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Asymptotics of resolvent integrals: The suppression of crossings for analytic lattice dispersion relations

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Abstract

We study the so-called crossing estimate for analytic dispersion relations of periodic lattice systems in dimensions three and higher. Under a certain regularity assumption on the behaviour of the dispersion relation near its critical values, we prove that an analytic dispersion relation suppresses crossings if and only if it is not a constant on any affine hyperplane. In particular, this applies to any dispersion relation which is an analytic Morse function. We also provide two examples of simple lattice systems whose dispersion relations do not suppress crossings in the present sense.

Résumé

On étudie la borne de croisement pour des relations de dispersion analytiques de systèmes sur réseaux périodiques en dimension $d \geqslant 3$. En supposant une certaine régularité de la relation de dispersion au voisinage des valeurs critiques, on démontre qu'une relation de dispersion analytique élimine les contributions de croisements si et seulement si elle n'est pas constante sur n'importe quel hyperplan affine. C'est le cas si la relation de dispersion est une fonction de Morse analytique. Enfin, on présente deux exemples de systèmes simples sur réseau pour lesquels la relation de dispersion n'élimine pas les croisements au sens défini ici.

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1. Introduction

Time-dependent perturbation theory has proven to be a useful tool in studying the behaviour of systems where a free, wave-like, evolution in three dimensions is perturbed by a weak random potential. An important set of tools for rigorous estimation of such a perturbation series was developed by Erdős and Yau in [1] to study the kinetic limit of the random Schrödinger evolution. These methods have later been extended to cover also the low density limit of the random Schrödinger evolution [2], as well as the kinetic limits of an electron coupled to a phonon field [3], of the discrete random Schrödinger equation (the Anderson model) [4,5], and of certain discrete wave equations with a

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weak, random mass-disorder [6]. There is also a recent, remarkable result where the methods have been reworked to allow going beyond the kinetic time-scales for the continuum and discrete random Schrödinger evolutions [7–9].

An important element in all of these results is an estimate proving that all so called *crossing graphs* are suppressed. For the discrete random Schrödinger equation this was proven in [4] by showing that for every sufficiently small $\beta > 0$,

$$\sup_{\alpha\in\mathbb{R}^3,\ k_0\in\mathbb{T}^3} \int\limits_{(\mathbb{T}^3)^2} \mathrm{d}k_1\,\mathrm{d}k_2\, \frac{1}{|\alpha_1-\omega(k_1)+\mathrm{i}\beta||\alpha_2-\omega(k_2)+\mathrm{i}\beta|}\frac{1}{|\alpha_3-\omega(k_1-k_2+k_0)+\mathrm{i}\beta|} \leqslant c_1\langle\ln\beta\rangle^{n_1}\beta^{\gamma-1}. \tag{1.1}$$

Here $\omega(k)=\sum_{\nu=1}^3(1-\cos(2\pi k_\nu))$ is *the dispersion relation* of the free discrete Schrödinger equation, and $c_1>0$, $n_1\geqslant 0$ and $\gamma>0$ are constants depending only on the function ω.

We call (1.1) the *crossing estimate*. The validity of the corresponding estimate in the earlier continuum Schrödinger case (when $\omega(k)=\frac{1}{2}k^2$, $k\in\mathbb{R}^3$) was fairly straightforward to prove, but the proof turned out to be involved in the discrete case, $\omega(k)=\sum_{\nu=1}^3(1-\cos(2\pi k_\nu))$. There are now two independent proofs of this result: the bound in (1.1) was shown to hold with $\gamma=1/5$ and $n_1=2$ in Lemma 3.11 of [4] and with $\gamma=1/4$ and $n_1=6$ in Appendix A.3 of [7]. The case of more general dispersion relations ω is not covered by the earlier results. However, in a very recent preprint by Erdős and Salmhofer [10] the related "four denominator estimate", which involves four resolvent terms instead of three and which was required in [7], has been studied using an approach different from ours.

For very small β, each of the factors in (1.1) is sharply concentrated around some level set of ω. However, the arguments of ω in the factors are not allowed to vary independently of each other, and the magnitude of the integral for small β is thus determined by the overlap of the different level sets depending on the constants α_j. Therefore, to prove (1.1) it will be necessary to consider the worst case scenario for the level sets, and then try to estimate the overlap between the three levels sets as k_1 and k_2 are varied.

However, it is not obvious how to carry out such an argument in the general setup. This raises the question: for what kind of dispersion relations ω is it possible to derive the estimate (1.1)? This question is particularly relevant in the context of microscopic models for lattice vibrations in a crystal where the dispersion relation is determined by the elastic couplings, and can be fairly arbitrary (we refer to the survey [11], and for a related mathematical treatment of the purely harmonic system to [12], for further details on the topic). In an earlier work [6], where the perturbation methods were applied to a simplified model of the lattice vibrations, the estimate (1.1) was in fact elevated to an assumption, denoted by (DR4) in the paper.

Here our main aim is to show that the technical assumption (DR4) of the earlier work [6] can be replaced by a much simpler geometric condition. However, the methods used here should have wider applications in analysis involving time-dependent perturbation expansions or relying on resolvent techniques. We will introduce the problem in detail and present the main results in Section 2, with the main notations collected to Section 2.1. Before proceeding to the more involved proof of validity of the crossing estimate, we first prove the converse and discuss a few counterexamples in Section 3. The proofs of the main theorems have been divided into Sections 4–6. Section 4 collects the main technical lemmas, with some of the more well-known details being reproduced for the sake of completeness in Appendices A and B. We prove in Section 5 that the technical assumption made about the nature of the set of singular points of the dispersion relation leads to a property similar to the usual dispersivity. To show that the assumptions are fairly general, we have also included in Appendix C a proof which shows that real-analytic Morse functions are covered by the main theorems. The proof of the suppression of crossings is the content of Section 6, where the first part gives a certain uniform estimate on the minimal curvature of the level sets of ω, and the second part exploits this to provide for the extra decay of the crossing integral.

2. Main results

Let us call a dispersion relation ω *semi-dispersive*, if the integral over the modulus of its resolvent diverges at most logarithmically, that is, if there are $c_0\in\mathbb{R}_+$ and $n_0\in\mathbb{N}$ such that for all $0<\beta\leqslant 1$, and $\alpha\in\mathbb{R}$,

$$\int\limits_{\mathbb{T}^d}\mathrm{d}k\,\frac{1}{|\alpha-\omega(k)+\mathrm{i}\beta|}\leqslant c_0\langle\ln\beta\rangle^{n_0}. \tag{2.1}$$

We will be here mainly interested in real-analytic dispersion relations which have this property. We aim at proving (1.1), and thus we need to consider the "three-resolvent[1] crossing integrals" defined by:

$$I_{3\mathrm{cr}}(\alpha, k_0, \beta) = \int\limits_{(\mathbb{T}^d)^2} \mathrm{d}k_1\, \mathrm{d}k_2\, \frac{1}{|\alpha_1 - \omega(k_1) + \mathrm{i}\beta||\alpha_2 - \omega(k_2) + \mathrm{i}\beta|} \frac{1}{|\alpha_3 - \omega(k_1 - k_2 + k_0) + \mathrm{i}\beta|}, \tag{2.2}$$

for $\alpha \in \mathbb{R}^3$, $k_0 \in \mathbb{T}^d$ and $0 < \beta \leqslant 1$. For any semi-dispersive ω, we immediately obtain a bound for the integral by estimating the third factor trivially by $1/\beta$, which yields:

$$\sup_{\alpha, k_0} I_{3\mathrm{cr}}(\alpha, k_0, \beta) \leqslant c_0^2 \langle \ln \beta \rangle^{2n_0} \beta^{-1}. \tag{2.3}$$

We call this the *basic estimate*. We shall say that the *dispersion relation suppresses crossings*, if it is possible to improve the basic estimate by some positive power of β, i.e., if there are constants $\gamma > 0$, $c_1 \in \mathbb{R}_+$, and $n_1 \in \mathbb{N}$ such that

$$\sup_{\alpha, k_0} I_{3\mathrm{cr}}(\alpha, k_0, \beta) \leqslant c_1 \langle \ln \beta \rangle^{n_1} \beta^{\gamma - 1}. \tag{2.4}$$

We note that this implies, in particular, that $\sup_{\alpha, k_0}(\beta I_{3\mathrm{cr}}(\alpha, k_0, \beta)) \to 0$ when $\beta \to 0^+$.

The following collects the precise assumptions made about ω here.

Assumption 2.1. Let $d \geqslant 3$, and let $\omega : \mathbb{R}^d \to \mathbb{R}$ be real-analytic and $\mathbb{Z}^d$-periodic. Define for all $s > 0$,

$$f_\omega(s) = \int\limits_{\mathbb{T}^d} \mathrm{d}k\, \frac{1}{|\nabla \omega(k)|^3} \mathbb{1}\big(|\nabla \omega(k)| \geqslant s\big). \tag{2.5}$$

We assume that there are $p_0, c_0 \geqslant 0$ such that for all $s > 0$,

$$f_\omega(s) \leqslant c_0 \langle \ln s \rangle^{p_0}. \tag{2.6}$$

Since obviously $f_\omega(s) \leqslant s^{-3}$, the assumption is in reality only about the nature of the singularity of the integrand near the set of singular points of ω, i.e., about the behaviour of ω near the points k for which $\nabla \omega(k) = 0$.

The first of the following theorems, Theorem 2.2, proves that every such ω is semi-dispersive with $n_0 = 1$. In particular, this is the case for every real-analytic Morse function ω on $\mathbb{T}^d$, and we have included a proof of this property in Appendix C. In the assumptions, for $d = 3$ we then need to take $p_0 = 1$, otherwise $p_0 = 0$ suffices. In the second theorem, Theorem 2.3, we present a simple geometric classification of whether such a dispersion relation suppresses crossings or not.

Theorem 2.2. *Let Assumption* 2.1 *be satisfied. Then for every* $0 \leqslant p \leqslant 1$ *there is a constant* C_p *with the following property*: *for all* $\alpha \in \mathbb{R}$, $0 < \beta \leqslant 1$, *if* $0 \leqslant p < 1$,

$$\int\limits_{\mathbb{T}^d} \frac{\mathrm{d}k}{|\nabla \omega(k)|^p} \frac{1}{|\alpha - \omega(k) + \mathrm{i}\beta|} \leqslant C_p \langle \ln \beta \rangle, \tag{2.7}$$

and, if $p = 1$,

$$\int\limits_{\mathbb{T}^d} \frac{\mathrm{d}k}{|\nabla \omega(k)|} \frac{1}{|\alpha - \omega(k) + \mathrm{i}\beta|} \leqslant C_1 \langle \ln \beta \rangle^{p_0 + 2}. \tag{2.8}$$

Theorem 2.3. *Let Assumption* 2.1 *be satisfied. Then* ω *suppresses crossings if and only if it is not a constant on any affine hyperplane.*

[1] This is to distinguish the estimate from the related integral involving four resolvent factors which was needed in [7] for the analysis going beyond the kinetic regime.

Thus, we can now conclude that there is a large class of functions for which the main theorem in [6] is satisfied:

Corollary 2.4. *If $\omega : \mathbb{T}^3 \to \mathbb{R}$ is a Morse function, whose periodic extension to $\mathbb{R}^3$ is real-analytic and the extension is not a constant on any affine hyperplane, then it satisfies the assumptions* (DR3) *and* (DR4) *of* [6].

The property called (DR3) was already shown to be valid for Morse functions in [6]. We have included it in the corollary only to allow for easier use of the result. With some effort, it should now also be possible to generalize the results about the Anderson model [4] accordingly to more general dispersion relations.

2.1. Notations

We use the standard notations S^d and $\mathbb{T}^d$ for the d-dimensional unit sphere and the unit torus, respectively. S^d is the surface of the unit ball in $\mathbb{R}^{d+1}$, with the topology and metric inherited from it, and $\mathbb{T}^d$ is identified with the topological space $\mathbb{R}^d/\mathbb{Z}^d$. We denote the equivalence class mapping $\mathbb{R}^d \to \mathbb{T}^d$ by $[\cdot]$, and its inverse on $(-1/2, 1/2]^d$ by $[\cdot]'$. The topology of the torus is then compatible with the metric d_{T} defined by $d_{\mathrm{T}}([y],[x]) = \min_{n\in\mathbb{Z}^d} |y-x+n| = |[[y-x]]'|$. Let us also remark that, in general, we do not make a distinction between a $\mathbb{Z}^d$-periodic function f and its unique representative as a function on $\mathbb{T}^d$, defined by $[x] \mapsto f(x)$.

