Dynamic stress intensity factors for two parallel interface cracks between a nonhomogeneous bonding layer and two dissimilar elastic half-planes subject to an impact load

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Some composite materials are constructed of two dissimilar half-planes bonded by a nonhomogeneous elastic layer. In the present study, a crack is situated at the interface between the upper half-plane and the bonding layer of such a material, and another crack is located at the interface between the lower half-plane and the bonding layer. The material properties of the bonding layer vary continuously from those of the upper half-plane to those of the upper half-plane. Incoming shock stress waves impinge upon the two interface cracks normal to their surfaces. Fourier transformations were used to reduce the boundary conditions for the cracks to two pairs of dual integral equations in the Laplace domain. To solve these equations, the differences in the crack surface displacements were expanded in a series of functions that are zero-valued outside the cracks. The unknown coefficients in the series were solved using the Schmidt method so as to satisfy the conditions inside the cracks. The stress intensity factors were defined in the Laplace domain and were inverted numerically to physical space. Dynamic stress intensity factors were calculated numerically for selected crack configurations.

1. Introduction

Ceramic-coated materials are increasingly being used for machine parts due to their good anti-oxidation characteristics at high temperatures. To improve adhesion between a stainless-steel substrate and a plasma-sprayed ceramic coating, the substrate may be pretreated by depositing a NiCr intermediate layer on the substrate. The material properties of this thin layer vary continuously from those of the substrate metal to those of the ceramic. Moreover, the layer may contain defects, such as voids and microcracks, which may grow into cracks. Therefore, it is useful to solve for the stress intensity factors around a crack in the nonhomogeneous thin layer between the dissimilar elastic half-planes. A similar thin layer also appears in the bonding of dissimilar materials by the diffusion method or ultrasonic welding, for example.

The two-dimensional problem for a cracked interfacial layer between two dissimilar elastic half-planes has been solved by Delale and Erdogan (1988). The stress and displacement fields for internal pressure on the crack surfaces were obtained by assuming that the material properties vary continuously in the interfacial layer from the material properties of the upper half-plane to those of the lower half-plane. Axisymmetric solutions have been determined for a penny-shaped crack in an interfacial nonhomogeneous layer between two dissimilar elastic half-planes by Ozturk and Erdogan (1995, 1996). In these studies (Delale and Erdogan, 1988; Ozturk and Erdogan, 1995, 1996), the material properties of the nonhomogeneous layer were assumed to vary exponentially.

Using a different method from that of Delale and Erdogan (1988), Itou and Shima (1997) solved for the stress intensity factors for a crack in an interfacial layer between two dissimilar elastic half-planes. More specifically, the interfacial layer was divided into several sublayers with different material properties. Subsequently, Itou and Shima (1999) solved the axisymmetric problem for a cylindrical crack in an interfacial zone between an elastic circular cylinder and an infinite elastic medium for mode I loading. A similar method was also applied to determine the thermal stresses for a crack in a nonhomogeneous layer between two dissimilar elastic half-planes (Itou, 2004a) and the thermal stresses for a crack in a nonhomogeneous layer between a coating plate and an elastic half-plane (Itou, 2005a).

For composite materials joined by a cracked nonhomogeneous layer that are suddenly loaded, it is necessary to clarify the transient dynamic stress intensity factors. Babaei and Łukasiwicz (1998) solved the transient dynamic problem for a crack in a nonhomogeneous layer between two dissimilar elastic half-planes. The crack surfaces were suddenly loaded by anti-plane shear traction and the corresponding dynamic stress intensity factor for mode III loading was obtained. Using a similar method to that employed for solving the static problem (Itou and Shima, 1997), the corre-
sponding dynamic stresses were solved for a crack in a nonhomo-
geneous interfacial layer between two dissimilar elastic half-
planes for mode I impact loading (Itou, 2001). Li et al. (2006a)
solved for the mode III dynamic stress intensity factors for an inter-
face crack located at the interface between two dissimilar half-
planes made of a functionally gradient material (FGM) during the
passage of an incident stress wave simulated by the Dirac delta
function. Li et al. (2006b) also obtained the mode I and II stress
intensity factors for an interface crack situated at the interface be-
tween a FGM substrate and a FGM coating plate. Yoffe-type moving
cracks propagate with a constant length, exhibiting a constant
velocity in an elastic medium (Yoffe, 1951). This model was ap-
p lied to a moving Griffith crack in an infinite nonhomogeneous
layer sandwiched between two dissimilar half-planes (Wang and

As for three-dimensional problems, solutions were obtained for
two dissimilar homogeneous half-spaces bonded by a nonhomo-
geneous layer weakened by a penny-shaped crack under a tor-
sional impact load (Li et al., 2002; Li and Weng, 2002). In fiber-
reinforced composite materials, a nonhomogeneous interlayer ex-
ists between the cylindrical fiber and the matrix. Xue-Li and Duo
(1996) obtained the stress intensity factors around a cylindrical
 crack in the interlayer under static torsional loading. The corre-
sponding dynamic solution has been obtained for a cylindrical
 crack in a nonhomogeneous interlayer under torsional impact
(Li and Weng, 2001). The stress intensity factors have been solved
for a cylindrical crack at the interface between a nonhomogeneous
interlayer and an infinite elastic medium subjected to torsional im-
pact (Li et al., 2001). A similar problem was solved for a cracked
 cylindrical interlayer between a solid cylinder and a hollow cylind-
er (Feng et al., 2005). Stresses have been determined for a Yoffe-
type moving cylindrical crack that propagates with a constant
velocity in a nonhomogeneous interlayer between an infinite elas-
tic medium and an elastic circular cylinder produced from another
material (Itou, 2005b). Transient dynamic stresses were solved for
two rectangular cracks in a nonhomogeneous interfacial layer be-
tween two dissimilar elastic half-spaces during the passage of inci-
dent impact stress waves (Itou, 2007a).

