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Concurrence Geometries*

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Recent application of combinatorial geometry to research on structure and form in architecture and structural engineering has revealed a number of fascinating open problems in pure projective geometry. The present paper deals with such a problem: to describe combinatorially the variety of configurations c of hyperplanes which can be constructed, given that the section of the configuration c on a fixed hyperplane H be a given geometric figure.

This problem arises when we try to associate with any plane bar-and-joint structure a combinatorial object descriptive of the space of infinitesimal motions (equivalently: parallel redrawings) of that structure. The solution of this problem is, in my view, crucial to advancing the exemplary work begun over a century ago by James Clerk Maxwell and Luigi Cremona, under the title of *graphical statics*. We return to this application in Section 6.

We begin with a detailed study of plane configurations of lines. The natural generalization of this theory to configurations of hyperplanes in higher-dimensional spaces is reserved for the final sections of this article, where we shall also see how the construction of a *concurrence geometry* solves the related problem of deriving from any vector geometry G a geometry $C(G)$ whose points are the *circuits* of G.

1. INTRODUCTION

Imagine a finite set of variable lines in a plane, each line running in a particular fixed direction (not all parallel), but otherwise free to move in the plane. Using these lines, a variety of combinatorially distinct plane configurations can be formed. This paper is devoted to a combinatorial investigation of this variety of configurations.

We shall refer to plane configurations simply as *drawings*. For each drawing we pay attention only to which subsets of its set of lines are *concurrent*. There is a natural order on the set D of possible drawings. We

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say a drawing f is a *specialization* of a drawing e, and write $e \leqslant f$, if and only if every set of lines concurrent in e are also concurrent in f. We shall show that the set D, with the preorder $\leqslant$ of specialization, is a *geometric lattice*. It will follow that the set D of possible drawings is itself a real projective configuration of the same dimension, the *concurrence geometry* of the given set of fixed directions. It follows also that every drawing, being a flat of some dimension in this concurrence geometry, will have a well-defined *rank*. Rank, circuits, bonds, dependence, bases, strong and weak maps, indeed all the ideas and theorems of matroid theory can thus be brought to bear on our analysis of configurations.

2. Example

In order to fix these ideas, let's start with an example. Consider five lines $L_1, \ldots, L_5$ in the plane, such that L_1 and L_2 are parallel, as are L_3 and L_4. Fix the three directions of these five lines. We arrange all the possible drawings in Fig. 1.[1] There is a most general sort of drawing "0," in which no three lines are concurrent: all other drawings are specializations of it. A typical elementary specialization of "0" involves making three lines, such as 1, 3 and 5, concurrent. Another type of elementary specialization is to make lines 1, 2 and 3 concurrent. This can only be done by making lines 1 and 2 coincident, forcing 1, 2, 4 and 1, 2, 5 also to be concurrent. There are six distinct elementary drawings (elementary specializations of "0").

Each elementary drawing can be further specialized in a number of ways, to yield drawings of rank 2. Each drawing of rank 2 could have been obtained by elementary specialization of a number of different drawings of rank 1. The connecting lines between drawings in Fig. 1 indicate all such elementary specializations. Finally, the most extreme specialization occurs when all five lines pass through a point. This drawing has rank 3.

The *order of specialization* linking these six elementary drawings and seven rank 2 drawings agrees with the *order of incidence* linking six points and seven lines in a certain plane projective configuration, the *concurrence geometry* shown in Fig. 2. Three types of lines in the concurrence geometry are there singled out, and their associated drawings are indicated.

[1] During the 16th OSU–Denison Mathematics Conference in May 1981, I conjectured that drawings with given edge directions, ordered by specialization, form a geometric lattice. Following the problem session at which this problem was posed, Denis Higgs, Neil Robertson, Paul Seymour, Neil White, Tom Brylawski, Tom Dowling, Tom Zaslavsky and others engaged in a lively discussion of the subject, and made a number of very helpful suggestions. It was the example in Fig. 1, worked through in detail with Denis Higgs, which convinced me of the truth of the conjecture.

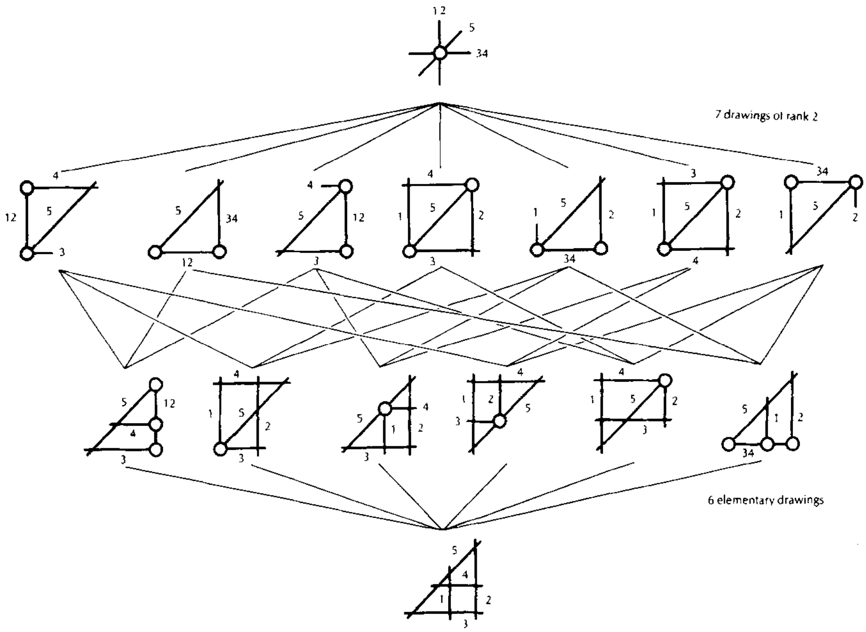


FIGURE 1

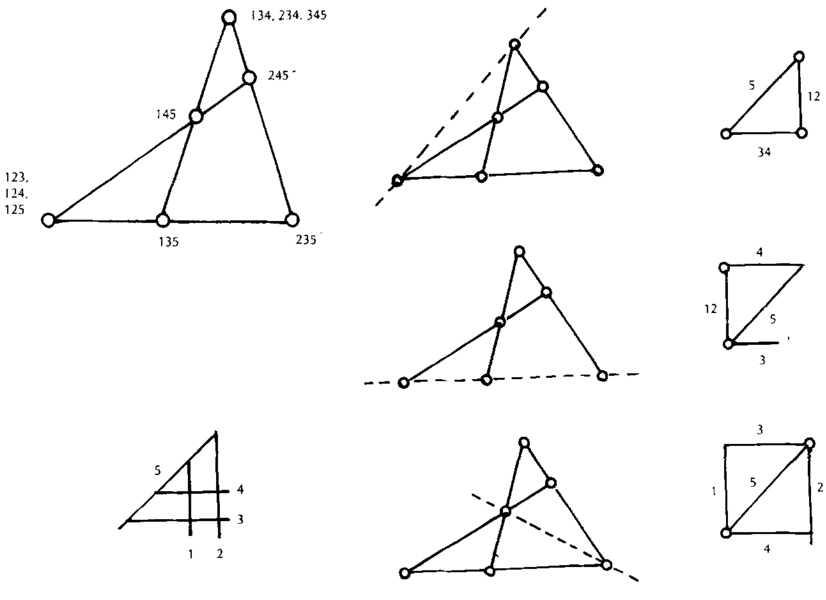


FIGURE 2

3. Concurrence Geometries: Some Basic Facts

A concurrence geometry has certain underlying *elements*, namely, the *triples of lines*, which are combined into *flats*, each flat being the set of triples of lines which are concurrent in an associated drawing. These flats are called "0," *points*, *lines*, *planes*, *etc.*, according to their rank. By extension every set of triples has a *rank*, equal to the rank of the most general drawing in which those triples are concurrent.

