ there exists a solution of (3.1) satisfying $u(0) = u^0$, $u'(0) = v^0$ if and only if u^0 and v^0 satisfy a linear equation. Moreover, if u^0, v^0 are such that there exists a solution of (3.1) satisfying $u(0) = u^0$, $u'(0) = v^0$ then this solution is not unique, since $u(t) = 4b^5t^2 + (40 - 5b^4)\, t^3$ is a solution of (3.1) for which $u(0) = 0 = u'(0)$.

To illustrate another possibility for integro-differential systems that is different from the situation occurring for differential systems, consider the case of (2.1), (2.2) wherein $n = 1$, $A \equiv 0$, $B \equiv 1$, $C \equiv -1$, $N(t, s) \equiv (4\pi)^{-1}$, $a = 0$, $b = 4\pi$, and $\mathscr{S}$ is the zero-dimensional subspace of $\mathbf{C}_2$. The system (2.1), (2.2) is then equivalent to the scalar integro-differential boundary problem

$$\begin{aligned} &\text{(i)} \quad u''(t) + u(t) - (4\pi)^{-1} \int_0^{4\pi} u(s)\, ds, \qquad 0 \leqslant t \leqslant 4\pi, \\ &\text{(ii)} \quad u(0) = 0 = u(4\pi). \end{aligned} \tag{3.2}$$

Equation (3.2i) has the nonvanishing real solution $u(t) \equiv 1$, but the corresponding functional

$$J[\eta{:}\, 0, 4\pi] = \int_0^{4\pi} \{\eta'^2 - \eta^2\}\, dt + (4\pi)^{-1} \left(\int_0^{4\pi} \eta\, ds\right)^2 \tag{3.3}$$

is negative for certain values of η satisfying the prescribed end conditions $\eta(0) = 0 = \eta(4\pi)$. In particular, for $\eta(t) = \sin(t/2)$ we have

$$J[\eta: 0, 4\pi] = -3\pi/2.$$

A partial explanation of the differences between the theory of integro-differential systems and ordinary differential systems is afforded by the special case of systems (2.1), (2.2) wherein $N(t, s)$ is a "degenerate kernel," or a "kernel of finite rank." In particular, suppose that

$$N(t, s) = M^*(t)\, RM(s) \tag{3.4}$$

where $M(t)$ is a $k \times n$ matrix function locally of class $\mathfrak{L}^{\infty}_{kn}$ on I, and R is a nonsingular, constant, hermitian $k \times k$ matrix. An integro-differential boundary problem (2.1), (2.2) is then equivalent to a differential boundary problem in $(n + k)$-dimensional vector functions $\mathbf{u}(t) = (\mathbf{u}_\alpha(t))$, $\mathbf{v}(t) = (\mathbf{v}_\alpha(t))$, $(\alpha = 1,\dots, n + k)$, with $\mathbf{u}_\alpha(t) = u_\alpha(t)$, $\mathbf{v}_\alpha(t) = v_\alpha(t)$, $(\alpha = 1,\dots, n)$. For brevity we write $\mathbf{u}(t) = (u(t); u^1(t))$, $\mathbf{v}(t) = (v(t); v^1(t))$, where $u^1(t)$ and $v^1(t)$ are k-dimensional vector functions. Specifically, let $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$ denote the $(n + k) \times (n + k)$ matrix functions defined by

$$\mathbf{A}(t) = \begin{bmatrix} A(t) & 0 \\ RM(t) & 0 \end{bmatrix}, \qquad \mathbf{B}(t) = \operatorname{diag}\{B(t), 0\}, \qquad \mathbf{C}(t) = \operatorname{diag}\{C(t), 0\}, \tag{3.5}$$

and if the $2n \times 2n$ hermitian matrix Q is represented in terms of $n \times n$ matrices as

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & Q_3 \end{bmatrix}, \tag{3.6}$$

let $\mathbf{Q}$ denote the $2(n + k) \times 2(n + k)$ matrix represented in terms of corresponding $(n + k) \times (n + k)$ matrices

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{Q}_2^* & \mathbf{Q}_3 \end{bmatrix}, \tag{3.7}$$

where $\mathbf{Q}_1 = \operatorname{diag}\{Q_1, R^{-1}\}$, $\mathbf{Q}_2 = \operatorname{diag}\{Q_2, 0\}$, and $\mathbf{Q}_3 = \operatorname{diag}\{Q_3, 0\}$. Moreover, let $\mathbf{D} = \operatorname{diag}\{-E_{n+k}, E_{n+k}\}$. Finally, if the subspace $\mathscr{S}$ of $\mathbf{C}_{2n}$ in (2.2) is of dimension d, let $\mathbf{S}$ denote the subspace of $\mathbf{C}_{2(n+k)}$ specified by $\{\mathbf{u}: \hat{u} \in \mathscr{S},\ u_{n+\beta}(b) = 0, \beta = 1,\dots, k\}$. When $\mathbf{u}(t)$, $\mathbf{v}(t)$ is a solution of the differential boundary problem

$$\begin{aligned} &\text{(a)} \quad -\mathbf{v}'(t) + \mathbf{C}(t)\,\mathbf{u}(t) - \mathbf{A}^*(t)\,\mathbf{v}(t) = 0, \\ &\qquad\quad \mathbf{u}'(t) - \mathbf{A}(t)\,\mathbf{u}(t) - \mathbf{B}(t)\,\mathbf{v}(t) = 0, \\ &\text{(b)} \quad \hat{\mathbf{u}} \in \mathbf{S}, \qquad \mathbf{Q}\hat{\mathbf{u}} + \mathbf{D}\hat{\mathbf{v}} \in \mathbf{S}^{\perp}, \end{aligned} \tag{3.8}$$

then $v^1(t)$ is constant on $[a, b]$, and

$$u^1(t) = -\int_t^b RM(s)\, u(s)\, ds, \qquad t \in [a, b]. \tag{3.9}$$

Also, in terms of the component boundary vectors $\hat{u}$, $\hat{u}^1$, $\hat{v}$, $\hat{v}^1$ the boundary conditions (3.8b) are

$$\begin{aligned} &\text{(a)} \quad \hat{u} \in \mathscr{S}, \qquad \underset{\sim}{Q}\hat{u} + D\hat{v} \in \mathscr{S}^{\perp}, \\ &\text{(b)} \quad u^1(b) = 0, \qquad R^{-1}u^1(a) - v^1(a) = 0. \end{aligned} \tag{3.10}$$

In particular,

$$v^1(t) \equiv R^{-1}u^1(a) = -\int_a^b M(s)\, u(s)\, ds, \tag{3.11}$$

and from (3.8a) and (3.10a) it follows that $(u(t), v(t))$ is a solution of (2.1), (2.2). Conversely, if $(u(t); v(t))$ is a solution of (2.1), (2.2), then

$$\mathbf{u}(t) = (u(t); u^1(t), \mathbf{v}(t) = (v(t); v^1(t))$$

with $u^1(t)$ and $v^1(t)$ defined by (3.9) and (3.11), respectively, is a solution of (3.8).

