On Planar Elliptic Structures with Infinite Type Degeneracy

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0. INTRODUCTION

This paper studies global properties of a class of planar vector fields that are elliptic except along a simple and closed curve. With each such vector field, we associate a complex number. This complex number characterizes the structure and bears heavily on the solvability of the associated pde equations.

Let $L$ be a subbundle of the complexified tangent bundle $\mathbb{C}T\mathbb{R}^2$. Suppose that $L$ is generated by a $C^\infty$ vector field

$$L = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$$

such that $L$ is elliptic everywhere on $\mathbb{R}^2$ except along a simple and closed curve $\Sigma$. We assume that on $\Sigma$ the vector field $L$ is of infinite type and that $L \wedge L$ vanishes to first order (see section 1 for definitions). Such a structure can be viewed as follows. For each $p \in \mathbb{R}^2 \setminus \Sigma$, $L$ is equivalent near $p$ to a multiple of the CR operator $\frac{\partial}{\partial \bar{z}}$, and for each $p \in \Sigma$, $L$ is equivalent near $p$, to a multiple of

$$\frac{\partial}{\partial y} - i x \frac{\partial}{\partial x}.$$

The vector field $L$ is therefore locally integrable and it satisfies the Nirenberg-Treves condition $\mathcal{P}$. The local solvability is well understood (see [NT] and [T2]). In this paper, we focus on the global aspect of such structures. The questions addressed here are within the spirit of those contained in the following papers [BCH], [BHS], [BM1, 2], [ChT], [CoT], [HJ].
The subbundle \( \mathcal{V} = \mathcal{L} \) of \( CT^*R^2 \) is generated by the differential form

\[
\omega = b\, dx - a\, dy.
\]  

(2)

Since \( \omega \neq 0 \), then

\[
d\omega = \omega \wedge \alpha
\]

(3)

for some differential form \( \alpha \). We prove in Section 2 that the complex number

\[
v = \exp \int_{\Sigma} \alpha \in C\setminus R
\]

(4)

is an invariant that characterizes \( \mathcal{L} \). Let \( \lambda \in R^* + iR \) be such that

\[
v = \exp \frac{2\pi i}{\lambda}.
\]

(5)

When \( \text{Im} \lambda \neq 0 \), we prove that, for every \( k \in Z^+ \), the vector field \( L \) is equivalent, under an \( C^k \) diffeomorphism defined near \( \Sigma \), to a multiple of the vector field

\[
T_{\lambda} = \lambda \frac{\partial}{\partial \theta} - ir \frac{\partial}{\partial r}
\]

(6)

defined in \( R \times S^1 \). When the structure is real analytic, the above equivalence holds under a real analytic diffeomorphism. In the case \( \text{Im} \lambda = 0 \), the above equivalence still holds but only under a \( C^{k,\sigma} \) diffeomorphism for some \( 0 < \sigma < 1 \).

Let \( \Omega_+ \) and \( \Omega_- \) be the connected components of \( R^2 \setminus \Sigma \), with \( \Omega_- \) bounded. In Section 3, we prove that \( L \) is equivalent on \( \Omega_- \) (under a \( C^\infty \) diffeomorphism) to a multiple of the vector field \( X_{\lambda}^+ \) defined in the unit disc \( D \) by

\[
X_{\lambda}^+ = [1 + (\lambda - 1)\, r] \frac{\partial}{\partial \theta} - ir(1 - r) \frac{\partial}{\partial r},
\]

(7)

where \((r, \theta)\) are the polar coordinates. In \( \Omega_- \), we show that, depending on whether the complex structure defined by \( \mathcal{L} \) on \( \Omega_- \) is parabolic or
In hyperbolic (see Section 3), the vector field $L$ is equivalent to a multiple of one of the following two vector fields

$$X_{1, \lambda}^- = \lambda \frac{\partial}{\partial \theta} - i(r - 1) \frac{\partial}{\partial r}$$  \hspace{0.5cm} \text{(8)}$$

$$X_{2, \lambda}^- = \lambda \frac{\partial}{\partial \theta} - ir(r - 1) \frac{\partial}{\partial r}$$  \hspace{0.5cm} \text{(9)}$$

In the remaining Sections 4 to 9, various associated pde are addressed. In Section 5 we consider the equation

$$T_\lambda u = f.$$  \hspace{0.5cm} \text{(10)}$$

We prove that if $f$ is Hölder continuous, then (10) has a Hölder continuous solution if and only if

$$\int_0^{2\pi} f(0, \theta) \, d\theta = 0.$$  \hspace{0.5cm} \text{(11)}$$

When $f$ is $C^\infty$ and satisfies (11), we show that for every $k \in \mathbb{Z}^+$, equation (10) has $C^k$ solutions. When $f$ does not satisfy (11), we prove that (10) has distribution solutions, provided that $f \in L^p$ with $p > \max(2, \Re \lambda)$. It should be noted that equation (10) was studied in [BM2], using Fourier series, and that existence of only continuous solutions was established there under the strong assumption that $f \in C^\infty$ and satisfies (11).

In Sections 6 and 8, we establish a version of the similarity principle for solutions of

$$T_\lambda u = pu + qu,$$  \hspace{0.5cm} \text{(12)}$$

with $q = 0$ on $\Sigma$. We prove that (12) has a continuous solution $u$ with $u \neq 0$ on $\Sigma$ if and only if

$$\int_0^{2\pi} p(0, \theta) \, d\theta = \lambda \mathbb{Z}.$$  \hspace{0.5cm} \text{(13)}$$

In which case $u$ has the form $e^{s w}$, with $s, w$ continuous and $T_\lambda w = 0$. When (13) fails, we show that there exists a unique $\mu \in \mathbb{C}$ with

$$0 < \Re \mu \leq \Re \lambda$$  \hspace{0.5cm} \text{(14)}$$

such that any continuous solution has the form $r^\mu e^{s w}$, with $s$ and $w$ as before. Conversely, for any such $w$ there exists a function $s$ so that $e^{s w}$ or $r^\mu e^{s w}$ (depending on whether or not (13) holds) is a solution of (12).
In Sections 7 and 9, particular solutions of the equations

\[ T_1 u = pu + f \]  
(15)
\[ T_2 u = pu + q\bar{u} + f \]  
(16)
are constructed.

1. PRELIMINARIES

Let \( L \) be a 1-dimensional \( C^\infty \) subbundle of the complexified tangent bundle \( \mathbb{C}T\mathbb{R}^2 \). Suppose that \( L \) is generated by a \( C^\infty \) vector field

\[ L = a(x, y) \partial_x + b(x, y) \partial_y \]  
(1.1)
where \( a \) and \( b \) are \( \mathbb{C} \)-valued, \( C^\infty \) functions defined on \( \mathbb{R}^2 \), such that

\[ |a(x, y)| + |b(x, y)| > 0 \quad \forall (x, y) \in \mathbb{R}^2. \]  
(1.2)
The orthogonal bundle \( \nu = L^\perp \) (for the duality between tangent and cotangent vectors) is a 1-dimensional subbundle of \( \mathbb{C}T^*\mathbb{R}^2 \) generated by the 1-form

\[ \omega = b(x, y) \, dx - a(x, y) \, dy. \]  
(1.3)
The characteristic set of the structure defined by \( L \) (or \( \nu \)) is the set \( \Sigma \subset \mathbb{R}^2 \) where the structure fails to be elliptic. That is,

\[ \Sigma = \{ p \in \mathbb{R}^2 : L_p \text{ and } \bar{L}_p \text{ are dependent} \} \]
\[ = \{ p \in \mathbb{R}^2 : \omega(p) \wedge \bar{\omega}(p) = 0 \}, \]  
(1.4)
where \( \bar{L} \) and \( \bar{\omega} \) are the complex conjugates of \( L \) and \( \omega \):

\[ \bar{L} = a(x, y) \frac{\partial}{\partial x} + \bar{b}(x, y) \frac{\partial}{\partial y} \quad \text{and} \quad \bar{\omega} = \bar{b}(x, y) \, dx - \bar{a}(x, y) \, dy. \]  
(1.5)
The structure \( L \) is said to be of finite type \( k - 1 \) at a point \( p \in \Sigma \) if there exists a vector field

\[ T^k = [X_1, [X_2, [... [X_{k-1}, X_k ]]]], \]
such that $L$ and $T^k$ are independent at $p$, where $T^k$ is a Lie bracket of length $k-1$, with each of the $X_j$'s is either $L$ or $\overline{L}$, and where $k$ is the smallest such integer. If no such vector field $T^k$ exists for all integers $k$, the structure is said to be of infinite type at the point $p$. In this paper we consider structures $\mathcal{L}$ for which

H1. $\Sigma$ is a compact and connected 1-dimensional submanifold of $\mathbb{R}^2$;
H2. $\mathcal{L}$ is of infinite type along $\Sigma$; and
H3. $\mathcal{L} \wedge \overline{\mathcal{L}}$ vanishes to first order along $\Sigma$.

Note that for $L$ as in (1.1), the 2-vector

$$L \wedge \overline{L} = (ab - \overline{ab}) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = 2i \text{Im}(a \overline{b}) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

(1.6)

is a generator of $\mathcal{L} \wedge \overline{\mathcal{L}}$.

If $p \in \Sigma$, then by a change of coordinates, we can assume $p = (0, 0)$ and $\Sigma$ is given (locally near 0) by $x = 0$. The above conditions mean that $\mathcal{L}$ is generated near 0 by a vector field

$$L = \frac{\partial}{\partial y} - i c(x, y) \frac{\partial}{\partial x},$$

(1.7)

for some $C^\infty$ function $c$. Condition H3 implies that locally $\Sigma$ is given by $\text{Re} c = 0$ and that the gradient of $\text{Re} c$ does not vanish on $\Sigma$. Thus $\Sigma$ is a $C^\infty$ one dimensional submanifold of $\mathbb{R}^2$. This together with the compactness of $\Sigma$ prove the following lemma

**Lemma 1.1.** There exists a $C^\infty$ change of coordinates of $\mathbb{R}^2$ that transforms $\Sigma$ into the circle $S^1$.

After a change of variables, we can assume that the function $c$ of (1.7) satisfies

$$\text{Re} c(0, y) = 0 \quad \text{and} \quad \frac{\partial}{\partial x} \text{Re} c(0, 0) \neq 0.$$  

(1.8)

The vector field (1.7) satisfies the Nirenberg-Treves condition ($P$) (see [NT] or [T2]). After a further change of coordinates, we can assume that near 0 we have

$$L = \frac{\partial}{\partial y} - ix \frac{\partial}{\partial x}.$$  

(1.9)
Hence, a structure $\mathcal{L}$ satisfying conditions H1, H2, and H3 can be thought of as a structure for which $\Sigma = S^1$ is the unit circle. Near each point $p \notin \Sigma$, we can find $(x, y)$ coordinates so that $L$ is generated by the CR operator

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

(1.10)

Near each point $p \in \Sigma$, we can find $(x, y)$ coordinates so that $L$ is generated by

$$\frac{\partial}{\partial y} - ix \frac{\partial}{\partial x}.$$  

(1.11)

**Remark 1.1.** Thanks to Lemma 1.1 we will assume throughout the remainder of this paper that $\Sigma = S^1$.

2. NORMAL FORMS NEAR THE CHARACTERISTIC SET

In this section, we prove that appropriate coordinates $(r, \theta)$ can be found near the characteristic set $S^1$ so that $\mathcal{L}$ is generated by a standard vector field

$$i \lambda \frac{\partial}{\partial \theta} - ir \frac{\partial}{\partial r}$$

for some complex number $\lambda$ with $\text{Re} \lambda > 0$.

Let $\mathcal{L}$ be a $C^\infty$ subbundle of $\mathbb{C}T\mathbb{R}^2$ with characteristic set $\Sigma = S^1$ and satisfying conditions H1, H2, and H3. Let $L$ and $\omega$ be as in (1.1) and (1.3). Let $i : S^1 \to \mathbb{R}^2$ be the natural injection. Conditions H1, H2, and H3 mean that $i^*\omega = 0$ on $S^1$, and that $\omega \wedge \omega$ vanishes to first order along $S^1$. Since the 1-form $\omega$ does not vanish in $\mathbb{R}^2$, then we can write

$$d\omega = \omega \wedge \alpha$$

(2.1)

for some $C^\infty$ differential form $\alpha$.