The space dimension is denoted by d, and for any $r > 0$, we denote the ball of radius r in $\mathbb{R}^d$ by B_r. In addition, we will reserve the notation e_j to the jth coordinate vector of $\mathbb{R}^d$, i.e., $(e_j)_\nu = \delta_{j\nu}$, where δ denotes the Kronecker delta. An affine hyperplane $M \subset \mathbb{R}^d$ is a set for which there exists a vector $x_0 \in \mathbb{R}^d$ such that $M - x_0$ is a hyperplane, i.e., a $(d-1)$-dimensional subspace of $\mathbb{R}^d$. Then there are a direction $u \in S^{d-1}$ and $r_0 \in \mathbb{R}$ such that with $x_0 = r_0 u$, $M = \{x \in \mathbb{R}^d \mid x\cdot u = r_0\} = \{x - (x\cdot u)u + x_0 \mid x \in \mathbb{R}^d\}$. We also denote the projection onto the hyperplane orthogonal to u by Q_u, when explicitly

$$Q_u x = x - (u\cdot x)u. \tag{2.9}$$

We use here the following standard shorthand notation:

$$\langle x\rangle = \sqrt{1+x^2}, \tag{2.10}$$

for $x \in \mathbb{R}$. This will be the main tool for handling the various power-law dependencies appearing later, and we have collected a few basic properties of $\langle\cdot\rangle$ into Appendix B. For any sufficiently many times differentiable function $f : X \to \mathbb{C}$, X an open subset of $\mathbb{R}^d$, we employ the notations:

$$\|f\|_N = \sup_{|\alpha|\leqslant N} \|\partial^\alpha f\|_\infty \quad \text{and} \quad \|f\|'_N = \sup_{0\leqslant n\leqslant N} \|D^n f\|_\infty \tag{2.11}$$

where, for a multi-index α, $\partial^\alpha f$ is the corresponding partial derivative of f, and, for a positive integer n, $D^n f|_x$ denotes the linear operator on $\mathbb{R}^{d\times n}$ corresponding to the nth derivative of f at x. Then $\|D^n f\|_\infty = \sup_{x,|v_k|=1} |\prod_{k=1}^n (v_j\cdot\nabla) f(x)|$. In particular, $\|f\|'_0 = \|f\|_\infty$, $\|f\|'_1 = \max(\|f\|_\infty, \sup_x |\nabla f(x)|)$, and $\|f\|'_2 = \max(\|f\|'_1, \sup_x \|D^2 f(x)\|)$, where $D^2 f(x)$ is the Hessian of f at x and the norm is its matrix norm.

Finally, $\mathbb{1}(P)$ denotes here a characteristic function of a statement P. That is, it takes the value 1, if P is true, and 0 otherwise.

3. Counterexamples

3.1. Proof of "only if" in Theorem 2.3

For this part of the proof, we do not need the dispersivity properties following from Assumption 2.1, or the full smoothness of the dispersion relation. Instead of the assumptions of Theorem 2.3, let us consider in this subsection the following, more general, case: let $d \geqslant 2$ and assume that $\omega : \mathbb{R}^d \to \mathbb{R}$ is $\mathbb{Z}^d$-periodic and Lipschitz. Let C' denote a Lipschitz constant of ω, i.e., it is positive and $|\omega(x') - \omega(x)| \leqslant C'|x'-x|$ for all $x', x \in \mathbb{R}^d$.

To complete the "only if" part of Theorem 2.3, we assume that there is an affine hyperplane $M \subset \mathbb{R}^d$ and $\alpha \in \mathbb{R}$ such that $\omega(x) = \alpha$ for all $x \in M$. Then there are $u \in S^{d-1}$ and $r_0 \in \mathbb{R}$ such that

$$M = \{x \in \mathbb{R}^d \mid x\cdot u = r_0\} = \{x - (x\cdot u)u + x_0 \mid x \in \mathbb{R}^d\},$$

where $x_0 = r_0 u$. We shall prove that ω cannot suppress crossings by showing that then there is $c > 0$ such that for all $0 < \beta \leqslant 1$,

$$I(\beta) = I_{3\mathrm{cr}}\big((\alpha,\alpha,\alpha),[x_0],\beta\big) \geqslant \frac{c}{\beta}. \tag{3.1}$$

By the remark after (2.4), this suffices, as then $\sup_{\alpha',k_0'}(\beta I_{3\mathrm{cr}}(\alpha',k_0',\beta)) \geqslant c$.

We will derive the bound by considering the integral only over a certain neighbourhood of $[M] \times [M] \subset \mathbb{T}^d \times \mathbb{T}^d$. Let for any $\delta \geqslant 0$

$$M_\delta' = \big\{[x] \mid x \in \mathbb{R}^d,\ |x \cdot u - r_0| \leqslant \delta\big\}. \tag{3.2}$$

Then $M_0' = [M] \subset M_\beta'$ and there is $C > 0$ such that $\int_{M_\delta'} \mathrm{d}k \geqslant C\delta$ for all $0 \leqslant \delta \leqslant 1$. If $k \in M_\delta'$, there is x such that $k = [x]$ and $|x \cdot u - r_0| \leqslant \delta$. Then $x' = x - (x \cdot u - r_0)u \in M$, and

$$\big|\alpha - \omega(k)\big| = \big|\omega(x') - \omega(x)\big| \leqslant C'|x \cdot u - r_0| \leqslant C'\delta. \tag{3.3}$$

Therefore, $|\alpha - \omega(k) + \mathrm{i}\beta| = \beta\langle(\alpha - \omega(k))\beta^{-1}\rangle \leqslant \beta\langle C'\delta\beta^{-1}\rangle$ for all $[x] \in M_\delta'$. If $k_1, k_2 \in M_\beta'$, then there are x_1, $x_2 \in \mathbb{R}^d$ such that $[x_j] = k_j$ and $|x_j \cdot u - r_0| \leqslant \beta$. Since $(x_1 - x_2 + x_0)\cdot u - r_0 = x_1 \cdot u - x_2 \cdot u$, then $[x_1 - x_2 + x_0] \in M_{2\beta}'$. Therefore, for all $0 < \beta \leqslant 1$,

$$I(\beta) \geqslant \int_{M_\beta'} \mathrm{d}k_1 \int_{M_\beta'} \mathrm{d}k_2 \frac{1}{\langle C'\rangle^2 \langle 2C'\rangle} \beta^{-3} \geqslant \frac{C^2}{2\langle C'\rangle^3}\beta^{-1}. \tag{3.4}$$

This proves (3.1), and finishes the proof of the "only if" part of Theorem 2.3.

3.2. *The first counterexample: NN-interaction in $d = 2$*

As the first counterexample, we consider the dispersion relation of the standard 2-dimensional classical harmonic crystal with nearest neighbour (NN) interactions. Although it does not satisfy Assumption 2.1, as $d < 3$ and ω is not analytic, it is a standard example used in perturbative analysis of 2-dimensional crystals. We therefore find it worth the diversion to stress the special nature of this dispersion relation. See, for instance, Sections 2.1 and 6 in [6] for more details on the subject.

Let $\omega : \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$\omega(x) = \sqrt{2 - \cos(2\pi x_1) - \cos(2\pi x_2)}. \tag{3.5}$$

It is $\mathbb{Z}^2$-periodic and has a cusp singularity at every $x \in \mathbb{Z}^2$, but it is straightforward to check that ω is nevertheless Lipschitz. On the other hand, if x is any point on the affine hyperplane $x_1 + x_2 = \frac{1}{2}$, then

$$\omega(x)^2 = 2 - \cos(2\pi x_1) - \cos(\pi - 2\pi x_1) = 2. \tag{3.6}$$

Therefore, we can apply the previous proof, and conclude that the dispersion relation ω does not suppress crossings. The same conclusion naturally holds also for the dispersion relation ω^2.

3.3. *Second example: A Morse function in $d = 3$*

To show that the extra condition in Theorem 2.3 cannot be dropped, let us also provide an example which satisfies Assumption 2.1 but which is nevertheless a constant on a certain hyperplane. Define:

$$\omega(x) = 5 - \cos(2\pi x_1)\big(3 + \cos(2\pi x_2) + \cos(2\pi x_3)\big) \tag{3.7}$$

which is $\mathbb{Z}^3$-periodic, real-analytic, and positive. If we denote $s_j = \sin(2\pi x_j)$ and $c_j = \cos(2\pi x_j)$, then

$$\frac{1}{2\pi}\nabla\omega(x) = \big(s_1(3 + c_2 + c_3), c_1 s_2, c_1 s_3\big). \tag{3.8}$$

Since $3 + c_2 + c_3 \geqslant 1$, ω has 8 critical points which are the points with $s_j = 0$ for all j, i.e., the points $x_j \in \{0, \frac{1}{2}\}$ for $j = 1, 2, 3$. The Hessian is:

$$\frac{1}{(2\pi)^2} D^2\omega(k) = \begin{pmatrix} c_1(3 + c_2 + c_3) & -s_1 s_2 & -s_1 s_3 \\ -s_1 s_2 & c_1 c_2 & 0 \\ -s_1 s_3 & 0 & c_1 c_3 \end{pmatrix} \tag{3.9}$$

and, since at all critical points $|c_j| = 1$, $|\det D^2\omega(x)| \geqslant (2\pi)^2 > 0$ at every critical point x. Therefore, ω is a Morse function, and thus satisfies Assumption 2.1. On the other hand, $\omega(\pm\frac{1}{4}, x_2, x_3) = 5$ for all x_2 and x_3, and ω is a constant, for instance, on the hyperplane $x_1 = \frac{1}{4}$.

As ω is positive, it is a dispersion relation of a certain classical harmonic crystal. The corresponding elastic couplings of the crystal can be obtained by taking the inverse Fourier transform of ω^2. Since ω^2 is a trigonometric polynomial, these elastic couplings correspond to a translation invariant harmonic interaction which is mechanically stable and has a finite range. Therefore, this example shows that even quite simple elastic couplings can lead to violation of the condition for suppression of crossings.

4. Main technical lemmas

This section collects the technical material which will be needed in derivation of the main results. We start with a few straightforward, but frequently applied, estimates. In the second subsection we derive estimates for the asymptotics of one-dimensional "resolvent integrals". The final subsection contains a derivation of the parameterisation of the level sets of ω, and most of it will be consumed by the more involved estimates about the higher order curvature induced by the parameterisation.

4.1. Basic estimates

For application of the following lemmas, let us note that if ω satisfies the Assumption 2.1, then it is $\mathbb{Z}^d$-periodic and smooth, and thus $\|\omega\|'_n < \infty$ for all n.

Lemma 4.1. *Suppose d and ω satisfy Assumption* 2.1. *Then for all* $0 < p < 3$,

$$\int_{\mathbb{T}^d} \mathrm{d}k \, \frac{1}{|\nabla\omega(k)|^p} < \infty. \tag{4.1}$$

Proof. Let $M = (\|\omega\|'_1)^{3-p}$. Then we can apply a "layer cake representation" to the integral:

$$\begin{aligned} \int_{\mathbb{T}^d} \mathrm{d}k \, \frac{1}{|\nabla\omega(k)|^p} &= \int_{\mathbb{T}^d} \mathrm{d}k \, \frac{1}{|\nabla\omega(k)|^3} \int_0^M \mathrm{d}s \, \mathbb{1}\big(\big|\nabla\omega(k)\big|^{3-p} \geqslant s\big) \\ &= \int_0^M \mathrm{d}s \, f_\omega\big(s^{1/(3-p)}\big) \leqslant c_0 \left\langle \frac{1}{3-p} \right\rangle^{p_0} \int_0^M \mathrm{d}s \, \langle \ln s \rangle^{p_0}, \end{aligned} \tag{4.2}$$

where we have used Fubini's theorem and the general property $\langle ab\rangle \leqslant \langle a\rangle\langle b\rangle$. By the change of variables to $y = -\ln s$, the remaining integral over s is easily shown to be finite, which proves (4.1). □

Lemma 4.2. *Let $a > 0$ and $\omega : \mathbb{R}^d \to \mathbb{R}$, with $M_2 = \|\omega\|'_2 < \infty$, be given. Then for all $x, x_0 \in \mathbb{R}^d$ with $|x - x_0| \leqslant \frac{a}{M_2}|\nabla\omega(x_0)|$,*

$$\big|\nabla\omega(x) - \nabla\omega(x_0)\big| \leqslant a\big|\nabla\omega(x_0)\big|. \tag{4.3}$$

Proof. Let x and x_0 be such that $|x - x_0| \leqslant \frac{a}{M_2}|\nabla\omega(x_0)|$. Choose an arbitrary $h \in \mathbb{R}^d$, then by Taylor formula:

$$\left|h \cdot \left(\nabla\omega(x) - \nabla\omega(x_0)\right)\right| \leqslant \int_0^1 \mathrm{d}t \left|D^2\omega|_{x_0+t(x-x_0)}(h, x - x_0)\right|$$
$$\leqslant |h||x - x_0|\|\omega\|_2' \leqslant a\left|\nabla\omega(x_0)\right||h|. \tag{4.4}$$

This implies (4.3). □

Lemma 4.3 *(Argument shift). Let ω be such that $M_2 = \|\omega\|_2' < \infty$, and assume that $s, p > 0$ and $0 < a < 1$ are given. Then for any $0 < \lambda \leqslant as/M_2$, and $x, y \in \mathbb{R}^d$,*

$$\frac{\mathbb{1}(|x-y| < \lambda)\mathbb{1}(|\nabla\omega(y)| \geqslant s)}{|\nabla\omega(y)|^p} \leqslant (1+a)^p \frac{\mathbb{1}(|x-y| < \lambda)\mathbb{1}(|\nabla\omega(x)| \geqslant (1-a)s)}{|\nabla\omega(x)|^p}. \tag{4.5}$$