Although the above examples do not illustrate the phenomenon in its full generality, one of the greatest differences between the theory of self-adjoint integro-differential equations (2.1) and the corresponding ordinary differential boundary problems occurring when $N(t, s) \equiv 0$ is that for the latter we have the concept of conjugate or conjoined solutions, whereas for (2.1) this condition is essentially lacking. Specifically, if $(u_1; v_1)$ and $(u_2; v_2)$ are two solutions of (2.1) then the function $v_2^*(t)\, u_1(t) - u_2^*(t)\, v_1(t)$ is a.c. and a.e. on $[a, b]$ we have

$$\begin{aligned} &[v_2^*(t)\, u_1(t) - u_2^*(t)\, v_1(t)]' + u_2^*(t) \left(\int_a^b N(t, s)\, u_1(s)\, ds\right) \\ &\quad - \left(\int_a^b u_2^*(s)\, N(s, t)\, ds\right) u_1(t) = 0. \end{aligned} \tag{3.12}$$

For $N(t, s) \equiv 0$ this relation implies that the function $v_2^*(t)\, u_1(t) - u_2^*(t)\, v_1(t)$ is constant, and when the value of this constant is zero the solutions $(u_1; v_1)$, $(u_2; v_2)$ are called mutually conjugate or conjoined. For the integro-differential system (2.1), however, Eq. (3.12) yields only that

$$v_2^*(b)\, u_1(b) - u_2^*(b)\, v_1(b) = v_2^*(a)\, u_1(a) - u_2^*(a)\, v_1(a).$$

4. The Green's Matrix for Integro-differential Systems

If λ is not an eigenvalue of (2.5) then using methods of Tamarkin [18] and Jonah [4], as in [11, Sect. 8], one may establish the existence of a Green's matrix for this integro-differential boundary problem. In this connection, it is to be noted that in [11, Sect. 8, paragraph 3] the final phrase "$y_i(x)\, z_i(x)$ is constant on ab" should be replaced by "$y_i(b)\, z_i(b) = y_i(a)\, z_i(a)$." In the case of a normal boundary problem (2.5) which satisfies hypotheses ($\mathfrak{H}$i, ii, iii: a, b) and ($\mathfrak{H}$iv: a, b); however, a more ready proof of existence and basic properties of a partial Green's matrix is afforded by the proof presented below of the following theorem.

Theorem 4.1. *If the boundary problem* (2.5) *satisfies hypotheses* ($\mathfrak{H}$i, ii, iii: a, b), ($\mathfrak{H}_K$: a, b), ($\mathfrak{H}$iv: a, b), *and is normal, then for λ not an eigenvalue of this problem there exist $n \times n$ matrix functions $G(t, s; \lambda)$, $G_0(t, s; \lambda)$ for $(t, s) \in \square = [a, b] \times [a, b]$ such that:*

(i) *$G(t, s; \lambda)$ is continuous in (t, s) on $\square$, is a.c. in each of these arguments on $[a, b]$ for fixed values of the other, and $G(t, s; \lambda) = [G(s, t; \bar{\lambda})]^*$* on $\square$.

(ii) *$G_0(t, s; \lambda)$ is continuous in (t, s) on each of the triangular domains $\Delta_1 = \{(t, s): (t, s) \in \square, s < t\}$ and $\Delta_2 = \{(t, s): (t, s) \in \square, t < s\}$, is bounded on $\square$, and the restriction of G_0 to Δ_α, ($\alpha = 1, 2$), has a finite limit at each point (s, s) with $s \in [a, b]$.*

(iii) *If $s \in [a, b]$, and ξ is an arbitrary vector in $\mathbf{C}_n$, then*

$$(u(t); v(t)) = (G(t, s; \lambda)\, \xi; G_0(t, s; \lambda)\, \xi)$$

is a solution of $L_2[u, v]\,(t) = 0$ on each of the nondegenerate subintervals $[a, s)$ and $(s, b]$; also, $\hat{u} \in \mathscr{S}$ and therefore $u \in \mathscr{D}[\mathscr{B}]: v$.

(iv) *If $f \in \mathfrak{L}_n^2[a, b]$, then the unique solution of the integro-differential system*

$$\begin{aligned} &(a) \quad L_1[u, v; \lambda]\,(t) = f(t), \qquad L_2[u, v]\,(t) = 0, \qquad t \in [a, b], \\ &(b) \quad \hat{u} \in \mathscr{S}[\mathscr{B}], \qquad T[u, v] \in \mathscr{S}^{\perp}[\mathscr{B}], \end{aligned} \tag{4.1}$$

is given by

$$u(t) = \int_a^b G(t, s; \lambda)\, f(s)\, ds, \qquad v(t) = \int_a^b G_0(t, s; \lambda)\, f(s)\, ds, \qquad t \in [a, b]. \tag{4.2}$$

(v) *If $\eta \in \mathscr{D}[\mathscr{B}]: \zeta$, and $(u(t); v(t))$ is the unique solution of the differential system*

$$\begin{aligned} &(a) \quad L_1[u, v; \lambda]\,(t) = K(t)\,\eta(t), \qquad L_2[u, v]\,(t) = 0, \qquad t \in [a, b], \\ &(b) \quad \hat{u} \in \mathscr{S}[\mathscr{B}], \qquad T[u, v] \in \mathscr{S}^{\perp}[\mathscr{B}], \end{aligned} \tag{4.3}$$

then

$$\hat{J}[u, \eta; \lambda: a, b] = K[\eta: a, b], \qquad \hat{J}[u; \lambda: a, b] = K[\eta, u: a, b]; \tag{4.4}$$

moreover, there exists a positive $k = k[\lambda: a, b]$ *such that*

$$K[u: a, b] \leqslant k^2 K[\eta: a, b].$$

Now for $k = k[a, b]$ as defined by (2.9) and $c = c[a, b]$ as in hypotheses ($\mathfrak{H}$iv: a, b), whenever λ_0 is a value satisfying conclusion (i) of Lemma 2.7 and $l < \lambda_0 - k/c$ the functional

$$\hat{J}^0[\eta; l: a, b] = \hat{\eta}^* Q \hat{\eta} + \int_a^b \{\zeta^* B \zeta + \eta^* C \eta\}\, dt - lK[\eta: a, b] \tag{4.5}$$

is positive definite on $\mathscr{D}[a, b]$, and hence, all eigenvalues λ of the differential boundary problem

$$\begin{aligned} &\text{(a)} \quad L_1^0[u, v; \lambda](t) \equiv -v'(t) + C(t)\, u(t) - A^*(t)\, v(t) - \lambda K(t)\, u(t) = 0, \\ &\qquad\; L_2^0[u, v; \lambda](t) \equiv L_2[u, v](t) \equiv u'(t) - A(t)\, u(t) - B(t)\, v(t) = 0, \\ &\text{(b)} \quad \hat{u} \in \mathscr{S}, \qquad T[u, v] \in \mathscr{S}^{\perp}, \end{aligned} \tag{4.6}$$

are real and satisfy $\lambda \geqslant \lambda_0 - k/c$.