**Lemma 2.1.** The following complex number is an invariant of $\mathcal{L}$

$$v = \exp \int_{S^1} \alpha$$

(2.2)
Proof. If $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism and $\omega' = \Phi^*\omega$ has characteristic set $\Sigma' = \Phi^{-1}(\Sigma)$, then
\[d\omega' = \Phi^* d\omega = \Phi^* \omega \wedge \Phi^* \alpha = \omega' \wedge \alpha'.\]

Thus
\[\nu' = \exp \int_{\Sigma'} \omega' = \exp \int_{\Phi^{-1}(\Sigma)} \Phi^* \alpha = \exp \int_{S^1} \alpha = \nu.\]

Also, if $\omega''$ is another generator of $\mathscr{L}$ near $S^1$, then $\omega'' = g\omega$ for some nonvanishing function $g$. We have then
\[d\omega'' = g\, d\omega - \omega \wedge dg = g\omega \wedge \alpha - g\omega \wedge \frac{dg}{g} \]

Therefore,
\[\exp \int_{S^1} (\alpha - d \log g) = \exp \int_{S^1} \alpha.\]

To study $\mathscr{L}$ near the characteristic set, we use convenient coordinates. The map
\[\mathbb{R} \times S^1 \to \mathbb{R}^2\]
\[(r, \theta) \mapsto ((1 + r) \cos \theta, (1 + r) \sin \theta) = (1 + r) e^{i\theta}\]
which is a diffeomorphism from a neighborhood of $\{0\} \times S^1$ to a neighborhood of the unit circle in $\mathbb{R}^2$. The pullback of $\nu' = \mathscr{L}^{-1}$ to a neighborhood of $\{0\} \times S^1$ is generated by a differential form
\[A(r, \theta) \, dr + B(r, \theta) \, d\theta, \quad (2.3)\]
where $A$ and $B$ are $2\pi$ periodic in $\theta$. It follows at once from conditions H1, H2, and H3 that
\[A(0, \theta) \neq 0, \quad \forall \theta;\]
\[\text{Im}(B\overline{A}(r, \theta)) = rK(r, \theta); \quad (2.4)\]
\[K(0, \theta) \neq 0, \quad \forall \theta,\]
for some real valued function \(K\). We can therefore assume that \(\nu\) is generated near \([0] \times S^1\) by the differential form

\[ \omega = dr + irC(r, \theta)\,d\theta, \]

for some \(C\)-valued function \(C\) with \(\text{Re} \,C(0, \theta) > 0\) for all \(\theta\).

**Lemma 2.2.** The invariant \(\nu\) of (2.2) is given by

\[ \nu = \exp \left( i \int_0^{2\pi} C(0, \theta)\,d\theta \right). \]

**Proof.** We use the form \(\omega\) in (2.5) to write

\[ d\omega = \omega \wedge \left[ iC(r, \theta) + irC(r, \theta) \right] d\theta. \]

Thus

\[ \nu = \exp \left( \int_{[0] \times S^1} \left[ iC(r, \theta) + irC(r, \theta) \right] d\theta = \exp \left( i \int_0^{2\pi} C(0, \theta)\,d\theta \right). \]

Set \(\nu = \exp(2\pi i\lambda^{-1})\), where

\[ \lambda = \left[ \frac{1}{2\pi} \int_0^{2\pi} C(0, \theta)\,d\theta \right]^{-1} \in \mathbb{R}^+ + i\mathbb{R}. \]

The main result of this section is the following theorem:

**Theorem 2.1.** Let \(\omega\) and \(\lambda\) be as in (2.5) and (2.7) respectively. Suppose that \(\text{Im} \lambda \neq 0\). Then, for every \(k \in \mathbb{Z}^+\), there exists a \(C^k\) diffeomorphism \(\Phi_k: U \to V\) where \(U\) and \(V\) are open neighborhoods of \([0] \times S^1 \subset \mathbb{R} \times S^1\) such that \(\Phi_k(S^1) = S^1\) and

\[ \Phi_k^* \omega = g(r, \theta) (\lambda \,dr + id\theta). \]

Before we proceed with the proof, we need some preliminary results. Consider the Taylor expansion of \(C\) with respect to \(r\),

\[ C(r, \theta) = \sum_{j=0}^N c_j(\theta) r^j + r^N C_{N+1}(r, \theta), \]

where

\[ c_j(\theta) = \frac{1}{j!} \frac{\partial^j C}{\partial r^j}(0, \theta) \in C^{\infty}(S^2). \]
and \( c_{N+1}(r, \theta) \in C^\infty(\mathbb{R} \times S^1) \). Set
\[
\omega_N = dr + ir \sum_{j=0}^N c_j(\theta) r^j d\theta. \tag{2.10}
\]

Note that
\[
\omega = \omega_N + ir^{N+2} c_{N+1}(r, \theta) d\theta \tag{2.11}
\]
and the coefficient of \( \omega_N \) are polynomials in \( r \). Let
\[
\hat{\omega}_N = d\hat{r} + ir \sum_{j=0}^N c_j(\theta) \hat{r}^j d\theta, \tag{2.12}
\]
where \( \hat{r} = r + ir' \in \mathbb{C} \). Let \( \hat{\omega}_N^1 \) and \( \hat{\omega}_N^2 \) be the real and imaginary parts of \( \hat{\omega}_N \) (note that \( \hat{\omega}_N \) is the complexification of \( \omega_N \)). From
\[
d\hat{\omega}_N \wedge \hat{\omega}_N = 0
\]
we have
\[
d\hat{\omega}_N^j \wedge \hat{\omega}_N^1 \wedge \hat{\omega}_N^2 = 0 \quad j = 1, 2. \tag{2.13}
\]

Hence \( \hat{\omega}_N \) defines (via \( \hat{\omega}_N^1 \) and \( \hat{\omega}_N^2 \)) a 1-dimensional foliation \( \mathcal{F}_N \) in \( \mathbb{C} \times S^1 \).

The leaves of \( \mathcal{F}_N \) are the integral curves of the vector field
\[
X_N = (r'x + r\beta) \frac{\partial}{\partial r} - (r'x - r\beta) \frac{\partial}{\partial \theta}, \tag{2.14}
\]
where \( x \) and \( \beta \) are the real and imaginary parts of \( \sum_{j=0}^N c_j(\theta) \hat{r}^j \). Denote by \( \Gamma_p \) the integral curve through the point \( p \) of \( X_N \). Note that \( \Gamma_{(0, 0)} = \{ 0 \} \times S^1 \). Now we study the holonomy of this circle. For \( \varepsilon > 0 \), let
\[
D_{\varepsilon} = \{ (z, 0) \in \mathbb{C} \times S^1 : |z| < \varepsilon \}.
\]

The disc \( D_{\varepsilon} \) is transversal to \( \mathcal{F}_N \). Let \( \psi \) be the first return map associated with the leaf \( \Gamma_{(0, 0)} \). The map \( \psi \) is a germ of a diffeomorphism at \( 0 \in D_{\varepsilon} \).

**Lemma 2.3.** \( \psi \) is holomorphic and
\[
\psi'(0) = v^{-1} = \exp(-2\pi i/\lambda^{-1}), \tag{2.15}
\]
where \( v \) is defined in (2.2).
Proof. The integral curve of $X_N$ through the point $(z, 0)$ is the trajectory of the system of equations

\[
\frac{dr}{d\theta} = r'\alpha(r, \theta) + r\beta(r, \theta)
\]
\[
\frac{dr'}{d\theta} = -r\alpha(r, \theta) + r'\beta(r, \theta)
\]  
(2.16)
\[
\dot{r}(0) = z
\]
or equivalently of the system

\[
\frac{d\dot{r}}{d\theta} = -i\sum_{j=0}^{N} \dot{r}c_j(\theta)
\]
\[
\dot{r}(0) = z.
\]  
(2.17)
Since (2.17) is holomorphic in $\dot{r}$, then the unique solution $\dot{r}(\theta, z)$ depends holomorphically on the initial condition $z$. The first return map is

\[
\psi(z) = \dot{r}(2\pi, z).
\]  
(2.18)
It is therefore holomorphic in $z$ and

\[
\psi'(0) = \frac{\partial \dot{r}}{\partial z}(2\pi, 0).
\]  
(2.19)
If we expand

\[
\dot{r}(\theta, z) = \alpha_1(\theta) z + \alpha_2(\theta) z^2 + \cdots,
\]  
(2.20)
then using (2.17), we see that

\[
\alpha_1(\theta) = -iz_1(\theta) c_0(\theta).
\]  
(2.21)
Hence,

\[
\alpha_1(\theta) = \exp \left( -i \int_{0}^{\theta} c_0(s) \, ds \right).
\]  
(2.22)
This together with (2.19) and Lemma 2.2 complete the proof of Lemma 2.3.
Lemma 2.4. Let $\omega_N$ be the 1-form defined in (2.12). Then there exists a $C^\infty$ diffeomorphism

$$\hat{\Psi}_N: \hat{U} \to \hat{V},$$

where $\hat{U}$ and $\hat{V}$ are open neighborhoods of $\{0\} \times S^1 \subset \mathbb{C} \times S^1$ such that $\hat{\Psi}_N(z, \theta)$ is holomorphic in $z$ and

$$\hat{\Psi}_N^* \omega_N = g(z, \theta) (\hat{\lambda} \ d\hat{t} + i \hat{t} \ d\theta),$$  \hspace{1cm} (2.23)

where $g \in C^\infty(\hat{U})$ and $\hat{\lambda} \in \mathbb{C}$ is defined in (2.7) with $\text{Im} \ \hat{\lambda} \neq 0$.

Proof. Let $\varepsilon > 0$ be small enough so that the first return map $\hat{\psi}$ associated with $\{0\} \times S^1$ is defined in the disc $D_{\varepsilon}$. Thus $\hat{\psi}: D_{\varepsilon} \to \hat{\psi}(D_{\varepsilon})$ is a biholomorphism with

$$|\psi(0)| = \exp \left( -2\pi \frac{\text{Im} \ \hat{\lambda}}{|\hat{\lambda}|^2} \right) \neq 1.$$  \hspace{1cm} (2.24)

$\psi$ is therefore conjugate to its linear part (see for example [AA]). That is, there exists a germ of a biholomorphism $h$ at $0 \in \mathbb{C}$ such that

$$\psi(z) = h^{-1} \cdot A \cdot h(z),$$  \hspace{1cm} (2.25)

where $A(\zeta) = e^{-2\pi i \zeta} - 1 = -1 \zeta$ is the linear part of $\psi$.