In the aforementioned studies (Ozturk and Erdogan, 1995,
1996; Li et al., 2002, 2006a,b; Li and Weng, 2001; Feng et al.,
2005), a crack (or cracks) was located at the interface between dis-
similar materials. This is because the interface may be somewhat
weaker than the nonhomogeneous layer and the homogeneous
elastic half-space, and because cracks are more likely to appear
at interfaces. These composite materials are considered to be
weakened by cracks that appear at individual interfaces between
dissimilar materials. Using the finite element method, static stres-
ses have been solved for two parallel cracks in composite materials
composed of two dissimilar elastic plates bonded by a nonhomo-
geneous interlayer (Zhang et al., 2008). Here, one of the cracks
was situated at the upper interface between the interlayer and
the upper elastic plate, while the other crack was located at the
lower interface between the interlayer and the lower elastic plate.

Wang et al. (2000) identified the dynamic stress intensity fac-
tors for a crack (or cracks) in nonhomogeneous composite materi-
als. Their method was similar to that used by Itou and Shima
(1997) to obtain static solutions for a crack in a nonhomogeneous
layer between two dissimilar elastic half-planes as well as that
used by Itou and Shima (1999b) to obtain static solutions for a cylin-
 drical crack in a nonhomogeneous cylindrical layer between an
elastic cylinder and an infinite elastic medium. Wang et al.
(2000) numerically calculated the dynamic stress intensity factors
for two parallel interface cracks, one situated at the interface be-
tween an aluminum plate and a nonhomogeneous plate and the
other located at the interface between the nonhomogeneous plate
and a ceramic plate. Numerical calculations were performed for
the case in which impact pressures were simultaneously applied
to the surfaces of the two cracks (Wang et al., 2000).

Ma et al. (2004) solved the dynamic mode III problem for two
parallel cracks, one of which was situated at the interface between
the lower metallic half-plane and a FGM layer and the other of
which was located at the interface between the FGM layer and
the upper ceramic half-plane. The material properties were found
to vary exponentially within the layer, and the time-harmonic inci-
dent displacement wave was found to impinge normal to the
interfaces.

When incident shock stress waves pass through a nonhomogene-
ous material, both the dilatational and shear wave velocities vary
in the material. To the best of the author’s knowledge, such a problem
has not been solved for a nonhomogeneous material weakened by
two parallel cracks. In the present study, the transient dynamic prob-
lem is solved for composite materials that consist of two dissimilar
elastic half-planes bonded by a nonhomogeneous thin elastic layer.
One crack is situated at the interface between the upper half-plane
and the bonding layer, and the other crack is located between the
lower half-plane and the bonding layer. Transient dynamic stresses
are solved during the passage of shock stress waves that propagate
from the lower half-plane toward the upper half-plane. The material
properties in the bonding layer vary arbitrarily. To solve this prob-
lem, the bonding layer was divided into several homogeneous sub-
layers that have different material properties. If the number of
sublayers, \( m \), is sufficiently large, then the stresses and displace-
ments approximate those of the nonhomogeneous layer.

Using Fourier and Laplace transforms, the boundary conditions
were reduced to dual integral equations in the Laplace domain. To
solve these equations, the differences between the crack surface
placements were expanded as a series of functions that were zero-
valued outside the cracks. The unknown coefficients in the
series are solved using the Schmidt method so as to satisfy the
boundary conditions inside the cracks (Itou and Haliding, 1997).
The stress intensity factors defined in the Laplace domain were in-
verted to physical space using the numerical technique described
by Miller and Guy (1966).

2. Fundamental equations

With reference to Fig. 1, the nonhomogeneous elastic layer (A)
lies within the region \(-h/2 < y < h/2\). A crack parallel to the x-axis
is located between \(-a\) and \(a\) at \(y = h/2\), and a second crack is located
between \(-b\) and \(b\) at \(y = -h/2\). The upper half-plane (C) and
lower half-plane (B) fall within the regions \(h/2 < y < -h/2\), respec-
tively. Plane strain is assumed. The shear modul-
lus, Poisson’s ratio, and density of layer (A) are represented by
\(\mu_A\), \(v_A\), and \(\rho_A\), respectively, and those of the lower half-plane (B)
and upper half-plane (C) are similarly represented by $\mu_B$, $\nu_B$, and $\rho_B$, and $\mu_C$, $\nu_C$, and $\rho_C$, respectively. The material properties ($\mu$, $\nu$, and $\rho$) are assumed to vary continuously with respect to $y$ in the interfacial layer, as shown in Fig. 2.