Now let l be a fixed real value less than $\lambda_0 - k/c$, and denote by $G^0(t, s)$, $G_0^0(t, s)$ the partial Green's matrix of (4.6) as established in [15, Theorem VII.8.2]. Then these matrix functions possess properties of the sort specified in the above statement of Theorem 4.1, and for $\phi \in \mathfrak{L}_n^2[a, b]$ the unique solution of

$$\begin{aligned} &\text{(a)} \quad L_1^0[u, v; l](t) = \phi(t), \qquad L_2[u, v](t) = 0, \qquad t \in [a, b], \\ &\text{(b)} \quad u \in \mathscr{S}, \qquad T[u, v] \in \mathscr{S}^{\perp}, \end{aligned} \tag{4.7}$$

is given by

$$u(t) = \int_a^b G^0(t, s)\, \phi(s)\, ds, \qquad v(t) = \int_a^b G_0^0(t, s)\, \phi(s)\, ds. \tag{4.8}$$

As the integro-differential system (2.6) is of the form (4.7) with

$$\phi(t) = f(t) + (\lambda - l)\, K(t)\, u(t) - \int_a^b N(t, s)\, u(s)\, ds, \tag{4.9}$$

it follows that $(u(t); v(t))$ is a solution of (2.6) if and only if $u(t)$ is a solution of the vector Fredholm integral equation

$$u(t) = \int_a^b M(t, s; \lambda)\, u(s)\, ds + \int_a^b G^0(t, s)\, f(s)\, ds, \tag{4.10}$$

where

$$M(t, s; \lambda) = (\lambda - l)\, G^0(t, s)K(s) - \int_a^b G^0(t, \xi)\, N(\xi, s)\, d\xi, \qquad (4.11)$$

and $v(t)$ is given by the second equation of (4.8) with $\phi(t)$ expressed in terms of $u(t)$ as in (4.9).

Now the condition that λ is not an eigenvalue of (2.5) is equivalent to the condition that the homogeneous Fredholm equation

$$u(t) = \int_a^b M(t, s; \lambda)\, u(s)\, ds \qquad (4.12)$$

has only the identically zero solution, and hence, there exists a resolvent kernel matrix $H(t, s; \lambda)$ such that the solution of (4.10) is given by

$$u(t) = \int_a^b G^0(t, s) f(s)\, ds - \int_a^b H(t, \xi; \lambda) \left(\int_a^b G^0(\xi, s) f(s)\, ds \right) d\xi. \qquad (4.13)$$

Consequently, if we set

$$G(t, s; \lambda) = G^0(t, s) - \int_a^b H(t, \xi; \lambda)\, G^0(\xi, s)\, d\xi, \qquad (4.14)$$

$$\begin{aligned} G_0(t, s; \lambda) = G_0{}^0(t, s) &+ (\lambda - l) \int_a^b G_0{}^0(t, \xi)\, K(\xi)\, G(\xi, s; \lambda)\, d\xi \\ &- \int_a^b \int_a^b G_0{}^0(t, \rho)\, N(\rho, \xi)\, G(\xi, s; \lambda)\, d\xi\, d\rho, \end{aligned} \qquad (4.15)$$

the solution of (2.6) is given by (4.2). In view of corresponding properties of $G^0(t, s)$ and $G_0{}^0(t, s)$ as established in [15, Theorem VII.8.2], for fixed λ the matrix functions $G(t, s; \lambda)$ and $G_0(t, s; \lambda)$ are seen to have the continuity properties of conclusions (i) and (ii), as well as the solution properties of (iii) and (iv). Since under the stated hypothesis of Theorem 4.1 all eigenvalues of (2.5) are real, if λ is not an eigenvalue, then its complex conjugate $\bar{\lambda}$ is also not an eigenvalue, and the identity $G(t, s; \lambda) \equiv [G(s, t; \bar{\lambda})]^*$ may be established in a classical manner using the resolvent properties of $G(t, s; \lambda)$ and $G(t, s; \bar{\lambda})$ as stated in conclusion (iv) and the identity of Corollary 1 to Lemma 2.5. This result, which in view of (4.14) is equivalent to the property $F(t, s; \lambda) = [F(s, t; \bar{\lambda})]^*$ of the matrix function

$$F(t, s; \lambda) = \int_a^b H(t, \xi; \lambda)\, G^0(\xi, s)\, d\xi,$$

also may be derived directly using the resolvent equations satisfied by $M(t, s; \lambda)$ and $H(t, s; \lambda)$, together with the fact that

$$F_1(t, s; \lambda) = \int_a^b M(t, \xi; \lambda)\, G^0(\xi, s)\, d\xi$$

possesses the analogous property $F_1(t, s; \lambda) \equiv [F_1(s, t; \bar{\lambda})]^*$.

Finally, the results of conclusion (v) are ready consequences of the identity of Lemma 2.1.

Although the conclusions of Theorem 4.1 contain no statements on the character of the matrix functions $G(t, s; \lambda)$ and $G_0(t, s; \lambda)$ as functions of λ, from the general theory of integral equations it follows that they are meromorphic functions of λ in the complex plane with poles at the eigenvalues of (2.5). Also, if $\mathbf{C}$ is a compact set in the complex λ plane not containing an eigenvalue of (2.5), then $G(t, s; \lambda)$ is continuous in (t, s, λ) on $[a, b] \times [a, b] \times \mathbf{C}$, and $G_0(t, s; \lambda)$ is bounded for such valus of (t, s, λ), while for $\alpha = 1, 2$ the matrix function $G_0(t, s; \lambda)$ is continuous on $\Delta_\alpha \times \mathbf{C}$.

For a normal problem (2.5) satisfying a strengthened form of the hypotheses of Theorem 4.1, Theorem 5.3 presents a series expansion of $G(t, s; \lambda)$ in terms of the eigenfunctions of this problem.

