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# Time periodic solution of the viscous Camassa–Holm equation ☆

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**Abstract**

This paper discusses the viscous Camassa–Holm equation with a periodic boundary condition. The existence and uniqueness of a time periodic solution are investigated by using the Galerkin method and Leray–Schauder fixed point theorem.


*Keywords:* Camassa–Holm equation; Time periodic solution; Galerkin method; Leray–Schauder fixed point theorem

## 1. Introduction and preliminaries

Recently, a considerable amount of research activity has focussed on the study of the Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x = uu_{xxx} + 2u_x u_{xx},$$

which models the unidirectional propagation of shallow water waves over a flat bottom, $u(t,x)$ stands for the fluid velocity at time $t$ in the spatial $x$ direction (see [1]). Various works for the system have been investigated in [2–10] and their references, such as the existence and stability of solitary waves and various other types of solution of the Camassa–Holm equation. In [11], C. Foias et al. considered the three-dimensional viscous Camassa–Holm equation

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$$\frac{\partial}{\partial t}\big(\alpha_0^2 u - \alpha_1^2 \Delta u\big) - \gamma \Delta\big(\alpha_0^2 u - \alpha_1^2 \Delta u\big) - u \times \big(\nabla \times \big(\alpha_0^2 u - \alpha_1^2 \Delta u\big)\big) + \frac{1}{\rho_0}\nabla p = f,$$
$$\nabla \cdot u = 0,$$

where $\gamma > 0$ is the constant viscosity. In their paper, the existence of a global regular solution was obtained and the estimates for the Hausdorff and fractal dimensions of its global attractor were provided.

In this paper, we are interested in the one-dimensional viscous Camassa–Holm equation

$$u_t - u_{xxt} - \gamma(u_{xx} - u_{xxxx}) + 3uu_x = uu_{xxx} + 2u_x u_{xx} + f(t,x), \quad t \in R,\ x \in R, \tag{1}$$

$$u(t, x+L) = u(t,x), \quad t \in R,\ x \in R, \tag{2}$$

$$u(t+\omega, x) = u(t,x), \quad t \in R,\ x \in R, \tag{3}$$

where $\gamma > 0$ is the constant viscosity and the forcing term $f$ is $\omega$-periodic in time $t$ and $L$-periodic in spatial $x$. Without loss of generality, we assume further $\int_\Omega f\,dx = 0$, $\Omega = [0, L]$. When system is periodically dependent on time $t$, a natural problem is caused, that is whether there exists time-periodic solution with the same period for the system. In many nonlinear evolution equations, the study of time-periodic solution has attracted considerable interest (for example [12,13]). In this paper, we shall prove that Eqs. (1)–(3) have a solution by using the Galerkin method (see [15]) and Leray–Schauder fixed point theorem (see [13]).

Our organization for this paper is as follows. In Section 2, we prove the existence of the approximate solution and give uniform a priori estimates needed where we prove the convergence of sequence of the approximate solution. Section 3 is devoted to the proof of existence and uniqueness of time-periodic solution for Eqs. (1)–(3).

Before starting our work, we first introduce some notations and inequalities which will be needed later.

Let $X$ be a Banach space, we denote by $C^k(\omega; X)$ the set of $\omega$-periodic $X$-valued functions on $R^1$ with continuous derivatives up to order $k$. We define the norm:

$$\|u\|_{C^k(\omega;X)} = \sup_{0 \leqslant t \leqslant \omega}\left\{\sum_{i=0}^{k}\big\|D_t^i u(t)\big\|_X\right\}.$$

We denote $L^p(\omega; X)$ $(1 \leqslant p \leqslant \infty)$ the set of $\omega$-periodic $X$-valued measurable functions on $R$ such that

$$\|u\|_{L^p(\omega;X)} = \left(\int_0^\omega \|u\|_X^p\,dt\right)^{1/p} < \infty \quad (1 \leqslant p < \infty),$$
$$\|u\|_{L^\infty(\omega;X)} = \sup_{0 \leqslant t \leqslant \omega}\|u\|_X < \infty.$$

We denote by $W^{k,p}(\omega; X)$ the set of functions which belong to $L^p(\omega; X)$ together with their derivatives up to order $k$, and in particular we write $H^k(\omega; X) = W^{2,k}(\omega; X)$ when $X$ is a Hilbert space. $L^p(\Omega)$, $H^m(\Omega)$ are usual Sobolev spaces. For simplicity, we denote $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$ as $p \neq 2$ and $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$.