If the displacement components $u$ and $v$ are expressed by two functions $\phi(x,y,t)$ and $\psi(x,y,t)$ such that

$$u = \partial \phi / \partial x - \partial \psi / \partial y, \quad v = \partial \phi / \partial y + \partial \psi / \partial x,$$

then the equations of motion reduce to

$$\partial^2 \phi / \partial x^2 + \partial^2 \phi / \partial y^2 = 1/c_1^2 \times \partial^2 \phi / \partial t^2,$$

$$\partial^2 \psi / \partial x^2 + \partial^2 \psi / \partial y^2 = 1/c_2^2 \times \partial^2 \psi / \partial t^2,$$

where $t$ is time, and the dilatational wave velocity $c_1$ and the shear wave velocity $c_2$ can be expressed as:

$$c_1^2 = 2(1 - \nu)\mu/[(1 - 2\nu)\rho], \quad c_2^2 = \mu/\rho.$$

The stresses can be expressed as follows:

$$\tau_{yy} = -2\mu\partial^2 \phi / \partial x^2 + \rho\partial^2 \phi / \partial t^2 + 2\mu\partial^2 \psi / \partial x \partial y,$$

$$\tau_{xx} = -2\mu\partial^2 \psi / \partial y^2 + \rho\partial^2 \psi / \partial t^2 - 2\mu\partial^2 \phi / \partial x \partial y,$$

$$\tau_{xy} = 2\mu\partial^2 \phi / \partial x \partial y + \mu(\partial^2 \phi / \partial x^2 - \partial^2 \psi / \partial y^2).$$

The incident stress wave, which propagates parallel to the $y$-axis in the positive direction through the lower half-plane (B), can be expressed as:

$$\tau_{yy}\text{inc} = \phi H(c_B t - (y + h/2)),$$

where $H$ is a constant, $H(t)$ is the Heaviside unit step function, and the subscript B indicates the variables for the lower half-plane (B). Time $t$ is considered to be zero when the wavefront reaches the interface $y = -h/2$. If the incident stress wave impinges on the nonhomogeneous interlayer, the wave is reflected and refracted in layer (A) in a complicated manner. However, it is possible that a stress wave similar to that given by Eq. (5) will pass through the nonhomogeneous layer, varying the wave velocity $c_2$ in the layer. This phenomenon is described in Appendix A. The incident stress wave can be expressed in layer (A) by

$$\tau_{yy}\text{inc} = \phi H \int_0^{c_A t} dT - (y + h/2),$$

where $c_A$ is the dilatational wave velocity at $y = -h/2 + \zeta$, as shown in Fig. 3. Let $t_h$ be the time when the wavefront reaches the interface $y = h/2$, which is given by

$$t_h = \int_0^{h/2} (1/c_{LA}) d\zeta.$$

The incident stress can be expressed in the upper half-plane (C) by

$$\tau_{yy}\text{inc} = \phi H(c_C t - t_h - (y - h/2)).$$

Therefore, the boundary conditions for the present problem can be expressed as

$$\tau_{yy} = \tau_{yyA}, \quad \tau_{yy} = \tau_{yyC} \text{ at } y = -h/2, \quad |x| \leq a,$$

$$\tau_{yy} = \phi H(t), \quad \tau_{yy} = 0 \text{ at } y = h/2, \quad |x| \leq b,$$

$$u_A = u_C, \quad u_B = u_A \text{ at } y = -h/2, \quad b \leq |x|,$$

$$\tau_{yy} = \phi H(t - t_h), \quad \tau_{yy} = 0 \text{ at } y = h/2, \quad |x| \leq a,$$

$$u_A = u_C, \quad u_B = u_A \text{ at } y = h/2, \quad a \leq |x|.$$
and (2), respectively. For \( m = 3 \), the shear moduli \( \mu_i (i = 3, 4, 5, 6, 7) \) for the five homogeneous layers are as shown in Fig. 5 and are given by

\[
\begin{align*}
\mu_4 &= \mu_b, \\
\mu_5 &= \mu_3 = (\mu_A \text{ at } y = 2h/6), \\
\mu_6 &= (\mu_A \text{ at } y = 0), \\
\mu_7 &= \mu_3 = \mu_C.
\end{align*}
\]

(15)

Here, \( \mu_2 \) and \( \mu_1 \) are given as follows:

\[
\mu_2 = \mu_b, \quad \mu_1 = \mu_C.
\]

(16)

Poisson's ratios \( \nu_i \) and the densities \( \rho_i (i = 3, 4, 5, 6, 7) \) have relationships that are similar to those expressed by Eq. (15).

