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The essence of ideal completion in quantitative form

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Abstract

If a poset lacks joins of directed subsets, one can pass to its *ideal completion*. But doing this means also changing the setting: The universal property of ideal completion of posets suggests that it should be regarded as a functor from the category of posets with monotone maps to the category of dcpos with Scott-continuous functions as morphisms. The same applies for the quantitative version of ideal completion suggested in the literature. As in the case of posets, it seems advantageous to consider a different topology with the completed spaces. We introduce *topological $\mathscr{V}$-continuity spaces* and their Smyth completion and show that this is an adequate setting to consider ideal completion of quantitative domains: Performing the Smyth completion of a $\mathscr{V}$-continuity space regarded as topological $\mathscr{V}$-continuity space gives the ideal completion of the original space together with its Scott topology. © 2002 Published by Elsevier Science B.V.

1. Introduction

This paper is part of the ongoing foundational work on quantitative domain theory [1, 4, 10, 11, 14, 17], which refines ordinary domain theory by replacing the *qualitative* notion of approximation by a *quantitative* notion of degree of approximation (cf. the introduction of [5]). We investigate the generalization of ideal completion of posets for quantitative domains suggested in [1, 4].

The use of ideal completion in ordinary domain theory is the Representation Theorem: Every algebraic domain arises as ideal completion and all ideal completions of posets are algebraic domains. In particular, this is a typically non-idempotent operation.

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But the quantitative version of ideal completion also generalizes Cauchy completion of metric spaces, which is idempotent. The explanation for this clash is the following: Idempotency of Cauchy completion depends on uniquely extending all uniformly continuous functions into complete spaces to uniformly continuous functions on the completion. The universal property of ideal completion of posets, however, refers to monotone functions extending to Scott-continuous functions on the ideal completion. Thus directed complete posets should be (and in fact usually are) considered to carry their Scott topologies rather than just as posets. Now the Scott topology does *not* necessarily coincide with the ε-ball topology defined by the generalized quasi-metric on the space – for example, it does not in the poset case, where the latter is the Alexandroff topology. It was Mike Smyth, who first realized that, accordingly, one has to consider a different topology on the space; he also introduced a quantitative version of the Scott topology [11]. Hence, the objects of our studies are *topological $\mathscr{V}$-continuity spaces*, quantitative domains carrying an additional topology. We define Smyth completeness for these and construct the completion, which is idempotent. Moreover, Smyth completion of a quantitative domain with its Alexandroff topology is exactly the essence of ideal completion: It yields the ideal completion together with the Scott topology. For the case of posets, this theorem reduces to a result by Hoffmann [6]: The sobrification of a poset in its Alexandroff topology equals the ideal completion in its Scott topology.

In the appendix, we give an example of a quantitative domain for the reals. The domain arises as completion of the binary tree endowed with a suitable structure. It is essential to perform the Smyth completion in order to end up with the usual topology on the reals.

As setting for our exposition, we choose *$\mathscr{V}$-continuity spaces* [3]. The reader not familiar with this concept may think of $\mathscr{V}$ as $[0,\infty]$, of $\succ$ as $>$, of $\dot{-}$ as truncated subtraction of non-negative reals, and of a $\mathscr{V}$-continuity space as a quasi-metric space.

2. Directed complete continuity spaces

In this section we recall some basic results about directed completeness and algebraic $\mathscr{V}$-domains from [4] which will be needed below.

2.1. Value quantals and continuity spaces

The first concept is that of a *value quantale*, which the reader might think of being the extended non-negative reals $[0,\infty]$. It is a structure $(\mathscr{V},\leqslant,+)$ where $(\mathscr{V},\leqslant)$ is a completely distributive complete lattice and $+$ an associative, commutative binary operation on $\mathscr{V}$ which preserves arbitrary infima. Moreover, $0\neq\infty$ is required to hold, where 0 is the least and ∞ the largest element of $\mathscr{V}$. Also, 0 is demanded to be an identity for $+$ and the set of all elements well above 0 is supposed to be filtered. Here an element x is *well above* y (abbreviated as $x\succ y$), if $y\geqslant \inf A$ implies that

$x \in \uparrow A$. It is well known that complete distributivity is equivalent to the condition $x = \inf\{y \in \mathscr{V} \mid y \succ x\}$ for all $x \in \mathscr{V}$ [9]. Value quantales are studied in detail in [3].

The main examples for value quantals are the booleans $\mathscr{V} = \{0, 1\}$ with $0 \leqslant 1$ and max as addition and the extended non-negative reals $\mathscr{V} = [0, \infty]$ with usual order and addition.

For the following definitions, it is necessary to fix the value quantale $\mathscr{V}$. A *distance function* on a set X is a function $d : X \times X \to \mathscr{V}$ such that (1) $d(x, x) = 0$, (2) if $d(x, y) = 0 = d(y, x)$ then $x = y$, and (3) such that the triangle inequality $d(x, y) + d(y, z) \geqslant d(x, z)$ holds for all $x, y, z \in X$. A *$\mathscr{V}$-continuity space* is a pair (X, d), where X is a set and d a distance function. For $\mathscr{V} = \{0, 1\}$, these are exactly preordered sets, for $\mathscr{V} = [0, \infty]$ quasi-metric spaces. A function $f : X \to Y$ between $\mathscr{V}$-posets is *non-expansive* if $d(x, y) \geqslant d(f(x), f(y))$ holds for all $x, y \in X$.

Suppose that $\mathscr{V}$ is a value quantale and (X, d) is a $\mathscr{V}$-continuity space. We denote for $x \in X$ with $B_\varepsilon(x) = \{y \in x \mid \varepsilon \succ d(x, y)\}$ the open ε-ball around x and for $A \subset X$ with $B_\varepsilon[A] = \bigcup_{x \in A} B_\varepsilon(x)$ the ε-neighborhood of A. The topology with neighborhoods generated by ε-balls is referred to as the *Alexandroff topology* and denoted by α_d. We employ the notation $X^\star$ for the dual of X, its distance is given by $d^\star(x, y) = d(y, x)$. The *(pre)order derived from d* is defined by $x \sqsubseteq_d y$ if and only if $d(x, y) = 0$. In this paper, we will only consider *separated* spaces, i.e. spaces where $d(x, y) = d(y, x) = 0$ implies $x = y$. Alternatively, a space is separated if and only if α_d is T_0 if and only if $\sqsubseteq_d$ is antisymmetric, i.e. a partial order.

Recall that $\mathscr{V}$ itself is a $\mathscr{V}$-continuity space with the distance function

$$d(p, q) = q \dot{-} p = \inf\{r \in \mathscr{V} \mid p + r \geqslant q\}.$$

Note that the order $\sqsubseteq_d$ derived from this distance is the opposite of $\leqslant$.

2.2. Characters and ideals

A *character* on the $\mathscr{V}$-continuity space X is a non-expansive map $\varphi : X^* \to \mathscr{V}$. Each element x of X induces a character, $[x]$, defined by

$$[x](y) = d(y, x), \quad y \in X.$$

These characters are called *representable*. We write $\hat{X}$ for the collection of all characters on X with the *sup-distance*:

$$\hat{d}(\varphi, \psi) = \sup_{x \in X} d(\varphi(x), \psi(x)) = \sup_{x \in X}(\psi(x) \dot{-} \varphi(x)).$$

$(\hat{X}, \hat{d})$ is easily seen to be a $\mathscr{V}$-continuity space.

Since continuity spaces are simply categories enriched in value quantales, the Yoneda lemma, which was previously used in quantitative domain theory in [8, 1], provides a fundamental tool for our investigations.

Lemma 1 (Yoneda). *For $x \in X$ and $\varphi \in \hat{X}$, $\hat{d}([x], \varphi) = \varphi(x)$.*

From the Yoneda lemma it follows at once that the mapping $\eta = x \mapsto [x] : X \to \hat{X}$ is an isometric embedding.

An element a of X is the *supremum* of the character φ, denoted by $a = \bigvee \varphi$, if for all $b \in X$, we have $d(a, b) = \hat{d}(\varphi, [b]) = \sup_{x\in X}(d(x, b) \dot{-} \varphi(x))$. We will call a an *upper bound* for φ if for all $x \in X$ the relation $\varphi(x) \geqslant d(x, a)$ holds. This is the case if and only if $\varphi \sqsubseteq [a]$ holds.

Lemma 2. *An element a of X is the supremum of the character φ if and only if a is an upper bound of φ and* $\sup_{y\in X}(d(y, x) \dot{-} \varphi(y)) \geqslant d(a, x)$ *for all $x \in X$.*

Assume X and Y are $\mathscr{V}$-continuity spaces, $f : X \to Y$ is a non-expansive map and φ is a character on X. Then the *direct image of φ under f* [8], denoted by $\hat{f}(\varphi)$, is the function from Y to $\mathscr{V}$ defined by

$$\hat{f}(\varphi)(y) = \inf_{x\in X}(\varphi(x) + d(y, f(x))) \quad \text{for } y \in Y.$$

It is easy to show that $\hat{f}(\varphi)$ is a character on Y, that the resulting map $\hat{f} : \hat{X} \to \hat{Y}$ is non-expansive, that the assignment $f \mapsto \hat{f}$ is functorial, and that η with $\eta_X(x) = [x]$ is a natural transformation from the identity to this functor.