In this paper the following inequalities (see [14]) will be used in the proofs later:

$$\|u\|_\infty \leqslant k_1\|u\|_{H^1}, \tag{4}$$

$$\big\|D^j u\big\|_p \leqslant k_2\|u\|_{H^m}^a\|u\|^{1-a}, \tag{5}$$

where $D^j u = \frac{\partial^j u}{\partial x^j}$, $\frac{1}{p} = j + a(\frac{1}{2} - m) + (1-a)\frac{1}{2}$ as $0 \leqslant j < m$, $\frac{j}{m} \leqslant a \leqslant 1$.

$$\|u\| \leqslant k_3 \|u_x\|, \qquad \int_\Omega u(x)\,dx = 0. \tag{6}$$

## 2. A priori estimates

In this section we first prove that Eqs. (1)–(3) have a sequence of approximate solutions $\{u_n\}_{n=1}^\infty$, then give a priori estimates about $\{u_n\}_{n=1}^\infty$.

We set the unbounded linear operator $Au = -u_{xx}$ on $X = L^2 \cap \{u \mid u(x+L) = u(x), \int_\Omega u\,dx = 0\}$, then the set of all linearly independent eigenvectors $\{w_j\}_{j=0}^\infty$ of $A$, i.e., $Aw_j = \lambda_j w_j$, with $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \to \infty$, is an orthonormal basis of $L^2(\Omega)$. For any number $n$ and a group of function $\{a_{jn}(t)\}_{j=1}^n$, where $a_{jn}$ $(j = 1, \ldots, n) \in C^1(\omega; R)$, the function $u_n(t) = \sum_{j=1}^n a_{jn}(t) w_j \in C^1(\omega; H_n)$ is called an approximate solution to (1)–(3) if it satisfies the equation system as follows:

$$\big(u_{nt} - u_{nxxt} - \gamma(u_{nxx} - u_{nxxxx}), w_j\big) = (Nu_n + f, w_j), \quad j = 1, \ldots, n, \tag{7}$$

where $Nu_n = -3u_n u_{nx} + u_n u_{nxxx} + 2u_{nx}u_{nxx}$ and $H_n = \operatorname{span}\{w_1, w_2, \ldots, w_n\}$. By the classical theory of ordinary differential equations, for any fixed $v_n(t) = \sum_{j=1}^n b_{jn}(t) w_j \in C^1(\omega; H_n)$ the system

$$\big(u_{nt} - u_{nxxt} - \gamma(u_{nxx} - u_{nxxxx}), w_j\big) = (Nv_n + f, w_j), \quad j = 1, \ldots, n,$$

has a unique $\omega$-periodic solution $u_n$ and the mapping $F: v_n \to u_n$ is continuous and compact in $C^1(\omega, H_n)$. Hence by Leray–Schauder fixed point theorem, we want to prove the existence of an approximate solution only to show the boundedness $\sup_{0 \leqslant t \leqslant \omega} \|u_n\|^2 \leqslant c$ for all possible solutions of (7) replaced by $\lambda N_n$ $(0 \leqslant \lambda \leqslant 1)$ instead of nonlinear term $Nu_n$, where $c$ is a constant which only depends on $L, \lambda_1, \omega, \gamma$ and $f$.

**Lemma 2.1.** *If $f \in C^1(\omega; H^{-1}(\Omega))$, then*

$$\sup_{0 \leqslant t \leqslant \omega} \big(\|u_n\|^2 + \|u_{nx}\|^2\big) \leqslant \left(\frac{1}{d} + \omega\right) \frac{M}{\gamma},$$

*where $M = \sup_{0 \leqslant t \leqslant \omega} \{\|f(x,t)\|_{H^{-1}(\Omega)}^2\}$ and $d = \min\{2\gamma\lambda_1, \gamma\}$.*

**Proof.** Multiplying (7) by $a_{jn}(t)$ and summing up over $j$ from 1 to $n$, we have

$$\big(u_{nt} - u_{nxxt} - \gamma(u_{nxx} - u_{nxxxx}), u_n\big) = (Nu_n + f, u_n).$$

This gives us

$$\frac{1}{2}\frac{d}{dt}\big(\|u_n\|^2 + \|u_{nx}\|^2\big) + \gamma\big(\|u_{nx}\|^2 + \|u_{nxx}\|^2\big) = (Nu_n + f, u_n).$$

Notice that

$$\int_\Omega u_n^2 u_{nx}\,dx = 0, \qquad \int_\Omega u_n^2 u_{nxxx}\,dx + \int_\Omega u_n u_{nx} u_{nxx}\,dx = 0,$$

$$\|u_{nxx}\|^2 = \int_\Omega \left| \left( \sum_{j=1}^n a_{jn}(t) w_j \right)_{xx} \right|^2 dx = \int_\Omega \left| \sum_{j=1}^n \lambda_j a_{jn}(t) w_j \right|^2 dx \geqslant \lambda_1 \|u_n\|^2,$$

and by Young inequality

$$\int_{\Omega} f u_n \, dx \leqslant \frac{\gamma}{2}\|u_{nx}\|^2 + \frac{M}{2\gamma}.$$