5. Existence and Properties of Eigenvalues

In view of the results of Sections 2 and 4, for integro-differential boundary problems (2.5) the proofs of the existence and properties of eigenvalues may be carried out in a manner completely analogous to that employed for differential boundary problems in [15, Chap. VII, Sects. 10–12]. A basic existence theorem is the following result.

THEOREM 5.1. *If the boundary problem* (2.5) *satisfies hypotheses* ($\mathfrak{H}$i, ii, iii: a, b), ($\mathfrak{H}_K$: a, b), ($\mathfrak{H}$iv: a, b) *and is normal, then the eigenvalues of this problem may be ordered as a sequence* $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ *with corresponding eigenfunctions* $(u(t); v(t)) = (u_j(t); v_j(t))$ *such that:*

(a) $K[u_i, u_j: a, b] = \delta_{ij}, \quad (i, j = 1, 2, \ldots)$;

(b) $\lambda_1 = \hat{J}[u_1: a, b]$ *is the minimum of* $\hat{J}[\eta: a, b]$ *on the class*

$$\mathscr{D}_N[\mathscr{B}; K] = \{\eta \colon \eta \in \mathscr{D}[\mathscr{B}], K[\eta: a, b] = 1\}; \tag{5.1}$$

(c) *for* $j = 2, 3, \ldots$ *the class*

$$\mathscr{D}_{Nj}[\mathscr{B}; K] = \{\eta \colon \eta \in \mathscr{D}_N[\mathscr{B}; K], K[\eta, u_i: a, b] = 0, i = 1, \ldots, j-1\} \tag{5.2}$$

is nonempty and $\lambda_j = \hat{J}[u_j: a, b]$ *is the minimum of* $\hat{J}[\eta: a, b]$ *on* $\mathscr{D}_{Nj}[\mathscr{B}: K]$;

(d) $\{\lambda_j\} \to \infty$ *as* $j \to \infty$.

Indeed, if λ_1 is defined as the infimum of $\hat{J}[\eta: a, b]$ on $\mathscr{D}_N[\mathscr{B}: K]$, then λ_1 is not less than the constant λ_0 in conclusion (ii) of Lemma 2.8, and $\hat{J}[\eta; \lambda_1: a, b] \geqslant 0$ on $\mathscr{D}[\mathscr{B}]$. Moreover, if λ_1 were not an eigenvalue of (2.5), for an $\eta \not\equiv 0$ belonging to $\mathscr{D}[\mathscr{B}]$ let $(u; v)$ denote the solution of (4.3) for $\lambda = \lambda_1$, and as in conclusion (v) of Theorem 4.1 let k be a positive constant such that $K[u; a, b] \leqslant k^2 K[\eta: a, b]$. Then the inequality

$$\begin{aligned} 0 &\leqslant \hat{J}[\eta - (1/k)\, u; \lambda_1: a, b] \\ &= \hat{J}[\eta; \lambda_1: a, b] - k^{-1}\hat{J}[\eta, u; \lambda_1: a, b] - k^{-1}\hat{J}[u, \eta; \lambda_1: a, b] + k^{-2}\hat{J}[u; \lambda_1: a, b] \\ &= \hat{J}[\eta; \lambda_1: a, b] - 2k^{-1}K[\eta: a, b] + k^{-2}K[\eta, u: a, b], \end{aligned}$$

together with the Schwarz inequality

$$| K[\eta, u: a, b]|^2 \leqslant K[\eta: a, b]\, K[u; a, b] \leqslant k^2\{K[\eta: a, b]\}^2,$$

imply the result $0 \leqslant \hat{J}[\eta; \lambda_1: a, b] - k^{-1}K[\eta: a, b]$, which contradicts the definitive property of λ_1.

Correspondingly, the proof of the existence of a sequence of eigenvalues and eigenfunctions satisfying conclusions (a), (b), (c) of Theorem 5.1 proceeds by induction as in the case of the differential boundary problem to which (2.5) reduces in case $N(t, s) \equiv 0$. In particular, for the integro-differential boundary problem one may establish a result corresponding to that of [15, Theorem VII.10.4], or one may reduce the consideration of higher eigenvalues to the consideration of the smallest eigenvalue of an associated integro-differential boundary problem by the device of [15, Problem VII.10.2].

A ready manner in which to establish conclusion (d) is to consider the extremizing properties of the eigenvalues of (2.5) and the differential boundary problem (4.6). For $k = k[a, b]$ defined by (2.9) and $c = c[a, b]$ in hypothesis ($\mathfrak{H}$iv: a, b) we have for $\eta \in \mathscr{D}[\mathscr{B}]$ the inequality $\hat{J}[\eta: a, b] \geqslant \hat{J}^0[\eta; k/c: a, b]$, where the latter functional is defined by (4.5). Consequently, if $\{\lambda_j^0, u_j^0; v_j^0\}$, $(j = 1, 2, \ldots)$ denotes a sequence of eigenvalues and corresponding eigenfunctions for (4.6) which satisfies the corresponding conditions (a), (b), (c) of Theorem 5.1, then the extremizing property of eigenvalues implies $\lambda_j \geqslant \lambda_j^0 - k/c$, $(j = 1, 2, \ldots)$, and the conclusion (d) for the sequence $\{\lambda_j\}$ is a consequence of the corresponding result for the sequence $\{\lambda_j^0\}$.

It is to be noted that once the existence of a partial Green's matrix is established for (2.5) as in Theorem 4.1 the theory of this integro-differential problem is reducible to that of an associated vector integral equation with symmetrizable kernel. Specifically, if l is a real value not exceeding the λ_0 of (i) of Lemma 2.8, then $(u(t), v(t))$ is a solution of (2.5) if and only if $u(t)$ is a solution of the integral equation

$$u(t) = (\lambda - l) \int_a^b G(t, s; l)\, K(s)\, u(s)\, ds, \tag{5.3}$$

and $v(t)$ is defined by

$$v(t) = (\lambda - l) \int_a^b G_0(t, s; l) K(s) u(s) \, ds. \tag{5.4}$$

Since $G(t, s; l) = [G(s, t; l)]^*$ and $K(t) \equiv K^*(t)$, we then have satisfied the symmetrizability condition: $\mathscr{K}(t, s) = K(t) G(t, s; l) K(s)$ is such that $\mathscr{K}(t, s) \equiv [\mathscr{K}(s, t)]^*$, and $K[u: a, b] > 0$ for an arbitrary eigenfunction $u(t)$ of (5.3). In this connection, the reader is referred to [12] and associated references therein. In particular, the integral equation (5.3) presents a special fully symmetrizable transformation of the type II discussed in [12, Sect. 7].