If the directions of the variables lines are fixed in such a way that no two lines are parallel, then the points of the concurrence geometry are simply all the single triples of lines. This is the case in Fig 3, which the reader can easily verify is the concurrence geometry for five lines in fixed directions (no two parallel) in the plane. We have labelled the major lines of this geometry $N_1, \ldots, N_5$. Notice that each line N_i contains exactly those triples not containing the index i. Note also how the concurrence geometry in Fig. 2 can be obtained from this geometry simply by moving the line N_5 until the two shaded triangles are reduced to single points. This is best accomplished in

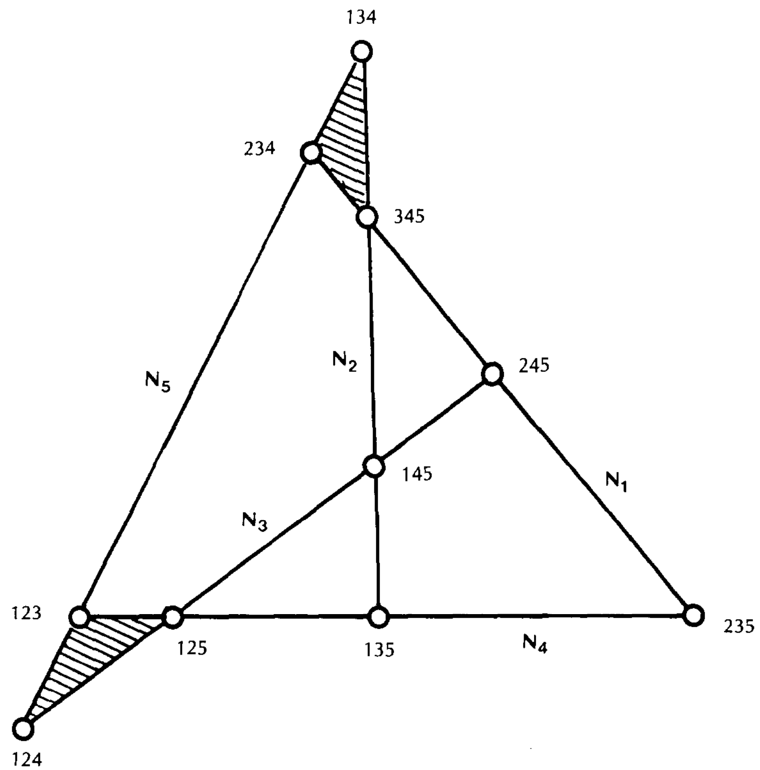


FIGURE 3

two steps. To obtain the concurrence geometry for the situation in which lines 1 and 2 are parallel, make N_3, N_4, N_5 concurrent. Then, if lines 3 and 4 are also to be parallel, make N_1, N_2 and N_5 pass through a point.

Geometric *dependence* in the concurrence geometry is a result of logical *implication* concerning concurrence of lines in drawings. For example, to say in Fig. 1 that the element 145 depends on the pair of elements 123 and 245 is to say that 145 is on the smallest flat containing 123 and 245 (namely, the line {123, 124, 125, 145, 245}). In terms of concurrences, this means that in every drawing in which 123 and 245 are concurrent triples (1 and 2 being parallel, 3 and 4 also), then 145 is also a concurrent triple. The three triples 123, 245, 145 here form a *circuit* (a minimal dependent set of elements): any two of these three concurrences implies the third. Note that 123 and 124 also form a circuit, each implying the other, both being equivalent to the condition that lines 1 and 2 coincide.

A *basis* of the concurrence geometry is any minimal set of triples such that only the trivial drawing "1" (all lines passing through a single point) makes all those triples concurrent. If 1 and 2 are the indices of two lines which are not parallel, then for n variable lines in fixed directions in the plane,

$$123,\ 124, \ldots,\ 12n$$

form a basis: no triple in this list is a consequence of its predecessors, and all told, they imply that all lines pass through the point of intersection of lines 1 and 2 (wherever that may be). Consequently, the concurrence geometry for n variable lines in fixed directions in the plane, not all parallel, invariably has overall rank $n-2$, independent of the choice of fixed directions.

The basic property of geometric lattices (and matroids), which is crucial for understanding concurrences, is the following. Any elementary specialization (of any drawing) is generated by a single triple, and a single triple can generate at most an elementary specialization of any drawing. There is also a closely related *exchange property*. Consider two triples s and t not concurrent in a drawing D. If in all specializations of D in which s is concurrent t is also concurrent, then the same is true with s and t interchanged: if t is concurrent, so is s.

4. Projective Relations

So far. we have given the reader no reason to believe that concurrence geometries depend on anything more than the *topology* of the given set of line directions For instance, one might expect that every set of n *distinct* directions for n lines would give the same concurrence geometry. This is far from true. *Projective* relations between the fixed directions of a set of lines are strongly reflected in the resulting concurrence geometry. Consider, for instance, a plane drawing of a tetrahedral graph (Fig. 4). We cut the six lines of the tetrahedral figure with a single line, which we think of as the line at infinity, so the six points of intersection are the "directions" of the lines of the tetrahedron. These six directions form what is called a *quadrilateral set.* Any one direction can be computed from the other five by simply drawing a tetrahedron with five edges in the correct directions, then drawing in the last edge to connect two known points. That the resulting direction is independent of the choice of tetrahedron (Fig. 4b) is a theorem of projective geometry (see Baker [1, Vol. I, p. 15]). A related theorem [1, Vol. I, p. 61], also illustrated in Fig. 4b, states that the dual tetrahedron can also be drawn with the same edge directions, as a *reciprocal figure.*

It is clear that the set of four triples 136, 145, 234, 256 will be geometrically different in the two concurrence geometries C_6 and C_Q, that for six lines in general directions, and that for six lines whose directions form a quadrilateral set permitting the drawing of a tetrahedron with those four vertices. What is the difference? In C_6, the first three triples form a flat of rank 3 (a plane): the triple 256 is not on that plane, and 256 is not a concurrence in a drawing such as Fig. 4c. Only the trivial drawing, with all lines through one point, has all four of these triples concurrent. In C_Q,

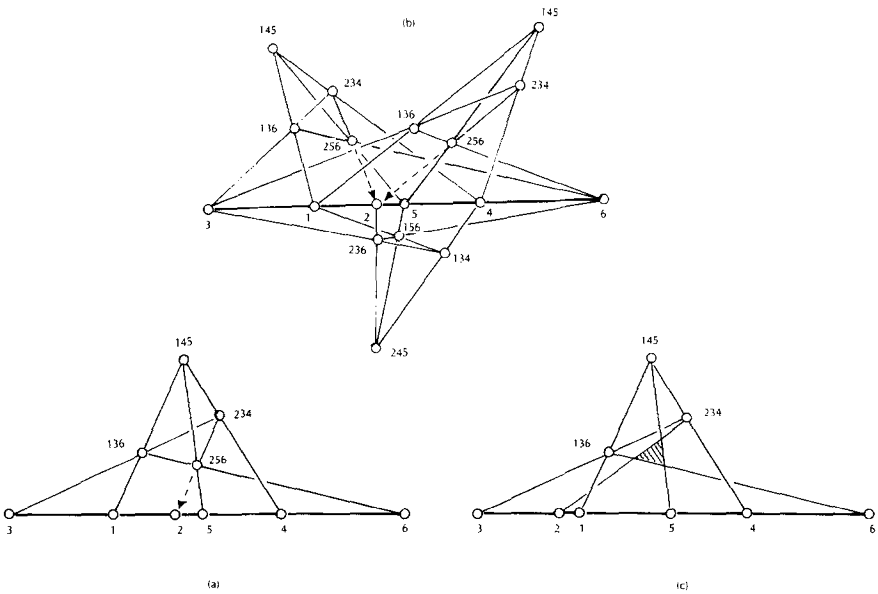


FIGURE 4

however, the four triples are coplanar, forming a circuit of rank 3. The concurrence of any three of these triples implies concurrence of the fourth.