From the above relations, we have

$$\frac{d}{dt}\big(\|u_n\|^2 + \|u_{nx}\|^2\big) + d\big(\|u_n\|^2 + \|u_{nx}\|^2\big) \leqslant \frac{M}{\gamma}. \tag{8}$$

Considering the time periodicity of $u_n$ and integrating (8) over $[0,\omega]$, we obtain

$$d\int_0^{\omega} \big(|u_n\|^2 + \|u_{nx}\|^2\big) \leqslant \frac{\omega M}{\gamma}.$$

Hence, there exists $t^* \in [0,\omega)$ such that

$$\big\|u_n(t^*)\big\|^2 + \big\|u_{nx}(t^*)\big\|^2 \leqslant \frac{M}{d\gamma}.$$

From (8), we can get

$$\big\|u_n(t)\big\|^2 + \big\|u_{nx}(t)\big\|^2 \leqslant \big\|u_n(t^*)\big\|^2 + \big\|u_{nx}(t^*)\big\|^2 + \frac{\omega M}{\gamma} \leqslant \left(\frac{1}{d} + \omega\right)\frac{M}{\gamma}.$$

Hence

$$\sup_{0\leqslant t\leqslant\omega} \big(\|u_n\|^2 + \|u_{nx}\|^2\big) \leqslant \left(\frac{1}{d} + \omega\right)\frac{M}{\gamma} =: c_1, \tag{9}$$

which concludes our proof. □

From Lemma 2.1 and Leray–Schauder fixed point theorem, Eq. (7) has solution $\{u_n\}_{n=1}^{\infty}$, that is also a sequence of approximation solutions of Eqs. (1)–(3). In order to obtain the convergence of $\{u_n\}_{n=1}^{\infty}$, we need to give a priori estimates for the high order derivatives of $\{u_n\}_{n=1}^{\infty}$.

**Lemma 2.2.** *If $f \in C^1(\omega; H^{-1}(\Omega))$, then*

$$\sup_{0\leqslant t\leqslant\omega} \big(\|u_{nx}\|^2 + \|u_{nxx}\|^2\big) \leqslant c_2,$$

*where $c_2$ is a constant which only depends on $L, \lambda_1, \omega, \gamma$ and $f$.*

**Proof.** Multiplying (7) by $-\lambda_j a_{jn}(t)$ and summing up over $j$ from 1 to $n$, we have

$$\big(u_{nt} - u_{nxxt} - \gamma(u_{nxx} - u_{nxxxx}), u_{nxx}\big) = (Nu_n + f, u_{nxx}),$$

which implies that

$$-\frac{1}{2}\frac{d}{dt}\big(\|u_{nx}\|^2 + \|u_{nxx}\|^2\big) - \gamma\big(\|u_{nxx}\|^2 + \|u_{nxxx}\|^2\big) = (Nu_n + f, u_{nxx}). \tag{10}$$

Using (4), (5), (9), Hölder inequality and Young inequality, we can obtain the following inequalities:

$$\left|\int_\Omega f u_{nxx}\,dx\right| \leqslant \varepsilon\|u_{nxxx}\|^2 + \frac{1}{4\varepsilon}M, \tag{11}$$

$$\left|\int_\Omega u_n u_{nx} u_{nxx}\,dx\right| \leqslant \|u_n\|_\infty \int_\Omega |u_{nx}u_{nxx}|\,dx \leqslant k_1\|u_n\|_{H^1}\int_\Omega |u_{nx}u_{nxx}|\,dx$$
$$\leqslant k_1 c_1^{1/2}\left(\frac{\varepsilon}{k_1 c_1^{1/2}}\|u_{nxx}\|^2 + \frac{k_1 c_1^{1/2}}{4\varepsilon}\|u_{nx}\|^2\right) \leqslant \varepsilon\|u_{nxx}\|^2 + \frac{k_1^2 c_1^2}{4\varepsilon}, \tag{12}$$

$$\left|\int_\Omega u_n u_{nxx} u_{nxxx}\,dx\right| = \left|\frac{1}{2}\int_\Omega u_n u_{nxx}^2\,dx\right| \leqslant \frac{1}{2}\|u_{nx}\|\|u_{nxx}\|_4^2$$
$$\leqslant \frac{1}{2}c_1^{1/2}\|u_n\|^{1/2}\|u_n\|_{H^3}^{3/2} \leqslant \frac{3}{4}\varepsilon\|u_n\|_{H^3}^2 + \frac{c_1^3}{64\varepsilon^3}$$
$$\leqslant \frac{3}{4}\varepsilon\left(\|u_n\|^2 + \|u_{nx}\|^2 + \|u_{nxx}\|^2 + \|u_{nxxx}\|^2\right) + \frac{c_1^3}{64\varepsilon^3}$$
$$\leqslant \frac{3}{4}\varepsilon\|u_{nxx}\|^2 + \frac{3}{4}\varepsilon\|u_{nxxx}\|^2 + \frac{3}{4}\varepsilon c_1 + \frac{c_1^3}{64\varepsilon^3}. \tag{13}$$