Now we construct a diffeomorphism $\hat{\Psi}_N$ that sends the vector field $X_N$ into a multiple of

$$X_0 = \frac{\partial}{\partial \theta} - i \hat{\lambda}^{-1} \frac{\partial}{\partial \xi}.$$  \hspace{1cm} (2.26)

The integral curve of $X_0$ through the point $(0, z)$ is given by

$$\hat{r}_0(\hat{\theta}, z) = (\hat{\theta}, z e^{-i \hat{\lambda}^{-1}})$$  \hspace{1cm} (2.27)

and the first return map is $z e^{-2\pi i \hat{\lambda}^{-1}}$ (the linear part of $\Psi$). Define $\hat{\Psi}_N$ by

$$\hat{\Psi}_N(\hat{\theta}, \hat{r}(\hat{\theta}, z)) = (\theta, h(z) e^{-i \hat{\lambda}^{-1}}).$$  \hspace{1cm} (2.28)

It follows from (2.25) and Lemma 2.3 that $\hat{\Psi}_N$ is a diffeomorphism (near $\{0\} \times S^1$) such that

$$\hat{\Psi}_N^* X_N = g X_0.$$  \hspace{1cm} (2.29)

Furthermore $\hat{\Psi}_N$ is the identity on $\{0\} \times S^1$ and it is holomorphic in $z$. \ \blacksquare
Lemma 2.5. There exists a $C^\infty$ diffeomorphism
\[ \Psi_N: U \to V, \]
where $U$ and $V$ are open neighborhoods of $\{0\} \times S^1 \subset \mathbb{R} \times S^1$ such that
\[ \Psi_N^* \omega_N \wedge (\lambda \, dr + ir \, d\theta) = 0. \tag{2.30} \]
Furthermore, $\Psi_N$ is real analytic in $r$ and
\[ \Psi_N(\{0\} \times S^1) = \{0\} \times S^1. \]
Proof. Let $\Psi_N(\theta, R) = (\Theta, R)$ be the diffeomorphism constructed in the proof of the previous lemma. Note that $\Theta = 0$, $R = \tilde{Q}(\tilde{r}, 0)$, with $Q(0, \theta) \neq 0$ and that $\lambda^4 e^{i\theta}$ is a multivalued first integral of $\lambda \, dR + iR \, d\theta$. Let
\[ f(r, \theta) = rQ(r, \theta). \tag{2.31} \]
Since $f^4 e^{i\theta}$ is a (multivalued) first integral of $\omega_N$, then
\[ \omega_N = g(\lambda \, df + if \, d\theta). \tag{2.32} \]
A direct calculation shows that the change of variables
\[ \rho = r \exp \left( -\frac{\text{Im} \lambda}{\text{Re} \lambda} \arg Q(r, \theta) \right) \]
\[ \phi = \theta + \frac{|\lambda|^2}{\text{Re} \lambda} \arg Q(r, \theta) \tag{2.33} \]
transforms $\omega_N$ into a multiple of $\lambda \, dr + ir \, d\theta$.

Proof of Theorem 2.1. Let $\omega$ be as in (2.5). For $N \in \mathbb{Z}^+$ let
\[ \omega = \omega_N + i r^{N+1} \epsilon_{N+1}(r, \theta) \, d\theta, \tag{2.34} \]
where $\omega_N$ and $\epsilon_{N+1}$ are defined in (2.9) and (2.10), respectively. Let $\Psi_N$ be the diffeomorphism of Lemma 2.5 that transforms $\omega_N$ into a multiple of $\lambda \, dr + ir \, d\theta$. The diffeomorphism $\Psi_N$ can be written as
\[ \Psi_N(\rho, \phi) = (x(r, \theta), \beta(r, \theta)), \]
where the functions $x, \beta$ are $2\pi$-periodic in $\theta$ and $x(0, \theta) \neq 0$, $\beta(0, \theta) \neq 0$, for all $\theta$. We can assume, after a change of variables, that
\[ \omega = \lambda \, dr + ir \, d\theta + i r^{N+1} \epsilon_{N+1}(r, \theta) \, d\theta. \tag{2.35} \]
Consider the map $A(r, \theta) = (\rho, \phi)$ defined (for $r > 0$) by
\[
\rho = r^a \\
\phi = \theta + b \log r,
\]
where $a = \Re \lambda$ and $b = \Im \lambda$. Note that $A$ is a diffeomorphism. The pullback of $\omega$ via $A^{-1}$ is a multiple of
\[
\Omega = \left(1 - i\rho^{(\alpha + 1)/a} \frac{b}{a} E \right) d\rho + i\rho (1 + i\rho^{(N + 1)/a} E) d\phi,
\]
where
\[
E(\rho, \phi) = c_{N+1} \left(\rho^{1/a}, \phi - \frac{b}{a} \log \rho\right)
\]
is a bounded function defined near $\{0\} \times S^1 \subset \mathbb{R}^+ \times S^1$.

Now we prove that $\Omega$ has a first integral in $\rho > 0$. Consider the blowing down of $\Omega$ along $\{0\} \times S^1$. Let
\[
\pi: \mathbb{R}^+ \times S^1 \to \mathbb{R}^2; \quad \pi(\rho, \phi) = z = \rho e^{i\phi}.
\]
It follows at once from
\[
2\rho \, d\rho = \bar{z} \, dz + z \, d\bar{z} \quad \text{and} \quad 2i\rho^2 \, d\phi = \bar{z} \, dz - z \, d\bar{z}
\]
that the blowing down of $\Omega$ is a multiple of the differential form
\[
\sigma = dz - i \frac{|z|^{N+1/2}}{2i(b-a)} (b + a) E \frac{z}{E} \bar{z} \, d\bar{z}.
\]
Let $N \in \mathbb{Z}^+$ be large enough so that
\[
\frac{N+1}{a} > k + 1,
\]
where $k$ is a preassigned integer. The 1-form $\sigma$ is then elliptic near $0 \in \mathbb{R}^2$ and coincides to order $k + 1$ with $dz$. It follows from the theory of quasi-conformal mappings (see for instance [LV]) that there exist a $C^{k+2}$ diffeomorphism $F(z, \bar{z})$ near $0 \in \mathbb{R}^2$ such that
\[
F(z, \bar{z}) = z + |z|^{k+1} K(z, \bar{z}) \quad \text{and} \quad \sigma \wedge dF = 0,
\]
where $K$ is a bounded function. This means that the form $\Omega$ has a first integral defined by

$$\rho (1 + \rho^{k+1} K(\rho, \phi)) e^{i\phi}. \quad (2.40)$$

Hence, the function

$$r^\mu (1 + r^{a(k+1)} K(r^a, \theta + b \log r)) e^{i(\theta + b \log r)} \quad (2.41)$$

is a first integral of $\omega$ in $r > 0$. An analogous argument leads to

$$|r|^{a} (1 + |r|^{a(k+1)} K(|r|^a, \theta + b \log |r|)) e^{i(\theta + b \log |r|)} \quad (2.41')$$

a first integral of $\omega$ in $r < 0$.

Finally, with respect to the coordinates

$$R = r \, |1 + |r|^{a(k+1)} M(r, \theta)|^{\frac{1}{a}}$$

$$\Theta = \theta - \arg (1 + |r|^{a(k+1)} M(r, \theta)) - \frac{b}{a} \log |1 + |r|^{a(k+1)} M(r, \theta)|, \quad (2.42)$$

where $M = K$ in $r > 0$ and $M = K'$ in $r < 0$, the form $\omega$ achieves the standard form $\lambda R \, dR + iR \, d\Theta$. Moreover, if $k$ is chosen large enough ($a(k+1) > l$) the above diffeomorphism is of class $C^l$ near $\{0\} \times S^1$.

**Remark 2.1.** When $\text{Im} \, \lambda = 0$ the first return map $\psi$ of the circle $\{0\} \times S^1$ satisfies $|\psi'(0)| = 1$. The function $\psi$ is not in general conjugate to its linear part. In this case the differential form $\omega$ is not necessarily conjugate via $C^l$ diffeomorphisms to the standard form $\lambda \, dr + i \, d\theta$. However, a weaker result can be proved in this situation. More precisely, we have the following theorem.

**Theorem 2.2.** Let $\omega$ and $\lambda$ be as in (2.5) and (2.7). Suppose that $\text{Im} \, \lambda = 0$. Then there exists $\sigma > 0$ and a $C^{1+\sigma}$ diffeomorphism $\Phi: U \to V$, where $U$ and $V$ are open neighborhoods of $\{0\} \times S^1 \subset \mathbb{R} \times S^1$ such that $\Phi(S^1) = S^1$ and

$$\Phi^* \omega = g(r, \theta)(\lambda \, dr + i \, d\theta). \quad (2.43)$$

**Proof.** Let's write $\omega$ as

$$\omega = \lambda \, dr + i(1 + \alpha(\theta) + i\beta(\theta)) \, d\theta + i r^2 e(r, \theta) \, d\theta, \quad (2.44)$$

where $\alpha, \beta$ are $\mathbb{R}$-valued functions and

$$\int_{\alpha}^{2\pi} \alpha(\theta) \, d\theta = \int_{\beta}^{2\pi} \beta(\theta) \, d\theta = 0.$$
Let
\[
A(\theta) = \int_0^\theta \alpha(\sigma) \, d\sigma \quad \text{and} \quad B(\theta) = \int_0^\theta \beta(\sigma) \, d\sigma.
\] (2.45)

With respect to the coordinates
\[
R = r \exp \frac{-B(\theta)}{a},
\]
\[
\Theta = \theta + A(\theta)
\] (2.46)
(where \( \lambda = a \in \mathbb{R}^+ \)), the form \( \omega \) has the expression
\[
\omega = \lambda \, dR + iR(1 + iRe(\lambda, \Theta)) \, d\Theta,
\] (2.47)
for some \( C^\infty \) function \( e' \). As in the proof of Theorem 2.1, we use the transformation
\[
p = R^a
\]
\[
\phi = \Theta
\] (2.48)
to pullback \( \omega \) into a multiple of
\[
\Omega = dp + ip(1 + ip^{1/a}E) \, d\phi,
\] (2.49)
where
\[
E(p, \phi) = e'(p^{1/a}, \phi)
\] (2.50)
is bounded. The rest of the proof is similar to that of Theorem 2.1. This time the blowing down of \( \Omega \) is a multiple of
\[
dz - i \left| z \right|^{1/a} \frac{E}{2 + i \left| z \right|^{1/a} E} \frac{z}{2} \, dz
\] (2.51)
which is elliptic near \( 0 \in \mathbb{R}^2 \) and of class \( C^{(1/a) - \varepsilon} \) for any \( 0 < \varepsilon < \frac{1}{2} \). The diffeomorphism of class \( C^{1+(1/a) - \varepsilon} \) that transforms \( \omega \) into a multiple of \( \lambda \, dr + i r \, d\theta \) is constructed in a similar fashion as that in the proof of Theorem 2.1.

**Remark 2.2.** It is clear that the form \( \lambda \, dr + i r \, d\theta \) and \( -\lambda \, dr + i r \, d\theta \) are equivalent. The invariant \( \lambda \) is therefore defined up to sign. In the next section, we will find it more convenient to use \( \text{Re} \lambda < 0 \).
When the structure given by \( \omega \) is real analytic in \( r \) and \( \text{Im} \lambda \neq 0 \), the conjugacy between \( \omega \) and \( \lambda \, dr + i r \, d\theta \) can be achieve via a \( C^\infty \) diffeomorphism that is real analytic in \( r \). To see why, we consider the complexification \( \hat{\omega}(r, \theta) \) of \( \omega(r, \theta) \) with respect to \( r \). One can show directly (as was done with \( \hat{\omega}_N \)) that \( \hat{\omega} \) is conjugate to \( \lambda \, dr + i r \, d\theta \), via a \( C^\infty \) diffeomorphism that is holomorphic in \( r \). An analogous result to that of Lemma 2.5 gives the following theorem.

**Theorem 2.3.** Let \( \omega \) and \( \lambda \) be as in (2.5) and (2.7). Suppose that \( \text{Im} \lambda \neq 0 \) and that \( \omega \) is real analytic in the \( r \) variable. Then \( \omega \) is conjugate (near the characteristic set) to \( \lambda \, dr + i r \, d\theta \) under a \( C^\infty \) diffeomorphism that is real analytic in \( r \).

### 3. GLOBAL NORMAL FORMS

In this section, we determine global representatives in each component of \( \mathbb{R}^2 \setminus \Sigma \) for structures satisfying conditions H1, H2, and H3.