Fig. 5 shows the concurrence geometry C_6 for six lines in fixed but general directions, not forming a quadrilateral set. Two drawings associated with typical lines in C_6 are indicated. Fig. 4c is the drawing corresponding to the plane 136, 145, 234, which does not contain any other point of the configuration. The overall geometry has a very simple structure, not sufficiently emphasized in this sort of drawing. The major planes $N_1,\ldots, N_6$ are just *six planes in general position* in 3-dimensional space. As before, the plane N_i contains precisely those points whose labels do not contain the index "i." The point 136, for instance, lies at the intersection of planes N_2, N_4 and N_5; in this way the complete configuration is determined. This geometry, by the way, is also known from another source. It is the *Dilworth completion* $D_2(B_6)$ of the Boolean algebra B_6, the latter being truncated from below so as to remove its one- and two-element sets.

The concurrence geometry C_Q, for six lines in directions forming a quadrilateral set, is shown in Fig. 6. Here, the points 136, 145, 234, 256 are coplanar, so the tetrahedral drawing with those vertices is of rank 3. By the

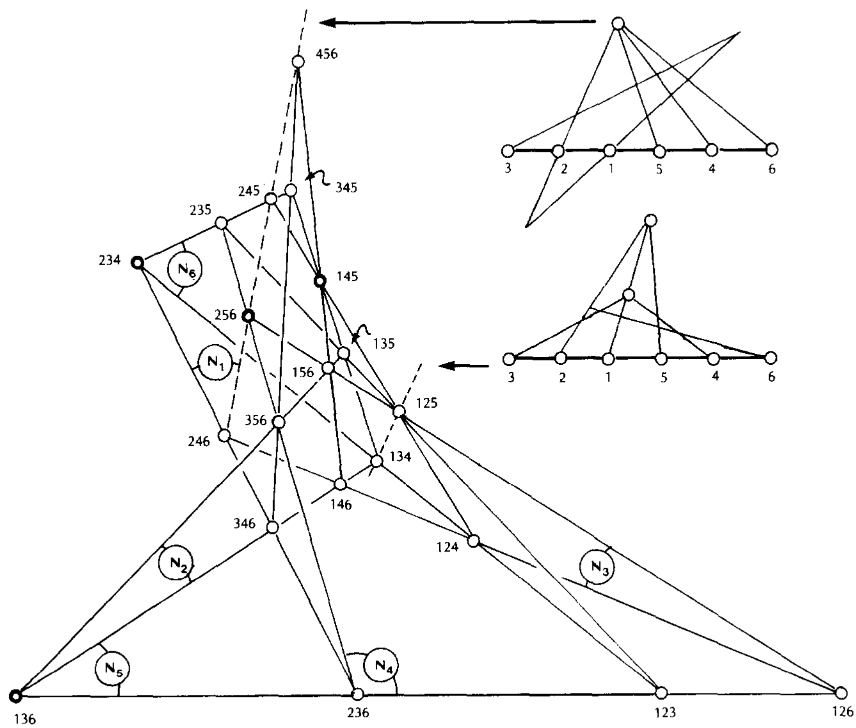


FIGURE 5

theorem of reciprocal figures, quoted above, it is also possible to draw the dual tetrahedron using the same edge directions, so the vertices 134, 156, 236, 245 must consequently also be coplanar as points in C_Q. This concurrence geometry as a whole is not hard to describe. The six planes $N_1,\ldots,N_6$ together with the two derived planes $N_7 = 136, 145, 234, 256$ and $N_8 = 134, 156, 236, 245$ form the face planes of a *Möbius pair* of tetrahedra. The vertices of the first tetrahedron lie on the faces of the second, the vertices of the second on the faces of the first. (To be precise, the vertices 136, 156, 134, 236 of one tetrahedron lie on faces N_7, N_3, N_6, N_1 of the other, while the faces N_8, N_5, N_4, N_2 contain the vertices 245, 234, 256, 145 of the other, respectively.)

Such a configuration is constructed as follows. Choose distinct support planes N_7 and N_8 freely in space, then cut them with two general planes N_1 and N_3. The line $N_1 \cap N_3$ pierces the planes N_7 and N_8. Choose a plane N_4 through the piercing point on N_7 and a plane N_6 through the piercing point on N_8. Finally, choose a plane N_2 through the meeting point of N_3 and N_6 on N_7, and also through the meeting point of N_3 and N_4 on N_8. The meeting points on N_7 of N_1 and N_6, N_4 and N_2, and the meeting points on N_8 of N_1

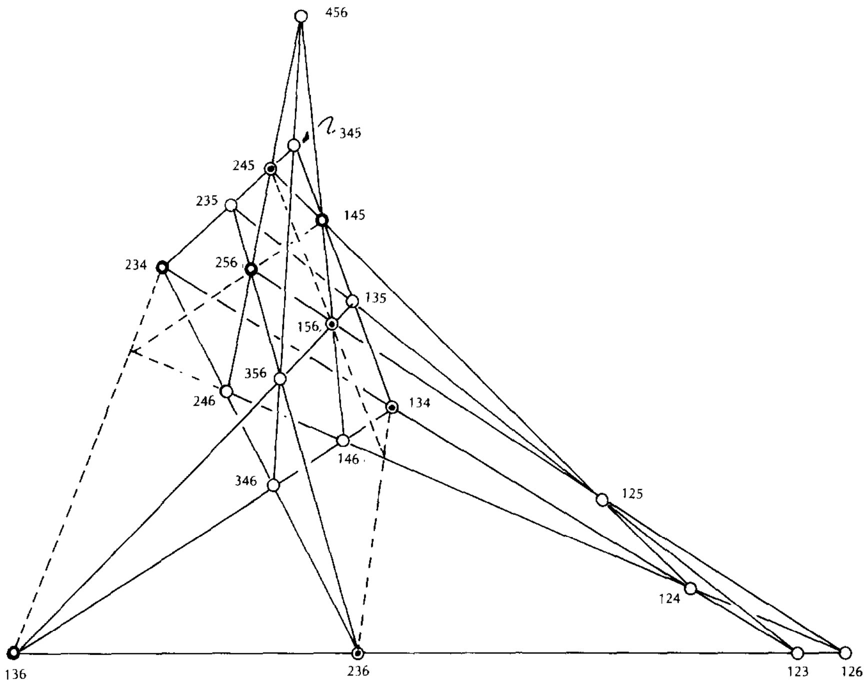


FIGURE 6

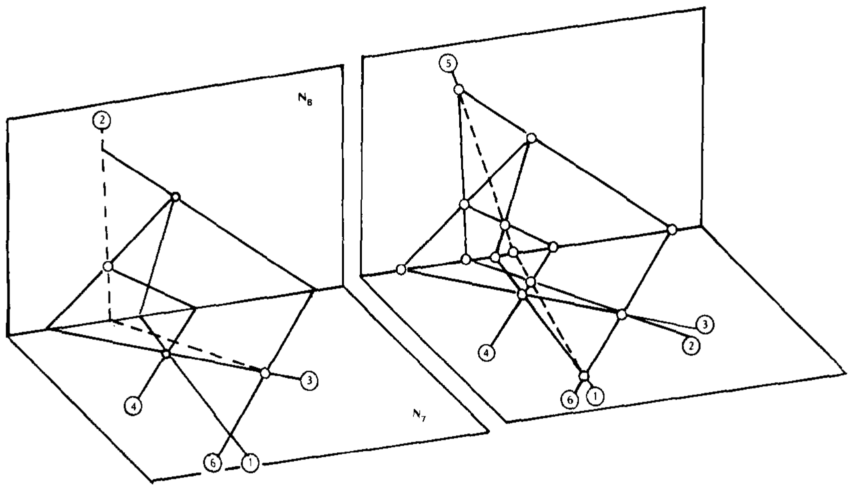


FIGURE 7

and N_4, N_2 and N_6 are all coplanar. We pass N_5 through these four points (Fig. 7).