Taking (11)–(13) into account and choosing $\varepsilon$ small enough such that $6\varepsilon + \frac{3}{4}\varepsilon < \frac{\gamma}{2}$, then (10) yields

$$\frac{d}{dt}\left(\|u_{nx}\|^2 + \|u_{nxx}\|^2\right) + \gamma\left(\|u_{nxx}\|^2 + \|u_{nxxx}\|^2\right) \leqslant \frac{M}{4\varepsilon} + \frac{3k_1^2c_1^2}{4\varepsilon} + \frac{15\varepsilon c_1}{4} + \frac{5c_1^3}{64\varepsilon^3} =: \widetilde{C}. \tag{14}$$

Integrating (14) about $t$ from 0 to $\omega$ and considering the time periodicity of $u_n$, we obtain that there exists $t^* \in [0,\omega)$ such that

$$\left\|u_{nxx}(t^*)\right\|^2 + \left\|u_{nxxx}(t^*)\right\|^2 \leqslant \frac{\widetilde{C}}{\gamma}.$$

From (14), we have

$$\left\|u_{nx}(t)\right\|^2 + \left\|u_{nxx}(t)\right\|^2 \leqslant \left\|u_{nx}(t^*)\right\|^2 + \left\|u_{nxx}(t^*)\right\|^2 + \widetilde{C}\omega.$$

Hence, we can get

$$\sup_{0\leqslant t\leqslant\omega}\left(\|u_{nx}\|^2 + \|u_{nxx}\|^2\right) \leqslant \frac{\widetilde{C}}{\gamma} + \widetilde{C}\omega$$
$$= \left(\frac{1}{\gamma} + \omega\right)\left(\frac{M}{4\varepsilon} + \frac{3k_1^2c_1^2}{4\varepsilon} + \frac{15\varepsilon c_1}{4} + \frac{5c_1^3}{64\varepsilon^3}\right) =: c_2. \tag{15}$$

Which concludes our proof. □

In the following, we continue to establish a priori estimates for the high order derivatives of the approximate solution $\{u_n\}_{n=1}^\infty$ by an inductive argument. For simplicity, in the proofs later, we denote by $c$ a generic constant that varies from line to line.

**Lemma 2.3.** *For any $k \geqslant 0$, if $f \in C^1(\omega; H^{k-1}(\Omega))$, then*

$$\sup_{0\leqslant t\leqslant \omega}\left(\left\|D^{k+1}u_n\right\|^2+\left\|D^{k+2}u_n\right\|^2\right)\leqslant c,$$

*where $c$ is a constant which only depends on $L, \lambda, \omega, \gamma$ and $f$.*

**Proof.** By Lemma 2.2, the conclusion of Lemma 2.3 holds for $k = 0$. Assume that for $k \leqslant m-1$ ($m \geqslant 2$) the conclusion of Lemma 2.3 holds; we want to prove that the same statement also holds for $k = m$.

Multiplying (7) by $(-1)^{m+1}\lambda_j^{m+1}a_{jn}(t)$ and summing up $j$ from 1 to $n$, we have

$$(-1)^{m+1}\frac{1}{2}\frac{d}{dt}\left(\left\|D^{m+1}u_n\right\|^2+\left\|D^{m+2}u_n\right\|^2\right)+(-1)^{m+1}\gamma\left(\left\|D^{m+2}u_n\right\|^2+\left\|D^{m+3}u_n\right\|^2\right)$$
$$=\left(Nu_n+f, D^{2(m+1)}u_n\right). \tag{16}$$

Since

$$\left|\int_\Omega fD^{2(m+1)}u_n\,dx\right|=\left|\int_\Omega D^{m-1}fD^{m+3}u_n\,dx\right|\leqslant\varepsilon\left\|D^{m+3}u_n\right\|+\frac{1}{4\varepsilon}\left\|D^{m-1}f\right\|^2, \tag{17}$$

$$\begin{aligned}
&\left|\int_\Omega u_nu_{nx}D^{2(m+1)}u_n\,dx\right|\\
&\quad=\left|\int_\Omega\left(\sum_{j=0}^{m+1}C_{m+1}^jD^ju_nD^{m+1-j}u_{nx}\right)D^{m+1}u_n\,dx\right|\\
&\quad\leqslant\int_\Omega\left|u_n(D)^{m+2}u_nD^{m+1}u_n\right|dx+\int_\Omega\left|\left(\sum_{j=0}^{m+1}C_{m+1}^jD^ju_nD^{m+1-j}u_{nx}\right)D^{m+1}u_n\right|dx\\
&\quad\leqslant\varepsilon\left\|D^{m+2}u_n\right\|^2+c(\varepsilon)\left\|D^{m+1}u_n\right\|^2+c\\
&\quad\leqslant\varepsilon\left\|D^{m+2}u_n\right\|^2+c(\varepsilon),
\end{aligned}\tag{18}$$