Let \( \omega \) be a \( C^\infty \)-1-form in \( \mathbb{R}^2 \) with characteristic set \( \Sigma \) and satisfying H1, H2, and H3 and let \( L \) be a generator of \( \mathcal{L} \). We have

\[
\mathbb{R}^+ = \Omega_+ \cup \Sigma \cup \Omega_-,
\]

where \( \Omega_\pm \) are the connected components of \( \mathbb{R}^2 \setminus \Sigma \), and where \( \Omega_+ \) is the simply connected and bounded component. Let \( U \) be an open neighborhood of \( \Sigma \). Define

\[
U^+ = \Omega_+ \cap U \quad \text{and} \quad U^- = \Omega_- \cap U.
\]

For \( \varepsilon > 0 \), define

\[
A^+ = (0, \varepsilon) \times S^1 \quad \text{and} \quad A^- = (-\varepsilon, 0) \times S^1.
\]

**Proposition 3.1.** Let \( \omega \) be as above and \( \lambda \in \mathbb{R}^+ + i \mathbb{R} \) be the associated invariant. Suppose that \( \text{Im} \lambda \neq 0 \). Then there exists a neighborhood \( U \) of \( \Sigma \) in \( \mathbb{R}^2 \), \( \varepsilon > 0 \), and \( C^\infty \) diffeomorphisms

\[
\phi^+: U^+ \cup \Sigma \to A^+ \cup (\{0\} \times S^1)
\]

\[
\phi^-: U^- \cup \Sigma \to A^- \cup (\{0\} \times S^1)
\]

such that

\[
\omega \wedge (\phi^+)\ast (\lambda \, dr + i r \, d\theta) = 0.
\]
Proof. It follows from Theorem 2.1 that there is an open neighborhood $U$ of $\Sigma$ and a $C^0$ function $z: U \to D_\sigma = \mathbb{C}$, where $D_\sigma$ is the disc with radius $\sigma$ centered at 0, such that
\[ z: U^+ \to D_\sigma \setminus 0 \quad \text{and} \quad z: U^- \to D_\sigma \setminus 0 \] (3.5)
are $C^\infty$ diffeomorphisms, $z(\Sigma) = 0$ and $\omega \wedge dz = 0$.
Let $\varepsilon = \sigma^{1/\alpha}$, where $\lambda = \alpha + ib$ and let
\[ F: (-\varepsilon, \varepsilon) \times S^1 \to D_\sigma, \quad F(r, \theta) = r^\lambda e^{i\theta}. \] (3.6)
Note that $dF \wedge (\lambda \, dr + ir \, d\theta) = 0$ (for $r \neq 0$). Define
\[ \phi^+: U^+ \to A_\varepsilon^+ \quad \text{by} \quad \phi^+(r, \theta) = F^{-1} \cdot z(r, \theta). \] (3.7)
$\phi^+$ is then a $C^\infty$ diffeomorphism and satisfies (3.4) in $U^+$. We need to show that $\phi^+$ extends as a $C^\infty$ diffeomorphism up to $\Sigma$. For this we use again Theorem 2.1 as follows. Let $k \in \mathbb{Z}^+$ and $\phi_k: U \to A_k$ be a $C^k$ diffeomorphism that conjugates $\omega$ with $\lambda \, dr + ir \, d\theta$. The function $z \circ \phi_k^{-1}$ is then a solution of the equation $L_0h = 0$, where
\[ L_0 = \zeta \frac{\partial}{\partial \theta} - ir \frac{\partial}{\partial r} \]
is orthogonal to $\lambda \, dr + ir \, d\theta$. We have
\[ z \circ \phi_k^{-1} = h^+ \circ F \quad \text{in} \quad U^+, \] (3.8)
where $h^+$ is a holomorphic function defined in $D_\sigma \setminus 0$. Since $z$ is continuous in $U$ and is a diffeomorphism in $U^+$, then $h^+$ is holomorphic in $D_\sigma$ and $(h^+)'(0) \neq 0$. We can assume that $(h^+)'(0) = 1$. Now we have,
\[ \phi^+(r, \theta) = F^{-1} \cdot h^+ \cdot F \circ \phi_k(r, \theta). \] (3.9)
We can write
\[ \phi_k(r, \theta) = rA(r, \theta) \exp(iB(r, \theta)) \quad \text{and} \quad h^+(\zeta) = \zeta(1 + \zeta K(\zeta)), \] (3.10)
where $A(0, \theta) \neq 0$, $A$, $B$ are $R$-valued of class $C^k$ and $2\pi$ periodic in $\theta$, and $K$ is holomorphic. A direct calculation shows that

$$\phi^+(r, \theta) = rA |1 + rK|^{1/2} \exp \left( i \left( B + \arg(1 + rK) - \frac{b}{a} \log(1 + rK) \right) \right).$$

(3.11)

It is therefore clear that $\phi^+$ extends as a $C^k$ function up to $\Sigma$. Since $k \in \mathbb{Z}^+$ is arbitrary, then $\phi^+$ is in fact $C^\infty$ up to $\Sigma$. An analogous argument shows that $\phi^-$ is also $C^\infty$ up to $\Sigma$.}

**Lemma 3.1.** Let $u$ be a solution of $Lu = 0$ in $U^+$ (respectively in $U^-$). If $u$ is bounded, then it is continuous in $U^+ \cup \Sigma$ (respectively in $U^- \cup \Sigma$) and $u$ is constant on $\Sigma$.

**Proof.** Let $\phi$ be a diffeomorphism near $\Sigma$ that conjugates $\omega$ with $\tilde{\omega} = dr + i d\theta$. The function $F \circ \phi$, where $F$ is given by (3.6), is a solution of $Lu = 0$. Moreover, $F \circ \phi$ is a first integral of $\omega$ in $U^\pm$. Hence, any solution $u$ of $Lu = 0$ in $U^+$ can be factored through $F \circ \phi$, via a holomorphic function, defined in a punctured disc. That is

$$u(r, \theta) = h \circ F \circ \phi(r, \theta),$$

where $h(\zeta)$ is holomorphic for $\zeta \neq 0$. Now, if $u$ is bounded, then so is $h$. Therefore $h$ extends holomorphically to 0. This means that $u \in C^0(U^+ \cap \Sigma)$ and $u(\Sigma) = h(0)$. An analogous argument holds for a bounded solution in $U^-$. 

The differential form $\omega$ defines a complex structure in $\Omega^+$ and a complex structure in $\Omega^-$. By the uniformization theorem (see [S]), there exist diffeomorphisms

$$z_{\pm} : \Omega_\pm \to \mathbb{C}$$

such that

$$\omega \wedge dz_{\pm} = 0 \quad \text{in } \Omega_\pm.$$

(3.13)

Furthermore, $z_+(\Omega_+)$ is either the unit disc $D$ or $\mathbb{C}$ and $z_-(\Omega_-)$ is an annulus.

**Lemma 3.2.** The differential form $\omega$ defines a parabolic structure in $\Omega_+$. That is, the uniformizing function of (3.12) satisfies $z_+(\Omega_+) = \mathbb{C}$. 


Proof. If the structure on $\Omega_+$ were hyperbolic ($z_+(\Omega_+) = D$), then the function $z_+$ would be a bounded solution of $Lu = 0$ in $\Omega_+$. By Lemma 3.1 the function $z_+$ would then be continuous up to $\Sigma$ and constant on $\Sigma$. This is a contradiction.

Lemma 3.3. There exists a uniformizing function

$$z_- : \Omega_- \to \mathbb{C}$$

such that $z_-$ is continuous up to $\Sigma$, $z_- (\Sigma) = 0$ and either

$$z_-(\overline{\Omega}_-) = C \quad \text{or} \quad z_-(\overline{\Omega}_-) = D.$$ 

Proof. Let $z_-$ be as in (3.12). The image $z_-(\Omega_-)$ is one of the following annuli

$$C^* = \{ z : |z| > 0 \} \quad \text{or} \quad A_\rho = \{ z : \rho < |z| < 1 \},$$

for some unique $0 \leq \rho < 1$. Let $R > 0$ be large enough so that

$$\Sigma \subset \{ (x, y) : x^2 + y^2 < R^2 \} = D_R.$$ 

Without loss of generality, we can assume that $z_- (\Omega_- \cap D_R)$ is bounded (otherwise replace $z_-$ by $1/z_-$. The function $z_-$, bounded near $\Sigma$, extends then continuously up to $\Sigma$ and is constant on $\Sigma$ (Lemma 3.2). This implies that $z_- (\Sigma) = 0$ and so the conclusion of the lemma holds.

For $\lambda = a + ib \in \mathbb{C}$ with $a < 0$, let

$$\omega_+^\lambda = [1 + (\lambda - 1) r] \, dr + ir(1 - r) \, d\theta. \tag{3.14}$$

Note that $\omega_+^\lambda$ is real analytic in $\mathbb{R}^2$, and satisfies H1, H2, H3 with characteristic set $S^1$ and associated invariant $\lambda$. Let

$$F : D \to \mathbb{C}, \quad F(re^{i\theta}) = r(1 - r)^i \, e^{i\theta}. \tag{3.15}$$

Note that since $\Re \lambda < 0$, then $F$ is a real analytic diffeomorphism.

Theorem 3.1. Let $\omega$ be as above with associated invariant $\lambda$. Suppose that $\Re \lambda < 0$ and $\Im \lambda \neq 0$. Then there exists a $C^\infty$-diffeomorphism

$$\Phi^+ : \overline{\Omega}_+ \to \overline{D}$$

such that

$$\left( \Phi^+ \right)^* \omega_+^\lambda \wedge \omega = 0. \tag{3.16}$$
Proof. Let \( z_+ : \Omega_+ \to \mathbb{C} \) be as in Lemma 3.2. Define
\[
\Phi_+ : \Omega_+ \to D \quad \text{by} \quad \Phi_+(p) = f^{-1} \circ z_+(p),
\]
where \( F \) is defined in (3.15). Clearly, \( \Phi_+ \) is a diffeomorphism and it satisfies (3.16) in \( \Omega_+ \). That \( \Phi_+ \) extends smoothly up to \( \Sigma \) follows from Theorem 2.1 and Proposition 3.1.  

Let
\[
\omega_{1,\lambda} = \lambda \, dr + i(r-1) \, d\theta
\]
(3.18)
\[
\omega_{2,\lambda} = \lambda \, dr + i(r-1) \, d\theta.
\]
The differential forms \( \omega_{1,\lambda} \) and \( \omega_{2,\lambda} \) are real analytic in \( \mathbb{R}^2 \setminus 0 \) with characteristic set \( S^1 \), satisfy conditions H1, H2, H3, and have invariant \( \lambda \). Let \( D' = \mathbb{R}^2 \setminus \bar{D} \) and let
\[
G_1 : D' \to C; \, G_1(r, \theta) = \frac{1}{(r-1)^2} e^{-\theta}
\]
(3.19)
\[
G_2 : D' \to D; \, G_2(r, \theta) = \left( \frac{r}{r-1} \right)^\lambda e^{-\theta}.
\]
The functions \( G_1 \) and \( G_2 \) are diffeomorphisms, continuous up to the boundary \( S^1 \) and satisfy
\[
dG_i \wedge \omega_{i,\lambda} = 0 \quad i = 1, 2.
\]
(3.20)

Arguments similar to those used in the proof of Theorem 3.1 give

**Theorem 3.2.** Let \( \omega \) be as above satisfying H1, H2, H3 and with invariant \( \lambda \in \mathbb{R}^- + i\mathbb{R} \). Suppose that \( \text{Im} \, \lambda \neq 0 \). If \( \omega \) defines a parabolic structure in \( \Omega_- \), then there exists a \( C^\infty \) diffeomorphism
\[
\Phi_1 : \bar{\Omega}_- \to \bar{D}', \quad \text{such that} \quad \Phi_1^* \omega_{1,\lambda} \wedge \omega = 0.
\]
(3.21)
If \( \omega \) defines a hyperbolic structure in \( \Omega_- \), then there exists a \( C^\infty \) diffeomorphism
\[
\Phi_2 : \bar{\Omega}_- \to \bar{D}', \quad \text{such that} \quad \Phi_2^* \omega_{2,\lambda} \wedge \omega = 0.
\]
(3.22)
When the associated invariant \( \lambda \in \mathbb{R} \), the results of the above theorems still hold but only under \( C^{1+\sigma} \) diffeomorphisms. More precisely,

**Theorem 3.3.** Let \( \omega \) be as in Theorem 3.1 with \( \lambda \in \mathbb{R} \). Then (3.16) holds with \( \Phi^+ \in C^{1+\sigma} \) for some \( \sigma > 0 \).
**Theorem 3.4.** Let $\Omega$ be as in Theorem 3.2 with $\lambda \in \mathbb{R}$. Then (3.21) or (3.22) holds with $\Phi_i \in C^{1+\sigma}$ for some $\sigma > 0$.

The proofs are analogous to those of Theorems 3.1 and 3.2 and will not be repeated here.