In view of the definitive extremizing properties of the eigenvalues of (2.5), under the hypotheses of Theorem 5.1 one has for the system $\{\lambda_j, u_j; v_j\}$ of that theorem the well-known property that if r is a positive integer and $d_1, \ldots, d_r$ are constants such that $|d_1|^2 + \cdots + |d_r|^2 = 1$, then

$$\eta = d_1 u_1(t) + \cdots + d_r u_r(t), \qquad \zeta = d_1 v_1(t) + \cdots + d_r v_r(t)$$

are such that $\eta \in \mathscr{D}[\mathscr{B}]: \zeta$ and $\hat{J}[\eta: a, b] = \lambda_1 |d_1|^2 + \cdots + \lambda_r |d_r|^2 \leqslant \lambda_r$. Also, for $k = 1, 2, \ldots$ the eigenvalue λ_{k+1} possesses the maximum–minimum property that if $\mathscr{F} = \{f_1(t), \ldots, f_k(t)\}$ is a set of n-dimensional vector functions $f_j(t)$ of class $\mathfrak{L}_n{}^2[a, b]$, and $\lambda[\mathscr{F}]$ denotes the minimum of $\hat{J}[\eta: a, b]$ on the class of $\eta \in \mathscr{D}_N[\mathscr{B}; K]$ satisfying $E[\eta, f_j: a, b] = 0$, $(j = 1, \ldots, k)$, then $\lambda[\mathscr{F}] \leqslant \lambda_{k+1}$ and $\lambda[\mathscr{F}] = \lambda_{k+1}$ for the particular set $f_j(t) = K(t) u_j(t)$, $(j = 1, \ldots, k)$. Also, corresponding to [15, Theorems 11.3, 11.4], one may establish the following results.

THEOREM 5.2. *If the hypotheses of Theorem* 5.1 *are satisfied, then:*

(i) *if* $\eta \in \mathscr{D}[a, b]$, *and* $c_j[\eta] = K[\eta, u_j: a, b]$, $(j = 1, 2, \ldots)$, *then the infinite series* $\sum_{j=1}^{\infty} |c_j[\eta]|^2$ *and* $\sum_{j=1}^{\infty} \lambda_j |c_j[\eta]|^2$ *converge, and*

$$\sum_{j=1}^{\infty} |c_j[\eta]|^2 = K[\eta: a, b], \tag{5.5}$$

$$\sum_{j=1}^{\infty} \lambda_j |c_j[\eta]|^2 \leqslant \hat{J}[\eta: a, b]; \tag{5.6}$$

(ii) *if* λ *is not an eigenvalue of* (2.5), *then*

$$\sum_{j=1}^{\infty} |\lambda_j - \lambda|^{-2} u_j(t) u_j^*(s) = \int_a^b [G(r, t; \lambda)]^* K(r) G(r, s; \lambda) \, dr \tag{5.7}$$

for $(t, s) \in [a, b] \times [a, b]$; *in particular,*

$$\sum_{j=1}^{\infty} |\lambda_j - \lambda|^{-2} |u_j(t)|^2 = \operatorname{Trace} \left\{ \int_a^b [G(r, t; \lambda)]^* K(r) G(r, t; \lambda) \, dr \right\}. \tag{5.8}$$

Now let $(\mathfrak{H}_K': a, b)$ denote the following hypothesis.

$(\mathfrak{H}_K': a, b)$. *$K(t)$ is a nonnegative hermitian matrix function of class $\mathfrak{L}^{\infty}_{nn}[a, b]$ and such that if $\eta(t) \not\equiv 0$ is an element of $\mathfrak{C}_n[a, b]$ then $K[\eta; a, b] > 0$.*

Clearly on a given compact subinterval $[a, b]$ of I, hypothesis $(\mathfrak{H}_K': a, b)$ is a stronger form of the condition $(\mathfrak{H}_K: a, b)$.

Corresponding to [15, Theorem 11.5], one has the following result.

THEOREM 5.3. *Suppose that for a given compact subinterval $[a, b]$ of I the boundary problem* (2.5) *is normal and satisfies hypotheses* ($\mathfrak{H}$i, ii, iii: a, b), $(\mathfrak{H}_B: a, b)$, *and* $(\mathfrak{H}_K': a, b)$.

(a) *If $\eta \in \mathscr{D}[\mathscr{B}]$, then the series $\sum_{j=1}^{\infty} c_j[\eta]\, u_j(t)$ converges to $\eta(t)$, uniformly on $[a, b]$; also,*

$$\left\{\int_a^b \left| \eta'(t) - \sum_{j=1}^{m} c_j[\eta]\, u_j'(t) \right|^2 dt\right\} \to 0 \qquad as \qquad m \to \infty \tag{5.9}$$

and

$$\hat{J}[\eta: a, b] = \sum_{j=1}^{\infty} \lambda_j \,|\, c_j[\eta]|^2. \tag{5.10}$$

(b) *If λ is not an eigenvalue of* (2.5), *then*

$$G(t, s; \lambda) = \sum_{j=1}^{\infty} (\lambda_j - \lambda)^{-2}\, u_j(t)\, u_j^*(s) \qquad for\ (t, s) \in [a, b] \times [a, b], \tag{5.11}$$

and the series in (5.11) *converges uniformly on* $[a, b] \times [a, b]$.

6. Comparison and Oscillation Theorems

With the results of Section 5 one may proceed to establish comparison theorems which are almost verbatim analogs of such theorems for differential systems as are presented [15, Chap. VII, Sect. 12], and consequently, even detailed statements of such theorems will be omitted here. For the special type of integro-differential systems considered in [11], such comparison theorems are presented in [11, Sect. 5]. It is important to note, however, that in view of inequality (2.9) one has for an integro-differential boundary problem (2.5) satisfying the hypotheses of Theorem 5.1 comparison theorems relating its eigenvalues to those of a corresponding differential boundary problem wherein $N(t, s) \equiv 0$ and $C(t)$ is replaced by $C(t) - k[a, b]\, E$, or by $C(t) - (k[a, b]/c[a, b])\, K(t)$. In particular, if there exist positive constants

c_1, c_2 such that $c_1E \leqslant K(t) \leqslant c_2E$ on a compact subinterval $[a, b]$ of I, then with the aid of such simple comparison theorems one establishes for a system satisfying the hypotheses of Theorem 5.1 that $\sum_j' 1/\lambda_j^p$ converges if $p > \frac{1}{2}$, but diverges for $p \leqslant \frac{1}{2}$, where $\sum_j'$ denotes summation over those values of j for which $\lambda_j \neq 0$.