The boundary conditions (9)-(14) may be represented as

\[
\begin{align*}
\tau_{yy}(1) &= \tau_{yy}(3), \quad \tau_{xy}(1) = \tau_{xy}(3), \quad \tau_{xy}(y) = \tau_{xy}(y), \quad u(y) = u(y), \\
\nu(1) &= \nu(3) \quad \text{at } y = h/2, \quad |x| \leq \infty, \\
\tau_{yy}(5) &= \tau_{yy}(7), \quad \nu(5) = \nu(7), \\
\nu(5) &= \nu(7) \quad \text{at } y = h/6, \quad |x| \leq \infty, \\
\tau_{yy}(7) &= \tau_{yy}(6), \quad \tau_{xy}(7) = \tau_{xy}(6), \quad u(7) = u(6), \\
\nu(7) &= \nu(6) \quad \text{at } y = -h/6, \quad |x| \leq \infty, \\
\tau_{yy}(4) &= \tau_{yy}(2), \quad \tau_{xy}(4) = \tau_{xy}(2), \quad u(4) = u(2), \\
\nu(4) = \nu(2) \quad \text{at } y = -h/2, \quad |x| \leq \infty, \\
\tau_{yy}(3) &= \tau_{yy}(5), \quad \tau_{xy}(3) = \tau_{xy}(5), \quad u(3) = u(5), \\
\tau_{yy}(6) &= \tau_{yy}(4), \quad \tau_{xy}(6) = \tau_{xy}(4), \quad u(6) = u(4), \\
\tau_{yy}(4) &= -pH(t), \quad \tau_{xy}(4) = 0 \quad \text{at } y = -h/3, \quad |x| \leq \infty, \\
u(4) = \nu(6) \quad \text{at } y = -h/3, \quad b \leq |x|, \\
\tau_{yy}(3) &= -pH(t - \tau_b), \quad \tau_{xy}(3) = 0 \quad \text{at } y = h/3, \quad |x| \leq a, \\
u(3) = \nu(5) \quad \text{at } y = h/3, \quad a \leq |x|.
\end{align*}
\]

(17-26)

where the subscript indicates the layer number, as previously defined.

4. Analysis

To obtain a solution, the following Laplace transforms are introduced:

\[
g^*(s) = \int_0^\infty g(t) \exp(-st)dt,
\]

\[
g(t) = 1/(2\pi i) \times \int_{\gamma i} g^*(s) \exp(st)ds,
\]

(27)

and the following Fourier transforms are also introduced:

\[
\tilde{f}(\xi) = \int_{-\infty}^{\infty} f(x) \exp(i\xi x)dx.
\]

\[
f(x) = 1/(2\pi i) \times \int_{-\infty}^{\infty} \tilde{f}(\xi) \exp(-i\xi x)d\xi.
\]

(28)

Applying Eqs. (27) and (28) to Eq. (2), we obtain

\[
(d^2/dy^2 - \xi^2 - s^2/C_1^2)\phi_i^* = 0.
\]

\[
(d^2/dy^2 - \xi^2 - C_2^2 s^2/C_1^2)\phi_i = 0.
\]

(29)

where

\[
C_1^2 = 2(1 - v_1)/(1 - 2v_1) \quad (i = 1, 2, \ldots, 7).
\]

(30)

For layers \( i = 3, 4, 5, 6, 7 \), the solutions of Eq. (29) take the following forms:

\[
\phi_i^* = A_i \sinh(\gamma_{1i}y) + B_i \cosh(\gamma_{1i}y),
\]

\[
\phi_i = B_i \sinh(\gamma_{2i}y) + B_i \cosh(\gamma_{2i}y).
\]

(31)

For half-planes (1) and (2), the solutions are:

\[
\phi_i^* = C_i \exp(-\gamma_{1i}y), \quad \phi_i^* = D_i \exp(-\gamma_{2i}y)
\]

(32)

and

\[
\phi_i^* = C_2 \exp(\gamma_{1i}y), \quad \phi_i^* = D_2 \exp(\gamma_{2i}y).
\]

(33)

In Eqs. (31)-(33),

\[
\gamma_{1i} = \left| \xi^2 + (s/C_1)^2 \right|^{1/2}, \quad \gamma_{2i} = \left| \xi^2 + C_2^2 (s/C_1)^2 \right|^{1/2}
\]

(34)

and \( A_{12}, A_{22}, B_{12}, \ldots, D_2 \) are unknown coefficients.

In the Fourier–Laplace transform domain, expressions of stresses and displacements can be obtained in a similar manner to that described in a previous paper (Itoh, 2001). Thereby, the boundary conditions, which are valid for \(-\infty < x < +\infty\), can be easily satisfied. Next, to satisfy Eqs. (24) and (26), the differences \( (u_{3b} - u_{5b}), (v_{3b} - v_{5b}), (u_{6b} - u_{4b}), \) and \( (v_{6b} - v_{4b}) \) are expanded as the following series:

Fig. 4. Interfacial layer represented by three sublayers.

Fig. 5. Shear moduli in the sublayers used to represent the bonding layer.
\[ \pi(u'_{(3a)} - u'_{(5a)}) = \sum_{n=1}^{\infty} a_n \sin[2n \sin^{-1}(x/a)] \quad \text{for } |x| \leq a, \]
\[ = 0 \quad \text{for } a \leq |x|, \]
\[ \pi(v'_{(3a)} - v'_{(5a)}) = \sum_{n=1}^{\infty} b_n \cos[(2n - 1) \sin^{-1}(x/a)] \quad \text{for } |x| \leq a, \]
\[ = 0 \quad \text{for } a \leq |x|, \]
\[ \pi(w'_{(6b)} - w'_{(4b)}) = \sum_{n=1}^{\infty} c_n \sin[2n \sin^{-1}(x/b)] \quad \text{for } |x| \leq b, \]
\[ = 0 \quad \text{for } b \leq |x|, \]
\[ \pi(v'_{(6b)} - v'_{(4b)}) = \sum_{n=1}^{\infty} d_n \cos[(2n - 1) \sin^{-1}(x/b)] \quad \text{for } |x| \leq b, \]
\[ = 0 \quad \text{for } b \leq |x|, \]
where \(a_n, b_n, c_n, d_n\) are unknowns.