An equivalent, and perhaps simpler description of these six planes $N_1,\ldots, N_6$ is as follows. In the original tetrahedral drawing, there are three pairs of edges which do not meet at vertices, namely, 12, 35 and 46. Arrange the six indices in cyclic order, taking care to have these three pairs opposite one another in the cycle: (1 3 4 2 5 6), for instance. Then the corresponding cycle of six planes, hinged along the six successive lines of intersection, form an infinitesimally movable ring of plane panels (see Crapo [4]).

5. Coordination, and the Main Theorem

We have yet to prove that drawings, with given directions for their lines, form a geometric lattice when ordered by specialization. We will prove directly that the elements of a concurrence geometry (that is, triples ijk of lines in the drawings) may be represented as vectors $[ijk]$ in an n-dimensional real space (n being the number of lines). In fact we shall show that each drawing corresponds to an n-vector c, which is orthogonal to a vector $[ijk]$ ($c \cdot [ijk] = 0$) if and only if the lines i, j, k are concurrent in the drawing indicated by the vector c. It then follows that the rank of any set of triples equals the vector space rank of the corresponding set of vectors (Cheung and Crapo [3]).

Assume the set of fixed directions of lines for a drawing are given by points $A_1, \ldots, A_n$ on the line at infinity. We may write the projective coordinates of these points in a 2-by-n matrix

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

where each line i has slope b_i/a_i. (There is at least one non-zero entry in each column of the matrix A.) For each 2-by-2 submatrix of A, compute the corresponding *Plücker coordinate*

$$d_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}.$$

Finally, for each triple ijk of directions, we make up an n-vector $[ijk]$ whose only non-zero components are the following: d_{jk} as the ith component, $-d_{jk}$ as jth component and d_{ij} as kth component. These, we shall see, are the correct vector representations of the elements of the concurrence geometry. Thus for six lines we have twenty vectors in 6-dimensional vector space:

	1	2	3	4	5	6
[123]	d_{23}	$-d_{13}$	d_{12}	0	0	0
[124]	d_{24}	$-d_{14}$	0	d_{12}	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮
[456]	0	0	0	d_{56}	$-d_{16}$	d_{45}

Equations of a line L_i with slope b_i/a_i may be written in the form

$$b_i x - a_i y + c_i = 0,$$

for any finite value of c_i. Such a line has x-intercept $-c_i/b_i$. If $b_i = 0$ (for lines parallel to the x-axis) it has y-intercept c_i/a_i. The coordinates $a_i = b_i = 0$ do not occur, so for each direction and each finite value of c_i there is a well-defined line (not at infinity) in the plane.

Any three lines L_i, L_j, L_k have a point in common if and only if the determinant

$$\begin{vmatrix} c_i & c_j & c_k \\ a_i & a_j & a_k \\ b_i & b_j & b_k \end{vmatrix}$$

is equal to zero; that is, if and only if

$$(c_i, c_j, c_k) \cdot (d_{jk}, -d_{ik}, d_{ij}) = 0,$$

if and only if

$$c \cdot [ijk] = 0.$$

We have a *Galois connection* between triples and drawings: each set of triples are concurrent in certain drawings, and each set of drawings have certain common concurrences. For each drawing, regard its vector c as a real-valued function defined on the set T of triples, the value $c([ijk])$ being the inner product $c \cdot [ijk]$. The *zero-set* of such a function c is the set of all triples on which it takes the value 0 (i.e., the set of triples concurrent in that drawing). Since the vectors c form a vector space, we know (Cheung and Crapo [3]) that the set of *intersections of zero-sets* forms a geometric lattice, this being a function-space geometry. Since, moreover, the vectors c form a *real finite-dimensional* space ($\cong R^n$), any intersection of zero-sets is also the zero-set of a single vector c. To see this, say we were given a finite list of vectors c with their various zero-sets. A linear combination of those vectors, using independent transcendentals as coefficients, will produce a single vector which has as zero-set exactly the intersection of the zero-sets of the given vectors. Thus, the zero-sets themselves already form a geometric lattice. Since the order of inclusion on zero-sets is isomorphic to the order of specialization on drawings, we have our main theorem. (We found the rank of the concurrence geometry in Section 3).

THEOREM 1. *Given a collection of n variable lines in fixed directions (not all parallel) in the plane, the set of all drawings which can be made with these n lines, ordered by specialization, forms a geometric lattice of rank* $n-2$. ∎

All the n-vectors $[ijk]$ are orthogonal to two fixed vectors, the n-vectors a and b, the two rows of the matrix A of projective coordinates, of the fixed directions for our variable lines. The concurrence geometry is thus a subgeometry of the orthogonal complement of the 2-dimensional space spanned by a and b. The principal hyperplanes N_i of the concurrence geometry have equations

$$x_i = 0 \qquad \text{and} \qquad a \cdot x = b \cdot x = 0$$

and rank $n-3$ in n-space. The reader may now wish to check that the 1-dimensional subspace generated by the vector $[ijk]$ is exactly the intersection of all hyperplanes N_h for $h \neq i, j$ or k, as we claimed above.

Let us see how some basic properties of concurrence geometries are reflected in this coordination:

(a) The four triples drawn from any quadruple are collinear as elements in the concurrence geometry. For instance, 123, 124, 134 (and 234) are collinear, there being the following linear relation between the first three of these triples:

$$d_{14}[123] - d_{13}[124] + d_{12}[134] = 0.$$

The cancellation giving a 0 in the first coordinate of this vector sum is

$$d_{14}d_{23} - d_{13}d_{24} + d_{12}d_{34} = 0,$$

the p-relation (Hodge and Pedoe [6, Vol. I, p. 309]) which holds between the six 2-by-2 determinants in any 2-by-4 matrix. A similar (or simpler) cancellation holds in each coordinate, and for each set of three of these four triples.

(b) If two of n variable lines are parallel, then all triples containing, along with those two lines, a third line not parallel to those two will be represented in the concurrence geometry by the same *projective point*. Say lines 1 and 2 are parallel, so $d^{12} = 0$. The two vectors

$$\begin{array}{cccccc} [123] & d_{23} & -d_{13} & 0 & \cdots & 0 \\ [124] & d_{24} & -d_{14} & 0 & \cdots & 0 \end{array}$$

are multiples of one another, that is the determinant of the four non-zero entries is zero, because

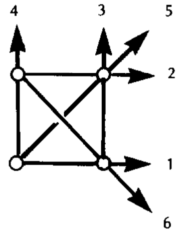
$$-d_{23}d_{14} + d_{13}d_{24} = d_{12}d_{34}$$

(the p-relation) and because $d_{12} = 0$. Note also that this projective point $[123] = [124] = \cdots = [12n]$ is a common point satisfying $x_3 = x_4 = \cdots = x_n = 0$, that is, a point common to the hyperplanes $N_3, \ldots, N_n$.

(c) Say six lines meet the line at infinity in a *harmonic set* of four points,

$$p = 1, 2; \qquad q = 3, 4; \qquad p + q = 5; \qquad p - q = 6.$$

For simplicity, take $p = (1, 0)$ and $q = (0, 1)$. Figure 8 exhibits the tetrahedral drawing one may draw with such a harmonic set of line directions, also the matrix A of projective coordinates of line directions, and finally the coordinates of the four triples used as vertices for the tetrahedron. These four vectors should be dependent, rank 3. We list, in the column to the right of the matrix, the coefficients of one dependence between those four vectors. The coplanarity of these four projective points, plus the fact that N_3 N_4 N_5 N_6 and N_1 N_2 N_5 N_6 both have points in common, due to the parallelism of 1 and 2, 3 and 4, respectively, is sufficient to characterize the concurrence geometry for these six fixed line directions.