$$\begin{aligned}
&\left|\int_\Omega u_nu_{nxxx}(D)^{2(m+1)}u_n\,dx\right|\\
&\quad=\left|\int_\Omega\left(\sum_{j=0}^{m}C_m^jD^ju_nD^{m-j}u_{nxxx}\right)D^{m+2}u_n\,dx\right|\\
&\quad\leqslant\int_\Omega\left|u_n(D)^{m+3}u_nD^{m+2}u_n\right|dx+\int_\Omega\left|C_m^1Du_n\left(D^{m+2}u_n\right)^2\right|dx\\
&\qquad+\int_\Omega\left|C_m^2D^2u_nD^{m+1}D^{m+2}u_n\right|dx+\int_\Omega\left|\left(\sum_{j=3}^{m}C_m^jD^ju_nD^{m+3-j}u_n\right)D^{m+2}u_n\right|dx\\
&\quad\leqslant\|u_n\|_\infty\int_\Omega\left|D^{m+3}u_nD^{m+2}u_n\right|dx+m\|Du_n\|_\infty\int_\Omega\left|D^{m+2}u_nD^{m+2}u_n\right|dx
\end{aligned}$$

$$+ C_m^2 \|D^2 u_n\|_\infty \int_\Omega |D^{m+1} u_n D^{m+2} u_n| \, dx$$
$$+ \sum_{j=3}^{m} C_m^j \|D^j u_n\|_\infty \|D^{m+3-j} u_n\|_\infty \int_\Omega |D^{m+2} u_n| \, dx$$
$$\leqslant c\bigg( \int_\Omega |D^{m+3} u_n D^{m+2} u_n| \, dx + \int_\Omega |D^{m+2} u_n D^{m+2} u_n| \, dx$$
$$+ \int_\Omega |D^{m+1} u_n D^{m+2} u_n| \, dx + \int_\Omega |D^{m+2} u_n| \, dx \bigg), \tag{19}$$

consider now

$$\begin{aligned} c\int_\Omega |D^{m+3} u_n D^{m+2} u_n| \, dx &\leqslant \varepsilon \|D^{m+3} u_n\|^2 + c(\varepsilon)\|D^{m+2} u_n\|^2 \\ &\leqslant \varepsilon \|D^{m+3} u_n\|^2 + c(\varepsilon)\|u_n\|_{H^{m+3}}^{2\frac{m+2}{m+3}} \|u_n\|^{2(1-\frac{m+2}{m+3})} \\ &\leqslant \varepsilon \|D^{m+3} u_n\|^2 + \varepsilon \|D^{m+3} u_n\|^2 + \varepsilon \|D^{m+2} u_n\|^2 + c(\varepsilon), \end{aligned} \tag{20}$$

$$\begin{aligned} c\int_\Omega |D^{m+1} u_n D^{m+2} u_n| \, dx &\leqslant \varepsilon \|D^{m+2} u_n\|^2 + c(\varepsilon)\|D^{m+1} u_n\|^2 \\ &\leqslant \varepsilon \|D^{m+2} u_n\|^2 + c(\varepsilon), \end{aligned} \tag{21}$$

$$c\int_\Omega |D^{m+2} u_n| \, dx \leqslant \varepsilon \|D^{m+2} u_n\|^2 + c(\varepsilon), \tag{22}$$

$$\begin{aligned} c\int_\Omega |D^{m+2} u_n D^{m+2} u_n| \, dx &\leqslant c\|D^{m+3} u_n\|^{2\frac{m+2}{m+3}} \|u_n\|^{2(1-\frac{m+2}{m+3})} \\ &\leqslant \varepsilon \|D^{m+3} u_n\|^2 + \varepsilon \|D^{m+2} u_n\|^2 + c(\varepsilon), \end{aligned} \tag{23}$$

where we have used Lemmas 2.1, 2.2, Young inequality, the assumption of Lemma 2.3 for $k \leqslant m-1$, (4) and (5).

Hence, from (19)–(23), we get

$$\bigg| \int_\Omega u_n u_{nxxx} D^{2(m+1)} u_n \, dx \bigg| \leqslant 3\varepsilon \|D^{m+3} u_n\|^2 + 4\varepsilon \|D^{m+2} u_n\|^2 + c(\varepsilon). \tag{24}$$

Similarly,

$$\bigg| \int_\Omega u_n u_{nxx} D^{2(m+1)} u_n \, dx \bigg| \leqslant 2\varepsilon \|D^{m+3} u_n\|^2 + c(\varepsilon). \tag{25}$$

Taking (16)–(25) into account, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\left\|D^{m+1}u_n\right\|^2+\left\|D^{m+2}u_n\right\|^2\right)+\gamma\left(\left\|D^{m+2}u_n\right\|^2+\left\|D^{m+3}u_n\right\|^2\right)$$
$$\leqslant 6\varepsilon\left\|D^{m+3}u_n\right\|^2+6\varepsilon\left\|D^{m+2}u_n\right\|^2+c(\varepsilon).$$