In the case of oscillation theorems, however, it is worthwhile to present specifically for a system (2.5) results analogous to those of [11, Sect. 6]. Suppose that hypotheses ($\mathfrak{H}$i, ii, iii) and $\mathfrak{H}_K$: a, b) are satisfied on arbitrary nondegenerate compact subintervals $[a, b]$ of I. Moreover, suppose that the end-form $\hat{\eta}^*Q\hat{\eta}$ involves only the values $\eta(a)$, that is, $Q = \text{diag}\{Q_{11}, 0\}$ where Q_{11} is an $n \times n$ hermitian matrix, and that Θ is an $n \times q$ matrix of rank q and column vectors $\Theta_1, \ldots, \Theta_q$. For $b > a$ and $[a, b] \subset I$, let $\mathscr{B}\{\Theta, b\}$ denote the normal boundary problem determined by the integro-differential equations (2.5i, ii), and the boundary conditions

$$\Theta^*\eta(a) = 0, \qquad \eta(b) = 0. \tag{6.1}$$

That is, the subspace $\mathscr{S}$ of $\mathbf{C}_{2n}$ belonging to this problem consists of those $\hat{\eta}$ satisfying

$$M\hat{\eta} = 0, \tag{6.2}$$

where M is the $(q + n) \times (2n)$ matrix of the form

$$M = \begin{bmatrix} \Theta^* & 0 \\ 0 & E_n \end{bmatrix}. \tag{6.3}$$

For $b > a$ and $[a, b] \subset I$, let r_b denote the order of abnormality of (2.5i, ii), and let $V(t)$ be an $n \times r_b$ matrix function whose column vectors form a basis for $\Lambda[a, b]$; that is, $V(t)$ is of constant rank r_b on $[a, b]$, and

$$V'(t) + A^*(t)\, V(t) = 0, \qquad B(t)\, V(t) = 0$$

on this interval, so that every solution $u(t) \equiv 0$, $v(t)$ of (2.5i, ii) on $[a, b]$ is of the form $v(t) = V(t)\,\rho$ for some constant r_b-dimensional vector ρ. If the $n \times (q + r_b)$ matrix $[\Theta \;\; V(a)]$ is of rank $q + r_b - k_b$, then $0 \leqslant k_b \leqslant q$ and there exist $p_b = q - k_b$ values $1 \leqslant \sigma_1 < \sigma_2 < \cdots < \sigma_{p_b}$ such that the $n \times (q + r_b - k_b)$ matrix

$$[\Theta_{\alpha\gamma} \;\; V_{\alpha j}(a)], \qquad (\gamma = \sigma_1, \sigma_2, \ldots, \sigma_{p_b}; j = 1, \ldots, r_b; \alpha = 1, \ldots, n) \tag{6.4}$$

is of rank $p_b + r_n = q - k_b + r_n$, so that for the normal boundary problem determined by (2.5i, ii) and (6.1) the subspace $\mathscr{S}$ of $\mathbf{C}_{2n}$ is specified by

$$\Theta_\gamma^*\eta(a) = 0, \qquad \eta(b) = 0, \qquad (\gamma = \sigma_1, \ldots, \sigma_{p_b}). \tag{6.5i}$$

Also, if Φ is an $n \times (n - p_b)$ matrix of rank $n - p_b$ such that $\Theta_\gamma{}^*\Phi = 0$, $(\gamma = \sigma_1, \ldots, \sigma_{p_b})$, then the end-values of a solution (u, v) of (2.5i, ii) are such that $T[u, v] \in \mathscr{S}^\perp$ if and only if

$$\Phi^*[Q_{11}u(a) - v(a)] = 0. \tag{6.5ii}$$

It is to be remarked that the choice of the $\sigma_1, \ldots, \sigma_{p_b}$ may be the same for all values of b on a subinterval $[a_0, b_0]$ of $I_a = \{t : t \in I, t > a\}$ and such that r_b is constant for $b \in [a_0, b_0]$. Thus the normal integro-differential boundary problem for $b \in I_a$ involves the integro-differential equations (2.5i, ii), and the two-point boundary conditions (6.5i, ii). Since for this problem the boundary condition at $t = b$ is $\eta(b) = 0$, it is to be noted that for $a < b < c$ and $[a, c] \subset I$, if $\eta \in \mathscr{B}[\Theta, b]: \zeta$ then $\eta(b) = 0$ and for

$$(\eta_0(t), \zeta_0(t)) = (\eta(t), \zeta(t))$$

on $[a, b]$, $(\eta_0(t), \zeta_0(t)) \equiv (0, 0)$ on $(b, c]$ we have that $\eta_0 \in \mathscr{B}[\Phi, c]: \zeta_0$.

A value $b \in I_a$ is called a *focal point* of $t = a$ relative to the system (2.5i, ii), (6.5i, ii) for $\lambda = \lambda_0$ if:

(i) $\lambda = \lambda_0$ *is an eigenvalue of* $\mathscr{B}\{\Theta, b\}$;

(ii) *there is at least one corresponding eigenfunction* $(u(t); v(t))$ *of* $\mathscr{B}\{\Theta, b\}$ *such that for* $c \in I_b$ *there exists no function* $v_0(t)$ *defined on* $[a, c]$ *and forming with* $u_0(t) \equiv u(t)$ *on* $[a, b]$, $u_0(t) \equiv 0$ *on* $[b, c]$ *an eigensolution* $u_0(t)$, $v_0(t)$ *of* $\mathscr{B}\{\Theta, c\}$.

If $(u(t), v(t))$ is a solution of $\mathscr{B}\{\Theta, b\}$ satisfying the above conditions (i), (ii), then $u = u(t)$, $t \in [a, b]$ is said to be an arc *determining* $t = b$ *as a focal point* of $t = a$ relative to the system (2.5i, ii), (6.5i, ii), and the dimension of the linear space of such determining arcs is called the order of $t = b$ as a focal point of $t = a$. If the matrix $[\Theta \;\; V(a)]$ is of rank n, so that an arc $\eta \in \mathscr{D}[\mathscr{B}\{\Theta, b\}]$ must satisfy $\eta(a) = 0$, then the corresponding focal points are called *conjugate points* of $t = a$, relative to the integro-differential system (2.5i, ii); in this case, for brevity, $\mathscr{B}\{\Theta, b\}$ is called the ***null end-point problem*** and denoted by $\mathscr{B}^0\{b\}$.

For $a \in I$ and $b \in I_a$, we shall denote by $\lambda_j(b)$, $(u, v) = (u_j(t : b), v_j(t : b))$, $(j = 1, 2, \ldots)$, a set of eigenvalues and eigenfunctions of $\mathscr{B}\{\Theta, b\}$, supposed ordered and orthonormal in the sense of Theorem 5.1.

Since, as noted above, the integro-differential boundary problem $\mathscr{B}\{\Theta, b\}$ may be compared with a differential boundary problem involving the same two-point boundary conditions, the following result for $\mathscr{B}\{\Theta, b\}$ is a consequence of the corresponding result for differential boundary problems.