	1	2	3	4	5	6	
	1	1	0	0	1	1	= A
	0	0	1	1	1	-1	

	1	2	3	4	5	6	
145	-1			-1	1		1
136	-1		1			1	-1
235		-1	-1		1		-1
246		-1		1		1	1

FIGURE 8

6. APPLICATIONS

The purely geometric problem discussed above arose as a question in statics and mechanics, in the context of a research program devoted to the application of mathematics to architecture and structural engineering. The Structural Topology research group in Montreal has worked to solve a variety of structural and morphological problems by reducing them to answerable questions in pure projective (possibly even combinatorial) geometry. Concurrence geometries in particular answer to a specific need: *to provide a combinatorial geometric representtion for the motion space of a structure.* The construction of a concurrence geometry permits us to associate a geometric lattice of rank m with each plane bar-and-joint structure having $m - 1$ internal (relative) degrees of freedom, and not all its joints collinear.

A plane *bar-and-joint structure* S is a linear graph whose *nodes* (vertices) are in given positions in the plane and whose *bars* (edges) are straight line segments connecting certain pairs of these nodes. Such a structure may be viewed as a mechanical structure composed of rigid bars, joined to one another by universal joints at the nodes.

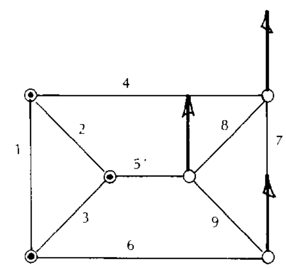
An *infinitesimal motion* v of a structure S is an assignment of vectors to its nodes, such that the difference between vectors assigned to the two ends of any bar is perpendicular to that bar. (This assures that the infinitesimal motion does not stretch or compress any bar.) These conditions being linear, the space V of infinitesimal motions of S form a vector space.

We now establish a connection between bar-and-joint structures, infinitesimal motions and drawings. Given n lines in fixed directions in the plane, and a drawing c satisfying those conditions, we associate with c a structure S whose nodes are the intersection points c_{ij} of pairs of lines in c, and whose bars are intervals (c_{ij}, c_{ik}) *along the lines* L_i, between pairs of intersection points. If h is any drawing with these same fixed directions, we can construct from h an infinitesimal motion of S as follows. Let the velocity v_{ij} of each node c_{ij} be $h_{ij}^{\perp}$, where $a^{\perp} = (-a_2, a_1)$ for any 2-vector $a = (a_1, a_2)$. Then for any bar (c_{ij}, c_{ik}), we have

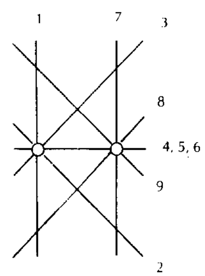
$$v_{ik} - v_{ij} = h_{ik}^{\perp} - h_{ij}^{\perp} = (h_{ik} - h_{ij})^{\perp},$$

which is perpendicular to $c_{ik} - c_{ij}$ because both $h_{ik} - h_{ij}$ and $c_{ik} - c_{ij}$ are in the fixed direction for the line L_i in both drawings. Thus v is an infinitesimal motion of S.

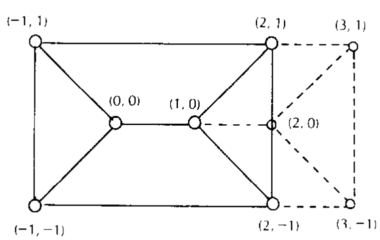
c, v



h



c→d



In drawing C, lines L_i for $i = 1, \ldots, 9$ are given by the equation $b_i x - a_i y + c_i = 0$.

	1	2	3	4	5	6	7	8	9
a	0	1	1	1	1	1	0	1	1
b	1	-1	1	0	0	0	1	1	-1
c	1	0	0	1	0	-1	-2	-1	1
d	1	0	0	1	0	-1	-3	-2	2
h	0	0	0	0	0	0	-1	-1	1
[124]	1	1		-1					
[136]	-1		1			-1			
[235]		-1	-1		2				
[478]				-1			-1	1	
[679]						-1	1		1
[589]					-2			1	1

FIGURE 9

If we are given a structure, not every drawing with the edge directions gives rise to an infinitesimal motion. The reason is that a velocity must be assigned which is consistent at each node, so that if bars ijk are concurrent at a node, the velocities v_{ij}, v_{ik}, v_{jk} will be equal. But this means merely, following the construction in the preceding paragraph, that $h_{ij} = h_{ik} = h_{jk}$. That is, all triples of bars concurrent at nodes of S must be concurrent in h. We call such a drawing an *S-drawing.*[2] Conversely, for any S-drawing h, $h^{\perp}$ is an infinitesimal motion. In Fig. 9 we show an infinitesimal motion v of a projected triangular prism, and the associated drawing h with $h^{\perp} = v$.

For any plane structure S with n bars in directions given by a 2-by-n matrix A, the set of S-drawings, ordered by specialization, forms an *interval* $[\bar{S}, 1]$ in the geometric lattice of drawings with those line directions, a *quotient* of the concurrence geometry $C(A)$. Here $\bar{S}$ is the closed set of concurrences which follow from the incidence of bars at nodes, and from the directions given by the matrix A. Certain S-drawings can be obtained as parallel redrawings of S in which certain bars have been *contracted* to length zero. Figure 10 shows the lattice of *contractions* of a plane structure (a projected triangular prism). Observe that the concurrence shown with broken lines *is* a triple in $\bar{S}$, as a consequence of the vertex concurrences and of two quadrilateral sets of directions. The drawing has rank 5, so we expect an interval of rank 2 before reaching the top at rank $n - 2 = 7$.

THEOREM 2. *The set of S-drawings of a plane bar-and-joint structure S is isomorphic to the space of infinitesimal motions of S.*

By Theorem 2, we know that for any plane structure S there is a relationship between the rank of the lattice of contractions and the dimension of the space of infinitesimal motions of the structure. These numbers always differ by 2, because there is a 2-dimensional subspace of *translations* which contributes to the dimension of the motion space, but not to the rank of the lattice. Two independent translations (translation by $(-1, 0)$ or by $(0, -1)$) are given by the drawings $h = a$ (all lines through $(0, 1)$) and $h = b$ (all lines through $(-1, 0)$), respectively, where a and b are the two rows of the matrix A which lists the directions of the lines L_i. There is a third independent *rigid motion*, namely, rotation about the origin, given by the vector $h = c$, where c is the drawing of the structure itself. Modulo the subspace of rigid motions, we have the space of *internal* (or *relative*) motions of S. If this space has rank r, we say the structure has *r internal degrees of freedom.*

[2] This treatment differs slightly from the connection between infinitesimal motions and parallel redrawings, familiar to readers of books on statics. If c is a drawing of a structure S, and if d is a parallel redrawing of c, let $h = d - c$. Then if $[ijk]$ is a vertex concurrence in S, so $c \cdot [ijk] = 0$, note that $d \cdot [ijk] = 0$ if and only if $h \cdot [ijk] = 0$. That is, those difference drawings h which are S-drawings yield exactly the parallel redrawings d of S.