Choosing $\varepsilon$ small enough such that $6\varepsilon<\gamma/2$, we have

$$\frac{d}{dt}\left(\left\|D^{m+1}u_n\right\|^2+\left\|D^{m+2}u_n\right\|^2\right)+\gamma\left(\left\|D^{m+2}u_n\right\|^2+\left\|D^{m+3}u_n\right\|^2\right)\leqslant c(\varepsilon). \tag{26}$$

Integrating (26) from 0 to $\omega$, there exists $t^*\in[0,\omega)$ such that

$$\left\|D^{m+2}u_n(t^*)\right\|^2+\left\|D^{m+3}u_n(t^*)\right\|^2\leqslant c. \tag{27}$$

Integrating $\frac{d}{dt}(\|D^{m+1}u_n\|^2+\|D^{m+2}u_n\|^2)\leqslant c(\varepsilon)$ again from $t^*$ to $t\in[t^*,t^*+\omega)$ and considering (27), we can easily get

$$\sup_{0\leqslant t\leqslant\omega}\left(\left\|D^{m+1}u_n\right\|^2+\left\|D^{m+2}u_n\right\|^2\right)\leqslant c.$$

The proof is completed by an inductive argument. □

**Lemma 2.4.** *For any $k\geqslant 0$, if $f\in C^1(\omega;H^{k+1}(\Omega))$, then*

$$\sup_{0\leqslant t\leqslant\omega}\left(\left\|D^k u_{nt}\right\|^2+\left\|D^{k+1}u_{nt}\right\|^2\right)\leqslant c,$$

*where $c$ is a constant which only depends on $L,\lambda,\omega,\gamma$ and $f$.*

**Proof.** We first prove that the conclusion of Lemma 2.4 holds for $k=0$. Multiplying (7) by $a'_{jn}(t)$ and summing up $j$ from 1 to $n$, we have

$$\|u_{nt}\|^2+\|u_{nxt}\|^2=\left(\gamma(u_{nxx}-u_{nxxxx})+Nu_n+f,u_{nt}\right). \tag{28}$$

By Lemma 2.3, if $f\in C^1(\omega;H^1(\Omega))$, then $\|u\|_{H^4}\leqslant c$. Hence

$$\left(\gamma(u_{nxx}-u_{nxxxx})+Nu_n+f,u_{nt}\right)\leqslant\left\|\gamma(u_{nxx}-u_{nxxxx})+Nu_n+f\right\|\|u_{nt}\|$$
$$\leqslant c\|u_{nt}\|. \tag{29}$$

Therefore, by (28) and (29), it is easy to know that

$$\sup_{0\leqslant t\leqslant\omega}\left(\|u_{nt}\|^2+\|u_{nxt}\|^2\right)\leqslant c,$$

i.e., the conclusion of Lemma 2.4 exists for $k=0$.

Now assume that the conclusion of Lemma 2.4 holds for $k\leqslant m$ $(m\geqslant 1)$, we want to prove that the conclusion of Lemma 2.4 also holds for $k=m+1$.

Multiplying (7) by $(-1)^{m+1}\lambda_j^{m+1}a'_{jn}(t)$ and summing up $j$ from 1 to $n$, we have

$$(-1)^{m+1}\left\|D^{m+1}u_{nt}\right\|^2+(-1)^{m+1}\left\|D^{m+2}u_{nt}\right\|^2$$
$$=\left(\gamma(u_{nxx}-u_{nxxxx})+Nu_n+f,D^{2(m+1)}u_{nt}\right). \tag{30}$$

By Lemma 2.3, if $f\in C^1(\omega;H^{m+2}(\Omega))$, then $\|D^k u_n\|\leqslant c$ for $k\leqslant m+5$. Hence

$$\left|\left(\gamma(u_{nxx}-u_{nxxxx})+Nu_n+f,D^{2(m+1)}u_{nt}\right)\right|$$
$$\leqslant\left\|D^{m+1}\left[\gamma(u_{nxx}-u_{nxxxx})+Nu_n+f\right]\right\|\left\|D^{m+1}u_{nt}\right\|\leqslant c\left\|D^{m+1}u_{nt}\right\|. \tag{31}$$

Therefore, from (30), (31), it follows

$$\sup_{0\leqslant t\leqslant\omega}\left(\left\|D^{m+1}u_n\right\|^2+\left\|D^{m+2}u_n\right\|^2\right)\leqslant C.$$

This completes the proof of Lemma 2.4 by an inductive argument. □

## 3. Existence and uniqueness of time periodic solution

We have proved that Eqs. (1)–(3) have a sequence of approximate solutions $\{u_n\}_{n=1}^{\infty}$. In this section, we want to prove that the sequence converges and the limit is a solution of Eqs. (1)–(3).