In similar manner to that employed in a previous paper (Itou and Haliding, 1997), we can see that the remaining boundary conditions (23) and (25) can be reduced to

\[ \sum_{n=1}^{\infty} a_n k_n(x) + \sum_{n=1}^{\infty} b_n m_n(x) + \sum_{n=1}^{\infty} c_n n_n(x) + \sum_{n=1}^{\infty} d_n o_n(x) = -u(x) \quad \text{for } |x| < a, \]
\[ \sum_{n=1}^{\infty} a_n k_n(x) + \sum_{n=1}^{\infty} b_n p_n(x) + \sum_{n=1}^{\infty} c_n q_n(x) + \sum_{n=1}^{\infty} d_n r_n(x) = 0 \quad \text{for } |x| < b, \]
with

\[ k_n(x) = 2n/\pi \times \int_0^\infty Q_1(\xi)/\xi \times J_{2n}(a \xi) \cos(\xi) d\xi, \]
\[ l_n(x) = (2n - 1)/\pi \times \left\{ \int_0^\infty Q_2(\xi)/\xi - Q_2^{1/2} J_{2n-1}(a \xi) \cos(\xi) d\xi \right\}, \]
\[ m_n(x) = 2n/\pi \times \int_0^\infty Q_3(\xi)/\xi \times J_{2n}(b \xi) \cos(\xi) d\xi, \]
\[ n_n(x) = (2n - 1)/\pi \times \int_0^\infty Q_4(\xi)/\xi \times J_{2n-1}(b \xi) \cos(\xi) d\xi, \]
\[ o_n(x) = 2n/\pi \times \left\{ \int_0^\infty Q_5(\xi)/\xi - Q_5^{1/2} J_{2n}(a \xi) \sin(\xi) d\xi \right\}, \]
\[ p_n(x) = (2n - 1)/\pi \times \int_0^\infty Q_6(\xi)/\xi \times J_{2n-1}(b \xi) \sin(\xi) d\xi, \]
\[ q_n(x) = 2n/\pi \times \left\{ \int_0^\infty Q_{15}(\xi)/\xi - Q_{15}^{1/2} J_{2n}(b \xi) \sin(\xi) d\xi \right\}, \]
\[ + Q_1^{1/2} \sin[2n \sin^{-1}(x/b)]/|b^2 - x^2|^{1/2} \right\}, \]
\[ r_n(x) = (2n - 1)/\pi \times \int_0^\infty Q_{16}(\xi)/\xi \times J_{2n-1}(b \xi) \sin(\xi) d\xi, \]
\[ \sin \left( \frac{\pi}{2} \right) = \sin \left( \frac{\pi}{2} \right), \]
\[ \sin \left( \frac{\pi}{2} \right) = \sin \left( \frac{\pi}{2} \right), \]
\[ \sin \left( \frac{\pi}{2} \right) = \sin \left( \frac{\pi}{2} \right), \]
\[ \sin \left( \frac{\pi}{2} \right) = \sin \left( \frac{\pi}{2} \right), \]
\[ \sin \left( \frac{\pi}{2} \right) = \sin \left( \frac{\pi}{2} \right), \]

where \(\xi\) is a large value of \(\xi\).

The unknowns \(a_n, b_n, c_n, d_n\) in Eqs. (39) and (40) can now be solved by the Schmidt method (Itou and Haliding, 1997).

5. Stress intensity factors

Using the relationships

\[ \int_0^\infty J_1(a \xi) \cos(\xi) d\xi = \left\{ -a^2 \left( x^2 - a^2 \right)^{1/2} \right\} \times \sin(\pi x) - a^2 \left( x^2 - a^2 \right)^{1/2} \cos(\pi x) \}
\[ \int_0^\infty J_1(b \xi) \sin(\xi) d\xi = \left\{ -b^2 \left( x^2 - b^2 \right)^{1/2} \right\} \times \cos(\pi x) - b^2 \left( x^2 - b^2 \right)^{1/2} \sin(\pi x) \}
\[ \text{for } a < x, \]

the stress intensity factors in the Laplace-transform domain can be expressed as

\[ K_a = \lim_{x \to a+} \sqrt{2 \pi (x-a)} \tau_{y(3a)}^{1/2} \]
\[ = \sum_{n=1}^{\infty} b_n \times (2n - 1) - (n)\sin(\pi/2) \]
\[ K_b = \lim_{x \to b-} \sqrt{2 \pi (x-a)} \tau_{y(4b)}^{1/2} \]
\[ = \sum_{n=1}^{\infty} a_n \times 2n \times (-1)^n \sin(\pi/2) \]
\[ K_a = \lim_{x \to a+} \sqrt{2 \pi (x-a)} \tau_{y(3a)}^{1/2} \]
\[ = \sum_{n=1}^{\infty} b_n \times (2n - 1) \times (n)\sin(\pi/2) \]
\[ K_b = \lim_{x \to b-} \sqrt{2 \pi (x-a)} \tau_{y(4b)}^{1/2} \]
\[ = \sum_{n=1}^{\infty} a_n \times 2n \times (-1)^n \sin(\pi/2) \]