Proof. Let us assume $|x - y| < \lambda$ and $|\nabla\omega(y)| \geqslant s$, otherwise the bound in (4.5) is trivial. Since then $|x - y| < a|\nabla\omega(y)|/M_2$, we can apply Lemma 4.2 and triangle inequality, yielding

$$\left|\left|\nabla\omega(x)\right| - \left|\nabla\omega(y)\right|\right| \leqslant \left|\nabla\omega(x) - \nabla\omega(y)\right| \leqslant a\left|\nabla\omega(y)\right|. \tag{4.6}$$

Therefore, $|\nabla\omega(x)| \geqslant (1-a)|\nabla\omega(y)| \geqslant (1-a)s$, and $(1+a)|\nabla\omega(y)| \geqslant |\nabla\omega(x)|$, which imply that (4.5) holds. □

Lemma 4.4. *For any $p \geqslant 0$ and $0 < \beta \leqslant 1$,*

$$\int_\beta^1 \mathrm{d}s \frac{\langle \ln s\rangle^p}{s} \leqslant \langle \ln\beta\rangle^{p+1}. \tag{4.7}$$

Proof. Now $0 \leqslant -\ln s \leqslant -\ln\beta$ for all $\beta \leqslant s \leqslant 1$. Therefore,

$$\int_\beta^1 \mathrm{d}s \frac{\langle \ln s\rangle^p}{s} \leqslant \langle \ln\beta\rangle^p \int_\beta^1 \frac{\mathrm{d}s}{s} = \langle \ln\beta\rangle^p |\ln\beta| \leqslant \langle \ln\beta\rangle^{p+1}, \tag{4.8}$$

proving (4.7). □

Lemma 4.5. *For any $\beta, \mu > 0$, and $x, h \in \mathbb{R}$ such that $|h| \leqslant 2\mu\beta$,*

$$\frac{1}{|x+h+\mathrm{i}\beta|} \leqslant \frac{\mu + \langle\mu\rangle}{|x+\mathrm{i}\beta|}. \tag{4.9}$$

Proof. By the triangle inequality, $|x+h|^2 \geqslant (|x| - |h|)^2$, and thus for any $0 < \lambda < 1$,

$$\begin{aligned}|x+h+\mathrm{i}\beta|^2 &\geqslant x^2 - 2|h||x| + h^2 + \beta^2 \\ &= \left(1-\lambda^2\right)\left(x^2+\beta^2\right) + \left(\lambda|x| - \frac{1}{\lambda}|h|\right)^2 - \left(\frac{1}{\lambda^2} - 1\right)|h|^2 + \lambda^2\beta^2 \\ &\geqslant \left(1-\lambda^2\right)\left(x^2+\beta^2\right) + \beta^2\left(\left(1 - \frac{1}{\lambda^2}\right)4\mu^2 + \lambda^2\right).\end{aligned} \tag{4.10}$$

By choosing $\lambda^2 = 1 - (\mu + \langle\mu\rangle)^{-2}$ the final term in the parenthesis vanishes. Since then $1 - \lambda^2 = (\mu + \langle\mu\rangle)^{-2}$, this proves (4.9). □

4.2. One-dimensional resolvent integrals

We derive here the required estimates for one-dimensional "resolvent" integrals. We start with polynomials, and then extend these results to functions f which are "almost polynomial" on the integration interval in the sense that the n_0th derivative of f is non-vanishing on the whole interval for some order n_0. The proof will be quite simple when $n_0 = 1$, and fairly involved when $n_0 > 1$. Although we are not aware of a reference to a derivation of these estimates in the literature, they could probably be pieced up from the known results. We point out, in particular, the similarity to Malgrange preparation theorem, see for instance Section 7.5 of [13]. The main point of reproducing the proofs in detail here is that we need to have some control on how the various constants in the estimates depend on the function f.

Proposition 4.6. *Let $n \geqslant 1$ and let $P_n(x) = \sum_{k=0}^{n} a_k x^k$, with $a_k \in \mathbb{R}$ and $a_n \neq 0$. If $n \geqslant 2$, then for all $\beta > 0$,*

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{|P_n(x) + \mathrm{i}\beta|} \leqslant \frac{2(n+2)}{|a_n|^{1/n}} \beta^{1/n-1}. \tag{4.11}$$

If $n = 1$, then for $\beta, \lambda > 0$, and $x_0 \in \mathbb{R}$,

$$\int_{|x-x_0|\leqslant\lambda} \frac{\mathrm{d}x}{|P_n(x) + \mathrm{i}\beta|} \leqslant \frac{6\langle \ln\langle \lambda a_1\rangle\rangle}{|a_1|} \langle \ln \beta\rangle. \tag{4.12}$$

Proof. Let first $n \geqslant 2$, and consider (4.11). Since P_n is a polynomial of nth degree, we can find $z \in \mathbb{C}^n$ such that for all x, $P_n(x) = a_n \prod_{\ell=1}^{n} (x - z_\ell)$. Fix then x, and let ℓ' be an integer such that $|x - z_\ell| \geqslant |x - z_{\ell'}|$ for all ℓ. Then, $|x - z_\ell| \geqslant |x - \operatorname{Re} z_{\ell'}|$, and

$$\frac{1}{|P_n(x) + \mathrm{i}\beta|} \leqslant \frac{1}{||a_n||x - \operatorname{Re} z_{\ell'}|^n + \mathrm{i}\beta|} \leqslant \sum_{\ell=1}^{n} \frac{1}{||a_n||x - \operatorname{Re} z_\ell|^n + \mathrm{i}\beta|}. \tag{4.13}$$

For any $y \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{||a_n||x-y|^n + \mathrm{i}\beta|} = \frac{\beta^{1/n-1}}{|a_n|^{1/n}} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\langle x^n\rangle}, \tag{4.14}$$

where $\int_{-\infty}^{\infty} \mathrm{d}x\, \langle x^n\rangle^{-1} \leqslant 2(1 + \int_1^\infty \mathrm{d}x\, x^{-n}) = 2n/(n-1) \leqslant 2(n+2)/n$, since $n \geqslant 2$. Thus (4.13) implies (4.11).

Assume then $n = 1$, when $P_n(x) = a_0 + a_1 x$. Changing variables to $y = (a_0 + a_1 x)/\beta$, we get:

$$\int_{|x-x_0|\leqslant\lambda} \frac{\mathrm{d}x}{|P_n(x) + \mathrm{i}\beta|} = \frac{1}{|a_1|} \int_{y_0-\lambda'}^{y_0+\lambda'} \frac{\mathrm{d}y}{|y + \mathrm{i}|}, \tag{4.15}$$

with $y_0 = (a_0 + a_1 x_0)/\beta$ and $\lambda' = |a_1|\lambda/\beta$. By differentiation with respect to y_0, we find that the second integral has a maximum at $y_0 = 0$. Therefore,

$$\int_{|x-x_0|\leqslant\lambda} \frac{\mathrm{d}x}{|P_n(x) + \mathrm{i}\beta|} \leqslant \frac{2}{|a_1|} \int_0^{\lambda'} \frac{\mathrm{d}y}{|y + \mathrm{i}|} \leqslant \frac{2}{|a_1|}\big(1 + |\ln \lambda'|\big) \leqslant \frac{2\sqrt{2}}{|a_1|} \langle \ln \lambda'\rangle. \tag{4.16}$$

If $\beta \leqslant \lambda|a_1|$, then $0 \leqslant \ln \lambda' \leqslant \ln\langle \lambda|a_1|\rangle + |\ln \beta|$, and, since $2\sqrt{2} < 3$, (4.16) implies:

$$\int_{|x-x_0|\leqslant\lambda} \frac{\mathrm{d}x}{|P_n(x) + \mathrm{i}\beta|} \leqslant \frac{3}{|a_1|} 2\big\langle \ln\langle \lambda a_1\rangle\big\rangle \langle \ln \beta\rangle, \tag{4.17}$$

where we have used the properties of $\langle\cdot\rangle$ given in Appendix B. This proves (4.12) for $\beta \leqslant \lambda|a_1|$. If $\beta > \lambda|a_1|$, then we can estimate trivially:

$$\int_{|x-x_0|\leqslant\lambda} \frac{\mathrm{d}x}{|P_n(x) + \mathrm{i}\beta|} \leqslant \frac{2\lambda}{\beta} < \frac{2}{|a_1|} < \frac{6}{|a_1|}\big\langle \ln\langle \lambda a_1\rangle\big\rangle \langle \ln \beta\rangle, \tag{4.18}$$

which proves (4.12) also for the remaining values of β. □

Proposition 4.7. ($n_0 = 1$) *Suppose $a, b \in \mathbb{R}$, with $a < b$. Denote $I = (a, b)$, and assume $f \in C^{(1)}(I, \mathbb{R})$ is such that $|f'(x)| \geqslant \varepsilon_0$ for some $\varepsilon_0 > 0$ and all $x \in I$, and that $m_0 = \sup_{x\in I} |f(x)| < \infty$. Then for all $\beta > 0$ and $\alpha \in \mathbb{R}$,*

$$\int_a^b \frac{\mathrm{d}x}{|f(x) - \alpha + \mathrm{i}\beta|} \leqslant \frac{6\langle \ln\langle m_0\rangle\rangle}{\varepsilon_0} \langle \ln \beta\rangle. \tag{4.19}$$

Proof. Since f' is continuous, either $f' \geqslant \varepsilon_0$ or $f' \leqslant -\varepsilon_0$, and we only need to prove the result in the first case (applying it to $-f$ then proves the result in the second case). Since $f' > 0$, f is strictly increasing. In addition, $f(I) = (a', b')$, where $a' = \lim_{x\to a^+} f(x)$ and $b' = \lim_{x\to b^-} f(x)$ exist and are bounded by $m_0 < \infty$. Thus there is $g : f(I) \to I$, $g = f^{-1}$, for which $g'(y) = 1/f'(g(y)) \in (0, 1/\varepsilon_0]$. Therefore,

$$\int_a^b \frac{\mathrm{d}x}{|f(x) - \alpha + \mathrm{i}\beta|} = \int_{a'}^{b'} \mathrm{d}y\, \frac{g'(y)}{|y - \alpha + \mathrm{i}\beta|} \leqslant \frac{1}{\varepsilon_0} \int_{a'}^{b'} \frac{\mathrm{d}y}{|y - \alpha + \mathrm{i}\beta|}. \tag{4.20}$$

By Lemma 4.6, this is bounded by $6\langle \ln\langle (b' - a')/2\rangle\rangle \langle \ln \beta\rangle/\varepsilon_0$. However, as $|b' - a'|/2 \leqslant m_0$, this bound implies also (4.19). □

Proposition 4.8. ($n_0 > 1$) *Suppose $a, b \in \mathbb{R}$, with $a < b$, and $n_0 \geqslant 2$ are given. Denote $I = (a, b)$, and assume $f \in C^{(n_0+1)}(I, \mathbb{R})$ is such that $|f^{(n_0)}(x)| \geqslant n_0!\varepsilon_0$ for some $\varepsilon_0 > 0$ and all $x \in I$, and that $m_0 = \sup_{x\in I} |f^{(n_0+1)}(x)|/(n_0 + 1)! < \infty$. Define $M = \max(m_0, 1)$, $C_{n_0} = 2^{n_0+1}(n_0 + 1)^{n_0}$, and*

$$\varepsilon' = \frac{\varepsilon_0}{M C_{n_0}} > 0. \tag{4.21}$$

If $0 < \beta \leqslant (\varepsilon')^{n_0+1}$, then

$$\int_a^b \frac{\mathrm{d}x}{|f(x) + \mathrm{i}\beta|} \leqslant C_{n_0}\left(\frac{b - a}{\varepsilon_0} \beta^{1/(n_0+1)-1} + M\varepsilon_0^{-1/n_0} \beta^{1/n_0 - 1}\right). \tag{4.22}$$

Proof. We need to find the local minima of $|f|$, which coincide with the local minima of f^2. Since $f^{(n_0)}$ has no zeroes, $f^{(m)}$ has maximally $n_0 - m$ zeroes for $m \leqslant n_0$. Let X be the union of the set of zeroes of f, of the zeroes of f' and of the end-points a and b, when $|X| \leqslant n_0 + n_0 - 1 + 2 = 2n_0 - 1$. Since $\mathrm{d}(f^2)/\mathrm{d}x = 2ff'$, X partitions (a, b) into subintervals on which f^2—and thus also $|f|$—is strictly monotonic: if $a' < b'$ are such that $(a', b') \subset (a, b) \setminus X$, then f^2 is either strictly increasing or decreasing on $[a', b'] \cap (a, b)$.