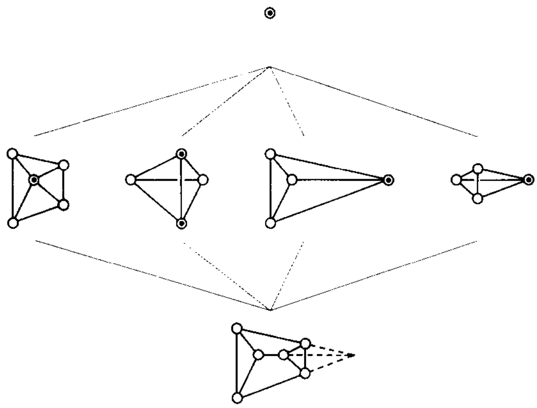


FIGURE 10

THEOREM 3. *If the lattice of contractions of a plane bar-and-joint structure S has rank m, then the structure S has $m-1$ internal degrees of freedom.*

Proof. In the Galois connection between triples and drawings, we have a cover-preserving isomorphism between the lattice of closed sets of triples and the inverted lattice of closed subspaces of drawings. So if a closed set of triples has rank d, the dimension of its drawing space is $n-d$. Since the rank of the overall concurrence geometry (for n lines in the directions of the edges of the structure S) is $n-2$, we know the lattice interval $[\bar{S}, 1]$ is of rank $n-2-d$, a number exactly 2 less than the rank of the drawing space. The lattice of contractions has the same rank $n-2-d$, being a *spanning* subgeometry of the interval $[\bar{S}, 1]$. By Theorem 2, this drawing space is isomorphic to the space of infinitesimal motions of S. Taking the quotient of this space modulo the subspace of rigid motions (a reduction in rank by 3), we have the required result. ∎

Observe that if the graph in Fig. 10 had been drawn in such a way as to destroy the concurrence indicated with the broken lines, its only possible specialization would be the one-point drawing. In this case, $m=1$ and there is no internal degree of freedom.

7. CONCURRENCE IN HIGHER DIMENSIONS

We describe concurrence geometries of higher-dimensional figures briefly as follows. In a k-dimensional projective space, fix one hyperplane H. On that hyperplane, which has rank k, select certain flats A_j of rank $k-1$ to be the intersections $A_j = H_j \cap H$ of variable hyperplanes on the fixed hyperplane H. As in the case $k=2$ discussed above, each variable hyperplane H_i has one degree of freedom of motion, subject to the intersection conditions, and its position can be determined by a single scalar c_i. The n-vector c determines a specific *configuration* of hyperplanes (a *drawing*, for $k=2$). The concurrence of $k+1$ hyperplanes is given by the condition $c \cdot f = 0$ where f is an n-vector having non-zero entries only in the coordinates corresponding to those $k+1$ hyperplanes. These possibly non-zero entries are again Plücker coordinates drawn from the k-by-n matrix A which coordinatizes the intersections A_i on H.

The details are as follows. Let H be the hyperplane at infinity in k-dimensional projective space. If the hyperplanes H_j are given by the equations

$$a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{kj}x_k = c_j$$

with c_j variable, then each variable hyperplane H_j has a fixed intersection with the hyperplane H at infinity, an intersection indicated by the jth column of the matrix $A = (a_{ij})$.

For any configuration of hyperplanes H_j satisfying the intersection conditions of H, and given by a vector c, let A' be the $(k+1)$-by-n matrix formed by adding c to A as a $k+1$st row. A set of $k+1$ hyperplanes H_j meet at a point (are *concurrent*) in configuration c if and only if the corresponding $k+1$ columns of A' have determinant zero. This is the case if and only if the vector c is orthogonal to a vector whose jth entry is non-zero only if the jth hyperplane is one of the selected hyperplanes, in which case the jth entry, up to an alternating sign, is the determinant of the k-by-k matrix formed from the selected columns of the matrix A, omitting the jth column (the cofactor of c_j in the determinant expansion). In the resulting Galois connection between $(k+1)$-tuples of hyperplanes H_j and configurations c, we have

THEOREM 4. *Given a collection of n variable hyperplanes with fixed intersections with a fixed hyperplane H (intersections whose union does not lie in any proper subflat of H), the set of all configurations which can be made with those n hyperplanes, ordered by specialization, is a geometric lattice of rank $n-k$.*

Proof. We check the rank, the rest having been proven in the preceding paragraphs. If $H_1,\ldots, H_k$ are hyperplanes with intersections $A_1,\ldots, A_k$ whose union is not in any proper subflat of H, then the concurrences

$$[1,\ldots, k, k+1], [1,\ldots, k, k+2],\ldots, [1,\ldots, k, n]$$

form a basis for the concurrence geometry, which consequently has rank $n-k$. ∎

A further generalization of these ideas results from the Laplace expansion of the determinant of an $n+p$-by-$n+p$ matrix in terms of complementary minors of size p by p and n by n. This algebra arises in the concurrence problem where we replace the fixed hyperplane H by a fixed flat H of corank p, letting the A_i be fixed intersections of variable hyperplanes H_i with the fixed flat H. $(k+p)$-tuples of hyperplanes are then coordinatized by $\binom{n}{p}$-vectors whose non-zero entries are Plücker coordinates drawn from matrices of size k by $k+p$. We shall not deal further with this generalization in the present paper.

THEOREM 5. *If n variable hyperplanes H_i in k-dimensional space have fixed intersections $A_i = H_i \cap H$ which are in general position on a fixed hyperplane H, the concurrence geometry of this intersection pattern consists of n hyperplanes N_i on general position in a space of projective dimension $n-k$, all points of intersection of those hyperplanes and all joins of those points.*

Proof. This result follows easily from Theorem 7, in the next section. We give such a proof here, and ask the reader to look back at it later. $S(A)$ is the geometry of n points in general position, rank k, $S^*(A)$ the geometry of n points (or better, hyperplanes) in general position, rank $n-k$. Its adjoint, $C(A)$, is thus isomorphic to the set of joins of points of intersection of those n hyperplanes. ∎

This concurrence geometry is isomorphic to the Dilworth completion $T_k(B_n)$ of the k-times lower truncated Boolean algebra B_n. Proofs of this and of several related theorems appear in (Crapo [5]).

Just as in the previously considered case of drawings with fixed line directions in the plane, special positions of the intersections A_i result in specializations of the concurrence geometry. The simplest of these specializations is as follows.

THEOREM 6. *A set of k fixed intersections, say, $A_1,\dots, A_k$, are concurrent in the hyperplane H if and only if the hyperplanes $N_{k-1},\dots, N_n$ (with the complementary set of indices) are concurrent in the concurrence geometry.*

Proof. $H_1,\dots, H_k$ have concurrent intersections on the hyperplane H if and only if the first k columns of the matrix A have determinant zero. That determinant, the Plücker coordinate $d_{1\dots k}=0$, occurs as the jth coordinate of all $(k+1)$-tuples $[1,\dots, k,j]$ for j such that $k+1 \leqslant j \leqslant n$. These points (which shall turn out to be equal!) all lie on the intersection of hyperplanes $N_{k+1},\dots, N_n$ in the concurrence geometry.

We prove that $[1,\dots, k, s]$ and $[1,\dots, k, t]$ are the same projective point, for any indices s and t with $k+1 \leqslant s < t \leqslant n$. Only the first k coordinates of these points can be non-zero, because $d_{1\ldots k}=0$. Choose any two indices p, q such that $1\leqslant p<q\leqslant k$. The 2-by-2 matrix of pth and qth coordinates of the two points has determinant zero, that is,

$$d_{1\ldots\hat{p}\ldots ks}d_{1\ldots\hat{q}\ldots kt}-d_{1\ldots\hat{q}\ldots ks}d_{1\ldots\hat{p}\ldots kt}=0$$

because of the quadratic p-relation $F_{Kp/Kqst}=0$, K being the sequence of indices from 1 to k, omitting p and q, together with the fact that $d_{1\ldots k}=0$.