We say that a character φ is an *ideal on X* if there is an x such that $\infty \succ \varphi(x)$ and whenever $\varepsilon_1 \succ \varphi(x_1)$, $\varepsilon_2 \succ \varphi(x_2)$ and $\delta \succ 0$, there is an x such that $\delta \succ \varphi(x)$, $\varepsilon_1 \succ d(x_1, x)$ and $\varepsilon_2 \succ d(x_2, x)$. The space (X, d) is *directed complete*, if for every ideal φ on X, its supremum, $\bigvee\varphi$, exists. Directed complete $\mathscr{V}$-continuity spaces are also called *$\mathscr{V}$-domains*.

2.3. Scott topology and ideal completion

Assume X and Y are $\mathscr{V}$-domains and $f : X \to Y$ is a non-expansive map. If φ is an ideal on X, then the direct image of φ under f is an ideal on Y. We say that f is *Scott continuous* if for all ideals φ on X, $\bigvee \hat{f}(\varphi) = f(\bigvee\varphi)$. As in the poset case, Scott continuity may also be characterized as continuity with respect to the Scott topologies on source and target spaces.

Definition 3 (*Smyth* [11]). Assume that (X, d) is a $\mathscr{V}$-domain. A subset U of X is *Scott open* if for all ideals φ on X, if $\bigvee\varphi \in U$, then there is an $\varepsilon \succ 0$ and an x such that $\varepsilon \succ \varphi(x)$ and $B_\varepsilon(x) \subseteq U$. The collection of all Scott open subsets of X is a topology, called the *Scott topology* and denoted by σ_d.

Every $\mathscr{V}$-continuity space X has a completion. Let $\mathscr{I}(X)$ denote the set of ideals on X with the sup-distance. Representable characters are easily seen to be ideals, so the Yoneda embedding η_X co-restricts to a map from X to $\mathscr{I}(X)$.

Theorem 4 (Flagg et al. [4]). *Assume X is a $\mathscr{V}$-continuity space. Then $\mathscr{I}(X)$ is a $\mathscr{V}$-domain. Moreover, for any $\mathscr{V}$-domain Y and any non-expansive map $f : X \to Y$*

there exists a unique Scott continuous and non-expansive map $\bar{f}:\mathscr{I}(X)\to Y$ *such that* $\bar{f}\circ\eta_X = f$.

2.4. Algebraic $\mathscr{V}$-domains

An element a in a $\mathscr{V}$-domain X is *compact* if $d(a, \bigvee\varphi) \geqslant \varphi(a)$ for all ideals φ on X. Let $K = K(X)$ denote the subspace of X consisting of the compact elements and let $\imath : K \to X$ be the isometric embedding of K into X.

Definition 5. Assume X is a $\mathscr{V}$-domain and $Y \subseteq X$. Then, Y generates X if for all $x \in X$ there is an ideal φ on Y such that $x = \bigvee \widehat{\imath_Y}(\varphi)$, where $\imath_Y$ is the isometric embedding of Y into X.

We say that a $\mathscr{V}$-domain is *algebraic* if it is generated by its compact elements.

Theorem 6 (Flagg et al. [4], Bonsangue et al. [1]). *Assume that* (X, d) *is an algebraic* $\mathscr{V}$*-domain. Then* $(X, d) \cong \mathscr{I}(K(X), d)$. *Conversely, for every* $\mathscr{V}$*-continuity space* (X, d), $\eta_X(X)$ *generates* $\mathscr{I}(X, d)$ *and* $\eta_X(X) \subseteq K(I(X, d))$; *in particular*, $\mathscr{I}(X, d)$ *is algebraic and* $\{B_\varepsilon([x])\}_{\varepsilon\succ 0, x\in X}$ *is a base for the Scott topology on* $\mathscr{I}(X, d)$.

Note that, unlike in the poset case, the inclusion $\eta_X(X) \subseteq K(\mathscr{I}(X, d))$ may be strict, i.e. a non-trivial subset of the set of compact elements may be sufficient to generate an algebraic $\mathscr{V}$-domain. If, for example, the distance d is symmetric, then all elements are compact. This implies in particular that all symmetric spaces are algebraic. The set of rational numbers as subset of the real line in its usual metric is an example of a generating set which does not contain all compact elements.

3. Topological continuity spaces

The universal property of the ideal completion (Theorem 4) suggests that one should not consider $\mathscr{V}$-domains with their usual (Alexandroff) topologies, but rather with their Scott topologies: Every non-expansive map to a complete space has a non-expansive *Scott-continuous* extension to the completion. This is in analogy to the poset case, where dcpos are usually considered to carry the Scott – rather than the Alexandroff topology. Mike Smyth had the idea to incorporate this possibility in the theory right from the starting point. In [11, 12] he introduces a set of axioms for an additional topology for quasi-metric and quasi-uniform spaces. The appropriate definition for continuity spaces is as follows:

Definition 7. A *topological* $\mathscr{V}$*-continuity space* (t$\mathscr{V}$-cs) is a triple (X, d, τ), where (X, d) is a $\mathscr{V}$-continuity space and τ is a topology on X satisfying the following axioms.

(a1) If $x \in U \in \tau$ then there are $\varepsilon \succ 0$ and $V \in \tau$ such that $x \in V$ and $B_\varepsilon[V] \subseteq U$.
(a2) For all $\varepsilon \succ 0$ and all $x \in X$, the set $\bar{B}^\star_\varepsilon(x) = \{y \in X \mid \varepsilon \geqslant d(y, x)\}$ is τ-closed.
(a3) For all $\varepsilon \succ 0$ and all $U \in \tau$, the set $B_\varepsilon[U]$ is τ-open.

A topology τ on X satisfying (a1)–(a3) is called *admissible*.

Axioms (a1) and (a2) are exactly Smyth's axioms, whereas (a3) is a strengthening of his *interpolation property*. This axiom is even stronger than (A3) of [13], a strengthening of the interpolation property which was introduced to make completion of topological quasi-uniform spaces possible. Our new, stronger axiom appears to be essential to be able to perform the completion in the setting of continuity spaces.

Like Smyth's structures, topological continuity spaces are a special case of Császár's *syntopologies* [2]. Indeed, many of our arguments will be in this spirit, referring to the relations $\ll_\varepsilon$, where

$$A \ll_\varepsilon B \Leftrightarrow \exists U \in \tau. A \subseteq U \ \& \ B_\varepsilon[U] \subseteq B$$

for A and B subsets of a $\mathscr{V}$-continuity space X taken with topology τ.

We will also use the abbreviations $x \ll_\varepsilon A$ for $\{x\} \ll_\varepsilon A$ and $A \ll B$ for $\exists \varepsilon \succ 0. A \ll_\varepsilon B$. The following alternative version of (a2) is very useful (see also [12, Lemma 1]).

Lemma 8. *If* (X, d) *is a* $\mathscr{V}$*-continuity space and* τ *is a topology on* X *then* (a2) *holds if and only if*
($a2'$) $\forall x, y \in X.\ d(x, y) = \inf\{p \succ 0 \mid x \ll_p U \in \tau \Rightarrow y \in U\}$.

Proof (*if*). Suppose $y \in X \setminus \bar{B}^\star_\varepsilon(x)$. Then $\varepsilon \not\succcurlyeq d(y, x)$ and so there is an $\varepsilon' \succ \varepsilon$ such that $\varepsilon' \not\succcurlyeq d(y, x)$; in particular, $\varepsilon' \notin \{p \succ 0 \mid y \ll_p U \in \tau \Rightarrow x \in U\}$ by ($a2'$). Thus, there are open sets $U, V \in \tau$ with $y \in V$, $B_{\varepsilon'}[V] \subseteq U$, and $x \notin U$. Then $y \in V \subseteq X \setminus \bar{B}^\star_\varepsilon(x)$.

(*only if*). If $p \succ d(x, y)$, then certainly $x \ll_p U$ implies $y \in U$. Thus, $d(x, y) \geqslant \inf\{p \succ 0 \mid x \ll_p U \in \tau \Rightarrow y \in U\}$. To see the converse, suppose $p \in \{q \succ 0 \mid x \ll_q U \in \tau \Rightarrow y \in U\}$. For the sake of contradiction, we assume $p \not\succcurlyeq d(x, y)$. Then $x \in V = X \setminus \bar{B}^*_p(y)$, hence $x \ll_p B_p[V]$. Thus, $y \in B_p[V]$ which is absurd. $\square$

Note that the inequality $\geqslant$ in ($a2'$) is always true since $x \ll_p U$ implies $B_p(x) \subseteq U$ and thus every $p \succ d(x, y)$ is in the set whose infimum is taken.

We omit the proof of the following standard observations.