**4. THE HOMOGENEOUS EQUATION $Lu = 0$**

Most of the results in this section are implicitly contained in [BM2] and, therefore, no proofs will be given here. Let

$$L = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}$$

be a $C^\infty$ vector field in $\mathbb{R}^2$. Suppose that $L$ satisfies conditions H1, H2, H3 with invariant $\lambda = a + ib$ and with characteristic set $\Sigma = S^1$. For convenience, we will assume in this section that $a < 0$. Let $(r, \theta)$ be the polar coordinates of $\mathbb{R}^2 (x = r \cos \theta, y = r \sin \theta)$, and let

$$X_\lambda = \lambda \frac{\partial}{\partial \theta} - i(1-r) \frac{\partial}{\partial r}$$

$$X^+ = \left[1 - (\lambda + 1) r \right] \frac{\partial}{\partial \theta} - ir(1-r) \frac{\partial}{\partial r}$$

$$X^-_{1,\lambda} = \lambda \frac{\partial}{\partial \theta} - i(r-1) \frac{\partial}{\partial r}$$

$$X^-_{2,\lambda} = \lambda \frac{\partial}{\partial \theta} - ir(r-1) \frac{\partial}{\partial r}$$

The first integrals of these vector fields are, respectively,

$$z_\lambda = |1-r|^{-\lambda} e^{\lambda \theta} \quad \text{for} \quad 1-\varepsilon < r < 1+\varepsilon$$

$$z^+_\lambda = r(1-r)^{\lambda} e^{\lambda \theta} \quad \text{for} \quad 0 \leq r < 1$$

$$z^-_{1,\lambda} = (r-1)^{-\lambda} e^{-\lambda \theta} \quad \text{for} \quad r > 1$$

$$z^-_{2,\lambda} = \left( \frac{r}{r-1} \right)^{\lambda} e^{-\lambda \theta} \quad \text{for} \quad r > 1.$$  

By using the standardness results of the previous sections, the study of the equation $Lu = 0$ can be converted into an equation $Xu = 0$, where $X$ is one
of the vector fields given in (4.2). We can therefore prove the following results.

**Theorem 4.1.** If $u$ is a $C^0$ solution of $X_{\varepsilon}u = 0$ in the annulus

$$\Gamma_{\varepsilon} = \{(r, \theta); 1 - \varepsilon < r < 1 + \varepsilon\} \quad \text{with} \quad 0 < \varepsilon < 1$$

then there exist holomorphic functions $h^+$ and $h^-$ defined in the disc $|z| < e^{-a}$ with $h^+(0) = h^-(0)$ such that

$$u(r, \theta) = h^+ (|1 - r|^{-1} e^{i\theta}) \quad \text{if} \quad 1 - \varepsilon < r < 1,$$

$$u(r, \theta) = h^- (|1 - r|^{-1} e^{i\theta}) \quad \text{if} \quad 1 < r < 1 + \varepsilon.$$  \hspace{1cm} (4.5)

The following result is an easy consequence of Theorem 4.1.

**Theorem 4.2.** If $\lambda \notin \mathbb{Q}$ and $u \in C^k(\Gamma_{\varepsilon})$ (where $\Gamma_{\varepsilon}$ is the annulus in (4.4)) satisfies $X_{\varepsilon}u = 0$, then

$$u(r, \theta) = c + O(|r - 1|^{-na}) \quad \text{with} \quad -na > k \quad \text{and} \quad c \in \mathbb{C}. \quad (4.6)$$

Since $\Sigma$ is an orbit on $L$, then, a priori, it could be support of a distribution solution. It is however proved in [BM2] that, unless $\lambda \in \mathbb{Q}$ no such distributions exist. More precisely, we have the following result.

**Theorem 4.3.** The equation $Lu = 0$ has a solution $u \in \mathcal{D}(\mathbb{R}^2)$ supported by $\Sigma$ if and only if $\lambda \in \mathbb{Q}$.

The vector field $L$ defines a parabolic structure in the unit disc $D$ (see Section 3). It follows then, from Liouville’s Theorem, that the constants are the only bounded solutions of $Lu = 0$ in $D$. The same conclusion holds for the solutions in $\mathbb{R}^2 \setminus D$ when $L$ is equivalent to $X_{\varepsilon}$. We state this as

**Theorem 4.4.** If $u \in C^0(D)$ is bounded and satisfies $Lu = 0$, then $u$ is constant. If $L$ is equivalent to $X_{\varepsilon}$ in $\mathbb{R}^2 \setminus D$ and if $u \in C^0(\mathbb{R}^2 \setminus D)$ is a bounded solution of $Lu = 0$, then $u$ is constant.

In view of this result, it is more appropriate to widen the class of solutions and to consider $C$-valued solution. Let $O$ be an open subset of $\mathbb{R}^2$. A $C^0$ function $u: O \to \mathbb{C}$ is said to be a solution of $Lu = 0$ if $\forall p \in O$ such that $u(p) \neq \infty$, $Lu = 0$ near $p$, and $\forall p \in O$ such that $u(p) = \infty$, $Lu^{-1} = 0$ near $p$.

**Theorem 4.5.** Let $u: \mathbb{R}^2 \to \mathbb{C}$ be a $C^0$ solution of $Lu = 0$ and let $\phi_+$ and $\phi_-$ be the diffeomorphisms that transform, respectively, $L$ into $X^+_\varepsilon$ in $D$ and
Let $X_{\pm}^L$ in $\mathbb{R}^n \setminus \mathcal{D}$ (where $i = 1$ or $2$, depending on the structure defined by $L$). Then there exist meromorphic functions $R_+$ and $R_-$, respectively, on $\mathbb{C}$ and $z_{\pm}^L(\mathbb{R}^n \setminus \mathcal{D})$ such that $R_+(0) = R_-(0)$ and

$$u(r, \theta) = R_+ \cdot z_+^L \cdot \phi_+(r, \theta) \quad \text{if} \quad r \leq 1$$

$$u(r, \theta) = R_- \cdot z_-^L \cdot \phi_-(r, \theta) \quad \text{if} \quad r \geq 1.$$  (4.7)

**Remark 4.1.** It follows from these results that if a solution $u$ vanishes on a sequence of points in $r < 1$ (resp. $r > 1$) and if the sequence has an accumulation point on $\Sigma$, then $u$ is constant in $r < 1$ (resp. $r > 1$). It also follows from Theorem 4.5 that there exists an $N > 0$ such that $|1 - r|^N u$ is a bounded function near $\Sigma$.

### 5. THE EQUATION $Lu = f$

In this section, we study the solvability of the equation

$$Lu = f,$$  (5.1)

where $L$ is a vector field in $\mathbb{R}^2$ satisfying conditions H1, H2, H3, with characteristic set $\Sigma = S^2$, and invariant $\lambda = a + ib$ with $a > 0$. We will prove (Theorem 5.1) that if $f$ is Hölder continuous, then (5.1) has a Hölder continuous solution if and only if

$$\int_{\Sigma} f \, ds = 0.$$  (5.2)

When $f$ is $C^\infty$ and satisfies (5.2), we prove (Theorem 5.2) that for every $k \in \mathbb{Z}^+$ Eq. (5.1) has a $C^k$ solution. It should be noted that there are $C^\infty$ functions $f$ for which (5.1) has no $C^\infty$ solution $u$ (see [BM]). When condition (5.2) fails, then Eq. (5.1) has distribution solutions provided that $f \in L^p$ with $p > \max(2, a^{-1})$ (Theorem 5.3).

Before we proceed further, note that the study of Eq. (5.1) can be reduced to the study of the equation

$$T_\lambda u = f,$$  (5.3)

where

$$T_\lambda = \frac{\partial}{\partial \theta} - i r \frac{\partial}{\partial r}.$$  (5.4)
is defined on $\mathbb{R} \times S^1$, and where the function $f$ can be assumed to have compact support in the cylinder

$$A_\varepsilon = (-\varepsilon, \varepsilon) \times S^1 \quad \text{(with } 0 < \varepsilon < 1). \quad (5.5)$$

**Theorem 5.1.** Let $f \in C_0^\sigma(A_\varepsilon)$, with $0 < \sigma < 1$. Then Eq. (5.3) has a solution $u \in C^\sigma(A_\varepsilon)$, for some $0 < \tau < 1$, if and only if

$$\int_0^{2\pi} f(0, \theta) \, d\theta = 0. \quad (5.6)$$

**Proof.** Necessity. Suppose that there exist $0 < \tau < 1$ and $u \in C^\tau(A_\varepsilon)$ that solves (5.3). Then, the $C^\sigma(S^1)$ function $u(0, \theta) = u(0, \theta)$ solves

$$\lambda \frac{du}{d\theta}(0) = f(0, \theta). \quad (5.7)$$

It is easily seen that (5.7) has a $(2\pi$-periodic) solution if and only if (5.6) holds.

Sufficiency. Suppose that (5.6) holds. Define $u_0 \in C^{\tau+1}(S^1)$ by

$$u_0(\theta) = \frac{1}{\lambda} \int_0^\theta f(0, s) \, ds. \quad (5.8)$$

Then $T_\lambda u_0 = f(0, \theta)$. Write

$$f(r, \theta) - f(0, \theta) = r^\sigma g(r, \theta), \quad (5.9)$$

with $g \in C^\sigma(A_\varepsilon \setminus \{0\} \times S^1)$ bounded. Consider the equation

$$T_\lambda v(r, \theta) = r^\sigma g(r, \theta). \quad (5.10)$$

To solve (5.10), we use the first integral

$$z_\lambda(r, \theta) = r^\lambda e^{i\theta} \quad (5.11)$$

of $T_\lambda$ to transform (5.10) into a CR equation in the disc $D(e^\lambda)$ with radius $e^\lambda$ and center 0 in $\mathbb{C}$. Let

$$A_\lambda^+ = A_\varepsilon \cap \{r \geq 0\} \quad \text{and} \quad A_\lambda^- = A_\varepsilon \cap \{r \leq 0\}. \quad (5.12)$$

It follows from $T_\lambda z_\lambda = -2\lambda z_\lambda$ that the Eq. (5.11) is equivalent to the pair of equations

$$-2\lambda z \frac{\partial \tilde{g}}{\partial z} = |z|^\sigma \tilde{g}(z) \quad (5.13)$$
in the disc $D(a)$, where

\[
\begin{align*}
v(r, \theta) &= \tilde{v}^\pm(z_j(r, \theta)) \quad \text{in } A_*^\pm \\
g(r, \theta) &= \tilde{g}^\pm(z_j(r, \theta)) \quad \text{in } A_*^\pm.
\end{align*}
\] (5.14)

The functions $\tilde{g}^\pm$ are bounded in $C^\sigma(D(a)\setminus0)$. Therefore,

\[
|z|^{(\sigma/\alpha)-1} \tilde{g}^\pm \in L^p(D(a)),
\] (5.15)

for any $p > 2$ if $\sigma > \alpha$ and for $2 < p < (2\alpha/(\alpha - \sigma))$ if $\sigma < \alpha$. Hence, if $\alpha = \frac{\sigma + \alpha}{p}$, the solutions

\[
\tilde{v}^\pm(z) = \frac{1}{2\pi i} \int_{D(e^\zeta)} \frac{|\zeta|^{(\sigma/\alpha)-1} \tilde{g}^\pm(\zeta) e^{\theta \zeta}}{\zeta - z} d\zeta \wedge d\bar{\zeta}
\] (5.16)

are in $C^\sigma(D(a))$ and in $C^{1, \sigma}(D(a)\setminus0)$ (see for example [CH] or [V]). We can therefore write

\[
\tilde{v}^\pm(z) = c^\pm + |z|^\sigma \tilde{w}^\pm(z),
\] (5.17)

with $c^\pm \in C$ and $\tilde{w}^\pm$ bounded and in $C^{1, \sigma}(D(a)\setminus0)$. The functions

\[
v^\pm(r, \theta) = |r|^\sigma \tilde{w}^\pm(z_j(r, \theta))
\] (5.18)

satisfy Eq. (5.10) and $A_*^\pm$. Define

\[
u(r, \theta) = u_d(\theta) + v^\pm(r, \theta) \quad \text{in } A_*^\pm.
\] (5.19)

Then $u \in C^m(A_\ast)$ solves (5.3).