Let us define $\lambda = \beta^{1/(n_0+1)}$ when by assumption $0 < \lambda \leqslant \varepsilon'$. Suppose $x_0 \in (a, b)$, and let $I = I(x_0) = \{x \in (a, b) \mid |x - x_0| < \lambda\}$. We claim that, if $x_0 - \lambda > a$, then there is $x_0^- \in I$, $x_0^- < x_0$ such that $|f(x_0^-)| \geqslant M\varepsilon'\lambda^{n_0}$, and similarly, if $x_0 + \lambda < b$, then there is $x_0^+ \in I$, $x_0^+ > x_0$ such that $|f(x_0^+)| \geqslant M\varepsilon'\lambda^{n_0}$. Consider the Taylor expansion of f around x_0 to degree n_0,

$$f(x) = \sum_{n=0}^{n_0} a_n (x - x_0)^n + R_{n_0}(x; x_0) \quad \text{where } a_n = \frac{f^{(n)}(x_0)}{n!}. \tag{4.23}$$

For any x there is a point ξ between x and x_0, such that the remainder is:

$$R_{n_0}(x; x_0) = \frac{f^{(n_0+1)}(\xi)}{(n_0 + 1)!} (x - x_0)^{n_0+1}, \tag{4.24}$$

implying that $|R_{n_0}| \leqslant M\lambda^{n_0+1}$ on I. On the other hand, since $|a_{n_0}| \geqslant \varepsilon_0 > 0$, there is $z \in \mathbb{C}^{n_0}$ such that

$$P_{n_0}(x; x_0) = \sum_{n=0}^{n_0} a_n (x - x_0)^n = a_{n_0} \prod_{j=1}^{n_0} (x - z_j). \tag{4.25}$$

Let $y_j = \operatorname{Re} z_j$, when by $|x - z_j| \geqslant |x - y_j|$, we have for all $x \in I$,

$$\left|f(x)\right| \geqslant \left|P_{n_0}(x)\right| - \left|R_{n_0}(x)\right| \geqslant \varepsilon_0 \prod_{j=1}^{n_0} |x - y_j| - M\lambda^{n_0+1}. \tag{4.26}$$

Consider the set Y which consists of the endpoints of I and of all those y_j which are in I. Then $2 \leqslant |Y| \leqslant n_0 + 2$. The set $[x_0 - \lambda, x_0] \setminus Y \subset I$ consists of maximally $n_0 + 1$ intervals. If $x_0 - \lambda \geqslant a$, one of them must be at least of length $\lambda/(n_0 + 1)$, and let x_0^- be a middle point of such an interval. Then $x_0^- < x_0$ and $|x_0^- - y_j| \geqslant \frac{1}{2}\lambda/(n_0 + 1)$ for all j. Therefore, by (4.26) and $\lambda \leqslant \varepsilon'$,

$$\left|f(x_0^-)\right| \geqslant \varepsilon_0 \left(\frac{\lambda}{2(n_0 + 1)}\right)^{n_0} - M\lambda^{n_0+1} \geqslant (2M\varepsilon' - M\varepsilon')\lambda^{n_0} = M\varepsilon'\lambda^{n_0}. \tag{4.27}$$

If $x_0 + \lambda \leqslant b$, we can similarly find $x_0^+ \in (x_0, x_0 + \lambda]$ with $|f(x_0^+)| \geqslant M\varepsilon'\lambda^{n_0}$.

For each $x_0 \in X$, we can thus find $x_0^\pm$ with the property that $x_0 \in [x_0^-, x_0^+] \subset I(x_0)$, and either $x_0^\pm \in \{a, b\}$ or $|f(x_0^\pm)| \geqslant M\varepsilon'\lambda^{n_0}$. Let,

$$X' = \left\{x_0 \in X \mid \left|f(x_0)\right| < M\varepsilon'\lambda^{n_0}\right\} \quad \text{and} \quad J = \bigcup_{x_0 \in X'} \left(x_0^-, x_0^+\right). \tag{4.28}$$

We claim that if $x \in I \setminus J$, then $|f(x)| \geqslant M\varepsilon'\lambda^{n_0}$.

Suppose $x \in I \setminus J$. It then belongs to an interval I' whose endpoints lie in the set $\bigcup_{x_0 \in X'}\{x_0^\pm\} \cup \{a, b\}$. Assume x' is a local minimum point of $|f|$ on the closure of I'. If x' is not an endpoint of I', it must be a critical point of f^2, and thus $x' \in X$, when by construction, $|f(x')| \geqslant M\varepsilon'\lambda^{n_0}$. The same holds if $x' \in \{a, b\} \subset X$. The only possibility left is that x' is one of the points $x_0^\pm$, when again by construction $|f(x')| \geqslant M\varepsilon'\lambda^{n_0}$. This proves that $|f| \geqslant M\varepsilon'\lambda^{n_0}$ on I', in particular, also at x.

Therefore,

$$\int_a^b \mathrm{d}x \frac{1}{|f(x) + \mathrm{i}\beta|} = \int_{I \setminus J} \mathrm{d}x \frac{1}{|f(x) + \mathrm{i}\beta|} + \int_J \mathrm{d}x \frac{1}{|f(x) + \mathrm{i}\beta|} \leqslant \frac{b - a}{M\varepsilon'\lambda^{n_0}} + \sum_{x_0 \in X'} \int_{x_0^-}^{x_0^+} \mathrm{d}x \frac{1}{|f(x) + \mathrm{i}\beta|}. \tag{4.29}$$

Consider one of the terms in the sum over X', i.e., let $x_0 \in X'$. Denote $R(x) = R_{n_0}(x; x_0)$ and $P(x) = P_{n_0}(x; x_0) = a_{n_0} \prod_{j=1}^{n_0}(x - z_j)$. Since $(x_0^-, x_0^+) \subset I(x_0)$, for all $x \in (x_0^-, x_0^+)$,

$$\left|f(x) - P(x)\right| = \left|R(x)\right| \leqslant M\lambda^{n_0+1} = M\beta. \tag{4.30}$$

Therefore, by Lemma 4.5, on the whole integration region:

$$\frac{1}{|f(x) + \mathrm{i}\beta|} \leqslant \frac{\frac{1}{2}M + \langle\frac{1}{2}M\rangle}{|P(x) + \mathrm{i}\beta|} \leqslant \frac{2M}{|P(x) + \mathrm{i}\beta|} \tag{4.31}$$

to which we can apply Lemma 4.6 with $|a_{n_0}| \geqslant \varepsilon_0$. Since $|X| \leqslant 2n_0 - 1$, the results proven so far can be collected into the estimate:

$$\int_a^b \frac{\mathrm{d}x}{|f(x) + \mathrm{i}\beta|} \leqslant \frac{b - a}{M\varepsilon'} \beta^{1/(n_0+1)-1} + (2n_0 - 1)2M\beta^{1/n_0 - 1}\varepsilon_0^{-1/n_0} 2(n_0 + 2). \tag{4.32}$$

To get the bound in (4.22), we only need to use the fact that, as $n_0 \geqslant 2$, $C_{n_0} \geqslant 2^3(n_0 + 1)^2 \geqslant 2^2(2n_0 - 1)(n_0 + 2)$. □

4.3. Parameterisation of the level sets

The first of the results in this subsection states that, apart from the critical points, there exists a local diffeomorphism which transforms the level sets of ω into hyperplanes orthogonal to e_1. Although this is a straightforward consequence of the inverse mapping theorem, we need fairly detailed information about the inverse function, and we have included also some details of the proof here.

In all of the results in this subsection we assume that $d \geqslant 2$ and $\omega : \mathbb{R}^d \to \mathbb{R}$ is a smooth function such that $\|\omega\|'_n < \infty$ for all n. In particular, this covers all dispersion relations satisfying Assumption 2.1.

Lemma 4.9. *Let $x_0 \in \mathbb{R}^d$ and $\lambda > 0$ be such that $\nabla\omega(x_0) \neq 0$, and $\lambda \leqslant \frac{1}{8}\frac{|\nabla\omega(x_0)|}{\|\omega\|'_2}$. Then there is an open set $U \subset \mathbb{R}^d$ and a diffeomorphism $\psi : B_{2\lambda} \to U$ with the following properties*:

(1) *$\psi(0) = x_0$ and $x_0 + B_\lambda \subset \psi(B_{2\lambda}) \subset x_0 + B_{4\lambda}$.*
(2) *For all y with $|y| < 2\lambda$,*

$$\omega\big(\psi(y)\big) = \omega(x_0) + \big|\nabla\omega(x_0)\big| y_1, \tag{4.33}$$

and

$$\big|\nabla\omega\big(\psi(y)\big) - \nabla\omega(x_0)\big| < \frac{1}{2}\big|\nabla\omega(x_0)\big|. \tag{4.34}$$

(3) *Denote $A = D\psi(0)$ and $u_0 = \nabla\omega(x_0)/|\nabla\omega(x_0)|$. Then A is a rotation of $\mathbb{R}^d$ such that $u_0 = Ae_1$. In addition, $\frac{2}{3} \leqslant |\det(D\psi)| \leqslant 2$ on $B_{2\lambda}$, and*

$$D\psi|_y A^T v = v - u_0 \frac{\nabla\omega(x) \cdot v}{\nabla\omega(x) \cdot u_0}\bigg|_{x=\psi(y)} \quad \textit{whenever } v \cdot u_0 = 0. \tag{4.35}$$

Proof. Let us denote $U_a = x_0 + B_{a8\lambda}$, and define $f : \mathbb{R}^d \to \mathbb{R}^d$ by the formula:

$$f(x) = \frac{\nabla\omega(x) - \nabla\omega(x_0)}{|\nabla\omega(x_0)|}. \tag{4.36}$$

Then $f(x_0) = 0$ and, by Lemma 4.2, $|f(x)| < a$ for all $x \in U_a$, $a > 0$. As before, let Q_{u_0} be the projection onto the subspace orthogonal to u_0, and let O to be a rotation of $\mathbb{R}^d$ for which $Ou_0 = e_1$; in particular, $O^T = O^{-1}$ and $\det O = 1$. Define $\varphi : U_1 \to \mathbb{R}^d$ by:

$$\varphi(x) = \frac{\omega(x) - \omega(x_0)}{|\nabla\omega(x_0)|} e_1 + OQ_{u_0}(x - x_0). \tag{4.37}$$

Since $Q_{u_0} O^T e_1 = Q_{u_0} u_0 = 0$, then

$$\varphi(x)_1 = \frac{\omega(x) - \omega(x_0)}{|\nabla\omega(x_0)|}. \tag{4.38}$$

By an explicit computation,

$$D\varphi(x) = O + e_1 \otimes f(x) = O\big(\mathbb{1} + u_0 \otimes f(x)\big). \tag{4.39}$$

Since O is orthogonal and $u_0 \otimes f(x)$ has rank one, the determinant of $D\varphi(x)$ can be computed explicitly: with $u = f(x)$, $\det D\varphi(x) = \det(\mathbb{1} + e_1 \otimes (Ou)) = 1 + (Ou)_1 = 1 + u_0 \cdot u$ and thus for all $x \in U_a$,

$$1 - a < \big|\det D\varphi(x)\big| < 1 + a. \tag{4.40}$$

Therefore, $D\varphi(x)$ is invertible on U_1, and by the inverse function theorem, φ is a local diffeomorphism on U_1. Where we need to do the extra work here, is to show that we can find a neighbourhood U on which the inverse has the properties stated in the lemma.

Consider then the case $a = 1/2$ in the above estimates. Let $\phi(x) = O^T\varphi(x) - (x - x_0)$ for $x \in U_a$, when $\|D\phi(x)\| < a$. By the standard arguments used in the proof of the inverse function theorem (see for instance the proof of Theorem 10.39 in [14]), it follows that φ is one-to-one on U_a, $B_{2\lambda} \subset \varphi(U_a)$, and $\psi = \varphi^{-1}|_{B_{2\lambda}}$ is a diffeomorphism from $B_{2\lambda}$ to an open set $U \subset U_a = x_0 + B_{4\lambda}$. Also, for all y,

$$D\psi(y) = D\varphi\big(\psi(y)\big)^{-1} = \bigg(\mathbb{1} - \frac{1}{1 + u_0 \cdot u} u_0 \otimes u\bigg)_{u=f(\psi(y))} O^T. \tag{4.41}$$

We now only need to check that ψ has all the properties mentioned in the lemma. Since $\varphi(x_0) = 0$, now $\psi(0) = x_0$ and we already proved $U \subset x_0 + B_{4\lambda}$. To complete item (1), we need to prove that $U_{1/8} = x_0 + B_\lambda \subset U$. Since

$U_{1/8} \subset U_{1/2}$, on which φ is one-to-one, it is enough to prove $\varphi(U_{1/8}) \subset B_{2\lambda}$. This however holds now, since $\|D\varphi(x)\| < 1 + \frac{1}{8}$ for all $x \in U_{1/8}$, and thus $|\varphi(U_{1/8})| \leqslant \frac{9}{8}\lambda < 2\lambda$. Of the two statements in item (2), (4.33) follows from (4.38) by bijectivity of ψ, and, since $\psi(B_{2\lambda}) \subset U_{1/2}$, (4.34) also holds. For item (3), we note that $A = D\psi(0)$ is equal to the rotation O^T, and thus $Ae_1 = u_0$, and (4.41) implies (4.35). Finally, by (4.40) and $U \subset U_{1/2}$, we have $\frac{2}{3} \leqslant |\det(D\psi(y))| \leqslant 2$ for all y. □

Corollary 4.10. *Let $f : \mathbb{R}^d \to [0, \infty]$ be measurable. Then for any x_0, λ, and ψ as in the previous lemma,*

$$\int\limits_{|x-x_0|<\lambda} \mathrm{d}x\, f(x) \leqslant 2 \int\limits_{|y|<2\lambda} \mathrm{d}y\, f\bigl(\psi(y)\bigr). \tag{4.42}$$

Proof. By the properties of the diffeomorphism ψ stated in the lemma,

$$\int\limits_{|x-x_0|<\lambda} \mathrm{d}x\, f(x) \leqslant \int\limits_{\psi(B_{2\lambda})} \mathrm{d}x\, f(x) = \int\limits_{|y|<2\lambda} \mathrm{d}y \bigl|\det\bigl(D\psi(y)\bigr)\bigr| f\bigl(\psi(y)\bigr) \tag{4.43}$$

which is bounded by the right-hand side of (4.42). □

The final result in this section concerns the curvature induced on straight lines by the "level set diffeomorphism" ψ. In the following proposition we show that, if all derivatives of ω at x_0 in the direction of the curve are small up to a certain order, then also the corresponding "bending" of the curve remains small up to the same order. The main difficulty in deriving these estimates lies in finding sufficiently sharp estimates also when the parameterisation is nearly singular, i.e., when $|\nabla\omega(x_0)| \ll 1$.