Proposition 9. (1) *If* (X, d) *is a* $\mathscr{V}$*-continuity space, then* (X, d, α_d) *is a* $t\mathscr{V}$*-cs.*
(2) *If* (X, d, τ) *is a* $t\mathscr{V}$*-cs, then* $\tau \subseteq \alpha_d$.
(3) *If* (X, d, τ) *is a* $t\mathscr{V}$*-cs and* $x, y \in X$, *then* $x \sqsubseteq_\tau y$ *if and only if* $d(x, y) = 0$.

A function $f : (X, d_X, \tau_X) \to (Y, d_Y, \tau_Y)$ between topological $\mathscr{V}$-continuity spaces is a t$\mathscr{V}$-cs *morphism* if it is non-expansive as a mapping between the continuity spaces (X, d_X) and (Y, d_Y) and continuous as a mapping between the topological spaces (X, τ_X) and (Y, τ_Y). As in the case of topological quasi-uniform spaces, these functions may also be characterized as syntopological morphisms:

Proposition 10. *A function $f : X \to Y$ is a morphism of topological $\mathscr{V}$-continuity spaces X and Y if and only if we have that whenever $A \ll_\varepsilon B$ in Y for some $A, B \subseteq Y$ and $\varepsilon \succ 0$ then $f^{-1}(A) \ll_\varepsilon f^{-1}(B)$ in X.*

Proof (*only if*). Suppose f is a t$\mathscr{V}$-cs-morphism and $A \ll_\varepsilon B$. Then there exists $U \in \tau_Y$ with $A \subseteq U$ and $B_\varepsilon[U] \subseteq B$. By continuity, $f^{-1}(U)$ is τ_X-open. As f is non-expansive, $B_\varepsilon[F^{-1}(U)] \subseteq f^{-1}(B)$. Hence, $f^{-1}(A) \ll_\varepsilon f^{-1}(B)$.

(*if*). If f has the above property then $f(x) \in U \in \tau_Y$ implies that $f^{-1}(U)$ is a τ_X-neighborhood of x by axiom (a1). Hence, f is continuous. To see that f is non-expansive, observe that the condition $(x \ll_p f^{-1}(U) \Rightarrow y \in f^{-1}(U))$ by assumption implies $(f(x) \ll_p U \Rightarrow f(y) \in U)$. Hence $p \succ d(x, y)$ certainly implies $p \geqslant d(f(x), f(y))$ by $(a2')$. Therefore, $d(x, y) \geqslant d(f(x), f(y))$. □

4. Closed ideals and Smyth completeness

In order to relate directed and Smyth completeness, we introduce closed ideals and use them to characterize Smyth completeness and to construct the completion.

To motivate the definitions below, we consider first the special case when $\mathscr{V} = \mathbf{2}$. Here a $\mathscr{V}$-continuity space is just a poset $(X, \sqsubseteq_d)$, where $x \sqsubseteq_d y \Leftrightarrow d(x, y) = 0$. Axioms (a1)–(a3) for a topology τ are equivalent to the condition that $\sqsubseteq_d$ be the specialization order of τ. Hence every topological space (X, τ) gives a $\mathscr{V}$-continuity space by taking d as above with $\sqsubseteq_d$ the specialisation of τ. Here a function $\varphi : X \to \mathscr{V}$ is character if and only if it is the characteristic function of a lower set. Recall that a subset A of X is *irreducible* if $A \neq \emptyset$ and whenever $A \subseteq C_1 \cup C_2$, for closed sets C_1, C_2, then $A \subseteq C_1$ or $A \subseteq C_2$. If $\varphi = \chi_A$ and we write $\varphi \asymp U$ to indicate that $\exists x (x \in U \,\&\, \varphi(x) = 0)$, then clearly A is irreducible if and only if $\varphi \asymp X$ and $\forall U_1, U_2 \in \tau.\ (\varphi \asymp U_1 \,\&\, \varphi \asymp U_2 \Rightarrow \varphi \asymp U_1 \cap U_2)$. Moreover, $A = \mathrm{cl}\{a\}$ if and only if $\varphi = d(\cdot, a)$. Thus, X is *sober* if and only if for every character $\varphi = \chi_A$ such that $A = \{x \in X \mid \varphi(x) = 0\}$ is irreducible and closed, there is a unique point a such that $\varphi = d(\cdot, a)$. For general $\mathscr{V}$, where $0 \succ 0$ need not hold, we must formulate things in terms of arbitrary $\varepsilon \succ 0$.

Definition 11. We say that a character φ meets a set $A \subseteq X$ and write $\varphi \asymp A$, if there are $\varepsilon \succ 0$ and $x \in X$ such that $\varepsilon \succ \varphi(x)$ and $x \ll_\varepsilon A$.

Observe that the existence of a point x in an open set U with $\varphi(x) = 0$ trivially implies $\varphi \asymp U$ by Axiom (a1). Also, if U is open then $[a] \asymp U$ if and only if $a \in U$.

Lemma 12. *Suppose that φ is a character and $A \subseteq X$. If $\varphi \asymp A$, then there is $U \in \tau$ with $\varphi \asymp U$ and $U \ll A$.*

Proof. Suppose that $\varphi \asymp A$. Then there exist $\varepsilon \succ 0$ and $x \in X$ so that $x \ll_\varepsilon A$ and $\varepsilon \succ \varphi(x)$. Choose $\varepsilon', \delta \succ 0$ so that $\varepsilon' \succ \varphi(x)$ and $\varepsilon \succ \varepsilon' + \delta$. Let $V \in \tau$ be such that

$x \in V \ll_{\varepsilon} A$. Then $x \in V \ll_{\varepsilon'} B_{\varepsilon'}[V] \ll_{\delta} A$. Consequently, with $U := B_{\varepsilon'}[V]$ we have $\varphi \asymp U$ and $U \ll A$. $\square$

For φ a character and $\delta \succ 0$, let $\varphi_\delta = \{x \in X \mid \delta \geqslant \varphi(x)\}$.

Definition 13. Assume φ is a character on X.

(1) φ is *irreducible* if $\varphi \asymp X$ and $(\varphi \asymp A \,\&\, \varphi \asymp B \Rightarrow \varphi \asymp A \cap B)$ holds for all $A,\ B \subseteq X$.
(2) φ is *Cauchy* if $\forall A \subseteq X\ \forall \delta \succ 0.\ (\varphi \asymp A \Rightarrow \varphi_\delta \cap A \neq \emptyset)$.
(3) φ is a τ-*ideal* if it is both irreducible and Cauchy.
(4) φ is *closed* if for all $\varepsilon \succ 0$, φ_ε is τ-closed.

It will sometimes be necessary to make clear what topology the relations $\ll$ and $\asymp$ refer to. In these cases we will indicate the relevant topology with a superscript as in $\asymp^{\alpha_d}$.

Proposition 14. *Assume that X is a continuity space and φ is a character on X. Then φ is α_d-closed. Moreover, φ is an ideal on X if and only if φ is an α_d-ideal.*

Proof. Since $\downarrow\varepsilon = \{p \in \mathscr{V} \mid d(\varepsilon, p) = 0\}$ is $\alpha_{d^\star}$-closed in $\mathscr{V}$ and $\varphi : X \to \mathscr{V}^\star$ is non-expansive, $\varphi_\varepsilon = \varphi^{-1}(\downarrow\varepsilon)$ is α_d-closed. Hence φ is α_d-closed.

For the second assertion, first note that $\varphi \asymp^{\alpha_d} A$ if and only if there is an $\varepsilon \succ 0$ and an x such that $\varepsilon \succ \varphi(x)$ and $B_\varepsilon(x) \subseteq A$.

Suppose that φ is an ideal on X. Since there is an x such that $\infty \succ \varphi(x)$, $\varphi \asymp^{\alpha_d} X$. Suppose $\varphi \asymp^{\alpha_d} A_1$ and $\varphi \asymp^{\alpha_d} A_2$. Choose $\varepsilon_1 \succ 0$, x_1 and $\varepsilon_2 \succ 0$, x_2 so that $\varepsilon_1 \succ \varphi(x_1)$, $B_{\varepsilon_1}(x_1) \subseteq A_1$, $\varepsilon_2 \succ \varphi(x_2)$, and $B_{\varepsilon_2}(x_2) \subseteq A_2$. Choose ε_1', ε_2', $\delta \succ 0$ so that $\varepsilon_1' \succ \varphi(x_1)$, $\varepsilon_2' \succ \varphi(x_2)$, $\varepsilon_1 \succ \varepsilon_1' + \delta$, and $\varepsilon_2 \succ \varepsilon_2' + \delta$. Finally, choose x so that $\delta \succ \varphi(x)$, $\varepsilon_1' \succ d(x_1, x)$, and $\varepsilon_2' \succ d(x_2, x)$. Then $B_\delta(x) \subseteq A_1 \cap A_2$ and $\delta \succ \varphi(x)$, so $\varphi \asymp^{\alpha_d} A_1 \cap A_2$. It follows that φ is irreducible. If $\varphi \asymp^{\alpha_d} A$ and $\delta \succ 0$, there are $\varepsilon \succ 0$ and $x \in X$ so that $\varepsilon \succ \varphi(x)$ and $B_\varepsilon(x) \subseteq A$. Choose y so that $\delta \succ \varphi(y)$ and $\varepsilon \succ d(x, y)$. Then $y \in \varphi_\delta \cap A$. Hence φ is Cauchy.