**Theorem 5.2.** Suppose that $\lambda \notin Q$. Let $f \in C^0_0(A_\ast)$. If $f$ satisfies (5.6), then for every $k \in \mathbb{Z}^+$, there exists $u \in C^k_0(A_\ast)$ such that

\[
T_\lambda u = f.
\] (5.20)

**Proof.** Let $n \in \mathbb{Z}^+$. Expand $f(r, \theta)$ as

\[
f(r, \theta) = \sum_{j=0}^{n} f_j(\theta) r^j + r^{n+1} f_{n+1}(r, \theta),
\] (5.21)

where

\[
f_j(\theta) = \frac{1}{j!} \frac{\partial^j f}{\partial r^j}(0, \theta).
\] (5.22)
For each $j \leq n$, let
\[ u_j(\theta) = \frac{1}{\lambda} \int_{0}^{\theta} f_j(s) \exp \left( \frac{i j}{\lambda} (\theta - s) \right) \, ds + K_j \exp \left( \frac{i j}{\lambda} \theta \right), \] (5.23)

where
\[ K_j = \left( 1 - \exp \left( \frac{i j}{\lambda} 2\pi \right) \right)^{-1} \frac{1}{\lambda} \int_{0}^{2\pi} f_j(s) \exp \left( \frac{i j}{\lambda} (2\pi - s) \right) \, ds. \] (5.24)

Note that $K_j$ is well defined since $\lambda \notin \mathbb{Q}$. The functions $u_j \in C^\infty$ satisfy
\[ T_s(u_j(\theta) r') = f_j(\theta) r'. \] (5.25)

As in the proof of the previous theorem, the equation
\[ T_s v = r^{n+1} f_{n+1}(r, \theta) \] (5.26)
is transformed, via the first integral $z_1$ of (5.11), into the following CR equations in the disc $D(e^\theta)$
\[ \frac{\partial \tilde{v}}{\partial z} = \frac{i}{2a} |z|^{l(n+1)+1} \tilde{f}_{n+1}(z) \exp \theta. \] (5.27)

The solutions
\[ \tilde{v}(z) = \frac{1}{2\pi i} \int_{D(e^\theta)} \frac{i z^{l(n+1)+1} \tilde{f}_{n+1}(\zeta) \exp \theta}{\zeta - z} \, d\zeta \] (5.28)
are in $C^l(D(e^\theta))$ with $l \geq n+1$. We can then write
\[ \tilde{v}(z) = \sum_{j=0}^{l-1} \tilde{v}_j z^j + |z|^l \tilde{v}_l(z), \] (5.29)
where $\tilde{v}_j$ are continuous and $C^\infty$ outside 0. The functions $|z|^l \tilde{v}_l(z)$ satisfy also Eq. (5.27). Finally, the function $u$ defined in $A_\epsilon$ by
\[ u(r, \theta) = \sum_{j=0}^{n} u_j(\theta) r^j + |r|^m \tilde{w}(r^m \exp \theta) \quad \text{in } A_\epsilon^\pm, \] (5.30)
satisfies equation $T_s u = f$ and is in $C^k$ for $n$ large enough. \[ \blacksquare \]
Theorem 5.3. Let \( f \in L^p(A) \) with \( p > \max(2, a^{-1}) \). Then for each \( q < \min(p, 2a) \), there exists \( w \in C^\infty(A) \) with \( 0 < \sigma < 1 \) such that the function
\[
u(r, \theta) = r^\sigma w(r, \theta) \quad \text{with} \quad \tau = \frac{2a}{q}
\]
is in \( L^q(A) \) with \( l < q/2a \) and satisfies Eq. (5.3).

Before we proceed with the proof of the Theorem, we need the following lemma.

Lemma 5.1. Let \( f \in L^p(A) \) and let \( \tilde{f}^\pm \) be the pushforwards to \( D(e^a) \), via \( z_\lambda \), of \( f \) in \( A^\pm \), respectively. Then,
\[
\tilde{f}^\pm \in L^p(D(e^a)) \quad \text{if} \quad 2a < 1;
\tilde{f}^\pm \in L^q(d(e^a)) \quad \text{for any} \quad q < 2ap \quad \text{if} \quad 2a > 1.
\]

Proof. By the change of variable formula, we have
\[
\int_{D(e^a)} |\tilde{f}^\pm|^q \, dz \, d\bar{z} = -2a \int_{A^\pm} |f|^q \, r^{2a-1} \, dr \, d\theta. \quad (5.31)
\]
Thus, if \( 2a > 1 \), then \( r^{2a-1} \) is a continuous function in \( A^\pm \) and \( \tilde{f}^\pm \in L^p(D(e^a)) \), since \( f \in L^p(A) \). If \( 2a < 1 \), let \( 0 < \delta < 1 - 2a \) and let
\[
s = \frac{1 - \delta}{2a} \quad \text{and} \quad t = \frac{1 - \delta}{(1 - 2a) - \delta}. \quad (5.32)
\]
We have
\[
s > 1 \quad \text{and} \quad \frac{1}{l} + \frac{1}{s} = 1.
\]
For any \( 0 < q < 2ap \) we have
\[
qs < p \quad \text{and} \quad (2a - 1) \, t > -1. \quad (5.33)
\]
It follows at once from (5.31), (5.33), and the Hölder’ inequality that
\[
\int_{D(e^a)} |\tilde{f}^\pm|^q \, dz \, d\bar{z} \leq 2a \left( \int_{A^\pm} |f|^q \, dr \, d\theta \right)^{1/s} \left( \int_{A^\pm} r^{(2a-1)\, t} \, dr \, d\theta \right)^{1/t} < \infty.
\]
That is, \( \tilde{f}^\pm \in L^q(D(e^a)) \).
Proof of Theorem 5.3. Consider the equation

\[ T_\lambda v = z_\lambda f. \]  

(5.34)

This equation is transferred via \( z_\lambda \) into equations

\[ \tilde{v}^\pm \left( \frac{1}{2\pi i} \int_{D(\rho)} \frac{ie^{2a\theta} \tilde{f}^\pm(\zeta)}{2a(\zeta - z)} d\zeta \wedge d\zeta^* \right), \]  

(5.35)

where the right hand side is in \( L^q(D(e^\sigma)) \), with \( 2 < q < \min(\rho, 2\alpha p) \). Hence the solutions

\[ \tilde{v}^\pm(z) = \frac{1}{2\pi i} \int_{D(\rho)} \frac{ie^{2a\theta} \tilde{f}^\pm(\zeta)}{2a(\zeta - z)} d\zeta \wedge d\zeta^* \]  

(5.36)

are in \( C^\sigma(D(e^\sigma)) \) with \( \sigma = 1 = \frac{3}{4} \). The functions

\[ \tilde{v}^\pm(z) - \tilde{v}^\pm(0) = |z|^\sigma \tilde{w}^\pm(z), \]  

where \( \tilde{w}^\pm \) are bounded and in \( C^\sigma(D(e^\sigma)) \), also satisfies (5.35). The function \( v(r, \theta) \) defined in \( A_\sigma \) by

\[ v(r, \theta) = |r|^\sigma w(r^\sigma e^{i\theta}), \]  

(5.36)

where \( w(r^\sigma e^{i\theta}) = \tilde{w}^\pm(r^\sigma e^{i\theta}) \) in \( A_\sigma^\pm \), satisfies (5.34). Let

\[ u(r, \theta) = \frac{v(r, \theta)}{z_\lambda} = |r|^\sigma w(r, \theta) = \frac{w(r, \theta)}{|r|^{2\alpha q}}. \]  

(5.37)

The function \( u \) is in \( L^l(A_\sigma) \) for each \( l < \frac{2q}{\alpha} \). Now we will verify that \( u \) satisfies Eq. (5.3).

The transpose of \( T_\lambda \) is

\[ T_\lambda^t = -\lambda \frac{\partial}{\partial \theta} + ir \frac{\partial}{\partial r} + 2i. \]  

(5.38)

For \( 0 < \delta < \varepsilon \), let

\[ A^+_{r, \delta} = A_\varepsilon \cap \{ r > \delta \} \quad \text{and} \quad A^-_{r, \delta} = A_\varepsilon \cap \{ r < -\delta \}. \]  

(5.39)

Then for \( \phi \in C_0^\infty(A_\varepsilon) \) we have

\[ \langle T_\lambda u, \phi \rangle = \langle u, T_\lambda^t \phi \rangle \]

\[ = \lim_{\delta \to 0} \int_{A^+_{r, \delta}} uT_\lambda^t \phi r \, dr \, d\theta + \lim_{\delta \to 0} \int_{A^-_{r, \delta}} uT_\lambda^t \phi r \, dr \, d\theta. \]  

(5.40)
An integration by parts shows that
\[ \int_{A_{\xi \eta}} u T' \phi r \, dr \, d\theta = \int_{A_{\xi \eta}} f \phi r \, dr \, d\theta = \int_{A_{\xi \eta}} f \phi \, dr \, d\theta. \]

Since \( 2 + a(\sigma - 1) > 0 \), then
\[ \lim_{\delta \to 0} \int_{A_{\xi \eta}} u T' \phi r \, dr \, d\theta = \int_{A_{\xi \eta}} f \phi \, dr \, d\theta. \]

Similarly,
\[ \lim_{\delta \to 0} \int_{A_{\xi \eta}} u T' \phi r \, dr \, d\theta = \int_{A_{\xi \eta}} f \phi \, dr \, d\theta. \]

This shows that \( T_{\xi} u = f \).

6. THE EQUATION \( Lu = pu \)

Let \( L \) be a vector field in \( \mathbb{R}^2 \) satisfying conditions H1, H2, H3, with invariant \( \lambda = a + ib \) and \( a > 0 \). The study of the equation \( Lu = pu \) can be reduced to that of an equation of the form
\[ T_{\xi} u = pu \quad \text{in} \quad A_{\xi}, \quad (6.1) \]
where \( T_{\xi} \) is the vector field defined in (5.4) and \( A_{\xi} \) is the ring defined in (5.5).

**Theorem 6.1.** Let \( p \in C^0(A_{\xi}) \) (with \( \lambda = \alpha - \xi \eta \)). Then Eq. (6.1) has a solution \( u \in C^0(A_{\xi}) \) (for some \( \lambda < \xi < 1 \)) with \( u \neq 0 \) on the characteristic set \( \Sigma \) if and only if the function \( p \) satisfies
\[ \frac{1}{2\pi i} \int_{0}^{2\pi} p(0, \theta) \, d\theta \in \lambda \mathbb{Z}. \quad (6.2) \]

Furthermore, when \( p \) satisfies (6.2), any other continuous solution \( u' \) of (6.1) can be written as
\[ u'(r, \theta) = u(r, \theta) H^\pm(r^\xi e^{i\theta}) \quad \text{in} \quad A_{\xi}^\pm, \quad (6.3) \]
where \( H^\pm \) are holomorphic functions defined in the disc \( D(e^\alpha) \).
Proof. Necessity. If \( u \) is a Hölder continuous solution of (6.1) such that \( u \neq 0 \) on \( \Sigma \), then \( u_0(\theta) = u(0, \theta) \) satisfies
\[
\lambda u_0(\theta) = p(0, \theta) u_0(\theta). \tag{6.4}
\]
Equation (6.4) has a nontrivial solution if and only if (6.2) holds.

Sufficiency. Suppose that \( p \) satisfies (6.2). Let
\[
w_0(\theta) = \frac{1}{\lambda} \int_0^\theta p(0, s) \, ds. \tag{6.5}
\]
Consider the equation
\[
T_* w = p(r, \theta) - p(0, \theta) = p_1(r, \theta). \tag{6.6}
\]
Since \( p_1 \) is Hölder continuous and \( p_1 \equiv 0 \) on \( \Sigma \), then (6.6) has a Hölder continuous solution \( w(r, \theta) \) with \( w \equiv 0 \) on \( \Sigma \) (see Section 5). The function
\[
u(r, \theta) = \exp(w_0(\theta) + w(r, \theta)) \tag{6.7}
\]
satisfies (6.1) and \( u(0, \theta) \neq 0 \) for all \( \theta \).

To prove the second part of the theorem, suppose that \( p \) satisfies (6.2) and that \( u' \) is any solution of (6.1). The function \( v = \frac{u'}{u} \) (where \( u \) is the solution defined in (6.7)) satisfies \( T_* v = 0 \). The results in Section 4 imply that \( u' \) has the form given in (6.3).