A quick look at the coordinates of the $n-k$ $(k+1)$-tuples $[1,\ldots,k,j]$ for j such that $k+1\leqslant j\leqslant n$ will reveal that they are *distinct*, even *independent*, points if $d_{1\ldots k}\neq 0$. Since, for each j, $[1,\ldots,k,j]$ is the intersection of hyperplanes $N_{k+1},\ldots,\hat{N}_j,\ldots,N_n$, the hyperplanes $N_{k+1},\ldots,N_n$ are also independent, and have no common point. The converse is proven. ∎

8. Concurrence Geometries as Geometries of Circuits

In previous sections we have detected certain relationships between the pattern of fixed intersections A_i on a fixed hyperplane H and the resulting concurrence geometry. In this final section we establish a conceptual link between the vector geometry $S(A)$ given by the columns of the matrix A and the concurrence geometry $C(A)$. We shall show that $C(A)$ *is an adjoint of the orthogonal geometry* $S^*(A)$. More simply, it is the *geometry of circuits of* $S(A)$.

We define the adjoint of a representation of a vector geometry. The lattice of flats of a vector geometry can be equally well represented in the lattice of subspaces of a vector space either right-side-up as a set of 1-dimensional subspaces (projective points) and all their joins, or upside-down as a set of hyperspaces (projective hyperplanes) and all their meets. Without loss of generality, we may take the vector space to have dimension equal to the rank of G. Corresponding to each copoint of G we have a hyperplane intersection which is a *projective point*. The set of all such points itself forms a vector geometry, the *adjoint* G' of the vector geometry G in that representation. In this chapter we shall use this inverted form for the construction of adjoints: G will be given as a geometry of points in a projective space P, G' as the geometry of intersections *in* P of sets of hyperplanes of G.

Let G' be an adjoint of a vector geometry G in some representation. Then the lattice of flats of G is naturally embedded one-to-one and upside-down in the lattice of flats of G', in such a way that joins and the strict cover relation in G are preserved as meets and strict cover in G', with $0\to 1$ and $1\to 0$.[3]

[3] Cheung [2], introducing the notion of the adjoint of a geometry, showed that certain (non-representable) combinatorial geometries do not have adjoints, in the sense that they cannot be represented as geometries of hyperplane intersections in other combinatorial geometries.

In the case in question, where $C(A)$ will be shown to be an adjoint $(S^*(A))'$, the points of $S^*(A)$ are represented as certain subspaces covering $R(A)$, the row space of A, in the interval $[R(A), 1]$ of the lattice of subspaces of the n-dimensional vector space of configurations. These subspaces will in turn be the major hyperplanes of the adjoint $C(A)$, the hyperplanes N_i noted in several previous examples. (We see now that these hyperplanes should be indexed by the *points* of $S^*(A)$, not necessarily by single indices $1,..., n$.) This result, that $C(A)$ is simply the set of meets of joins of certain easily identifiable subspaces, will enable us more systematically to draw conclusions concerning relations between patterns of fixed intersections and their corresponding concurrence geometries. (See Theorems 5 and 6, above, for instance.)

As a next step, we consider representations of the orthogonal geometries $S(A)$, $S^*(A)$. We have assumed throughout that the intersections A_i do not themselves have a common point, i.e., the k rows of the matrix A are independent. We can easily complete A to a square (and non-singular) matrix by adding $n-k$ rows which are not only independent, but also orthogonal to all the rows of A. We call the added $n-k$-by-n matrix $A^{\perp}$. The vector geometry $S(A)$ given by columns of A is, on one hand, isomorphic to the geometry whose flats are the intersections (in rank n) of various sets of basic hyperplanes $x_j = 0$, with the row space of A. This geometry is also isomorphic to the geometry of standard basis vectors e_j (jth component equal to 1, the rest equal to 0) modulo the row space of the matrix $A^{\perp}$. By interchanging the roles of A and $A^{\perp}$ in the above discussion, we obtain three equivalent formulations of the orthogonal geometry $S^*(A) = S(A^{\perp})$.

In previous sections of this article we were accustomed to thinking of the major hyperplanes N_j of the concurrence geometries as certain large sets of concurrences: in certain cases simply as those concurrences not involving the jth hyperplane. (This is almost correct: on such hyperplanes are all those $(k+1)$-tuples which are *always* concurrent, together with those disjoint from a given point of $S^*(A)$.) Across the Galois connection between $(k+1)$-tuples and configurations, we find another interpretation of N_j: it is the set of configurations where all hyperplanes except possibly the jth go through a single point. This space of configurations is generated by a single configuration, where all hyperplanes except the jth go through the *origin*. The vector of this configuration is just the jth standard basis vector e_j.

For every subset $E' \subseteq \{1,..., n\}$, the corresponding subset of this standard basis, taken together with the row space $R(A)$, generates a subspace $\sigma(E)$ of R^n. Since the orthogonal geometry $S^*(A)$ is isomorphic to the geometry of standard basis vectors modulo the row space $R(A)$, the ordered set

$$\{\sigma(E) : E \subseteq \{e_1,..., e_n\}\}$$

is isomorphic to the lattice of flats of $S^*(A)$. We also know that, under the Galois connection, the lattice of flats of the concurrence geometry $C(A)$ has an inverted strong map embedding in this same lattice of subspaces of the n-dimensional vector space, with $0 \to 1$ and $1 \to R(A)$. To show that $C(A)$, thus represented, is adjoint to $S^*(A)$, it will suffice to show

(1) that for any subset E of the indices $1,..., n$, the subspace $\sigma(E)$ is closed in the Galois connection between $(k+1)$-tuples and configurations of hyperplanes, and

(2) that single $(k+1)$-tuples (those for which the corresponding intersections A_j have no common point) have as images under the Galois connection precisely the maximal proper closed subspaces of the form $\sigma(E)$, the hyperplanes of the geometry $S^*(A)$.

THEOREM 7. *The concurrence geometry $C(A)$ for a pattern of intersections given by a k-by-n matrix A of rank k is isomorphic to the geometry of circuits of $S(A)$, $S(A)$ being the vector geometry of columns of A, and is thus adjoint to the orthogonal geometry $S^*(A)$.*

Proof. We carry out the two steps listed just before the statement of the theorem.

(1) For any subset E of the standard basis, the subspace $\sigma(E)$ may be written as a join $\rho(\bar{E}) \vee R(A)$, where $\bar{E}$ is the closure of the subset E in the vector geometry $S^*(A)$, and where $\rho(\bar{E})$ is the subspace of n-space generated by the vectors e_j whose indices are in the subset $\bar{E}$. Note that $\sigma(E) = \sigma(\bar{E})$, so we need prove only that $\sigma(E)$ is closed in the Galois connection when E is closed in $S^*(A)$, in which case $\sigma(E) = \rho(E) \vee A$. Now, $\rho(E)$ is the space of configurations in which all hyperplanes H_j for $j \notin E$ pass through the origin, and, assuming E is closed in $S^*(A)$, $\sigma(E)$ is the space of configurations in which those same hyperplanes pass through a point (not necessarily the origin).

Let $c(E)$ be a general configuration in which hyperplanes H_j for $j \notin E$ pass through a point. The image of $\{c(E)\}$ under the Galois connection consists precisely of those $(k+1)$-tuples T of hyperplanes H_j such that $T \backslash E$ is dependent in $S(A)$. For any index $i \notin E$, i is an element of a circuit C disjoint from E. The broken circuit $C \backslash \{i\}$ can be extended to a basis; let T be that basis, together with the index i. This $(k+1)$-tuple is concurrent in $c(E)$ because the hyperplanes $\{H_j, j \in C\}$ have an intersection of corank $|C| - 1$. T is *not* concurrent in the configuration $c(\{i\})$, because the intersection of all hyperplanes with indices in $T \backslash \{i\}$ is exactly the common point through which all hyperplanes H_j $(j \neq i)$ pass, and H_i does not pass through that point. Thus $\sigma(E)$ is a closed space of configurations.