Stress intensity factors have been solved only for the case in which the nonhomogeneous layer is divided into three homogeneous layers (i.e., \(m = 3\)). As \(m\) increases, the numerical results for the stress intensity factors approach the true values for the nonhomogeneous bonding layer.

The inverse Laplace transforms of the stress intensity factors are performed using the numerical method described by Miller and Guy (1966). When the Laplace transform \(g(t)\) can be evaluated at discrete points given by
\[ s = (\beta + 1 + k), \quad k = 0, 1, 2, \ldots \]
the coefficients \(C_M\) are determined using
\[ \delta \times g^k[(k + \beta + 1)M] \delta = \sum_{M=0}^{k} C_M \delta^k[k + \beta + 1](k + \beta + 2) \ldots \]
\[ (k + \beta + 1 + M)(k - M)! \]
where \(\delta > 0\) and \(\beta > -1\). If the coefficients up to \(C_{M-1}\) are calculated, an approximate value of \(g(t)\) can be found, as follows:
\[ g(t) = \sum_{M=0}^{N-1} C_M P_M(z) [2 \exp(\beta t) - 1], \]  
(50)

where \( P_M(z) \) is a Jacobi polynomial. The parameters \( \delta, \beta, \) and \( N \) must be selected such that \( g(t) \) can best be described within a particular range of time \( t \).

We have the following relation between \( g(s) \) and \( g(t) \):

\[ \lim_{t \to s} g(s) = \lim_{t \to s} g(t). \]  
(51)

Therefore, the static results of the stress intensity factors in physical space can be obtained using Eq. (51).

6. Numerical examples

The dynamic stress intensity factors were calculated numerically with quadrule precision using a Fortran program for which overflow and underflow do not occur in the range \( 10^{-5500} - 10^{5500} \). The following three cases were considered for composite materials made from a ceramic half-plane, a steel half-plane, and a nonhomogeneous bonding layer:

Case 1: a steel lower half-plane (B) and a ceramic upper half-plane (C).

Case 2: a ceramic lower half-plane (B) and a steel upper half-plane (C).

Case 3: steel lower and upper half-planes and a steel-bonding layer (A).

The corresponding material properties are given in Table 1. In the interfacial layer (A), the material properties are assumed to vary linearly with \( y \). The dynamic stress intensity factors are calculated for \( b/a = 1.0 \) and \( h/a = 0.2 \). The numerical Laplace inversions are performed by setting \( \beta = 0.0, \delta = 0.2, N = 11 \) in Eq. (50). For Case 2, the values of \( K_{fa} \) and \( K_{fb} \) for \( m = 5, 10, 15, \) and 20 are shown in Fig. 6. The results have acceptable accuracy if the nonhomogeneous layer is approximated by 20 homogeneous sublayers. Therefore, all of the calculations herein have been performed using \( m = 20 \).

For Case 2, the values of \( Q_i(\alpha)/(\alpha) \) for \( \alpha = 1.0 \) and \( m = 20 \) are listed in Table 2. In the author’s opinion, the upper limit of integration should be set to a value greater than \( (a) = 100.01 \) (e.g., \( (\alpha) = 200.01 \)). In the execution of the Fortran program, numerical overflows appeared at \( (\alpha) = 120.01 \). Consequently, the value \( (\alpha) = 100.01 \) was chosen as the upper limit of integration. Table 2 shows that the numerical integrations can be performed with acceptable accuracy using Filon’s method. Table 3 gives the values for the left-hand side of Eq. (39) and \( u(x) \), for \( \alpha = 1.0 \) and \( m = 20 \). Similarly, Table 4 gives the values for Eq. (40) and \( s(x) \).

These tables show that the boundary conditions inside the cracks are also satisfied with acceptable accuracy.

The stress intensity factors are plotted with respect to \( c_L s/t(a) \) for Cases 1, 2, and 3 in Figs. 7–9, respectively. In these figures, for reference, the corresponding static values, calculated using Eq. (51), are shown by the straight-line segments on the right-hand side of the figure.

7. Fracture toughness of the interface crack

Consider two dissimilar elastic half-planes bonded by a nonhomogeneous layer. If the elastic properties are continuous at the interface, the stresses near the tip of the interface crack do have the oscillatory singularity that appears in the exact solution (England, 1965) or the contacting area at the crack end reported by Comuninius (1977). Consequently, the stress intensity factors are successfully defined, and the fracture toughness \( K_I \) can be obtained experimentally. To determine the value of \( K_I \) for the actual interface crack, the static stress intensity factor \( K_I^* \) must be solved for many material combinations beforehand.