Theorem 6.2. Suppose that \( p \in C^\infty(A_i) \) satisfies (6.2). Then for every \( k \in \mathbb{Z}^+ \), Eq. (6.1) has a solution \( u \in C^k(A_i) \) such that \( u \neq 0 \) on \( \Sigma \).

Proof. Let \( n \in \mathbb{Z}^+ \). Consider the Taylor expansion of \( p \) with respect to \( r \)
\[
p(r, \theta) = \sum_{j=0}^n p_j(\theta) r^j + r^{n+1}p_{n+1}(r, \theta). \tag{6.8}
\]
For each \( j \geq 1 \), there is a \( C^\infty \) function \( w_j(\theta) \) such that
\[
T_j(w_j(\theta) r^j) = p_j(\theta) r^j \tag{6.9}
\]
(see Section 5). If \( n \) is large enough, the equation
\[
T_j(w_{n+1}) = r^{n+1}p_{n+1}(r, \theta) \tag{6.10}
\]
has a solution that vanishes to order \( k \) along \( \Sigma \). The function

\[
u(r, \theta) = \exp \left( w_0(\theta) + \sum_{j=1}^{n} w_j(\theta) r^j + w_{n+1}(r, \theta) \right),\]

(6.11)

where \( w_0 \) is given by (6.4) is a \( C^k \) solution of (6.1) and \( u \) is nowhere 0 on the characteristic circle \( \Sigma \).

**Theorem 6.3.** Let \( p \) be a Hölder continuous function in \( A_* \). Suppose that

\[
\frac{1}{2\pi i} \int_0^{2\pi} p(0, \theta) d\theta \not\in \mathbb{Z}.
\]

(6.12)

Then there exist a Hölder continuous function \( v \) in \( A_* \), with \( v \) nowhere 0 on \( \Sigma \), and a unique \( \mu \in \mathbb{C} \) with

\[
0 < \Re \mu \leq a
\]

(6.13)

such that the function \( u = r^\mu v \) satisfies Eq. (6.1). Furthermore, any other Hölder continuous solution \( u' \) of (6.1) can be written as

\[
u'(r, \theta) = \nu(r, \theta) H^\pm(r^\mu \theta) \quad \text{in} \ A^\pm_*,
\]

(6.14)

where \( H^\pm \) are holomorphic functions defined in the disc \( D(e^a) \).

**Proof.** Let

\[
\alpha + i\beta = \frac{1}{2\pi i} \int_0^{2\pi} p(0, \theta) d\theta \in \mathbb{C}.
\]

(6.15)

Since \( p \) satisfies (6.12), then \( \alpha + i\beta \not\in \mathbb{Z} \). Let \( k = \left\lfloor \frac{\alpha}{2} \right\rfloor + 1 \in \mathbb{Z} \), where \( \left\lfloor x \right\rfloor \) denotes the integer part of the real number \( x \). Define the complex number \( \mu \) as

\[
\mu = (ka - \alpha) + i(kb - \beta) = k\lambda - (a + ib).
\]

(6.16)

Note that \( \mu \) satisfies (6.13). With this choice of \( \mu \), the function

\[
\hat{p}(r, \theta) = p(r, \theta) - i\mu
\]

(6.17)

satisfies condition (6.2). Let \( \nu \) be a Hölder continuous solution of \( T_\nu = \hat{p} \nu \) such that \( \nu \) is nowhere 0 on \( \Sigma \) (see Theorem 6.1). The function \( u = r^\mu \nu \) satisfies (6.1).

Now, suppose that \( u' \) is a Hölder continuous solution of (6.1). Then necessarily \( u'(0, \theta) \equiv 0 \), and so \( u' = r^\mu \nu' \) for some \( t > 0 \), and some bounded
function $v'$ not identically 0 on $\Sigma$. If $t \geq \text{Re} \mu$, then an argument similar to that used in the proof of Theorem 6.1 shows that $u'$ has the desired form (6.14). To finish the proof, we will show that the condition

$$0 < t < \text{Re} \mu$$

(6.18)
cannot occur. Indeed, if $t$ satisfies (6.18), then the function $w = \frac{z}{2}$ is a continuous solution of $T_*w = 0$. The function $w$ has therefore the form $H^+(r^2e^{i\theta})$ in $A_+^*$ with $H^+$ holomorphic in $D(e^a)$. It follows that

$$r^2u(r, \theta) = u(r, \theta) H^+(r^2e^{i\theta}) = r^2v'(r, \theta) H^+(r^2e^{i\theta}).$$

(6.19)

This together with (6.18) implies that $H^+(0) = 0$ and thus

$$\text{Re} \mu \geq t + \text{Re} \lambda > a.$$

This contradicts the choice of $\mu$ satisfying (6.13).

**Theorem 6.4.** Let $p$ be a function defined in $A_\varepsilon$ such that $r^{-\delta}p \in L^t(A_\varepsilon)$ for some $\delta > 0$ and $t > 2$. Then Eq. (6.1) has a H"older continuous solution $u$ with $u(0, \theta) \neq 0$ for all $\theta$. Furthermore, any other continuous solution $u'$ of (6.1) can be written as

$$u'(r, \theta) = u(r, \theta) H^+(r^2e^{i\theta}), \quad \text{in } A_+^*,$$

(6.20)

where $H^+$ are holomorphic functions defined in the disc $D(e^a)$.

**Proof.** Equation (6.1) leads, via $z = r^2e^{i\theta}$ to equations

$$\frac{\partial \tilde{u}^\pm}{\partial z} = \frac{i \tilde{v}^\pm}{2az} \tilde{w}^\pm$$

(6.21)
in the disc $D(e^a)$. Since $r^{-\delta}p \in L^t(A_\varepsilon)$, then $\frac{\partial \tilde{v}^\pm}{\partial z} \in L^t(D(e^a))$ for some $t > 2$. The equations

$$\frac{\partial \tilde{u}^\pm}{\partial z} = \frac{i \tilde{v}^\pm}{2az}$$

(6.22)
have then H"older continuous solutions. We can assume that $\tilde{w}^\pm(0) = 0$.

The function $u$ defined in $A_\varepsilon$ by

$$u(r, \theta) = \exp \langle \tilde{w}^\pm(r^2e^{i\theta}) \rangle \quad \text{in } A_+^*$$

(6.23)
is a H"older continuous solution of (6.1). The proof of the second part of the Theorem is similar to that of Theorem 6.1. $\blacksquare$
7. THE EQUATION \( Lu = pu + f \)

We will study in this section the equation
\[
T_z u = pu + f \quad \text{in } A_z,
\]  
(7.1)

where \( T_z \) is the vector field defined in (5.4), \( A_z \) is the ring defined in (5.5), and \( p \) and \( f \) are \( C^\infty \) functions in \( A_z \). Although the results given here can be generalized to a wider class of data, we will limit the study to only the smooth case.

**Theorem 7.1.** Let \( p, f \in C^\infty(A_z) \). Suppose that
\[
\frac{1}{2\pi i} \int_0^{2\pi} p(0, \theta) d\theta \notin \mathbb{Z}. 
\]  
(7.2)

Then for every \( k \in \mathbb{Z}^+ \), Eq. (7.1) has a solution \( u \in C^k(A_z) \).

**Proof.** For \( n \in \mathbb{Z}^+ \), consider the Taylor expansions with respect to \( r \)
\[
p(r, \theta) = \sum_{j=0}^n p_j(\theta) r^j + r^{n+1} p_{n+1}(r, \theta) 
\]  
(7.3)

\[
f(r, \theta) = \sum_{j=0}^n f_j(\theta) r^j + r^{n+1} f_{n+1}(r, \theta).
\]

Define
\[
u_0(\theta) = \left\{ \int_0^\theta f_0(s) \exp \left( -\frac{1}{\lambda} \int_0^s p_0(t) dt \right) ds + K_0 \right\} \exp \left( \frac{1}{\lambda} \int_0^\theta p_0(t) dt \right),
\]  
(7.4)

where
\[
K_0 = \left[ 1 - \exp \left( \frac{1}{\lambda} \int_0^{2\pi} p_0(t) dt \right) \right]^{-1} \times \int_0^{2\pi} f_0(s) \exp \left( -\frac{1}{\lambda} \int_0^s p_0(t) dt \right) ds.
\]  
(7.5)

Define functions \( u_j(\theta) \) (\( 1 \leq j \leq n \)) inductively by
\[
u_j(\theta) = \left\{ \int_0^\theta g_j(s) \exp \left( -\frac{1}{\lambda} \int_0^s p_0(t) dt + \frac{ij\theta}{\lambda} \right) ds + K_j \right\} \times \exp \left( \frac{1}{\lambda} \int_0^\theta p_0(t) dt + \frac{ij\theta}{\lambda} \right),
\]  
(7.6)
where
\[ K_j = \left[ 1 - \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \right) \right]^{-1} \]
\[ \times \int_0^{2\pi} g_j(s) \exp \left( \frac{1}{2\pi} \int_0^s p_i(t) \, dt + \frac{ijn}{2\pi} \right) \, ds \]  
(7.7)

and
\[ g_j(s) = f_j(s) + \sum_{i=1}^J p_i(s) u_{ij}(s). \]  
(7.8)

Note that the constants \( K_j \)'s have been chosen so that the functions \( u_j \)'s are \( 2\pi \)-periodic. Moreover the functions \( u_j \) satisfy the differential equations
\[ iu_j'(\theta) - ju_j(\theta) = p_j(\theta) u_j(\theta) + \sum_{i=1}^J p_i(\theta) u_{ij}(\theta) + f_j(\theta). \]  
(7.9)

The function \( v_n \) defined in \( A_n \) by
\[ v_n(r, \theta) = \sum_{j=0}^n u_j(\theta) r^j \]  
(7.10)

satisfies
\[ T_j v_n = \left( \sum_{j=0}^n p_j(\theta) r^j \right) v_n + \left( \sum_{j=0}^n f_j(\theta) r^j \right). \]  
(7.11)

Since the function \( p \) satisfies (7.2), then there exists \( \mu \in \mathbb{C} \) with \( 0 < \text{Re} \mu \leq \alpha \) such that the equation
\[ T_j \phi = -p \phi \]  
(7.12)

has a solution of the form \( \phi_l = r^l \psi_l \) with \( \psi_l \in C^l(A_n) \) and \( \psi_l \) is nowhere zero on \( \Sigma \), where \( l \) is any preassigned integer (see Theorem 6.3). The equation
\[ T_j w = \frac{r^{n+1} + f_{n+1}}{\phi_l} \]  
(7.13)

has a \( C^m \) solution (for any given \( m \)) provided that \( n \) and \( l \) are large enough. Moreover, a solution \( w \) of (7.13) can be chosen to vanish to an order \( > k + 1 + \text{Re} \mu \) on \( \Sigma \) (provided that \( n \) is large). The function
\[ u = \phi_l w + v_n \]  
(7.14)

is a \( C^k \) solution of (7.1).
Theorem 7.2. Let $p, f \in C^\alpha(A_x)$. Suppose that

$$
\frac{1}{2\pi i} \int_0^{2\pi} p(0, \theta) d\theta \in \lambda \mathbb{Z}.
$$

(7.15)

Then Eq. (7.1) has a Hölder continuous solution if and only if

$$
\int_0^{2\pi} f(0, s) \exp \left[ -\frac{1}{\lambda} \int_0^s p(0, t) dt \right] ds = 0.
$$

(7.16)

Moreover, when (7.16) holds, then for every $k \in \mathbb{Z}^+$, Eq. (7.1) has a solution $u \in C^k(A_x)$.

Proof. Suppose that (7.1) has a solution $u \in C^\tau(A_x)$ ($0 < \tau < 1$). If $f \equiv 0$ on $\Sigma$, then (7.16) holds. If $f \not\equiv 0$ on $\Sigma$, then the function $u_0(\theta) = u(0, \theta)$ satisfies

$$
\lambda u_0'(\theta) = p(0, \theta) u_0(\theta) + f(0, \theta).
$$

(7.17)

A direct integration shows that (7.17) has a $2\pi$-periodic solution if and only if (7.16) holds.