(2) For the second part of this proof, we must show that the points of

the concurrence geometry $C(A)$ match the copoints (hyperplanes) of the geometry $S^*(A)$, that is, they are embedded as exactly the same set of hyperspaces of the vector space of configurations. Now the complements of copoints of $S^*(A)$ are the circuits of $S(A)$. For each circuit F of $S(A)$, let $c(F)$ be the configuration in which hyperplanes $\{H_j; j \in F\}$ have an intersection of corank $|F| - 1$. (Such a configuration is given by a vector with components 0 on F, general off F.) Certain $(k+1)$-tuples D are concurrent in every configuration of hyperplanes H_j, and have image equal to "1" under the Galois connection. (They have rank 0 in the concurrence geometry.) These are the $(k+1)$-tuples D for which the corresponding intersections $\{A_j; j \in D\}$ have a point in common on the fixed hyperplane H, and the corresponding columns D of the matrix A have rank $<k$. For all *other* $(k+1)$-tuples D, and for each circuit F of $S(A)$, the $(k+1)$-tuple D is concurrent in the configuration $c(F)$ if and only if D contains the set F. We claim that the configurations $c(F)$ are exactly the elementary specializations of the zero configuration, so the sets $\{D; |D| = k+1 \text{ and } D \supseteq F\}$ are closed sets of $(k+1)$-tuples, precisely the points of the concurrence geometry.

To see this, let D be any $(k+1)$-tuple of rank 1 (not 0) in the concurrence geometry. Then the columns D of the matrix A have rank k. Since D has $k+1$ elements, there is a *unique* circuit F of $S(A)$ within the set D. The point generated by the singleton $\{D\}$ in the concurrence geometry consists of those $(k+1)$-tuples E which are concurrent whenever D is concurrent. Since D is concurrent in the configuration $c(F)$, only those $(k+1)$-tuples containing F can be in the closure of $\{D\}$. Whenever D is concurrent, say, at a point q, the join of q with the intersection of $\{A_i; i \in F\}$ has rank $|F| - 1$, so any $(k+1)$-tuple E containing F will also be concurrent. This completes the proof. ∎

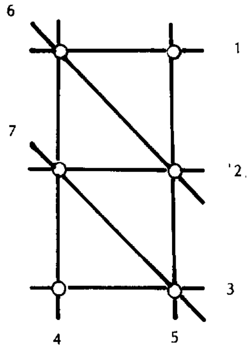
We provide an example to illustrate the concepts involved in the preceding proof. In Fig. 11 we show a general configuration for seven lines in fixed directions in the plane, together with the coordinatization matrix A for those fixed directions of lines. The circuits of the geometry $S(A)$ are

(i) all pairs of coincident directions: 12, 13, 23, 45, 67;

(ii) all triples of distinct directions: 146, 147,..., 357.

For a circuit F such as 146, the triples 146 and 123 are concurrent in the drawing $c(F)$, 123 because it has rank $< k = 2$ in $S(A)$, and 146 because it contains F. For a smaller circuit G, such as $G = 67$, the drawing $c(G)$ is special only in that lines 6 and 7 *coincide*, so the triples 167, 267,..., 567 and 123 are concurrent.

Another example, with enough degeneracies to make the combinatorics interesting, is given by the pattern of intersections in Fig. 12. The geometry $S(A)$ has four points in a plane, three of them (1, 2 and 345) collinear.

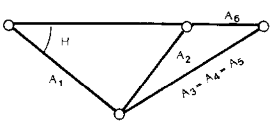


1	2	3	4	5	6	7
1	1	1	0	0	1	1
0	0	0	1	1	−1	−1

FIGURE 11

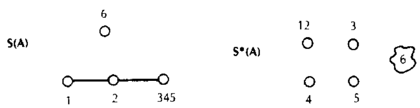
The orthogonal geometry $S^*(A)$ has one element, 6, which is in the closure of the empty set, and four points (126, 36, 46 and 56). The concurrence geometry has six elements in the closure of the empty set: 1234, 1235, 1245, 1345, 2345 and 3456, these sets of four hyperplanes being concurrent in all configurations with section A. The points of the concurrence geometry are the circuits of $S(A)$; 123, 124, 125, 34, 35 and 45. There are no spanning circuits (the element 6 being an *isthmus* in $S(A)$), so

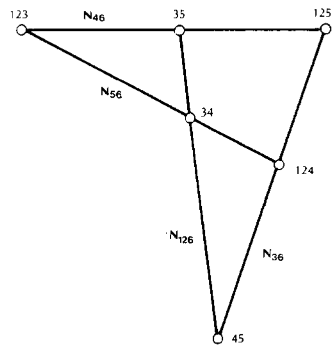
the pattern of intersections



the matrix A

1	2	3	4	5	6
1	1	1	1	1	0
0	1	−1	−1	−1	0
1	1	1	1	1	1





points of C(A)
=circuits of S(A)
123, 124, 125,
34, 35, 45.

major hyperplanes of C(A),
indexed by points of S*(A)
N_{126}, N_{36}, N_{46}, N_{56}.

4 - tuples which are always concurrent :
1234, 1235, 1245, 1345, 2345, 3456

	1	2	3	4	5	6
1236 generates point 123, with coordinates	−2	1	1	0	0	0
1246 " " 124, " "	−2	1	0	1	0	0
1256 " " 125, " "	−2	1	0	0	1	0
1346, 2346 each generate point 34 :	0	0	1	−1	0	0
1356, 2356 " " " 35 :	0	0	1	0	−1	0
1456, 2456 " " " 45 :	0	0	0	1	−1	0

FIGURE 12

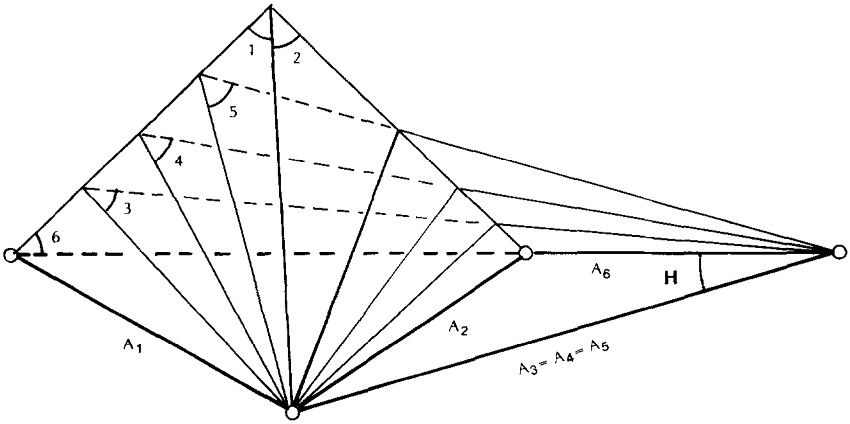


FIGURE 13

none of the points of $C(A)$ is a 4-tuple! Each point consists of either seven of eight 4-tuples. Every point contains the six 4-tuples in the closure of the empty set, and is then generated by one or by each of more than one additional 4-tuple, as listed in Fig. 12. A drawing of the "zero" configuration appears as Fig. 13. The points of $C(A)$ are the elementary specializations of this figure. In the configuration for point 123, plane H_3 is made to pass through the intersection point of planes H_1, H_2 and H_6. In the configuration for point 34, planes H_3 and H_4 are made to coincide.

In conclusion, we note that concurrence geometries also solve another longstanding problem in combnatorial geometry: to associate with each vector geometry G a combinatorial geometry $C(G)$ whose points are the *circuits* of G. Applications of this construction to the representation theory for matroids, and in the combinatorial investigation of invariant theory and homological algebra, are particularly intriguing. These matters are taken up again in more detail in (Crapo [5]).

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