Proposition 4.11. *Let ω, x_0 and λ satisfy the assumptions of Lemma 4.9, and let ψ, A, and u_0 be defined as in the conclusions of the lemma. Consider also some given $|y| < 2\lambda$ and $v \in S^{d-1}$, with $v \cdot u_0 = 0$.*

Let $v' = A^T v$ and define:

$$\gamma(t; y, v) = \psi(y + tv') \quad \textit{and} \quad \Gamma(t; y, v) = \gamma(t; y, v) - tv - \psi(y), \tag{4.44}$$

for all t with $|y + tv'| < 2\lambda$. Then for any such t, and $n \geqslant 1$,

$$\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Gamma(t) = -g_n(t)u_0, \tag{4.45}$$

where

$$g_n(t) = g_n(t; y, v) = \frac{1}{n!}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} g\bigl(\gamma(t; y, v)\bigr) \quad \textit{with } g(x) = \frac{v \cdot \nabla\omega(x)}{u_0 \cdot \nabla\omega(x)},\ x \in \mathbb{R}^d. \tag{4.46}$$

Denote $M_n = \|\omega\|'_n$, and $a_0 = \max(1, 8M_2)$. If $N \geqslant 2$, $0 < \varepsilon \leqslant 1$, $\mu > 0$ and $r_0 > 0$ are such that $\mu \leqslant (1 + 2^N + M_{N+1}2^{2N+1})^{-1}$, $r_0 \leqslant \min(1, |\nabla\omega(x_0)|)$, $\lambda \leqslant \varepsilon(r_0\mu)^N a_0^{-1}$, and for all $2 \leqslant n < N$,

$$\frac{1}{n!}\bigl|(v \cdot \nabla)^n \omega(x_0)\bigr| \leqslant \frac{1}{2}\varepsilon(\mu r_0)^{N-n}, \tag{4.47}$$

then, with $C = 1 + \frac{M_N}{N!}$,

$$\left|\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Gamma(t)\right| \leqslant \begin{cases} \varepsilon\mu^N r_0^{N-1}, & \textit{for } n = 1, \\ 2\varepsilon\mu^{N-n} r_0^{N-1-n}, & \textit{for } 2 \leqslant n < N, \\ 2Cr_0^{-1}, & \textit{for } n = N, \\ 2C\mu^{-1}r_0^{-2}, & \textit{for } n = N + 1. \end{cases} \tag{4.48}$$

The proof will be essentially a corollary of the following lemma, whose proof we will postpone until the end of this section:

Lemma 4.12. *Let the assumptions and definitions of the first paragraph of Proposition* 4.11 *be satisfied. Denote* $M_n = \|\omega\|'_n$, $a_0 = \max(1, 8M_2)$, *and assume* $0 < r_0 \leqslant \min(1, |\nabla\omega(x_0)|)$ *is given. If* $\lambda \leqslant r_0 a_0^{-1}$, *then all of the following results are valid*:

(1) $|\Gamma(t)| \leqslant |t| < 2\lambda$.
(2) *Let us define* $\tilde{g}_n = \tilde{g}_n(x_0, v)$ *by the following iterative procedure*: *Let* $\tilde{g}_1 = 0$, $\tilde{g}_2 = \frac{1}{2}|\nabla\omega(x_0)|^{-1}(v\cdot\nabla)^2\omega(x_0)$, *and for* $n > 2$, *define*

$$\tilde{g}_n = \frac{1}{|\nabla\omega(x_0)|}\Bigg[\frac{1}{n!}(v\cdot\nabla)^n\omega(x_0) + \sum_{k=2}^{n-1}\sum_{m\in\mathbb{N}_+^k} \mathbb{1}\Bigg(\sum_{j=1}^k m_j = n\Bigg)\prod_{j=1}^{k-1}\frac{m_j}{\sum_{j'=j}^k m_{j'}}\prod_{\substack{j=1\\ m_j>1}}^{k}\tilde{g}_{m_j}(-u_0\cdot\nabla)^{k-\ell}(v\cdot\nabla)^\ell\omega(x_0)\Big|_{\ell=|\{j|m_j=1\}|}\Bigg]. \tag{4.49}$$

Then $g_n(0; 0, v) = \tilde{g}_n$ *for all* $n \geqslant 1$.
(3) *Suppose* $0 < \varepsilon, \mu \leqslant 1$ *and* $N \geqslant 2$ *are such that for all* $2 \leqslant n < N$, *inequality* (4.47) *is satisfied. If* $\mu \leqslant 2^{-N}M_{N-1}^{-1}$, *then for all* $2 \leqslant m < N$,

$$|\tilde{g}_m| \leqslant \varepsilon\mu^{N-m}r_0^{N-m-1} \leqslant 1, \tag{4.50}$$

and, with $C = 1 + \frac{M_N}{N!}$ *defined as in* (4.48),

$$|\tilde{g}_N| \leqslant Cr_0^{-1} \quad \textit{and} \quad |\tilde{g}_{N+1}| \leqslant M_{N+1}\big(1+2^N C\big)r_0^{-2}. \tag{4.51}$$

Furthermore, if also $b \geqslant 1 + 2^N + M_{N+1}2^{2N+1}$, *then for all* $1 \leqslant n \leqslant N$ *and allowed* t,

$$\big|g_n(t) - \tilde{g}_n\big| \leqslant a_0 b^{n-1}\lambda r_0^{-n}, \tag{4.52}$$

and

$$\big|g_{N+1}(t) - \tilde{g}_{N+1}\big| \leqslant 5Ca_0 b^N\lambda r_0^{-N-2}. \tag{4.53}$$

Proof of Proposition 4.11. By Lemma 4.9,

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = v - g\big(\gamma(t)\big)u_0 \tag{4.54}$$

which implies (4.45). For the results in the second paragraph, let us note that under the assumptions of the proposition, we have $\lambda \leqslant r_0/a_0$, so that items (1) and (2) of Lemma 4.12 are immediately applicable. In addition, also $0 < \mu \leqslant 1$ with $\mu^{-1} \geqslant 2^N M_{N-1}$, so that if we define $b = \mu^{-1}$, then b and μ are small enough for applying the conclusions in item (3). Therefore, for $n = 1$, we have $|\Gamma'(t)| = |g_1(t)| \leqslant a_0 r_0^{-1}\lambda \leqslant \varepsilon\mu^N r_0^{N-1}$, and if $2 \leqslant n < N$, then

$$\left|\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Gamma(t)\right| = \big|g_n(t)\big| \leqslant \big|g_n(t) - \tilde{g}_n\big| + |\tilde{g}_n| \leqslant a_0 b^{n-1}\lambda r_0^{-n} + \varepsilon\mu^{N-n}r_0^{N-n-1} \leqslant 2\varepsilon\mu^{N-n}r_0^{N-n-1}. \tag{4.55}$$

For $n = N$, we get similarly a bound $a_0\mu^{1-N}\lambda r_0^{-N} + Cr_0^{-1} \leqslant 2Cr_0^{-1}$. Finally, for $n = N + 1$, we have:

$$\left|\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\Gamma(t)\right| \leqslant 5Ca_0\mu^{-N}\lambda r_0^{-N-2} + \mu^{-1}Cr_0^{-2} \leqslant 2C\mu^{-1}r_0^{-2}, \tag{4.56}$$

where we have used $C' \leqslant bC = \mu^{-1}C$ and $\mu \leqslant \frac{1}{5}$. This proves that all of the bounds given in (4.48) are valid. □

Proof of Lemma 4.12. For any $x = \gamma(t)$, we have in the definition of g:

$$\big|u_0\cdot\nabla\omega(x)\big| = \big|\big|\nabla\omega(x_0)\big| + u_0\cdot\big(\nabla\omega(x) - \nabla\omega(x_0)\big)\big| \geqslant \frac{1}{2}\big|\nabla\omega(x_0)\big|, \tag{4.57}$$

by (4.34). Similarly, $v\cdot u_0 = 0$ implies:

$$\big|v\cdot\nabla\omega(x)\big| = \big|v\cdot\big(\nabla\omega(x) - \nabla\omega(x_0)\big)\big| \leqslant \big|\nabla\omega(x) - \nabla\omega(x_0)\big| \leqslant 4\lambda M_2. \tag{4.58}$$

Therefore (using the definition of a_0 and the assumption made on λ)

$$\bigl|g_1(t)\bigr| \leqslant a_0\lambda r_0^{-1} \leqslant 1, \tag{4.59}$$

which implies $|\Gamma'(t)| \leqslant 1$. Since $\Gamma(0) = 0$, item (1) holds now.

Consider then item (2). In (4.49), the sum over m_j is restricted by $k \geqslant 2$ so that always $m_j \leqslant n - m_1 \leqslant n-1$. Thus the right-hand side depends only on $\tilde{g}_m$ with $2 \leqslant m \leqslant n-1$, and the sequence $\tilde{g}_n$ is thus uniquely determined from $\tilde{g}_2$ and it depends only on x_0 and v (and naturally also on ω). To complete the proof of the item, we need to show that $\tilde{g}_n = g_n(0; 0, v)$. We do this by induction: Since $g_1(0; 0, v) = 0 = \tilde{g}_1$, this holds for $n = 1$. Let us assume that the result is true for $1 \leqslant m < n$. By Lemma 4.9, we have for all t, $\omega(\gamma(t)) = \omega(x_0) + |\nabla\omega(x_0)|y_1$, which is independent of t. By Lemma A.1 the nth derivative of $\omega \circ \gamma$, which is zero, can be expressed in terms of differentials of γ. We separate the $k = 1$ term in the resulting sum, yielding:

$$-\frac{\gamma^{(n)}(t)}{n!}\cdot\nabla\omega\bigl(\gamma(t)\bigr) = \sum_{k=2}^{n}\sum_{m\in\mathbb{N}_+^k}\mathbb{1}\Biggl(\sum_{j=1}^{k} m_j = n\Biggr)\prod_{j=1}^{k-1}\frac{m_j}{\sum_{j'=j}^{k} m_{j'}}\prod_{j=1}^{k}\biggl[\frac{1}{m_j!}\gamma^{(m_j)}(t)\cdot\nabla\biggr]\omega\bigg|_{\gamma(t)}. \tag{4.60}$$

At $t = 0$ and $y = 0$, $\gamma(t) = x_0$ and $\gamma^{(1)}(t) = v$, and the left hand side evaluates to $g_n(0; 0, v)|\nabla\omega(x_0)|$. Since the induction assumption can be applied to all derivatives of γ in the right-hand side, we find that it evaluates to right-hand side of (4.49) times $|\nabla\omega(x_0)|$. This completes the induction step and proves $g_n(0; 0, v) = \tilde{g}_n$.