Suppose φ is an α_d-ideal. Since $\varphi \asymp^{\alpha_d} X$, there is an x such that $\infty \succ \varphi(x)$. Suppose $\varepsilon_1 \succ \varphi(x_1)$, $\varepsilon_2 \succ \varphi(x_2)$ and $\delta \succ 0$. Pick $\delta' \succ 0$ with $\delta \succ \delta'$. Then $\varphi \asymp^{\alpha_d} B_{\varepsilon_1}(x_1)$ and $\varphi \asymp^{\alpha_d} B_{\varepsilon_2}(x_2)$. Thus $\varphi \asymp^{\alpha_d} B_{\varepsilon_1}(x_1) \cap B_{\varepsilon_2}(x_2)$. Since φ is Cauchy, there is an $x \in B_{\varepsilon_1}(x_1) \cap B_{\varepsilon_2}(x_2)$ with $\delta' \geqslant \varphi(x)$. Then $\delta \succ \varphi(x)$. It follows that φ is an ideal. $\square$

We will now define a closure operator on characters.

Lemma 15. *Assume φ is a character on X. For $x \in X$, let*

$$\diamond\varphi(x) = \inf\{\varepsilon \succ 0 \mid x \in \mathrm{cl}_\tau\, \varphi_\varepsilon\}.$$

(1) *For all $\varepsilon \succ 0$ and all x, $\varepsilon \succ \diamond\varphi(x) \Rightarrow x \in \mathrm{cl}_\tau \varphi_\varepsilon \Rightarrow \varepsilon \geqslant \diamond\varphi(x)$.*
(2) *$\diamond\varphi$ is a closed character on X.*
(3) *For all x, $\varphi(x) \geqslant (\diamond\varphi)(x)$ and so $\hat{d}(\varphi, \diamond\varphi) = 0$, i.e. $\varphi \sqsubseteq \diamond\varphi$.*

(4) *For all* $U \in \tau$, $\varphi \asymp U$ *if and only if* $\diamond\varphi \asymp U$.
(5) *For any closed character* ψ *on* X, $\hat{d}(\diamond\varphi, \psi) = \hat{d}(\varphi, \psi)$. *Hence, if* $\varphi \sqsubseteq \psi$ *and* ψ *closed, then* $\diamond\varphi \sqsubseteq \psi$ *follows.*

Proof. (1) This follows at once from the definitions of $\diamond\varphi(x)$ and $\succ$.

(2) To show that $\diamond\varphi$ is a character it will suffice to show that whenever $\varepsilon \succ d(y, x)$ and $\delta \succ \diamond\varphi(x)$, then $y \in \mathrm{cl}_\tau\, \varphi_{\varepsilon+\delta}$. Suppose that $y \in U \in \tau$. Then $x \in B_\varepsilon[U]$ and so there is a $v \in B_\varepsilon[U]$ such that $\delta \geqslant \varphi(v)$. Choose $u \in U$ with $\varepsilon \succ d(u, v)$. Then $\varepsilon + \delta \geqslant d(u, v) + \varphi(v) \geqslant \varphi(u)$ so $u \in U \cap \varphi_{\varepsilon+\delta}$. Consequently, $y \in \mathrm{cl}_\tau\, \varphi_{\varepsilon+\delta}$. From (1) it follows easily that $(\diamond\varphi)_\varepsilon = \bigcap_{\varepsilon' \succ \varepsilon} \mathrm{cl}_\tau \varphi_{\varepsilon'}$ and so $\diamond\varphi$ is closed.

(3) Immediate from (1).

(4) The implication from left to right follows at once from (3). For the converse, assume $\diamond\varphi \asymp U$. Then there exist $\varepsilon \succ 0$ and $x \in X$ so that $\varepsilon \succ (\diamond\varphi)(x)$ and $x \ll_\varepsilon U$. Choose $\varepsilon' \succ 0$ so that $\varepsilon \succ \varepsilon' \succ (\diamond\varphi)(x)$ and $V \in \tau$ so that $x \in V \ll_\varepsilon U$. From (1), $V \cap \varphi_{\varepsilon'} \neq \emptyset$. Hence $\varphi \asymp U$.

(5) The inequality $\hat{d}(\diamond\varphi, \psi) \geqslant \hat{d}(\varphi, \psi)$ follows at once from (3). For the reverse inequality, it will suffice to show that if $p \succ \hat{d}(\varphi, \psi)$ and $\varepsilon \succ \diamond\varphi(x)$, then $p + \varepsilon \geqslant \psi(x)$. Suppose $x \in U \in \tau$. Since $\varepsilon \succ \diamond\varphi(x)$, by (1), there is a $u \in U$ such that $\varepsilon \geqslant \varphi(u)$. Hence $p + \varepsilon \geqslant \hat{d}(\varphi, \psi) + \varphi(u) \geqslant (\psi(u) \dot{-} \varphi(u)) + \varphi(u) \geqslant \psi(u)$. Thus $x \in \mathrm{cl}_\tau\, \psi_{p+\varepsilon}$. But ψ is closed, so $p + \varepsilon \geqslant \psi(x)$. $\square$

Proposition 16. *The map* $\diamond : (\hat{X}, \hat{d}) \to (\hat{X}, \hat{d})$ *is non-expansive and for all characters* φ *on* X, φ *is closed if and only if* $\diamond\varphi = \varphi$. *Moreover,* $\diamond$ *preserves Cauchyness and irreducibility.*

Proof. By (3) and (5) of Lemma 15, we have $d(\varphi, \psi) = d(\varphi, \psi) + d(\psi, \diamond\psi) \geqslant d(\varphi, \diamond\psi) = d(\diamond\varphi, \diamond\psi)$, hence $\diamond$ is non-expansive. From (2) and (5) of Lemma 15 it follows that φ is closed if and only if $\diamond\varphi = \varphi$. Finally, $\diamond$ preserves Cauchyness and irreducibility by (3) and (4) of Lemma 15. $\square$

Proposition 17. *Assume* φ *is a Cauchy character. Then for every* $x \in X$,

$$\diamond\varphi(x) = \inf\{\varepsilon \succ 0 \mid \forall U \in \tau(x \ll_\varepsilon U \Rightarrow \varphi \asymp U)\}.$$

Proof. Let S be the set whose infimum is considered. Assume $p \succ \diamond\varphi(x)$. Then $x \ll_p U$ implies $\diamond\varphi \asymp U$, hence $\varphi \asymp U$ by Lemma 15(4). It follows that $p \geqslant \inf S$ and since $p \succ \diamond\varphi(x)$ was arbitrary, $\diamond\varphi(x) \geqslant \inf S$. For the reverse inequality, suppose $p \succ \inf S$. Choose p', $\delta \succ 0$ so that $p \succ p' + \delta$ and $p' \succ \inf S$. Then $\forall U \in \tau(x \ll_{p'} U \Rightarrow \varphi \asymp U)$. If $x \in U \in \tau$, then $x \ll_{p'} B_{p'}[U]$ and so $\varphi \asymp B_{p'}[U]$. Since φ is Cauchy, there is a $v \in B_{p'}[U]$ such that $\delta \geqslant \varphi(v)$. Choose $u \in U$ so that $p' \succ d(u, v)$. Then $p \succ p' + \delta \geqslant d(u, v) + \varphi(v) \geqslant \varphi(u)$, so $U \cap \varphi_p \neq \emptyset$. Thus $x \in \mathrm{cl}_\tau \varphi_p$ and so by Lemma 15(1), $p \geqslant \diamond\varphi(x)$. Since $p \succ \inf S$ was arbitrary, $\inf S \geqslant \diamond\varphi(x)$. $\square$

Lemma 18. *Assume φ is a character. Then* $(1)\Leftrightarrow(2)\Leftrightarrow(3)\Rightarrow(4)$ *holds for the following statements. If φ is a Cauchy character, then all four statements are equivalent.*

(1) $\diamond\varphi=[x]$.

(2) $x=\bigvee\varphi$ *and for all* $\delta\succ 0$, $x\in \mathrm{cl}_\tau\varphi_\delta$.

(3) x *is an upper bound for* φ *and for all* $\delta\succ 0$, $x\in \mathrm{cl}_\tau\,\varphi_\delta$.

(4) *For all* $U\in\tau$, $x\in U$ *if and only if* $\varphi\asymp U$.