LEMMA 6.1. $\lambda_1(b) \to +\infty$ *as* $b \to a^+$.

For $\mathscr{B}\{\Theta, b\}$ we also have the following result, which may be proved by exactly the same method as that used to establish [11, Lemma 6.2]. The pertinent solvability results for certain systems occurring in the proof are now provided by conclusion (iv) of Theorem 4.1.

LEMMA 6.2. *Each of the eigenvalues $\lambda_j(b)$ of $\mathscr{B}\{\Theta, b\}$ is a continuous monotone nonincreasing function on I_0 and $\lambda_j(b) \to +\infty$ as $b \to a^+$.*

With the results of Lemmas 6.1 and 6.2, an argument similar to that used to establish [11, Theorem 6.1] now yields the corresponding result.

THEOREM 6.1. *For $c \in I_a$ and a given value $\lambda = \tilde{\lambda}$, the number of points on the open interval (a, c) which are focal points to $t = a$ relative to* (2.5i, ii), (6.5i, ii) *for $\lambda = \tilde{\lambda}$ is equal to the number of eigenvalues $\lambda_j(c)$ of $\mathscr{B}\{\Theta, c\}$ which are less than $\tilde{\lambda}$, where each focal point is counted a number of times equal to its order.*

For a given real value l let $V_l(b)$ denote the number of eigenvalues of $\mathscr{B}\{\Theta, b\}$ less than l. As a result of one of the simplest comparison theorems for such problems it then follows that the number of conjugate points of $t = a$ relative to (2.5i) for $\lambda = l$, and located in the open interval (a, b) is at least $V_l(b) - d$ and at most V_l, where d denotes the dimension of the null end-point problem $\mathscr{B}^0\{b\}$ as a subproblem of $\mathscr{B}\{\Theta, b\}$.

As an example, for $b > 0$ consider the canonical system

$$\begin{gathered} -v'(t) - u(t) + (4\pi)^{-1} \int_0^b u(s)\, ds = 0, \\ u'(t) - v(t) = 0, \qquad u(0) = 0, \qquad u(b) = 0, \end{gathered} \tag{6.6}$$

which for $b = 4\pi$ reduces to system (3.2). In this case, the kernel function $K(t, s) \equiv 1$ is of rank 1, and, as noted in Section 3, the determination of solutions of (6.6) is equivalent to solving the differential boundary problem

$$\begin{aligned} &\text{(a)} \quad -v'(t) - u(t) + (4\pi)^{-1} v^1(t) = 0, \qquad -v^{1\prime}(t) = 0, \\ &\qquad\ u'(t) - v(t) = 0, \qquad u^{1\prime}(t) + (4\pi)^{-1} u(t) = 0, \\ &\text{(b)} \quad u(0) = 0, \qquad 4\pi u^1(0) - v^1(0) = 0, \\ &\qquad\ u(4\pi) = 0, \qquad u^1(4\pi) = 0. \end{aligned} \tag{6.7}$$

In particular, for this differential system the conjoined basis for (6.7a) determined by the end conditions of (6.7b) at $t = 0$ is given by the 4×2 matrix $[U(x); V(x)]$ with

$$\begin{aligned} &\text{(a)} \quad U(t) = \begin{bmatrix} \sin t & (4\pi)^{-1}[1 - \cos t] \\ (4\pi)^{-1}[\cos t - 1] & (4\pi)^{-1} - (16\pi^2)^{-1}[t - \sin t] \end{bmatrix}, \\ &\text{(b)} \quad V(t) = \begin{bmatrix} \cos t & (4\pi)^{-1} \sin t \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{6.8}$$

The values $t \in I_0$ which define conjugate points to $t = 0$ for (6.6) are then the values for which the matrix $U(t)$ is singular, and an easy computation shows that these conjugate points are the values $t = 2s$, where $s > 0$ and either $\sin s = 0$ or $\tan s = s - 2\pi$. In particular, for this system there are three focal points to $t = 0$ on $(0, 4\pi]$: $\tau_1 = 2s_1$, where $s = s_1$ is the root on $(\pi/2, \pi)$ of the equation $\tan s = s - 2\pi$, $\tau_2 = 2\pi$, and $\tau_3 = 4\pi$.

It is to be emphasized that the problem of determining the points conjugate to $t = 0$ relative to the integro-differential system (6.6) is distinct from that of determining the points conjugate to $t = 0$ relative to the differential system (6.7a). For this latter problem the conjoined basis determining the points conjugate to $t = 0$ is the 4×2 matrix $(U_1(t); V_1(t))$ with

$$\begin{aligned} &\text{(a)} \quad U_1(t) = \begin{bmatrix} \sin t & (4\pi)^{-1}(1 - \cos t) \\ (4\pi)^{-1}(\cos t - 1) & -(16\pi^2)^{-1}(t - \sin t) \end{bmatrix}, \\ &\text{(b)} \quad V_1(t) = \begin{bmatrix} \cos t & (4\pi)^{-1}\sin t \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{6.9}$$

That is, the points conjugate to $t = a$ relative to this ordinary differential system are the values for which the matrix $U_1(t)$ is singular, and these values are $t = 2r$ where $r > 0$ and either $\sin r = 0$ or $\tan r = r$. Thus, on $(0, 4\pi]$ there are three such conjugate points: $\tau_1 = 2\pi$, $\tau_2 = 2r_1$, where $r = r_1$ is the root on $(\pi, 3\pi/2)$ of $\tan r = r$, and $\tau_3 = 4\pi$.

As in [11, Sect. 7], the results that have been described for integro-differential boundary problems linear in the parameter may be used to establish results on existence of eigenvalues, comparison and oscillation for similar problems nonlinear in the parameter, and wherein the functional corresponding to J satisfies certain monotoneity conditions. For brevity, however, such results will not be presented in detail.