We next prove the statements in item (3). If $N = 2$, then (4.50) is vacuously true, and (4.51) holds by an explicit computation. Consider then $N > 2$, when again an explicit computation proves that (4.50) holds for $n = 2$. We will prove its validity for higher values of n by induction. Let us thus assume that $2 \leqslant n \leqslant N$ is given and that (4.50) is valid for all $2 \leqslant m < n$. Suppose $2 \leqslant k \leqslant n-1$ and $\sum_{j=1}^{k} m_j = n$, and let $\ell = |\{j \mid m_j = 1\}|$. Then $0 \leqslant \ell \leqslant k-1$, and $\sum_{j, m_j>1}(1 - m_j) = \sum_j (1 - m_j) = k - n$. Therefore, since $0 < \varepsilon, \mu, r_0 \leqslant 1$, and $k \geqslant 2$,

$$\Biggl|\prod_{j=1,\ m_j>1}^{k}\tilde{g}_{m_j}\Biggr| \leqslant \bigl(\varepsilon\mu^{N-1}r_0^{N-2}\bigr)^{k-\ell}(\mu r_0)^{\sum_{j,m_j>1}(1-m_j)} \leqslant \varepsilon\mu^{N-1+k-n}r_0^{N-2+k-n} \leqslant \varepsilon\mu^{N+1-n}r_0^{N-n}. \tag{4.61}$$

Using this estimate in (4.49) yields:

$$\begin{aligned}\biggl|\tilde{g}_n - \frac{1}{|\nabla\omega(x_0)|}\frac{1}{n!}(v\cdot\nabla)^n\omega(x_0)\biggr| &\leqslant \frac{1}{r_0}\sum_{k=2}^{n-1}\sum_{m\in\mathbb{N}_+^k}\mathbb{1}\Biggl(\sum_{j=1}^{k} m_j = n\Biggr)\varepsilon\mu^{N+1-n}r_0^{N-n}M_k \\ &\leqslant \varepsilon\mu^{N-n}r_0^{N-n-1}\mu M_{N-1}2^{n-1} \leqslant \frac{1}{2}\varepsilon\mu^{N-n}r_0^{N-n-1},\end{aligned} \tag{4.62}$$

where we have applied the assumption made on μ, and the following equality, valid for all $n \geqslant 1$ and $1 \leqslant k \leqslant n$ (and provable, e.g., by induction, or by a combinatorial argument)

$$\sum_{m\in\mathbb{N}_+^k}\mathbb{1}\Biggl(\sum_{j=1}^{k} m_j = n\Biggr) = \binom{n-1}{k-1}. \tag{4.63}$$

If $n < N$, we can then apply the assumption (4.47) to (4.62) and obtain the bound:

$$|\tilde{g}_n| \leqslant \frac{1}{r_0}\frac{1}{2}\varepsilon(\mu r_0)^{N-n} + \frac{1}{2}\varepsilon\mu^{N-n}r_0^{N-n-1} \leqslant \varepsilon\mu^{N-n}r_0^{N-n-1}. \tag{4.64}$$

This completes the induction step and proves (4.50) for $2 \leqslant m < N$. However, then (4.62) is valid also for $n = N$, and thus also,

$$|\tilde{g}_N| \leqslant \frac{1}{r_0}\frac{M_N}{N!} + \frac{1}{2}\varepsilon r_0^{-1} \leqslant \biggl(\frac{M_N}{N!} + \frac{1}{2}\biggr)r_0^{-1}. \tag{4.65}$$

Finally, then for any $2 \leqslant k \leqslant N$ and $m \in \mathbb{N}_+^k$ such that $\sum_j m_j = N+1$,

$$\Biggl|\prod_{j=1,\ m_j>1}^{k}\tilde{g}_{m_j}\Biggr| \leqslant C r_0^{-1}. \tag{4.66}$$

To see this, note that $|\tilde g_N|$ can appear in the product only once, and the other factors are always less than one. Therefore, as in (4.62), we find:

$$|\tilde g_{N+1}| \leqslant \frac{1}{r_0}\frac{M_{N+1}}{(N+1)!} + \frac{1}{r_0}\sum_{k=2}^{N}\binom{N}{k-1} M_N C r_0^{-1} \tag{4.67}$$

which yields the bound in (4.51).

We still need to prove (4.52). By (4.59), it holds for $n=1$, so let us assume that $n \geqslant 2$. By (4.45), the left-hand side of (4.60) is then equal to $g_n(t)u_0 \cdot \nabla\omega(\gamma(t))$, implying:

$$\begin{aligned}\left| g_n(t)\big|\nabla\omega(x_0)\big| + \frac{\gamma^{(n)}(t)}{n!}\cdot\nabla\omega\big(\gamma(t)\big)\right| &\leqslant \big|g_n(t)\big|\big|\nabla\omega(x_0) - \nabla\omega\big(\gamma(t)\big)\big| \leqslant \big|g_n(t)\big| M_2 4\lambda \\ &\leqslant M_2 4\lambda|\tilde g_n| + \frac{1}{2}\big|\nabla\omega(x_0)\big|\big|g_n(t) - \tilde g_n\big|. \end{aligned}\tag{4.68}$$

Therefore, by employing the triangle inequality to change $g_n(t)$ to $\tilde g_n$ in the leftmost expression, we find that

$$\big|g_n(t) - \tilde g_n\big| \leqslant \frac{8M_2}{r_0}\lambda|\tilde g_n| + \frac{2}{r_0}\left|\big|\nabla\omega(x_0)\big|\tilde g_n + \frac{\gamma^{(n)}(t)}{n!}\cdot\nabla\omega\big(\gamma(t)\big)\right|. \tag{4.69}$$

We next need to bound the right-hand side of (4.60) minus $|\nabla\omega(x_0)|\tilde g_n$. Using the definition of $\tilde g_n$, we get a bound:

$$\begin{aligned}&\sum_{k=2}^{n}\sum_{m\in\mathbb{N}_+^k}\mathbb{1}\left(\sum_{j=1}^{k} m_j = n\right)\prod_{j=1}^{k-1}\frac{m_j}{\sum_{j'=j}^{k} m_{j'}} \\ &\quad\times\left|\prod_{j=1}^{k}\left[\frac{1}{m_j!}\gamma^{(m_j)}(t)\cdot\nabla\right]\omega\bigg|_{\gamma(t)} - \prod_{\substack{j=1\\ m_j>1}}^{k}(-\tilde g_{m_j}u_0\cdot\nabla)\prod_{\substack{j=1\\ m_j=1}}^{k}(v\cdot\nabla)\omega(x_0)\right|.\end{aligned}\tag{4.70}$$

Here the absolute value remains to be bounded, and we do this in two steps: first we shift $\gamma'(t)$ to v and higher derivatives to $\tilde g$ by using the induction assumption, and then we shift the evaluation point from $\gamma(t)$ to x_0.

To illustrate this, let us perform the estimates first for the case $k=n$, when the induction assumption is not needed, and we can therefore apply the result for any n. Then the absolute value is explicitly:

$$\begin{aligned}&\big|\big[\big(v - g_1(t)u_0\big)\cdot\nabla\big]^n\omega\big(\gamma(t)\big) - (v\cdot\nabla)^n\omega(x_0)\big| \\ &\quad\leqslant \big|\big[\big(v - g_1(t)u_0\big)\cdot\nabla\big]^n\omega\big(\gamma(t)\big) - (v\cdot\nabla)^n\omega\big(\gamma(t)\big)\big| + \big|(v\cdot\nabla)^n\omega\big(\gamma(t)\big) - (v\cdot\nabla)^n\omega(x_0)\big| \\ &\quad\leqslant \sum_{j=1}^{n}\binom{n}{j}\big|g_1(t)\big|^j M_n + M_{n+1}\big|\gamma(t) - x_0\big| \leqslant M_{n+1}\left[\sum_{j=1}^{n}\binom{n}{j}\big|g_1(t)\big| + 4\lambda\right] \leqslant a_0\lambda r_0^{-1}M_{n+1}2^{n+1}\end{aligned}\tag{4.71}$$

where we have used the Leibniz rule. But now (4.69) implies that for $n=2$,

$$\big|g_2(t) - \tilde g_2\big| \leqslant a_0\lambda r_0^{-1}|\tilde g_2| + 2a_0\lambda r_0^{-2}M_3 2^3. \tag{4.72}$$

If $N=2$, (4.51) implies then that

$$\big|g_2(t) - \tilde g_2\big| \leqslant a_0\lambda r_0^{-2}\left(\frac{1}{2}(1+M_2) + M_3 2^4\right) \leqslant a_0\lambda r_0^{-2} b|_{N=2}. \tag{4.73}$$

If $N>2$, by (4.50) $|\tilde g_2| \leqslant 1$, and thus

$$\big|g_2(t) - \tilde g_2\big| \leqslant a_0\lambda r_0^{-2}\big(1 + M_3 2^3\big) \leqslant a_0\lambda r_0^{-2} b. \tag{4.74}$$

This proves that (4.52) holds always for $n=2$.

Let us then make the induction assumption that $2 < n \leqslant N$ and (4.52) holds for all $2 \leqslant m < n$. The case $k=n$ has already been treated above, so let us assume $k<n$. We begin by estimating the result from the second step. Let $\ell = |\{j \mid m_j = 1\}|$, which now satisfies $\ell < k$. Since $k>1$, we also have $m_j \leqslant n-1$ for all j, and by (4.50), now $\prod_{j=1,\, m_j>1}^{k} |\tilde g_{m_j}| \leqslant 1$. Therefore,

$$\left| \prod_{\substack{j=1\\ m_j>1}}^{k} (-\tilde{g}_{m_j} u_0 \cdot \nabla)(v \cdot \nabla)^{\ell} \omega\big(\gamma(t)\big) - \prod_{\substack{j=1\\ m_j>1}}^{k} (-\tilde{g}_{m_j} u_0 \cdot \nabla)(v \cdot \nabla)^{\ell} \omega(x_0) \right|$$
$$\leqslant \big|\gamma(t) - x_0\big| M_{k+1} \prod_{\substack{j=1\\ m_j>1}}^{k} |\tilde{g}_{m_j}| \leqslant 4\lambda M_n \leqslant b\lambda \leqslant a_0 b^{n-2}\lambda. \tag{4.75}$$

To estimate the result from the first step, let $I_k = \{1, 2, \dots, k\}$. Using the commutativity of partial derivatives, the result can be bounded by:

$$\sum_{\substack{I\subset I_k\\ I\neq\emptyset}} \left| \prod_{j\in I} \big(g_{m_j}(t) - \tilde{g}_{m_j}\big) \prod_{\substack{j\notin I\\ m_j>1}} (-\tilde{g}_{m_j} u_0 \cdot \nabla) \prod_{\substack{j\notin I\\ m_j=1}} (-v \cdot \nabla)(-u_0 \cdot \nabla)^{|I|} \omega\big(\gamma(t)\big) \right|$$
$$\leqslant M_k \sum_{\substack{I\subset I_k\\ I\neq\emptyset}} \prod_{j\in I} \big|g_{m_j}(t) - \tilde{g}_{m_j}\big| \leqslant M_k \sum_{\substack{I\subset I_k\\ I\neq\emptyset}} \big(a_0\lambda r_0^{-1}\big)^{|I|} \prod_{j\in I} \big(b r_0^{-1}\big)^{m_j-1}$$
$$\leqslant M_{n-1} \sum_{\substack{I\subset I_k\\ I\neq\emptyset}} a_0\lambda r_0^{-1} \big(b r_0^{-1}\big)^{n-2} \leqslant M_{n-1} 2^k a_0 \lambda r_0^{1-n} b^{n-2}, \tag{4.76}$$

where we have applied,

$$\sum_{j\in I} (m_j - 1) \leqslant \sum_{j=1}^{k} (m_j - 1) = n - k \leqslant n - 2, \tag{4.77}$$

and, as $I \neq \emptyset$ and $a\lambda \leqslant r_0$, we have also $(a\lambda/r_0)^{|I|} \leqslant a\lambda/r_0$. Combining the above estimates, we then have obtained the following bound for (4.70):

$$\sum_{k=2}^{n} \binom{n-1}{k-1} a_0 b^{n-2} \lambda r_0^{1-n} \big(1 + M_n 2^n\big) \leqslant a_0 b^{n-2} \lambda r_0^{1-n} 2^{n-1} \big(1 + M_n 2^n\big). \tag{4.78}$$

Therefore, (4.69) now implies that for any $n < N$,

$$\big|g_n(t) - \tilde{g}_n\big| \leqslant \frac{a_0}{r_0}\lambda + a_0 b^{n-2} \lambda r_0^{-n} 2^n \big(1 + M_n 2^n\big)$$
$$\leqslant \frac{1 + 2^n + M_n 2^{2n}}{b} a_0 b^{n-1} \lambda r_0^{-n} \leqslant a_0 b^{n-1} \lambda r_0^{-n}, \tag{4.79}$$

by our choice of b. This completes the induction step and proves that (4.52) is valid for all $2 \leqslant n < N$. However, then we can still use the bound (4.78), together with (4.51), in (4.69) which shows that

$$\big|g_N(t) - \tilde{g}_N\big| \leqslant \frac{M_N + 1 + 2^N + M_N 2^{2N}}{b} a_0 b^{N-1} \lambda r_0^{-N} \leqslant a_0 b^{N-1} \lambda r_0^{-N}. \tag{4.80}$$

This proves that b is large enough for (4.52) to hold also for $n = N$. For $n = N + 1$, we repeat the above steps using (4.66), and the fact that (4.52) holds also for $n = N$. Then the left-hand sides of Eqs. (4.75) and (4.76) can be bounded by $4\lambda M_{N+1} C r_0^{-1}$ and $2^{N+1} M_{N+1} C r_0^{-1-N} \lambda a_0 b^{N-1}$, respectively. This yields a bound $2^{2N+2} M_{N+1} a_0 b^{N-1} C r_0^{-1-N} \lambda$ for (4.70). Then using the bound for $|\tilde{g}_{N+1}|$ given in (4.69) proves that

$$\big|g_{N+1}(t) - \tilde{g}_{N+1}\big| \leqslant \frac{a_0}{r_0} \lambda C' r_0^{-2} + C r_0^{-2-N} \lambda a_0 b^{N-1} 2^{2N+3} M_{N+1}, \tag{4.81}$$

where $C' = M_{N+1}(1 + 2^N C) \leqslant bC$. Finally, using $2^{2N+1} M_{N+1} \leqslant b$, proves (4.53). □

5. Semi-dispersivity (proof of Theorem 2.2)

Let $0 \leqslant p \leqslant 1$, $\alpha \in \mathbb{R}$, and $0 < \beta \leqslant 1$ be arbitrary, and denote $M_n = \|\omega\|_n'$ for all n. Let us define further $q = 1 + \frac{1}{2}(1-p)$, so that, if $p = 1$, also $q = 1$, and otherwise $q + p + 1 < 3$. We then apply the layer cake representation as

$$\begin{aligned}\int_{\mathbb{T}^d} \frac{\mathrm{d}k}{|\nabla\omega(k)|^p} \frac{1}{|\alpha - \omega(k) + \mathrm{i}\beta|} &= \int_0^{M_1^q} \mathrm{d}s \int_{\mathbb{T}^d} \frac{\mathrm{d}k}{|\alpha - \omega(k) + \mathrm{i}\beta|} \frac{\mathbb{1}(|\nabla\omega(k)| \geqslant s^{1/q})}{|\nabla\omega(k)|^{p+q}} \\ &\leqslant \int_0^{\beta} \frac{\mathrm{d}s}{\beta} \int_{\mathbb{T}^d} \frac{\mathrm{d}k}{|\nabla\omega(k)|^{p+q}} + \int_{\beta}^{M_1^q} \mathrm{d}s \int_{\mathbb{T}^d} \frac{\mathrm{d}k}{|\alpha - \omega(k) + \mathrm{i}\beta|} \frac{\mathbb{1}(|\nabla\omega(k)| \geqslant s^{1/q})}{|\nabla\omega(k)|^{p+q}}. \quad (5.1)\end{aligned}$$

Since $p + q \leqslant 2$, the first term is bounded by a β-independent constant by Lemma 4.1. To analyse the second term, let us define the following cut-off function $G : \mathbb{R}^d \times (0, 1/2] \to \mathbb{R}$,

$$G(x,\lambda) = \frac{N_d}{\lambda^d} \mathbb{1}\big(|x| < \lambda\big) \quad (5.2)$$

where $N_d = d/|S^{d-1}|$ is a normalisation constant such that $\int_{\mathbb{R}^d} \mathrm{d}x\, G(x,\lambda) = 1$ for all λ. We have restricted the range of λ in the above manner so that for all $k \in \mathbb{T}^d$ and λ we still have $\int_{\mathbb{T}^d} \mathrm{d}x\, G([x-k]', \lambda) = 1$ (we recall the definition of $[\cdot]'$ in Section 2.1).