Proof. $(1)\Rightarrow(2)$. By Lemma 15(3), for all y, $\varphi(y)\geqslant\diamond\varphi(y)=d(y,x)$, so x is an upper bound for φ. Assume $p\succ\sup_y(d(y,a)\dot{-}\varphi(y))$. Choose p', $\delta\succ 0$ so that $p\succ p'+\delta$ and $p'\succ\sup_y(d(y,a)\dot{-}\varphi(y))$. We will be using axiom (a2′). Suppose $x\ll_p U$ and let $V\in\tau$ be such that $x\in V\ll_p U$. Let $y\in V\cap\varphi_\delta$. Then $p\succ p'+\delta\geqslant(d(y,a)\dot{-}\varphi(y))+\varphi(y)\geqslant d(y,a)$ and so $a\in U$. From (a2′) it follows that $p\geqslant d(x,a)$. Thus $\sup_y(d(y,a)\dot{-}\varphi(y))\geqslant d(x,a)$. By Lemma 2, $x=\bigvee\varphi$. Note that $x\in\mathrm{cl}_\tau\varphi_\delta$ for all $\delta\succ 0$ is trivial.

$(2)\Rightarrow(3)$. Trivial.

$(3)\Rightarrow(1)$. Assume x is an upper bound for φ and for all $\delta\succ 0$, $x\in\mathrm{cl}_\tau\,\varphi_\delta$. By Lemma 15(1), $\diamond\varphi(x)=0$ and so for any $y\in X$, $d(y,x)=d(y,x)+\diamond\varphi(x)\geqslant\diamond\varphi(y)$. Since x is an upper bound for φ, $\hat{d}(\varphi,[x])=0$. But $[x]$ is closed by (a2), so by Lemma 15(5), $\hat{d}(\diamond\varphi,[x])=0$; that is, for any $y\in X$, $\diamond\varphi(y)\geqslant d(y,x)$. (1) follows.

$(3)\Rightarrow(4)$. Assume x is an upper bound for φ and for all $\delta\succ 0$, $x\in\mathrm{cl}_\tau\varphi_\delta$. Suppose $x\in U\in\tau$. Choose $\varepsilon\succ 0$ and $V\in\tau$ so that $x\in V\ll_\varepsilon U$. Let $y\in V\cap\varphi_\delta$, where $\varepsilon\succ\delta\succ 0$. Then $\varepsilon\succ\varphi(y)$ and $y\ll_\varepsilon U$, hence $\varphi\asymp U$. Conversely, suppose $\varphi\asymp U$ and choose $\varepsilon\succ 0$ and y that $\varepsilon\succ\varphi(y)$ and $y\ll_\varepsilon U$. Since $\varphi(y)\geqslant d(y,x)$, we have $\varepsilon\succ d(y,x)$ and so $x\in U$.

$(4)\Rightarrow(1)$. If φ is Cauchy and (4) holds, then by Proposition 17, $(\diamond\varphi)(y)=\inf\{\varepsilon\succ 0 \mid \forall U\in\tau.\ (y\ll_\varepsilon U\Rightarrow x\in U)\}$ which equals $d(y,x)$ by Axiom (a2′). □

We now approach a central definition of this paper, that of Smyth completeness.

Definition 19. If φ is a τ-ideal on the t$\mathscr{V}$-cs X, then a point $x\in X$ is the *strong supremum* of φ, denoted by $x=\bigvee^\tau\varphi$ if and only if the equivalent statements of Lemma 18 hold. The space is *Smyth complete* if every τ-ideal has a strong supremum (or, equivalently, if every closed τ-ideal is representable). Smyth complete topological $\mathscr{V}$-continuity spaces are also called *topological $\mathscr{V}$-domains*.

Every t$\mathscr{V}$-cs carries a natural quasi-uniformity induced by its distance function. This gives a topological quasi-uniform space which is Smyth complete in the sense of [12, 13] if and only if the original t$\mathscr{V}$-cs is Smyth complete as defined above. As a consequence, a t$\mathscr{V}$-cs which is a metric space in its metric topology is Smyth complete if and only if it is a complete metric space in the classical sense. Also, a topological spaces viewed as t$\mathscr{V}$-cs with $\mathscr{V}=\{0,1\}$ is Smyth-complete if and only if the topology is sober.

5. Smyth completion

This section is devoted to the construction of the Smyth completion of a t$\mathscr{V}$-cs X. The underlying set will be $\tilde{X}$, the collection of closed τ-ideals on X. The distance on $\tilde{X}$ is the usual distance for characters:

$$\tilde{d}(\varphi,\psi) = \hat{d}(\varphi,\psi) = \sup_{x\in X}(\psi(x) \dot{-} \varphi(x))$$

and the topology, $\tilde{\tau}$, is generated by the sets $\tilde{U} = \{\varphi \in \tilde{X} \mid \varphi \asymp U\}$, where $U \in \tau$. It is easily seen that $\tilde{U} \cap \tilde{V} = \widetilde{U \cap V}$, so the family $\{\tilde{U} \mid U \in \tau\}$ is actually a base for $\tilde{\tau}$.

Lemma 20. *Assume U, $V \in \tau$ and $\varepsilon \succ 0$.*
(1) *For $x \in X$ we have $[x] \in \tilde{U} \Leftrightarrow x \in U$.*
(2) $B_\varepsilon[\tilde{U}] \subseteq \widetilde{B_\varepsilon[U]}$.
(3) *If $\varepsilon \succ \varepsilon' \succ 0$, then $\widetilde{B_{\varepsilon'}[U]} \subseteq B_\varepsilon[\tilde{U}]$.*
(4) $U \subseteq V \Rightarrow \tilde{U} \subseteq \tilde{V}$.
(5) $U \ll_\varepsilon V \Leftrightarrow \tilde{U} \ll_\varepsilon \tilde{V}$.

Proof. (1) Lemma 18 gives us $x \in U \Leftrightarrow [x] \asymp U$. By definition, the latter means $[x] \in \tilde{U}$.

(2) Suppose $\varphi \asymp U$ and $\varepsilon \succ \tilde{d}(\varphi,\psi)$. Choose ε', $\delta \succ 0$ so that $\varepsilon \succ \varepsilon' + \delta$ and $\varepsilon' \succ \tilde{d}(\varphi,\psi)$. Since φ is Cauchy, there is a $u \in U$ such that $\delta \succ \varphi(u)$. It follows that $u \ll_\varepsilon B_\varepsilon[U]$ and $\varepsilon \succ \varepsilon' + \delta \geqslant \varepsilon' + \varphi(u) \geqslant \psi(u)$. Thus, $\psi \asymp B_\varepsilon[U]$.

(3) Assume $\varepsilon \succ \varepsilon' \succ 0$ and $\varphi \asymp B_{\varepsilon'}[U]$. Choose ε'' so that $\varepsilon \succ \varepsilon'' \succ \varepsilon'$ and let $v \in B_{\varepsilon'}[U]$ be such that $\varepsilon'' \succ \varepsilon' + \varphi(v)$. Choose $u \in U$ so that $\varepsilon' \succ d(u,v)$. Then $[u] \asymp U$ and for all x, $\varepsilon'' + d(x,u) \geqslant \varepsilon' + \varphi(v) + d(x,u) \geqslant d(u,v) + \varphi(v) + d(x,u) \geqslant d(x,v) + \varphi(v) \geqslant \varphi(x)$. Thus, $\varepsilon \succ \varepsilon'' \geqslant \sup_x(\varphi(x) \dot{-} d(x,u)) = \tilde{d}([u],\varphi)$, so $\varphi \in B_\varepsilon[\tilde{U}]$.

(4) Immediate.

(5) Suppose that $U \ll_\varepsilon V$. Then $B_\varepsilon[U] \subseteq V$. By (2) and (4), $B_\varepsilon[\tilde{U}] \subseteq \widetilde{B_\varepsilon[U]} \subseteq \tilde{V}$, so $\tilde{U} \ll_\varepsilon \tilde{V}$. Now suppose $\tilde{U} \ll_\varepsilon \tilde{V}$ and $y \in B_\varepsilon[U]$. Let $x \in U$ be such that $\varepsilon \succ d(x, y)$. Then $[x] \in \tilde{U}$ and $\varepsilon \succ \tilde{d}([x], [y])$. Thus, $[y] \in \tilde{V}$ and $y \in V$ by (1). Hence $B_\varepsilon[U] \subseteq V$ and so $U \ll_\varepsilon V$. $\square$

Theorem 21. *$(\tilde{X}, \tilde{d}, \tilde{\tau})$ is a topological $\mathscr{V}$-continuity space.*

Proof. (a1) This holds by Lemma 12 and Lemma 20(5).

(a2′) We have to show that if $p \succ 0$ satisfies

$$\forall V \in \tau \quad (\varphi \in \tilde{V} \Rightarrow \psi \in B_p[\tilde{V}]), \tag{1}$$

then $p \geqslant \tilde{d}(\varphi,\psi)$, i.e. $p + \varphi(x) \geqslant \psi(x)$ for all $x \in X$. So assume $q \succ \varphi(x)$. Then

$$\forall U \in \tau \quad (x \ll_q U \Rightarrow \varphi \asymp U) \tag{2}$$

by Proposition 17. Employing Lemma 20(2), we deduce from (1) and (2) the condition

$$\forall U \in \tau \quad (x \ll_{p+q} U \Rightarrow \psi \asymp U)$$

which implies $p + q \geqslant \psi(x)$ by Proposition 17 again.