8. Discussion

A Fortran program has been written that is applicable for any number of sublayers. This negates the need to estimate the true values of the stress intensity factors using the results for \( m = 3, 5, \)...
and 7, or those for \( m = 5, 7, \) and 9, as performed in previous studies (Itou and Shima, 1997, 1999; Itou, 2001, 2004a,b, 2005a,b, 2007a).

The author has attempted to obtain the numerical calculations for selected thicknesses of the bonding layer. However, this was unsuccessful even for \( h/a = 0.5 \) because the functions \( Q_i(\zeta)/|a(\zeta)\) in the integrands of Eqs. (41) and (42) overflow at a comparatively low value of the integration variable \( (\zeta) \). Despite this, numerical calculations for \( h/a = 0.2 \) have been performed, yielding useful conclusions in the field of the fracture mechanics of composite materials.

In the present study, the upper crack is situated in the layer nearest the upper half-plane, whereas the lower crack is situated in the layer nearest the lower half-plane. Therefore, it may not be appropriate to denote the cracks as interface cracks. As seen in Fig. 6, the curves of the dynamic stress intensity factors are approximately the same for \( m = 10, 15, \) and 20. Therefore, these curves are not expected to be affected in the range \( m > 20 \). If \( m \) is set to 100,000, then the resulting curves are almost the same as those for \( m = 20 \). Consequently, the author considers that the numerical results presented here closely approximate the results that would be obtained for the case in which the cracks are interface cracks.

As described in Section 2, the incident stress given by Eq. (5) is assumed to pass through the nonhomogeneous layer \((A)\), varying the wave velocity \( c_{IL} \) in that layer. It is not certain whether the reflected wave completely disappears when it passes through the nonhomogeneous layer. This can be verified by measuring the magnitudes of the normal strains at the upper and lower interfaces of the nonhomogeneous layer. If both strains are approximately equal, the reflected wave may be neglected. In the author’s opinion, this phenomenon should be verified by an experimentalist. If the nonhomogeneous layer is replaced by an infinite number of homogeneous sublayers, then the incident stress given by Eq. (5) passes through the nonhomogeneous layer \((A)\), varying the wave velocity \( c_{IL} \) in that layer.

The Schmidt method was first applied by the present author to solve the crack problem (Itou, 1978). When the Schmidt method is used to solve crack problems, the stress field, the displacement field, the temperature field, and the stress intensity factors can

---

**Table 3**

Values of the LHS and RHS of Eq. (39) for Case 2 \( [(a/b) = 1.0, m = 20, b/a = 1.0, h/a = 0.2] \).

<table>
<thead>
<tr>
<th>( x/a )</th>
<th>( u(x/a)/[a(\zeta)] )</th>
<th>( b(x/a)/[a(\zeta)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000100</td>
<td>-0.75423</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.041667</td>
<td>-0.75441</td>
<td>-0.00006</td>
</tr>
<tr>
<td>0.500000</td>
<td>-0.75415</td>
<td>0.00003</td>
</tr>
<tr>
<td>0.958333</td>
<td>0.75433</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.999900</td>
<td>0.75435</td>
<td>0.00000</td>
</tr>
</tbody>
</table>

---

**Table 4**

Values of the LHS and RHS of Eq. (40) for Case 2 \( [(a/b) = 1.0, m = 20, b/a = 1.0, h/a = 0.2] \).

<table>
<thead>
<tr>
<th>( x/a )</th>
<th>( v(x/a)/[a(\zeta)] )</th>
<th>( v(x/a)/[a(\zeta)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000100</td>
<td>-1.00013</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.041667</td>
<td>-0.99999</td>
<td>-0.00007</td>
</tr>
<tr>
<td>0.500000</td>
<td>-1.00025</td>
<td>0.00013</td>
</tr>
<tr>
<td>0.958333</td>
<td>-1.00119</td>
<td>0.00059</td>
</tr>
<tr>
<td>0.999900</td>
<td>-0.99880</td>
<td>-0.00111</td>
</tr>
</tbody>
</table>

---

**Fig. 7.** Stress intensity factors \( K_{IL}, K_{II}, K_{L}, \) and \( K_{III} \) with respect to \( c_{IL}a \) for Case 1, \( b/a = 1.0 \) and \( h/a = 0.2 \).

**Fig. 8.** Stress intensity factors \( K_{IL}, K_{II}, K_{L}, \) and \( K_{III} \) with respect to \( c_{IL}a \) for Case 2, \( b/a = 1.0 \) and \( h/a = 0.2 \).

**Fig. 9.** Stress intensity factors \( K_{IL}, K_{II}, K_{L}, \) and \( K_{III} \) with respect to \( c_{IL}a \) for Case 3, \( b/a = 1.0 \) and \( h/a = 0.2 \).
be obtained in a straightforward manner, even for three-dimensional rectangular cracks (Itou, 2007a), a single cylindrical crack (Itou, 2003, 2005b, 2007b), and two cylindrical cracks (Itou, 2007c, 2008). Takakuda solved the dynamic stress intensity factors around two parallel cracks during the passage of a time-harmonic stress wave (1982). Itou and Hasiet obtained the corresponding solution for an infinite orthotropic plane weakened by two parallel cracks (1997). They also reworked Takakuda’s problem and verified that the values of the dynamic stress intensity factors obtained using the Schmidt method are completely coincident with those obtained using the integral equation method (Takakuda, 1982). These results demonstrate the reliability of the Schmidt method.