Conversely, suppose that (7.16) holds. Define

$$
u_0(\theta) = \int_0^\theta f(0, s) \exp \left[ \frac{1}{2} \int_s^\theta p(0, t) dt \right] ds.
$$

(7.18)

The function $u_0$ is a $2\pi$-periodic solution of (7.17). Arguments analogous to those used in the proof of Theorem 7.1 can be adapted to construct a $C^k$ solution of (7.1).

8. THE EQUATION $Lu = pu + q\bar{u}$

In this section we will study the structure of the solutions of the equation

$$
T_x u = pu + q\bar{u} \quad \text{in } A_x,
$$

(7.1)

where $T_x$ and $A_x$ are defined in (5.4) and (5.5). We will show that a generalized form of the similarity principle holds for this type of equations.

Theorem 8.1. Let $p, q \in C^\alpha(A_x)$ with $q = 0$ on $\Sigma$. Suppose that

$$
\frac{1}{2\pi i} \int_0^{2\pi} p(0, \theta) d\theta \in \lambda \mathbb{Z}.
$$

(8.2)
Then every continuous solution of (8.1) has the form
\[ u(r, \theta) = e^{\alpha r, \theta} w(r, \theta), \]
where \( s \) is Hölder continuous and \( w \) satisfies
\[ T_s w = 0 \quad \text{in} \ A_\epsilon. \]

Furthermore, for every continuous solution \( w \) of (8.4) there exists a continuous function \( s \) such that the function \( e^s w \) satisfies (8.1).

**Remark 8.** This Theorem together with the results in Section 4 imply that a solution of (8.1) has the form
\[ u(r, \theta) = e^{i \alpha r, \theta} H^\pm(r^2 e^{i \theta}) \quad \text{in} \ A_\epsilon, \]
where \( H^\pm \) are holomorphic functions defined in the disc \( D(e^\alpha) \).

**Proof of Theorem 8.1.** Let \( \phi \in \mathcal{C}^1(A_\epsilon) \) such that \( \phi \neq 0 \) on \( \Sigma \) and
\[ T_s \phi = p \phi \]
(see Theorem 6.2). Consider the equation
\[ T_s v = g \phi \phi^{-1} \]
where
\[ g(r, \theta) = \frac{g(r, \theta)}{r} \frac{\phi(r, \theta)}{\phi(r, \theta)} \in \mathcal{C}^1(A_\epsilon). \]

The transfer of equation (8.7), via the first integral \( z = r^2 e^{i \theta} \) to the disc \( D(e^\alpha) \), gives rise to equations
\[ \frac{\partial \tilde{v}^\pm}{\partial z} = \frac{|z|^{1/a}}{z} \tilde{g}^\pm \tilde{v}^\pm. \]

Since \( \tilde{g}^\pm \) is bounded and
\[ \frac{|z|^{1/a}}{z} \in L^t(D(e^\alpha)) \]
for any \( t > 0 \) if \( a \leq 1 \) and for \( 2 < t < \frac{2a}{a-1} \) if \( a > 1 \), then Eqs. (8.8) have \( \mathcal{C}^\sigma \) solution with \( \sigma = \frac{t-1}{t} \). Moreover, the solutions \( \tilde{v}^\pm \) have the form
\[ \tilde{v}^\pm(z) = e^{z^{1/a}} H^\pm(z), \]
where $s^\pm \in C^\alpha$ with $s^\pm(0) = 0$ and $H^\pm$ are holomorphic (see [CH] or [V]). Conversely, for given holomorphic functions $H^\pm$, there exist $s^\pm$ as above so that the functions defined in (8.9) satisfy (8.8). In particular, if $H^+(0) = H^-(0)$, then the function

$$w(r, \theta) = H^+(r^4 e^{i\theta})$$

in $A^+_\pm$ (8.10)
satisfies (8.4) and the function

$$u(r, \theta) = \exp(s^\pm(r, \theta) + \log \phi(r, \theta)) w(r, \theta)$$

satisfies (8.1).

**Theorem 8.2.** Let $p, q \in C^\alpha(A_\pm)$ with $q = 0$ on $\Sigma$. Suppose that

$$\frac{1}{2\pi i} \int_0^{2\pi} p(0, \theta) \, d\theta \notin \mathbb{Z}. \quad (8.12)$$

Then there exists a unique $\mu \in \mathbb{C}$ with

$$0 < \Re \mu \leq a$$

(8.13)
such that every continuous solutions $u$ of (8.1) has the form

$$u(r, \theta) = r^a e^{\alpha r, \theta} w(r, \theta), \quad (8.14)$$

where $s$ is Hölder continuous and $w$ satisfies (8.4). Furthermore, for every continuous solution $w$ of (8.4) there exists a continuous function $s$ such that the function $u$ defined in (8.14) satisfies (8.1).

**Proof.** Let $\mu$ be the unique complex number defined in Theorem (6.3). Define

$$\phi(r, \theta) = r^a \psi(r, \theta),$$

with $\psi \in C^1(A_\pm), \psi$ nowhere zero on $\Sigma$, such that

$$T_x \phi = p \phi$$

(8.15)

(see Theorem 6.3). Consider the equation

$$T_x v = q \frac{\phi}{\psi},$$

(8.16)

It follows from the proof of the previous theorem that

$$v(r, \theta) = e^{\alpha r, \theta} w(r, \theta)$$

(8.17)
for some continuous functions \( s \) and \( w \) with \( w \) satisfying (8.4) and \( s \equiv 0 \) on \( \Sigma \). The function

\[
u(r, \theta) = r^\mu \exp(s(r, \theta) + \log \psi(r, \theta)) w(r, \theta) \tag{8.18}\]

is a continuous solution of (8.1) of the form (8.14).

If \( \nu' \) is any other solution of (8.1) such that \( r^{-\mu} \nu' \) is a bounded function, then

\[
u'(r, \theta) = \frac{\nu'(r, \theta)}{\phi(r, \theta)} = \frac{\mu'(r, \theta)}{r^\mu \psi(r, \theta)} \tag{8.19}\]

satisfies Eq. (8.16). Thus \( \nu' \) has the form (8.17) and therefore \( \nu' = \phi \nu' \) has the form (8.14).

To complete the proof, we will show that whenever \( \nu' \) is a solution of (8.1), then \( r^{-\mu} \nu' \) is bounded. By contradiction, suppose that \( \nu' \) is a solution of (8.1) such that

\[
|r|^{-\mu} |\nu'| \geq c \quad \text{for some} \quad c > 0 \quad \text{and} \quad 0 < t < \Re \mu. \tag{8.20}\]

The continuous function

\[
\nu(r, \theta) \quad \text{is such that} \quad |r|^{-\Re \mu} |\nu| \geq M > 0 \tag{8.22}\]

for some \( M > 0 \). Moreover, \( \nu \) is a solution of the equation

\[
T \nu = q |\nu|^2 \quad \text{for some bounded function} \quad g \quad (\text{we are using the fact that} \quad q = 0 \quad \text{on} \quad \Sigma). \quad \tag{8.23}\]

for some bounded function \( g \). If \( \tilde{\nu}^+ \) is defined in \( D(e^z) \) such that \( \tilde{\nu}^+ = \nu^+ \circ \widetilde{z} \) in \( A^+ \), then \( \tilde{\nu}^+(0) = 0 \) and it satisfies a CR equation

\[
\frac{\partial \tilde{\nu}^+}{\partial \bar{z}} = |z|^\alpha A(z) \tag{8.24}\]

with \( A \) a bounded function and \( \nu = \frac{1 + \Re \mu - t}{\mu} - 1 \). Therefore

\[
\tilde{\nu}^+(z) = H^+(z) + \frac{1}{2\pi i} \int_{\partial \nu(v)} \frac{|z'|^\alpha A(z)}{z' - z} \, d\zeta' \wedge \bar{d\zeta}' \tag{8.25}\]
with $H$ holomorphic. Since the integral appearing in (8.25) is in $C'$ with $\tau = \frac{2-n}{p}$ for any $p > 2$ if $v > 0$ and $2 < p < \frac{2-n}{2v}$ if $v < 0$, then

$$\int\int_{D(z,\tau)} |z|^n A(z) \left( \frac{1}{z-z} + \frac{1}{z} \right) d\zeta \wedge d\bar{\zeta} = O(|z|^\tau).$$

(8.26)

It follows from (8.26), (8.25), and the definition of $\bar{F}$ that

$$F = O(|\tau|^{\Re p - \frac{1}{2}})$$

(8.27)

which contradicts (8.22).

9. THE EQUATION $Lu = pu + q\bar{u} + f$

Consider the equation

$$T_s u = pu + q\bar{u} + f \quad \text{in} \ A_s,$$

(9.1)

where $T_s$ and $A_s$ are defined in (5.4) and (5.5). The general solution of (9.1) has the form $u + w$, where $u$ is a particular solution of (9.1) and $w$ is the general solution of (8.1). In this section, we construct a particular solution of (9.1) when $a < \frac{1}{2}$ (recall that $a = \Re \lambda$).

**Theorem 9.1.** Suppose that $a < \frac{1}{2}$. Then for every $p$, $q \in C^\infty(A_s)$ with $q = 0$ on $\Sigma$ and such that

$$\int_0^{2\pi} f(0, \theta) \exp \left[ -\frac{1}{\lambda} \int_0^\theta p(0, s) \, ds \right] d\theta = 0$$

(9.2)

whenever

$$\int_0^{2\pi} \frac{1}{2\pi i} \int_0^{2\pi} \frac{p(0, \theta)}{\lambda + 0} \, d\theta \in i \mathbb{Z},$$

(9.3)

Eq. (9.1) has a Hölder continuous solution in $A_s$.

**Proof.** Let $u_0(\theta)$ be a $C^\infty$ solution of

$$\lambda u_0(\theta) = p(0, \theta) u_0(\theta) + f(0, \theta).$$

(9.4)

Such a solution exists since $f$ satisfies (9.2), when $p$ satisfies (9.3) (see Section 7). Let

$$g(r, \theta) = f(r, \theta) - f(0, \theta) + (p(r, \theta) - p(0, \theta)) u_0(\theta) + q(r, \theta) \overline{u_0(\theta)}$$

(9.5)
Note that \( g(r, \theta) = r g_1(r, \theta) \) with \( g_1 \in C^\infty(A_\varepsilon) \). Let \( \phi \) be a Hölder continuous solution of

\[
T_{\lambda} \phi = p \phi. \quad (9.6)
\]

We can take \( \phi \) so that \( \phi \neq 0 \) on \( \Sigma \) if (9.3) does not hold and \( \phi = r^a \psi \) with \( \psi \neq 0 \) on \( \Sigma \) if (9.3) holds (see Section 6) with \( 0 < \Re \mu \leq a \). Since \( a < \frac{1}{2} \), then \( \frac{\lambda}{\phi} \) is Hölder continuous and vanishes on \( \Sigma \). Consider the equation

\[
T_{\lambda} v = q \frac{\phi}{\phi} \ddot{v} + \frac{R}{\phi}. \quad (9.7)
\]

This equation is transferred, via the first integral \( z_1 = r^a e^{i\theta} \) into the disc \( D(e^a) \), into equations of the form

\[
\frac{\partial \ddot{v}^\pm}{\partial z^2} = A^\pm \ddot{v}^\pm + B^\pm, \quad (9.8)
\]

where \( A^\pm \) and \( B^\pm \) are Hölder continuous and vanish at 0 (that \( A^\pm \) and \( B^\pm \) vanish at 0 follows from the fact that \( q \) and \( \frac{\lambda}{\phi} \) vanish to order 1 and \( \Re \mu \), respectively, and from the hypothesis that \( a < \frac{1}{2} \)) Equations (9.8) have then solutions \( \ddot{v}^\pm \) with \( \ddot{v}^\pm(0) = 0 \). Equation (9.7) has then a Hölder continuous solution \( v \) with \( v = 0 \) on \( \Sigma \). Finally, the function

\[
u(r, \theta) = \nu_0(\theta) + \phi(r, \theta) v(r, \theta)
\]

is a Hölder continuous solution of (9.1).}

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\[\text{REFERENCES}\]


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