(a3) By Lemma 20 we have $B_\varepsilon[\tilde{U}] = \bigcup_{\varepsilon \succ \varepsilon'} B_{\varepsilon'}[\tilde{U}] \subseteq \bigcup_{\varepsilon \succ \varepsilon'} \widetilde{B_{\varepsilon'}[U]} \subseteq B_\varepsilon[\tilde{U}]$. Thus $B_\varepsilon[\tilde{U}] = \bigcup_{\varepsilon \succ \varepsilon'} \widetilde{B_{\varepsilon'}[U]}$ is τ-open. $\square$

Theorem 22. *The map $\eta_X : x \mapsto [x] : X \to \tilde{X}$ is a $t\mathscr{V}$-cs embedding (i.e. an isometrie and homeomorphism onto its image) and the image of this embedding is $\tilde{\tau} \vee \alpha_{\tilde{d}^\star}$-dense in $\tilde{X}$.*

Proof. By the Yoneda lemma, η_X is an isometric embedding. Since $[x] \in \tilde{U}$ if and only if $x \in U$, η_X is also a topological embedding. To see that the image is $\tilde{\tau} \vee \alpha_{\tilde{d}^\star}$-dense, suppose that ω is open with respect to that topology and that $\varphi \in \omega$. Then there are $\varepsilon \succ 0$ and $U \in \tau$ such that $\varphi \in \tilde{U}$ and $\tilde{U} \cap B^*_\varepsilon(\varphi) \subseteq \omega$. Choose $x \in U$ so that $\varepsilon \succ \varphi(x)$. Then $[x] \in \tilde{U}$ and, by the Yoneda lemma, $\varepsilon \succ d([x], \varphi)$. Thus, $[x] \in \omega$. $\square$

Theorem 23. *The space $(\tilde{X}, \tilde{d}, \tilde{\tau})$ is Smyth complete, i.e. it is a topological $\mathscr{V}$-domain.*

Proof. Suppose that Ψ is a closed $\tilde{\tau}$-ideal on $\tilde{X}$. Let $\psi = \Psi \circ \eta$. Then ψ is a character on X and since $\psi_\varepsilon = \eta^{-1}(\Psi_\varepsilon)$, ψ is closed. To proceed, we first establish the

Claim. *For all $U \in \tau$ we have $\psi \asymp U$ if and only if $\Psi \asymp \tilde{U}$.*

Assume $\psi \asymp U$. Choose $\varepsilon \succ 0$ and $x \in X$ so that $\varepsilon \succ \psi(x) = \Psi([x])$ and $x \ll_\varepsilon U$. By Lemma 20(5) we have $[x] \ll_\varepsilon \tilde{U}$. Therefore, $\Psi \asymp \tilde{U}$. Conversely, if $\Psi \asymp \tilde{U}$ there exist $\varepsilon \succ 0$ and φ such that $\varepsilon \succ \Psi(\varphi)$ and $\varphi \ll_\varepsilon \tilde{U}$. Hence, there is $V \in \tau$ with $\varphi \in \tilde{V} \ll_\varepsilon \tilde{U}$. As $\varphi \in \tilde{V}$ means $\varphi \asymp V$, there are $\delta \succ 0$ and $x \in X$ with $\delta \succ \varphi(x)$ and $x \ll_\delta V$. We thus have $x \ll_{\varepsilon+\delta} U$ and are now going to prove that $\varepsilon + \delta \succ \psi(x)$ in order to establish $\psi \asymp U$ and verify the claim. As Ψ is non-expansive, we have $\varphi(x) = \tilde{d}([x], \varphi) \geqslant \Psi([x]) \dot{-} \Psi(\varphi)$. Thus, $\varepsilon + \delta \succ \Psi(\varphi) + \varphi(x) \geqslant \Psi([x]) = \psi(x)$.

From the claim and the fact that Ψ is irreducible, irreducibility is immediate for ψ. For the Cauchyness, assume that $\psi \asymp U$ and $\delta \succ 0$. Choose $\delta' \succ 0$ so that $\delta \succ 2\delta'$. Then $\Psi \asymp \tilde{U}$ and hence there is a $\varphi \in \Psi_{\delta'} \cap \tilde{U}$. By $\tilde{\tau} \vee \alpha_{\tilde{d}^\star}$-density of $\eta_X(X)$ in $\tilde{X}$, there is $x \in X$ with $[x] \in B^*_{\delta'}(\varphi) \cap \tilde{U}$. It follows that $x \in \psi_\delta \cap U$.

To see $\Psi = \tilde{d}(\cdot, \psi)$, it suffices, by Lemma 18, to show that (a) $\Psi(\varphi) \geqslant \tilde{d}(\varphi, \psi)$ for all $\varphi \in \tilde{X}$ and (b) $\psi \in \omega$ implies $\omega \cap \Psi_\delta \neq \emptyset$ for all $\omega \in \tilde{\tau}$ and $\delta \succ 0$. Now (a) holds since $\Psi(\varphi) + \varphi(x) \geqslant \Psi([x]) = \psi(x)$. To see (b), assume $\psi \in \omega$. Then there is some $U \in \tau$ with $U \subseteq \omega$ and $\psi \asymp U$. Then $\Psi \asymp \tilde{U}$ and so $\Psi_\delta \cap \tilde{U} \neq \emptyset$ by Cauchyness. Thus certainly $\Psi_\delta \cap \omega \neq \emptyset$, so (b) holds and the Theorem is proved. $\square$

Lemma 24. *If $f : X \to Y$ is a $t\mathscr{V}$-cs-morphism then so is $\tilde{f} := \diamond \circ \hat{f} : \tilde{X} \to \tilde{Y}$.*

Proof. We know from Section 2.2 that $\hat{f} : \tilde{X} \to \tilde{Y}$ is non-expansive. So is $\diamond$ (Proposition 16), thus the same is true of $\tilde{f}$. It remains to show that $\hat{f}$ preserves the property of being an ideal and that $\tilde{f}$ is continuous. Suppose $\hat{f}(\varphi) \asymp V \in \tau_Y$. Choose $\varepsilon \succ 0$, y so that $\varepsilon \succ \hat{f}(\varphi)(y)$ and $y \ll_\varepsilon V$. Let $x \in X$ be such that $\varepsilon \succ \varphi(x) + d(y, f(x))$ and choose

$\varepsilon_1,\ \varepsilon_2 \succ 0$ so that $\varepsilon \succ \varepsilon_1 + \varepsilon_2$, $\varepsilon_1 \succ \varphi(x)$, and $\varepsilon_2 \succ d(y, f(x))$. Let $W \in \tau_Y$ be such that $y \ll_{\varepsilon_2} W \ll_{\varepsilon_1} V$. Then $x \in f^{-1}(W) \ll_{\varepsilon_1} f^{-1}(V)$ and so $\varphi \asymp f^{-1}(V)$. We obtain:

$$\hat{f}(\varphi) \asymp V \Rightarrow \varphi \asymp f^{-1}(V). \tag{3}$$

If $\varphi \asymp f^{-1}(V')$ and $V' \ll V$, then there is an $\varepsilon \succ 0$ such that $V' \ll_\varepsilon V$ and an $x \in f^{-1}(V')$ such that $\varepsilon \succ \varphi(x)$. But then $f(x) \in V' \ll_\varepsilon V$ and $\varepsilon \succ \hat{f}(\varphi)(f(x))$, so $\hat{f}(\varphi) \asymp V$. We obtain:

$$\varphi \asymp f^{-1}(V') \ \&\ V' \ll V \Rightarrow \hat{f}(\varphi) \asymp V. \tag{4}$$

From (3) and (4) it easily follows that $\hat{f}(\varphi)$ is irreducible and Cauchy. For continuity, suppose $\varphi \in \tilde{f}^{-1}(\tilde{V})$, for $V \in \tau_Y$. Then $\tilde{f}(\varphi) \in \tilde{V}$. Choose $V' \ll V$ so that $\tilde{f}(\varphi) \in \tilde{V}'$. Then $\diamond \hat{f}(\varphi) \asymp V'$, hence $\hat{f}(\varphi) \asymp V'$ and so $\varphi \asymp f^{-1}(V')$. But then $\varphi \in \widetilde{f^{-1}(V')} \subseteq \tilde{f}^{-1}(\tilde{V})$. If follows that $\tilde{f}$ is continuous. □

Theorem 25. *Assume X is a t$\mathscr{V}$-cs, Y is a topological $\mathscr{V}$-domain and $f : X \to Y$ is a t$\mathscr{V}$-cs morphism. Then there is a unique t$\mathscr{V}$-cs morphism $f^\flat : \tilde{X} \to Y$ such that $f^\flat \circ \eta_X = f$.*

Proof. By completeness, η_Y is surjective and hence an isomorphism. Thus $f^\flat = \eta_Y^{-1} \circ \tilde{f}$ is a t$\mathscr{V}$-cs-morphism. Moreover, $\tilde{f}(d_X(\cdot, x)) = d_Y(\cdot, f(x))$ for all $x \in X$, thus $f^\flat \circ \eta_X = f$. Since t$\mathscr{V}$-cs morphisms are $\tilde{\tau} \vee \alpha_{\tilde{d}^\star}$-continuous, this extension is unique by Theorem 22. □

Theorems 22, 23 and 25 yield the desired result: The constructed t$\mathscr{V}$-cs $\tilde{X}$ contains X, is Smyth complete and enjoys the usual universal property of morphisms from X into complete spaces extending uniquely to $\tilde{X}$. Thus $(\tilde{X}, \tilde{d}, \tilde{\tau})$ is the Smyth completion of (X, d, τ).