By choosing $\lambda = \lambda(s) = \min(\frac{1}{4}, s^{1/q}/(9M_2))$, we then find:

$$\int_{\mathbb{T}^d} \frac{\mathrm{d}k}{|\alpha - \omega(k) + \mathrm{i}\beta|} \frac{\mathbb{1}(|\nabla\omega(k)| \geqslant s^{1/q})}{|\nabla\omega(k)|^{p+q}} = \int_{\mathbb{T}^d} \mathrm{d}x \int_{\mathbb{T}^d} \mathrm{d}k\, \frac{G([x-k]',\lambda)}{|\alpha - \omega(k) + \mathrm{i}\beta|} \frac{\mathbb{1}(|\nabla\omega(k)| \geqslant s^{1/q})}{|\nabla\omega(k)|^{p+q}}. \quad (5.3)$$

Applying Lemma 4.3 with $a = \frac{1}{9}$ shows that this is bounded by:

$$(1+a)^q \int_{\mathbb{T}^d} \mathrm{d}x \frac{\mathbb{1}(|\nabla\omega(x)| \geqslant (1-a)s^{1/q})}{|\nabla\omega(x)|^{p+q}} \frac{N_d}{\lambda^d} \int_{\mathbb{R}^d} \mathrm{d}k\, \frac{\mathbb{1}(|[x]' - k| < \lambda)}{|\alpha - \omega(k) + \mathrm{i}\beta|}. \quad (5.4)$$

Let $x_0 = [x]'$. Then inside the integral $\lambda \leqslant \frac{|\nabla\omega(x_0)|}{8M_2}$ since $9(1-a) = 8$. Therefore, Lemma 4.9 yields a diffeomorphism ψ, such that we can apply Corollary 4.10. This shows that

$$\begin{aligned}\frac{N_d}{\lambda^d} \int_{|x_0 - k| < \lambda} \frac{\mathrm{d}k}{|\alpha - \omega(k) + \mathrm{i}\beta|} &\leqslant \frac{2N_d}{\lambda^d} \int_{|y| < 2\lambda} \frac{\mathrm{d}y}{|\alpha - \omega(\psi(y)) + \mathrm{i}\beta|} \\ &\leqslant \frac{2^d N_d}{\lambda} \frac{|S^{d-2}|}{d-1} \int_{-2\lambda}^{2\lambda} \frac{\mathrm{d}y_1}{|\alpha - \omega(x_0) - |\nabla\omega(x_0)| y_1 + \mathrm{i}\beta|} \\ &\leqslant \frac{2^d N_d}{N_{d-1}} \frac{6\langle \ln\langle 2\lambda|\nabla\omega(x_0)|\rangle\rangle}{\lambda|\nabla\omega(x_0)|} \langle \ln\beta\rangle \leqslant \frac{2^d N_d}{N_{d-1}} \frac{6\langle \ln\langle M_1\rangle\rangle}{\lambda|\nabla\omega(x_0)|} \langle \ln\beta\rangle, \quad (5.5)\end{aligned}$$

where we have applied Lemma 4.6, and the properties of $\langle\cdot\rangle$ given in Appendix B together with $0 \leqslant 2\lambda|\nabla\omega(x_0)| \leqslant M_1$. Combining this with (5.4) and (5.3), we have proven that there is a constant $c' \geqslant 1$, which depends only on $M_1 = \|\omega\|_1'$, such that

$$\int_{\mathbb{T}^d} \frac{\mathrm{d}k}{|\alpha - \omega(k) + \mathrm{i}\beta|} \frac{\mathbb{1}(|\nabla\omega(k)| \geqslant s^{1/q})}{|\nabla\omega(k)|^{p+q}} \leqslant \frac{(1+a)^q c'}{\lambda(s)} \langle \ln\beta\rangle \int_{\mathbb{T}^d} \mathrm{d}x \frac{\mathbb{1}(|\nabla\omega(x)| \geqslant (1-a)s^{1/q})}{|\nabla\omega(x)|^{p+q+1}}. \quad (5.6)$$

If $p < 1$, then $p + q + 1 < 3$ and, by Lemma 4.1, the remaining integral over x can be bounded by a constant independent of s. After this, the integral over s only yields a factor:

$$\int_{\beta}^{M_1^q} \frac{\mathrm{d}s}{\lambda(s)} \leqslant \int_{0}^{M_1^q} \frac{\mathrm{d}s}{\lambda(s)} < \infty, \tag{5.7}$$

since $s^{-1/q}$ is integrable at zero, due to $q > 1$. This proves (2.7).

If $p = 1$, then $p + q + 1 = 3$, and, by Assumption 2.1, the integral over x is bounded by

$$c_0 \big\langle \ln(1-a) + \ln s \big\rangle^{p_0} \leqslant c_0 2^{p_0} \big\langle \ln(1-a) \big\rangle^{p_0} \langle \ln s \rangle^{p_0}.$$

Then the integral over s can be estimated by:

$$\int_{\beta}^{M_1} \frac{\mathrm{d}s}{\lambda(s)} \langle \ln s \rangle^{p_0} \leqslant 2\langle M_1 - s_c \rangle \max\big(\langle \ln M_1 \rangle^{p_0}, \langle \ln s_c \rangle^{p_0}\big) + 2s_c \langle s_c - 1 \rangle \langle \ln s_c \rangle^{p_0} + \int_{\beta}^{1} \mathrm{d}s\, \frac{2s_c}{s} \langle \ln s \rangle^{p_0}, \tag{5.8}$$

where $s_c = 9M_2/2$. By Lemma 4.4, the final integral can be bounded by a constant times $\langle \ln \beta \rangle^{p_0+1}$. Collecting the powers of $\langle \ln \beta \rangle$ together, and denoting the remaining factor by C_0 proves (2.8).

6. Suppression of crossings (proof of "if" in Theorem 2.3)

6.1. Uniform minimal curvature

Theorem 6.1. *Let $d \geqslant 2$, and let $\omega : \mathbb{R}^d \to \mathbb{R}$ be real-analytic and $\mathbb{Z}^d$-periodic. Then one and only one of the following alternatives is true*:

(1) *There is an affine hyperplane $M \subset \mathbb{R}^d$ such that ω is constant on M.*
(2) *There are an integer $n_0 \geqslant 2$ and a constant $\varepsilon_0 > 0$ with the following property*: *for any $k \in \mathbb{R}^d$ and $u \in S^{d-1}$, there is an integer n with $2 \leqslant n \leqslant n_0$, and a direction $v \in S^{d-1}$ orthogonal to u, such that*

$$\frac{1}{n!} \big| (v \cdot \nabla)^n \omega(k) \big| > \varepsilon_0. \tag{6.1}$$

We will use the remainder of the subsection for the proof. From now on, let d and ω satisfy the assumptions of the theorem. Let $X = C^\infty(\mathbb{R}^d, \mathbb{C})$ denote the topological vector space of smooth functions equipped with its usual Fréchet topology. The topology is uniquely determined by the local base given by the sets:

$$B^{(N)} = \left\{ f \in X \;\middle|\; p_N(f) < \frac{1}{N} \right\}, \tag{6.2}$$

with $N \in \mathbb{N}_+$ and p_N denoting the seminorm,

$$p_N(f) = \max\big\{ \big| D^\alpha f(x) \big| \;\big|\; |\alpha| \leqslant N,\ |x| \leqslant N \big\}. \tag{6.3}$$

For more details, see [14, Section 1.46].

We recall that if X and Y are two topological vector spaces with local bases $\mathcal{B}_X$ and $\mathcal{B}_Y$, respectively, then a function $F : X \to Y$ is continuous if and only if it has the following property: For all $B \in \mathcal{B}_Y$ and $x \in X$ there is $B' \in \mathcal{B}_X$ such that $F(x + B') \subset F(x) + B$. From this, it is straightforward to prove the continuity of the following two basic mappings: for any $v \in \mathbb{R}^d$, the mapping $f \mapsto v \cdot \nabla f$ is a continuous linear map $X \to X$, and for any $x \in \mathbb{R}^d$ the functional $f \mapsto f(x)$ is continuous on X. Therefore, also the functional $f \mapsto (v \cdot \nabla)^n f(0)$ is always continuous on X. This implies the following result:

Lemma 6.2. *For any $n \in \mathbb{N}_0$, $v \in \mathbb{R}^d$ and $\varepsilon \geqslant 0$, let*

$$U_{n,v,\varepsilon} = \left\{ f \in X \;\middle|\; \frac{1}{n!} \big| (v \cdot \nabla)^n f(0) \big| > \varepsilon \right\}. \tag{6.4}$$

Then every such $U_{n,v,\varepsilon}$ is open in X.

The proof of Theorem 6.1 will rely on compactness of $S^{d-1} \times \mathbb{T}^d$ and on the continuity of the following auxiliary mapping.

Definition 6.3. Let $F : S^{d-1} \times \mathbb{T}^d \to X$ be defined, for any $x_0 \in \mathbb{R}^d$, by

$$F\big(u, [x_0]\big)(x) = \omega\big(x - (x \cdot u)u + x_0\big) = \omega(Q_u x + x_0). \tag{6.5}$$

Since ω is periodic, $F(u, [x_0])$ does not depend on the choice of x_0, and, by smoothness of ω, $F(u, [x_0])$ is also always smooth. Thus F is a well-defined function $S^{d-1} \times \mathbb{T}^d \to X$, as claimed above. In addition, $F(u, k)$ is always real-analytic and constant in the direction u: $F(u, k)(x + su) = F(u, k)(x)$ for all $s \in \mathbb{R}$. Moreover, we have:

Proposition 6.4. *F is continuous.*

Proof. To prove the continuity of F, it is enough to show that for all $u_0 \in S^{d-1}$, $k_0 \in \mathbb{R}^d$ and $N \in \mathbb{N}_+$ there is $\delta > 0$ such that

$$p_N\big(F(u, k) - F(u_0, k_0)\big) < \frac{1}{N} \tag{6.6}$$

for all $u \in S^{d-1}$ and $k \in \mathbb{R}^d$ with $|k - k_0| < \delta$ and $|u - u_0| < \delta$. In order to prove this property, we first note that for any multi-index α there is a finite collection of constants $c_{\beta,\gamma}(\alpha)$, such that for all x, u, k,

$$D^\alpha F(u, k)(x) = \sum_{\beta:\ |\beta|=|\alpha|} \ \sum_{\gamma:\ |\gamma|\leqslant 2|\alpha|} c_{\beta,\gamma}(\alpha) u^\gamma D^\beta \omega\big(x - (x \cdot u)u + k\big), \tag{6.7}$$

which can be proven by straightforward induction in $|\alpha|$. Therefore,

$$\begin{aligned} \big|D^\alpha F(u, k)(x) - D^\alpha F(u_0, k_0)(x)\big| &\leqslant \sum_{\beta,\gamma} \big|c_{\beta,\gamma}(\alpha)\big| |u^\gamma - u_0^\gamma| \|\omega\|'_{|\alpha|} \\ &+ \sum_{\beta,\gamma} \big|c_{\beta,\gamma}(\alpha)\big| \big|D^\beta \omega\big(x - (x \cdot u)u + k\big) - D^\beta \omega\big(x - (x \cdot u_0)u_0 + k_0\big)\big|. \end{aligned} \tag{6.8}$$