By Lemmas 2.1–2.4 and standard compactness arguments we conclude that there is a subsequence which we denote also by $\{u_n\}$ such that for any $k\geqslant 0$, if $f\in C^1(\omega, H^{k+1}(\Omega))$, we have

$$\begin{aligned}
&u_n(t)\to u(t), \quad \text{weakly* in } L^\infty\big(\omega; H^{k+4}(\Omega)\big),\\
&u_n(t)\to u(t), \quad \text{strongly in } L^\infty\big(\omega; H^{k+3}(\Omega)\big),\\
&u_{nt}(t)\to u_t(t), \quad \text{weakly* in } L^\infty\big(\omega; H^{1+k}(\Omega)\big),\\
&u_{nt}(t)\to u_t(t), \quad \text{strongly in } L^\infty\big(\omega; H^{k}(\Omega)\big).
\end{aligned}$$

By the above lemmas we know that the nonlinear terms are well defined:

$$\begin{aligned}
\|u_nu_{nx}-uu_x\| &\leqslant \left\|u_n(u_{nx}-u_x)\right\|+\left\|u_x(u_n-u)\right\|\\
&\leqslant \|u_n\|_\infty\|u_{nx}-u_x\|+\|u_x\|_\infty-u\to 0, \quad n\to\infty, \ \text{uniformly in } t,
\end{aligned}$$

$$\begin{aligned}
\|u_nu_{nxxx}-uu_{xxx}\| &\leqslant \|u_n\|_\infty\|u_{nxxx}-u_{xxx}\|+\|u_{xxx}\|\|u_n-u\|_\infty\\
&\leqslant \|u_n\|_\infty\|u_{nxxx}-u_{xxx}\|+k_1\|u_{xxx}\|\|u_n-u\|_{H^1}\to 0,\\
&n\to\infty, \ \text{uniformly in } t.
\end{aligned}$$

Similarly,

$$\|u_{nx}u_{nxx}-u_xu_{xx}\|\to 0, \quad n\to\infty, \ \text{uniformly in } t.$$

Consequently, it follows

$$\big(u_t-u_{xxt}-\gamma(u_{xx}-u_{xxxx}),\eta\big)=(Nu+f,\eta), \quad \eta\in L^2_{\text{per}}.$$

Thanks to the estimates obtained in the previous section, we get

$$u_t-u_{xxt}-\gamma(u_{xx}-u_{xxxx})=Nu+f, \quad \text{a.e. on } R^1\times\Omega.$$

So we obtain the existence of time periodic solution for the viscous Camassa–Holm equations (1) and (2), that is the following result.

**Theorem 3.1.** *For any $k\geqslant 0$, if $f\in C^1(\omega; H^{k+1}(\Omega))$, then Eqs.* (1)–(3) *have a time periodic solution $u(t)$ which satisfies $u(t)\in L^\infty(\omega; H^{k+4}(\Omega))\cap W^{1,\infty}(\omega; H^k(\Omega))$.*

Under the assumption of Theorem 3.1 we are unable to prove the uniqueness of the solution for Eqs. (1)–(3). But if we assume that $M$ is sufficiently small, then the result can be obtained.

**Theorem 3.2.** *Suppose that the assumption in Theorem* 3.1 *holds. If $M$ is sufficiently small, then the time periodic solution of Eqs.* (1)–(3) *in Theorem* 3.1 *is unique.*

**Proof.** Let $u$ and $\bar{u}$ be any two time periodic solutions of Eqs. (1)–(3). Let us denote $v = u - \bar{u}$. Then from Eq. (1) we get

$$v_t - v_{xxt} - \gamma(v_{xx} - v_{xxxx}) = Nu - N\bar{u}. \tag{32}$$

Taking the inner product of (32) with $v$, we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|v\|^2 + \|v_x\|^2\big) + \gamma\big(\|v_x\|^2 + \|v_{xx}\|^2\big) = (Nu - N\bar{u}, v).$$

By the inequality (6),

$$\frac{1}{2}\frac{d}{dt}\big(\|v\|^2 + \|v_x\|^2\big) + \frac{\gamma}{2k_3^2}\|v\|^2 + \frac{\gamma}{2}\|v_x\|^2 + \gamma\|v_{xx}\|^2 = (Nu - N\bar{u}, v). \tag{33}$$