9. Conclusions

Based on the numerical calculations outlined above, we arrive at the following conclusions:

(1) For all cases, the significant stress intensity factor is \( K_{\text{peak}} \) at the end of the lower interface crack. Its peak value, \( K_{\text{peak}}^\text{int} \), is approximately 1.0. If the \( h/a \) ratio approaches infinity, the corresponding peak value approaches that for a crack in an infinite elastic plane. The dynamic stress intensity factor for a finite crack of length \( 2a \) in an infinite plane subjected to an impact load has been solved by Sih et al. (1972). Their solution revealed that the peak value \( K_{\text{peak}}^\text{int} \) is approximately 1.3. As such, the dynamic stress intensity factors around two parallel interface cracks are found to be considerably lower than that for a finite crack in an infinite plane.

(2) For a finite crack in an infinite plane, the dynamic stress intensity factor increases continuously, eventually reaching the upper peak value. The dynamic stress intensity factor then tends to decrease, eventually reaching a lower peak value. In contrast, the curves for the dynamic stress intensity factors around the nonhomogeneous bonding layer and the two dissimilar elastic half-planes.

Appendix A

Consider a small cross-sectional area \( \Delta A \) at the interface \( y = -h/2 \). The wavefront of the incident normal stress \( \tau^{\text{inc}}_{xy} = p \) reaches the interface at \( t = 0 \). After a short time \( \Delta t \), the incident stress is separated into the reflected wave \( \Delta p \) and the transmitted wave \( \beta \Delta p \), as shown in Fig. 10, in which PQRS indicates a small cubic element with length \( (c_{\text{iB}} \times \Delta t + c_{\text{iA}} \times \Delta t) \) and a small area \( \Delta A \). The forces acting on the cubic element are shown in Fig. 11. Then, the equation of equilibrium for forces in the \( y \)-direction is

\[
-p \times \Delta A - \Delta p \times \Delta A + \beta p \times \Delta A = 0, \quad \therefore \quad \alpha = \beta - 1. \tag{A1}
\]

At time \( \Delta t \), the energy of the incident wave must equal the sum of the energies of the reflected and transmitted waves, so that

\[
\frac{p^2}{2E_B} \times \Delta A \times (c_{\text{iB}} \times \Delta t) = \left(\frac{\Delta p}{2E_B}\right)^2 \times \Delta A \times (c_{\text{iB}} \times \Delta t) + \frac{(\beta p)^2}{2E_A} \times \Delta A \times (c_{\text{iA}} \times \Delta t), \tag{A2}
\]

where \( E_A \) and \( E_B \) are the Young moduli. From Eqs. (A1) and (A2), we find that

\[
\alpha = \frac{c_{\text{iB}} E_A - c_{\text{iA}} E_B}{c_{\text{iB}} E_A + c_{\text{iA}} E_B}, \quad \beta = \frac{2c_{\text{iB}} E_A}{c_{\text{iB}} E_A + c_{\text{iA}} E_B}. \tag{A3}
\]

Fig. 2 shows that \( c_{\text{iA}} \) and \( E_B \) in Eq. (A3) near the interface \( y = -h/2 \) are given by the following equations:

\[
c_{\text{iA}} = \frac{1}{2} \left[ c_{\text{iB}} + \left( c_{\text{iB}} + \frac{dc_{\text{iA}}}{dy} \right) \right] = c_{\text{iB}} + \frac{1}{2} \frac{dc_{\text{iA}}}{dy}, \tag{A4}
\]

\[
E_A = \frac{1}{2} \left[ E_b + \left( E_b + \frac{dE_A}{dy} \right) \right] = E_b + \frac{1}{2} \frac{dE_A}{dy}. \tag{A5}
\]

Substituting Eq. (A4) into Eq. (A3) and solving for \( \beta \), we obtain the following:

\[
\beta = \left[ 1 + \frac{1}{2E_b} \frac{dE_A}{dy} \right] \times \left[ 1 + \frac{1}{4E_b c_{\text{iB}}} \left( \frac{c_{\text{iB}}}{E_b} dE_A + E_b \frac{dc_{\text{iA}}}{dy} \right) \right]^{-1} = 1 + \frac{1}{4E_b c_{\text{iB}}} \left( \frac{c_{\text{iB}}}{E_b} \frac{dE_A}{dy} - E_b \frac{dc_{\text{iA}}}{dy} \right) \cdots \tag{A6}
\]

If the value of \( \frac{c_{\text{iB}}}{E_b} \frac{dE_A}{dy} - E_b \frac{dc_{\text{iA}}}{dy} \) is not very large at the interface \( y = -h/2 \), then \( \alpha \) and \( \beta \) can be approximated as

\[
\alpha = 0, \quad \beta = 1. \tag{A6}
\]

Based on the above considerations, it is believed that a stress wave similar to that given by Eq. (5) will pass through the nonhomogeneous layer, varying the wave velocity \( c_i \) in the layer.

References