7. General Comments

As noted in the examples of Section 3, in general for integro-differential equations there do not exist results on the existence of solutions satisfying given initial values, as hold in the case of differential equations. Consequently, for boundary problems involving integro-differential equations the methods of Morse [8–10] using "broken extremals" are no longer available for the derivation of oscillation and comparison theorems. In particular, for integro-differential systems (2.1) wherein $N(t, s)$ is of the degenerate form (3.4) the results involving a given problem and its subproblems are equivalent to corresponding problems and subproblems for the enlarged differential system (3.8), and thus in such cases the comparison theorems for the integro-differential systems are deducible from corresponding comparison theorems for the associated differential system. As illustrated by the example considered at the end of Section 6, for such integro-differential systems the problem of focal points is equivalent to a corresponding focal point problem for the associated differential system, although the specific conjugate point problem for the integro-differential system is not the same as the conjugate point problem for the related differential system. In this connection, however, a fundamental property of the treatment is that presented in Section 6, to the effect that if $a < b < c$ and $[a, c] \subset I$, then whenever $\eta \in \mathscr{B}[\Theta, b]: \zeta$ with $\eta(b) = 0$ then $(\eta_0(t), \zeta_0(t)) = (\eta(t), \zeta(t))$ on $[a, b]$, $(\eta_0(t), \zeta_0(t)) \equiv (0, 0)$ on $(b, c]$ is such that $\eta_0 \in \mathscr{B}[\Theta, c]: \zeta_0$. Because of this property, the oscillation theory for integro-differential systems of the form (2.5i, ii) may be considered in the setting of the general theory of Hestenes [3], although for the derivation of certain aspects of this problem in that context it appears that one needs much of the structure of the present discussion, especially that of Sections 2 and 4. As far as the consideration of comparison theorems, it also appears that a modified Weinstein method in the general character of Weinberger [19] may be used. Again, an important aspect of the treatment involves the partial Green's matrix $G(t, s; \lambda)$ and its expansion as presented in Theorem 5.3.

An important open question is the general relationship between the theory of a given integro-differential problem (2.5) and associated problems involving kernels of finite rank which approximate $N(t, s)$. Specifically, from the Hilbert–Schmidt theory of Fredholm equations with hermitian kernels it follows that the eigenvalues of the vector integral equation

$$\phi(t) = \sigma \int_a^b N(t, s)\, \phi(s)\, ds, \qquad t \in [a, b] \tag{7.1}$$

are all real, and the totality of eigenvalues and corresponding eigenfunctions may be arranged as a sequence $\{\sigma_j, \phi_j(t)\}$ with $|\sigma_j| \leqslant |\sigma_{j+1}|$, $(j = 1, 2, \ldots)$, $\int_a^b \phi_i^*(s)\, \phi_j(s)\, ds = \delta_{ij}$, $(i, j = 1, 2, \ldots)$, and each eigenvalue occurring in the sequence a number of times equal to its multiplicity. Also, $|\sigma_j| \to \infty$ as $j \to \infty$ and $N(t, s) = \sum_{j=1}^{\infty} (1/\sigma_j)\, \phi_j(t)\, \phi_j^*(s)$ in the sense of the Hilbert space $\mathfrak{L}^2\{[a, b] \times [a, b]\}$. Indeed, with the σ_j arranged in nondecreasing absolute value, if

$$N_m(t, s) = \sum_{j=1}^{m} (1/\sigma_j)\, \phi_j(t)\, \phi_j^*(s), \qquad m = 1, 2, \ldots \tag{7.2}$$

then the $\mathfrak{L}^2$-norm of $N(t, s) - N_m(t, s)$ on $[a, b] \times [a, b]$ is equal to $1/|\sigma_{m+1}|$, and thus tends to zero as $m \to \infty$. As $N_m(t, s)$ is of the form (3.4) with $M(t)$ the $m \times m$ matrix with row vectors $\phi_j^*(t)$, $(j = 1, \ldots, m)$, and R the real diagonal matrix $[(1/\sigma_j)\, \delta_{ij}]$, $(i, j = 1, \ldots, m)$, the modified integro-differential problem with $N(t, s)$ replaced by $N_m(t, s)$ is reducible to an ordinary differential equation problem in $(n + m)$-dimensional vector functions $\mathbf{u}(t)$, $\mathbf{v}(t)$ as defined in Section 3. If $\hat{J}_m[\eta: a, b]$ denotes the functional (2.3) with $N(t, s)$ replaced by $N_m(t, s)$, then for $\eta \in \mathscr{D}[a, b]$ we have

$$|\hat{J}[\eta: a, b] - \hat{J}_m[\eta: a, b]| \leqslant (1/|\sigma_{m+1}|)\, E[\eta: a, b]. \tag{7.3}$$

Also, whenever condition ($\mathfrak{H}$iv: a, b) is satisfied, we have

$$|\hat{J}[\eta: a, b] - \hat{J}_m[\eta: a, b]| \leqslant (1/[c\,|\sigma_{m+1}|])\, K[\eta: a, b], \tag{7.4}$$

for arbitrary $\eta \in \mathscr{D}[a, b]$. Consequently, whenever the hypotheses of Theorem 5.1 are satisfied, and for the problem involving $N_m(t, s)$ the eigenvalues and eigenfunctions $\{\lambda_j^m, u_j^m(t); v_j^m(t)\}$ are ordered as in that theorem, for the eigenvalues and eigenfunctions $\{\lambda_j, u_j(t); v_j(t)\}$ of the given problem we have for $m = 1, 2, \ldots$, the comparison result

$$|\lambda_j - \lambda_j^m| \leqslant (1/[c\,|\sigma_{m+1}|]) \qquad (j = 1, 2, \ldots). \tag{7.5}$$

Finally, it is to be noted that the theory of integro-differential systems as discussed herein may be extended to "generalized integro-differential systems" similar to the "generalized differential systems" considered by the author in [13, 16, 17]. Moreover, as noted in [17, Section 3], certain types of Fredholm–Stieltjes integral equations, including those considered by Krall [5], are equivalent to such generalized differential equations. For such problems the functional $\hat{J}[\eta: a, b]$ is replaced by a functional of the form

$$\begin{aligned} \hat{\eta}^* Q \hat{\eta} + \int_a^b \{\zeta^* B \zeta + \eta^* C \eta\}\, dt + \int_a^b \int_a^b \eta^*(t)\, N(t, s)\, \eta(s)\, ds\, dt \\ + \int_a^b \eta^*(t)\, [dM(t)]\, \eta(t), \end{aligned} \tag{7.6}$$

where $M(t)$ is an $n \times n$ hermitian matrix function of bounded variation on $[a, b]$. System (2.1) is correspondingly replaced by the system

$$\begin{aligned} -dv(t) + \left\{C(t)\, u(t) - A^*(t)\, v(t) + \int_a^b N(t, s)\, u(s)\, ds\right\} dt + [dM(t)]\, u(t) = 0, \\ u'(t) - A(t)\, u(t) - B(t)\, v(t) = 0. \end{aligned} \tag{7.7}$$

No detailed discussion of boundary problems associated with such systems will be presented, however, since under hypotheses of the sort considered in the preceding sections, system (7.7) is reducible to a corresponding system of the form (2.1) in the vector functions $\tilde{u}(t) = u(t)$, $\tilde{v}(t) = v(t) - M(t)\, u(t)$. In this connection, for differential problems the reader is referred to [13, Theorem 2.3].

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