Since

$$\begin{aligned}\big|(-3uu_x + 3\bar{u}\bar{u}_x, v)\big| &\leqslant \big|(-3uv_x, v)\big| + \big|(-3\bar{u}_x v, v)\big| \\ &\leqslant 3\|u\|_\infty\big(\|v\|^2 + \|v_x\|^2\big) + \frac{3}{2}\|\bar{u}\|_\infty\|v\|^2 \\ &\leqslant \left(\frac{3}{2}k_1 c_1^{1/2} + \frac{3}{2}k_1 c_2^{1/2}\right)\|v\|^2 + \frac{3}{2}k_1 c_1^{1/2}\|v_x\|^2,\end{aligned} \tag{34}$$

$$\begin{aligned}&\big|(uu_{xxx} - \bar{u}\bar{u}_{xxx}, v)\big| \\ &\quad\leqslant \big|(uv_{xxx}, v)\big| + |\bar{u}_{xxx}v, v| \\ &\quad\leqslant \int_\Omega |u_x v_{xx} v|\,dx + \int_\Omega |uv_x v_{xx}|\,dx + \frac{1}{2}\int_\Omega \big|\bar{u}_x v_x^2\big|\,dx + \frac{1}{2}\int_\Omega |\bar{u}_x v v_{xx}|\,dx \\ &\quad\leqslant \frac{1}{2}\|u_x\|_\infty\big(\|v\|^2 + \|v_{xx}\|^2\big) + \frac{1}{2}\|u_x\|_\infty\big(\|v_x\|^2 + \|v_{xx}\|^2\big) + \frac{1}{2}\|\bar{u}_x\|_\infty\|v_x\|^2 \\ &\qquad + \frac{1}{2}\|\bar{u}_x\|_\infty\big(\|v\|^2 + \|v_{xx}\|^2\big) \\ &\quad\leqslant \left(\frac{k_1}{2}c_2^{1/2} + \frac{k_1}{4}c_2^{1/2}\right)\|v\|^2 + \left(\frac{k_1}{2}c_1^{1/2} + \frac{k_1}{2}c_2^{1/2}\right)\|v_x\|^2 \\ &\qquad + \left(\frac{k_1}{2}c_2^{1/2} + \frac{k_1}{2}c_1^{1/2} + \frac{k_1}{4}c_2^{1/2}\right)\|v_{xx}\|^2,\end{aligned} \tag{35}$$

$$\begin{aligned}&\big|(2u_x u_{xx} - 2\bar{u}_x\bar{u}_{xx}, v)\big| \\ &\quad\leqslant \big|(2u_x v_{xx}, v)\big| + |2\bar{u}_{xx}v_x, v| \\ &\quad\leqslant 2\|u_x\|_\infty\big(\|v\|^2 + \|v_{xx}\|^2\big) + 2\int \big|\bar{u}_x v_x^2\big|\,dx + 2\int |\bar{u}_x v_{xx} v|\,dx \\ &\quad\leqslant k_1 c_2^{1/2}\big(\|v\|^2 + \|v_{xx}\|^2\big) + \|\bar{u}_x\|_\infty\|v_x\|^2 + \|\bar{u}_x\|_\infty\big(\|v\|^2 + \|v_{xx}\|^2\big) \\ &\quad\leqslant 2k_1 c_2^{1/2}\|v_x\|^2 + \big(2k_1 c_2\|v_{xx}\|^2\big).\end{aligned} \tag{36}$$

Hence, if $M$ is sufficiently small such that

$$\frac{\gamma}{4k_3^2} \geqslant \frac{3k_1}{2}c_1^{1/2} + \frac{17k_1}{4}c_2^{1/2}, \qquad \frac{\gamma}{4} \geqslant 2k_1 c_1^{1/2} + \frac{5k_1}{2}c_2^{1/2},$$
$$\frac{\gamma}{4} \geqslant \frac{k_1}{2}c_1^{1/2} + \frac{11k_1}{4}c_2^{1/2}, \tag{37}$$

then it follows from (33)–(37)

$$\frac{d}{dt}\big(\|v\|^2 + \|v_x\|^2\big) + \rho\big(\|v\|^2 + \|v_x\|^2\big) \leqslant 0,$$

where $\rho > 0$ is a suitable constant, so that

$$\big(\|v\|^2 + \|v_x\|^2\big)(t) \leqslant \big(\|v\|^2 + \|v_x\|^2\big)(0)\exp(-\rho t), \quad \text{for any } t \geqslant 0.$$

Since $v$ is $\omega$-periodic in $t$, then for any positive integer $N$ we have $(\|v\|^2 + \|v_x\|^2)(t) = (\|v\|^2 + \|v_x\|^2)(t + N\omega)$, so that

$$\big(\|v\|^2 + \|v_x\|^2\big)(t) \leqslant \big(\|v\|^2 + \|v_x\|^2\big)(0)\exp\big(-\rho(t + N\omega)\big),$$

which implies $\|v\|^2 + \|v_x\|^2 = 0$. This completes the proof of Theorem 3.2. □

Here we remark that this paper deals with the existence and uniqueness of time- and space-periodic solution for the viscous Camassa–Holm equation with the forcing terms. It seems that the method used in this paper does not apply to the Camassa–Holm equation, because a priori estimates for the high order derivatives of solution are difficult to obtain for the Camassa–Holm equation.

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