Smyth completeness in terms of filters:

The more traditional approach to Smyth completeness is via filters. In this section, we show that our approach yields the same notions.

We first need to introduce Cauchy filters. For a filter $\mathscr{F}$ on X and $\varepsilon \succ 0$, we use the notation $\mathscr{F}^{(\varepsilon)} = \{A \subseteq X \mid \exists B \in \mathscr{F}.\ B \ll_\varepsilon A\}$. Moreover, let $\mathscr{N}(x)$ stand for the filter of τ-neighborhoods of the point $x \in X$. A filter $\mathscr{F}$ on a t$\mathscr{V}$-cs (X, d, τ) is *Cauchy* [12] if for all $\varepsilon \succ 0$ and all $A \in \mathscr{F}$ there is an $x \in A$ such that $\mathscr{N}(x)^{(\varepsilon)} \subseteq \mathscr{F}$. The filter is *round* if $\mathscr{F} = \bigcup_{\varepsilon \succ 0} \mathscr{F}^{(\varepsilon)}$. A space is *complete* in this sense if every round Cauchy filter coincides with the neighborhood filter of some unique point.

Proposition 26. *Suppose that (X, d, τ) is a topological $\mathscr{V}$-continuity space.*

(1) *If φ is an ideal on X, then $\mathscr{F}_\varphi = \{A \subseteq X \mid \varphi \asymp A\}$ is a round Cauchy filter such that $x = \bigvee^\tau \varphi$ if and only if $\mathscr{N}(x) = \mathscr{F}_\varphi$.*
(2) *If $\mathscr{F}$ is a round Cauchy filter on X, then $\varphi_{\mathscr{F}}(x) := \inf\{\varepsilon \succ 0 \mid \mathscr{N}^{(\varepsilon)}(x) \subseteq \mathscr{F}\}$ defines an ideal such that $x = \bigvee^\tau \varphi_{\mathscr{F}}$ if and only if $\mathscr{N}(x) = \mathscr{F}$.*

Proof. (1) Clearly if $A \in \mathscr{F}_\varphi$ and $A \subseteq B$, then $B \in \mathscr{F}_\varphi$. From irreducibility it follows that $\mathscr{F}_\varphi$ is closed under finite intersections. Hence, $\mathscr{F}_\varphi$ is a filter. From Cauchyness of φ one easily shows that $\mathscr{F}_\varphi$ is Cauchy and by Lemma 12 $\mathscr{F}_\varphi$ is round. The equivalence $x = \bigvee^\tau \varphi$ if and only if $\mathscr{N}(x) = \mathscr{F}_\varphi$ follows from Lemma 18.

(2) To show that $\varphi_{\mathscr{F}} : X^* \to \mathscr{V}$ is non-expansive it will suffice to show that whenever $\varepsilon \succ d(y,x)$ and $\delta \succ \varphi_{\mathscr{F}}(x)$, then $\mathscr{N}^{(\varepsilon+\delta)}(y) \subseteq \mathscr{F}$. Suppose that $y \ll_{\varepsilon+\delta} U$. Choose V so that $y \ll_\varepsilon V \ll_\delta U$. Then $x \in V$ and so $U \in \mathscr{F}$. A simple calculation shows that for all $U \in \tau$, we have $\varphi_{\mathscr{F}} \asymp U$ if and only if $U \in \mathscr{F}$. From this it follows easily that $\varphi_{\mathscr{F}}$ is irreducible and Cauchy. Moreover, we conclude $\mathscr{F}_{\varphi_{\mathscr{F}}} = \mathscr{F}$, so the equivalence $x = \bigvee^\tau \varphi_{\mathscr{F}}$ if and only if $\mathscr{N}(x) = \mathscr{F}$ now follows from (1). $\square$

Using (1) and (2) in Proposition 26 one can freely switch between ideals and filters. As a round Cauchy filter is the neighborhood filter of a point if and only if that point is the strong supremum of the corresponding ideal, the two approaches to Smyth completeness coincide.

6. Ideal completion via Smyth completion

In this section, we prove the main result of the paper: The process of first performing the ideal completion and then considering the Scott topology may be replaced by just performing the Smyth completion.

Lemma 27. *Suppose that φ is a character on the t$\mathscr{V}$-cs (X,d,τ). If φ is an ideal on the $\mathscr{V}$-continuity space (X,d) then it is a τ-ideal.*

Proof. Observe that $A \ll_\varepsilon^\tau B \Rightarrow A \ll_\varepsilon^{\alpha_d} B$ and $\varphi \asymp^\tau A \Rightarrow \varphi \asymp^{\alpha_d} A$ hold. Moreover, if φ is Cauchy on (X,d,α_d) then $\varphi \asymp^{\alpha_d} U \ll^\tau V$ implies $\varphi \asymp^\tau V$: Suppose $U \ll_\delta^\tau V$. By Cauchyness and the assumption $\varphi \asymp^{\alpha_d} U$, there is $x \in U$ such that $\delta \succ \varphi(x)$. Hence $\varphi \asymp^\tau V$. The result follows easily from these observations and Proposition 14. $\square$

Proposition 28. *Assume that (X,d,τ) is a topological $\mathscr{V}$-domain. Then the space (X,d) is a $\mathscr{V}$-domain and $\tau \subseteq \sigma_d$.*

Proof. If φ is an ideal on (X,d) then, by the preceding lemma, it is a τ-ideal and hence its strong supremum exists by Smyth-completeness. In particular, φ has a supremum. Therefore, the space is directed complete. Now suppose $U \in \tau$ and $\bigvee \varphi \in U$ for some ideal φ. We have just seen that this ideal is indeed a τ-ideal and that its supremum is a strong supremum. Hence, $\varphi \asymp^\tau U$ by Lemma 18. This implies in particular, that there are $\varepsilon \succ 0$ and $x \in X$ such that $\varepsilon \succ \varphi(x)$ and $B_\varepsilon(x) \subseteq U$. Thus, U is Scott-open. $\square$

By Proposition 9, taking a $\mathscr{V}$-continuity space (X,d) with its Alexandroff topology α_d gives a t$\mathscr{V}$-cs. It is our main result that the Smyth Completion of this t$\mathscr{V}$-cs yields the ideal completion of (X,d) and its Scott topology.

Theorem 29. *Assume that* (X,d) *is a* $\mathscr{V}$*-continuity space and that* $(\tilde{X},\tilde{d},\widetilde{\alpha_d})$ *is the Smyth completion of* (X,d,α_d)*. Then* $(\tilde{X},\tilde{d}) \cong \mathscr{I}(X,d)$ *and* $\widetilde{\alpha_d} = \sigma_{\tilde{d}}$.

Proof. By Proposition 14 we may conclude that the underlying sets of both completions are identical. Clearly, the distance functions coincide. Proposition 28 asserts in particular that $\widetilde{\alpha_d} \subseteq \sigma_{\tilde{d}}$. So it remains to prove $\sigma_{\tilde{d}} \subseteq \widetilde{\alpha_d}$. By Theorem 6, it suffices to show that the sets $B_\varepsilon([x])$ are $\widetilde{\alpha_d}$-open. So assume $x \in X$, $\varepsilon \succ 0$, and $\varphi \in B_\varepsilon([x])$. Pick ε', $\delta \succ 0$ so that $\varepsilon \succ \varepsilon' + \delta + \delta$ and $\varepsilon' \succ d([x]), \varphi) = \varphi(x)$. We will see that then $\varphi \in \widetilde{B_{\varepsilon'+\delta}}(x) \subseteq B_\varepsilon([x])$ holds which finishes the proof.

$$\varphi \in \widetilde{B_{\varepsilon'+\delta}}(x): \text{ We have } x \in B_\delta(x) \ll_{\varepsilon'} B_{\varepsilon'+\delta}(x) \text{ and } \varepsilon' \succ \varphi(x).$$

$\widetilde{B_{\varepsilon'+\delta}}(x) \subseteq B_\varepsilon([x])$: Suppose $\psi \in \widetilde{B_{\varepsilon'+\delta}}(x)$, i.e. $\psi \asymp B_{\varepsilon'+\delta}(x)$. As ψ is Cauchy, there is $z \in \psi_\delta$ with $\varepsilon' + \delta \geqslant d(x,z) = d([x],[z])$. Now $\delta \geqslant \psi(z) = d([z]),\psi)$, and hence $\varepsilon \succ \varepsilon' + \delta + \delta \geqslant d([x],\psi)$. $\square$

Combining the Representation Theorem (Theorem 6) with Theorem 29, we obtain the following result on algebraic $\mathscr{V}$-domains.

Corollary 30. (1) *If* (X,d) *is a* $\mathscr{V}$*-domain such that* (X,d,α_d) *is a topological* $\mathscr{V}$*-domain, then* (X,d) *is algebraic.*

(2) *Assume that* (X,d) *is an algebraic* $\mathscr{V}$*-domain. Then* (X,d,σ_d) *is a topological* $\mathscr{V}$*-domain. In particular, the Scott topology* σ_d *is admissible and* (X,σ_d) *is a sober space.*

If a poset P is isomorphic to its ideal completion, then it does not contain any infinite ascending chain. As a consequence, every upper subset of P is Scott open. It is a corollary of Theorem 29 that a similar result holds for $\mathscr{V}$-continuity spaces.

Corollary 31. *If* (X,d) *is a* $\mathscr{V}$*-continuity space such that* $\mathscr{I}(X,d) \cong (X,d)$ *then the Scott- and Alexandroff topologies on* X *coincide.*

Proof. By Theorem 29, $X \cong \tilde{X}$ in this case. As any t$\mathscr{V}$-cs is isomorphically embedded in its Smyth-completion, this implies in particular $\widetilde{\alpha_d} = \alpha_{\tilde{d}}$. The theorem also asserts $\widetilde{\alpha_d} = \sigma_{\tilde{d}}$, hence the corollary follows. $\square$

Now we can return to the question of idempotency of the ideal completion.

Corollary 32. *Suppose that* (X,d) *is a* $\mathscr{V}$*-continuity space. Then* $\mathscr{I}\mathscr{I}(X,d) \cong \mathscr{I}(X,d)$ *if and only if the Scott- and Alexandroff topologies on* $\mathscr{I}(X,d)$ *coincide.*

Proof. Corollary 31 proves the (only if)-part. To see the (if)-part, observe that $\mathscr{I}(X,d) \cong (\tilde{X},\tilde{d})$, where $(\tilde{X},\tilde{d},\sigma_{\tilde{d}})$ is the Smyth completion of (X,d,α_d) by Theorem 29. By assumption, $\sigma_{\tilde{d}} = \alpha_{\tilde{d}}$, hence $\mathscr{I}\mathscr{I}(X,d) \cong (\tilde{\tilde{X}},\tilde{\tilde{d}})$ with $(\tilde{\tilde{X}},\tilde{\tilde{d}},\tilde{\tilde{\tau}})$ the Smyth completion of $(\tilde{X},\tilde{d},\tilde{\tau})$. But Smyth completion is idempotent, hence $\mathscr{I}\mathscr{I}(X,d) \cong \mathscr{I}(X,d)$. $\square$

This can be combined with the results of [16] to get a characterization of idempotency of the ideal completion in terms of nets. Recall that a net $(x_i)_{i\in I}$ is *forward Cauchy*, if for every $\varepsilon \succ 0$, there is $i \in I$ such that $i \leqslant j \leqslant k$ implies $\varepsilon \succ d(x_j, x_k)$. The net is *bi-Cauchy*, if for every $\varepsilon \succ 0$, there is $i \in I$ such that $\varepsilon \succ d(x_j, x_k)$ holds for all $j,\ k \geqslant i$ (regardless of their order).

Corollary 33. *Suppose that (X,d) is a $\mathscr{V}$-continuity space. Then $\mathscr{I}\mathscr{I}(X,d) \cong \mathscr{I}(X,d)$ if and only if every forward Cauchy net on (X,d) is bi-Cauchy.*

Proof. Theorem 5 of [16] asserts that all forward Cauchy nets are bi-Cauchy if and only if $\tilde{\alpha}_d = \alpha_{\tilde{d}}$, where $(\tilde{X}, \tilde{d}, \tilde{\alpha}_d)$ is the Smyth completion of (X, d, α_d). By Theorem 29, we have $\tilde{\alpha}_d = \sigma_{\tilde{d}}$, so Corollary 32 applies. $\square$

As mentioned before, topological **2**-continuity spaces correspond to topological spaces with their orders of specialization. Moreover, Smyth completion is just sobrification in this case, i.e. if $(\tilde{X}, \tilde{d}, \tilde{\tau})$ is the Smyth completion of $(X, d_{\leqslant}, \tau)$, then $(\tilde{X}, \tilde{\tau})$ is the sobrification of (X, τ) and $\tilde{d}$ is induced by $\sqsubseteq_{\tilde{\tau}}$. Hence, we may conclude from Theorem 29 the following corollary which was proved by Hoffmann in [6] (see also [7, Lemma VII-2.6]).

Corollary 34. *If $(X, \leqslant)$ is a poset, then the sobrification of $(X, \alpha_{\leqslant})$ is homeomorphic to the ideal completion of $(X, \leqslant)$ with its Scott topology.*

Appendix An example: the eager reals

Finding a convenient domain for the reals is one of the motivations for quantitative domain theory: By zero-dimensionality, there is no (ordinary) algebraic domain which contains the unit interval as set of maximal elements. The closest to the reals one can get is the Cantor set; the ideal completion of the full binary tree adds this as set of maximal elements. The elements of the binary tree represent partial binary expansions of real numbers, the maximal elements total expansions – and the latter are ambiguous. Quantitative domain theory provides the tools to overcome this problem. If the binary tree is equipped with a suitable quantitative structure, then the multiple representations are identified in the ideal completion. In [15], one of the authors gave an example of such a quantitative domain modeling the reals. We will now modify the involved structure as follows. In loc. cit., the T_n-neighborhood of a sequence did only depend on the first n elements of this sequence. On one hand, this is advantageous (cf. the definition of *finitary base* in [15]). But on the other hand, one could argue that we are not exploiting the whole amount of information available. Following this line of thought, we define a quasi-metric for the binary tree $\mathscr{B} = \{0,1\}^*$ as follows.[1] For

[1] For simplicity, we restrict ourselves to this case. The construction can easily be modified to work for signed digit representations as in [15].

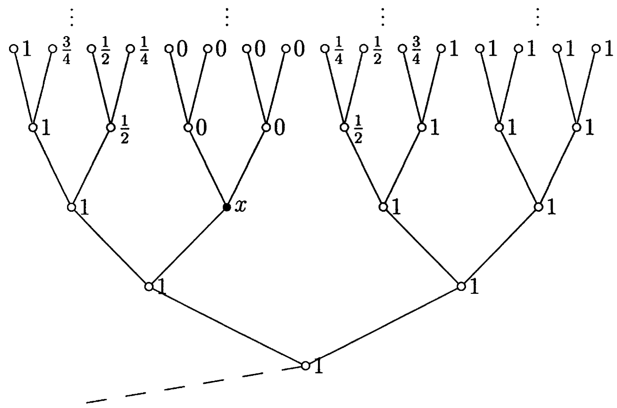

Fig. 1. Distances from the filled dot x. The values $d(x,\ y)$ do not depend on the absolute positions of x and y in the tree, just on their relative position. The essential difference to the model of [15] is the distance which is discrete on $[0,1]$: In defining the model, we were eager to differentiate as much as possible between different sequences; this property is carried over to the completion.

$x=(x_1,\dots,x_n)\in\mathscr{B}$ define $S^-(x)=\sum_{i=1}^n x_i2^{-i}$ and $S^+(x)=S^-(x)+2^{-n}$. (Hence, these are the least, respectively, largest possible reals with binary expansion starting with $(x_1,\dots,x_n)$.) Now we set

$$d(x,\ y)=\min\left(\frac{(S^-(x)\dot{-}S^-(y))+(S^+(y)\dot{-}S^+(x))}{S^+(x)-S^-(x)},1\right).$$

Fig. 1visualizes this definition. The essential point is that the indicated distances hold regardless of the actual position of x in the tree.

The Smyth completion of $(\mathscr{B},d,\alpha_d)$ is isomorphic to $\mathscr{B}\cup[0,1]$, where the distance is given by $d(\xi,\eta)=1$ for different elements $\xi,\eta\in[0,1]$ and by $d(\xi,x)=1$ and

$$d(x,\xi)=\min\left(\frac{(S^-(x)\dot{-}\xi)+(\xi\dot{-}S^+(x))}{S^+(x)-S^-(x)},1\right)$$

for $\xi\in[0,1]$ and $x\in\mathscr{B}$. The induced order is $x\leqslant\xi$ if and only if ξ has a binary expansion starting with x. The topology is the Scott topology on that poset (which coincides with the generalized Scott topology). Thus, its trace on $[0,1]$ is the usual topology.

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