Let $\delta > 0$, and choose any $|k - k_0| < \delta$, $|u - u_0| < \delta$. Then by the Leibniz rule and $|u|, |u_0| = 1$, we find $|u^\gamma - u_0^\gamma| \leqslant 2^{|\gamma|}\delta$. By expressing the difference as an integral over a derivative in the direction of the line connecting the points, we obtain the estimate:

$$\big|D^\beta \omega\big(x - (x \cdot u)u + k\big) - D^\beta \omega\big(x - (x \cdot u_0)u_0 + k_0\big)\big| \leqslant \|\omega\|'_{|\beta|+1} \big|(x \cdot u)u - (x \cdot u_0)u_0 + k - k_0\big|, \tag{6.9}$$

where, for all $|x| \leqslant N$,

$$\big|(x \cdot u)u - (x \cdot u_0)u_0 + k - k_0\big| \leqslant 2|x||u - u_0| + |k - k_0| \leqslant (2N + 1)\delta. \tag{6.10}$$

By (6.8), then

$$p_N\big(F(u, k) - F(u_0, k_0)\big) \leqslant \delta\big(4^N \|\omega\|'_N + (2N + 1)\|\omega\|'_{N+1}\big) \max_{|\alpha|\leqslant N} \sum_{\beta,\gamma} \big|c_{\beta,\gamma}(\alpha)\big|. \tag{6.11}$$

Since ω is periodic, $\|\omega\|'_n < \infty$ for all $n \in \mathbb{N}$, which implies that the factor multiplying δ on the right-hand side is always finite. Thus by choosing a small enough δ, the bound can be made less than $1/N$. □

Lemma 6.5. *Let $u \in S^{d-1}$, $k \in \mathbb{R}^d$ be given, and denote $f = F(u, k)$. Then either f is constant, or there is $n \geqslant 2$, $v \in S^{d-1}$, and $\varepsilon > 0$, such that $|(v \cdot \nabla)^n f(0)| > n!\varepsilon$.*

Proof. Suppose f is not constant. Then there is $x_0 \neq 0$ such that $f(x_0) \neq f(0)$. Let us define $v = x_0/|x_0|$, when $v \in S^{d-1}$, and let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(t) = f(tv) - tv \cdot \nabla f(0) - f(0)$. Then g is real-analytic with $g(0) = 0$ and $g'(0) = 0$. If $g^{(n)}(0) = 0$ for all $n \geqslant 2$, then $g = 0$ everywhere, i.e., $f(tv) = tv \cdot \nabla f(0) + f(0)$ for all $t \in \mathbb{R}$. Since $f(|x_0|v) \neq f(0)$, then necessarily $v \cdot \nabla f(0) \neq 0$, and thus $\lim_{t\to\infty} |f(tv)| = \infty$. However, this contradicts the obvious bound $\|f\|_0 \leqslant \|\omega\|'_0 < \infty$, and thus we can conclude that there is $n \geqslant 2$ such that $g^{(n)}(0) = (v \cdot \nabla)^n f(0) \neq 0$. Thus, for instance, $\varepsilon = |(v \cdot \nabla)^n f(0)|/(2n!) > 0$ suffices for the bound in the lemma. □

Proof of Theorem 6.1. Let us first note that for any $u \in S^{d-1}$, $x_0 \in \mathbb{R}^d$, the image of $x \mapsto x - (x \cdot u)u + x_0$ is exactly the affine hyperplane $\{x \in \mathbb{R}^d \mid x \cdot u = x_0 \cdot u\}$. Thus the first alternative is true if and only if there is $u \in S^{d-1}$, $x_0 \in \mathbb{R}^d$ such that $F(u, k)$, $k = [x_0]$, is constant. On the other hand, then also $g_v(t) = F(u, k)(tv) = \omega(tQ_u v + x_0)$ is constant for any $v \in S^{d-1}$, and thus $0 = g_v^{(n)}(0) = (Q_u v \cdot \nabla)^n \omega(k)$ for all $n \geqslant 2$. This proves that the second alternative is false when the first is true.

Suppose that the first alternative is false and let $K = S^{d-1} \times \mathbb{T}^d$. Then $F(u, k)$ is never a constant function. Thus by Lemma 6.5,

$$F(K) \subset \bigcup_{n,v,\varepsilon} U_{n,v,\varepsilon}, \tag{6.12}$$

where $U_{n,v,\varepsilon}$ is defined by (6.4), and the union is taken over all $n \in \mathbb{N}$, with $n \geqslant 2$, and $v \in S^{d-1}$, $\varepsilon > 0$. Since K is compact and, by Proposition 6.4, F is continuous, $F(K)$ is compact. Using Lemma 6.2, we can thus conclude that $U_{n,v,\varepsilon}$ form an open cover of the compact set $F(K)$, and thus there is a finite sequence $(n_i, v_i, \varepsilon_i)$ such that $(U_{n_i,v_i,\varepsilon_i})$ cover the whole image of F. Let

$$\varepsilon_0 = \min_i \varepsilon_i > 0 \quad \text{and} \quad n_0 = \max_i n_i \geqslant 2. \tag{6.13}$$

Let $u \in S^{d-1}$, $x_0 \in \mathbb{R}^d$ be arbitrary, and let $k = [x_0]$. There is an index i such that $f = F(u, k) \in U_{n,v',\varepsilon_i}$, with $v' = v_i \in S^{d-1}$ and $n = n_i$. Then $2 \leqslant n \leqslant n_0$, and $|(v' \cdot \nabla)^n f(0)| > n!\varepsilon_i \geqslant n!\varepsilon_0$. Since $(v' \cdot \nabla)^n f(x) = (Q_u v' \cdot \nabla)^n \omega|_{Q_u x + x_0}$, we have also $|(v \cdot \nabla)^n \omega(x_0)| > n!\varepsilon_0$ with $v = Q_u v'/|Q_u v'|$ (note that obviously $|Q_u v'| \neq 0$). As $v \cdot u = 0$, the pair n, v has the properties required by the second alternative. $\square$

6.2. *Crossing estimate*

Let us assume that ω is not a constant on any affine hyperplane. Then we can find constants $n_0 \geqslant 2$ and $0 < \varepsilon_0 \leqslant \frac{1}{2}$, for which the second alternative in Theorem 6.1 holds. As in Proposition 4.11, let $M_n = \|\omega\|'_n$, $a_0 = \max(1, 8M_2)$, and define:

$$\mu = \frac{1}{1 + 2^{n_0+3} + M_{n_0+1} 2^{4n_0+1}}, \tag{6.14}$$

when $0 < \mu \leqslant \frac{1}{33}$, and μ satisfies the conditions of the proposition for any $2 \leqslant N \leqslant n_0$. We also define for any given $0 < r \leqslant 1$ and $2 \leqslant N \leqslant n_0$,

$$\varepsilon(r, N) = \varepsilon_0 (r\mu)^{n_0 - N} \leqslant \varepsilon_0 \leqslant \frac{1}{2}. \tag{6.15}$$

Consider arbitrary given $k_0 \in \mathbb{T}^d$, $\alpha \in \mathbb{R}^3$, and $0 < \beta \leqslant 1$. We need to estimate:

$$I = I_{3\mathrm{cr}}(\alpha, k_0, \beta) = \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1\, \mathrm{d}k_2 \prod_{j=1}^{3} \frac{1}{|\alpha_j - \omega(k_j) + \mathrm{i}\beta|} \tag{6.16}$$

where $k_3 = k_3(k_1, k_2) = k_1 - k_2 + k_0$. By using a layer cake representation,

$$\begin{aligned} I &= \int_0^{M_1} \mathrm{d}s \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1\, \mathrm{d}k_2 \prod_{j=1}^{3} \frac{1}{|\alpha_j - \omega(k_j) + \mathrm{i}\beta|} \frac{\mathbb{1}(\min_j |\nabla\omega(k_j)| \geqslant s)}{\min_j |\nabla\omega(k_j)|} \\ &\leqslant 3 \int_0^{\beta^\gamma} \frac{\mathrm{d}s}{\beta} \int_{\mathbb{T}^d} \frac{\mathrm{d}k'}{|\alpha - \omega(k') + \mathrm{i}\beta|} \int_{\mathbb{T}^d} \frac{\mathrm{d}k}{|\nabla\omega(k)|} \frac{1}{|\alpha - \omega(k) + \mathrm{i}\beta|} \\ &\quad + \int_{\beta^\gamma}^{M_1} \mathrm{d}s \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1 \mathrm{d}k_2 \prod_{j=1}^{3} \frac{1}{|\alpha_j - \omega(k_j) + \mathrm{i}\beta|} \frac{\mathbb{1}(\min_j |\nabla\omega(k_j)| \geqslant s)}{\min_j |\nabla\omega(k_j)|}, \end{aligned} \tag{6.17}$$

where, to get the first term, we have used $\frac{1}{\min_j |\nabla\omega(k_j)|} \leqslant \sum_j \frac{1}{|\nabla\omega(k_j)|}$ and then estimated one of the factors trivially, followed by a change of variables. Using Theorem 2.2, the first term is bounded by:

$$3C_0C_1\langle \ln\beta\rangle^{p_0+3}\beta^{-(1-\gamma)}, \tag{6.18}$$

and so it is "harmless" for any $\gamma > 0$.

To estimate the second term, let us define for any $s > 0$,

$$r_0(s) = \min\left(1, \frac{s}{2}\right), \quad \lambda(s) = \frac{1}{4}\min\left(\frac{1}{2}, \frac{\varepsilon_0}{a_0}\big(r_0(s)\mu\big)^{n_0}\right) \quad \text{and} \quad \delta(s) = \lambda(s). \tag{6.19}$$

By employing the cut-off function G introduced in Section 5 inside the k integrals,

$$\begin{aligned} I_2 &= \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1\,\mathrm{d}k_2 \prod_{j=1}^{3} \frac{1}{|\alpha_j - \omega(k_j) + \mathrm{i}\beta|} \frac{\mathbb{1}(\min_j |\nabla\omega(k_j)| \geqslant s)}{\min_j |\nabla\omega(k_j)|} \\ &= \int_{(\mathbb{T}^d)^2} \mathrm{d}x_1\,\mathrm{d}x_2 \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1\mathrm{d}k_2 \prod_{j=1}^{2} \frac{G(x_j - k_j, \lambda(s))}{|\alpha_j - \omega(k_j) + \mathrm{i}\beta|} \frac{1}{|\alpha_3 - \omega(k_3) + \mathrm{i}\beta|} \frac{\mathbb{1}(\min_j |\nabla\omega(k_j)| \geqslant s)}{\min_j |\nabla\omega(k_j)|}. \end{aligned} \tag{6.20}$$

Let $x_3 = x_1 - x_2 + k_0$, when inside the integral, for $j = 1, 2, 3$,

$$\big|\big|\nabla\omega(k_j)\big| - \big|\nabla\omega(x_j)\big|\big| \leqslant |k_j - x_j| M_2 \leqslant 2\lambda M_2 \leqslant \frac{s}{2} \leqslant \frac{|\nabla\omega(k_j)|}{2}, \tag{6.21}$$

since $|k_j - x_j| \leqslant \lambda < 2\lambda$ for $j = 1, 2$, and $|k_3 - x_3| \leqslant 2\lambda$. Therefore, we have: $|\nabla\omega(x_j)| \geqslant \frac{1}{2}|\nabla\omega(k_j)| \geqslant \frac{s}{2}$ and $2|\nabla\omega(k_j)| \geqslant |\nabla\omega(x_j)|$, for all j, and thus

$$I_2 \leqslant 2 \int_{(\mathbb{T}^d)^2} \mathrm{d}x_1\,\mathrm{d}x_2 \frac{\mathbb{1}(\min_j |\nabla\omega(x_j)| \geqslant \frac{1}{2}s)}{\min_j |\nabla\omega(x_j)|} \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1\,\mathrm{d}k_2 \prod_{j=1}^{2} \frac{G(x_j - k_j, \lambda)}{|\alpha_j - \omega(k_j) + \mathrm{i}\beta|} \frac{1}{|\alpha_3 - \omega(k_3) + \mathrm{i}\beta|}. \tag{6.22}$$

In particular, now inside the x integrals $r_0(s) \leqslant \min(1, |\nabla\omega(x_j)|)$, for all $j = 1, 2, 3$, and, since $\lambda(s) \leqslant r_0(s)/a_0$, we can apply the results of Proposition 4.11 around any of the points x_j.

We next need to estimate, for given $\alpha \in \mathbb{R}^3$ and $x_j \in \mathbb{T}^d$ the integral,

$$J = \int_{(\mathbb{T}^d)^2} \mathrm{d}k_1\,\mathrm{d}k_2 \prod_{j=1}^{2} \frac{G(x_j - k_j, \lambda)}{|\alpha_j - \omega(k_j) + \mathrm{i}\beta|} \frac{1}{|\alpha_3 - \omega(k_3) + \mathrm{i}\beta|}, \tag{6.23}$$

assuming $\min_j |\nabla\omega(x_j)| \geqslant s/2 \geqslant \beta^\gamma/2 > 0$. Since then $\nabla\omega(x_j) \neq 0$, we can define for all $j = 1, 2, 3$,

$$u_j = \frac{\nabla\omega(x_j)}{|\nabla\omega(x_j)|} \in S^{d-1}. \tag{6.24}$$

We apply different estimates depending on whether all u_j are almost parallel or not. A sufficient degree of separation turns out to be determined by the parameter δ defined in (6.19), for which in particular $0 < \delta \leqslant \frac{1}{2}$. The first of the estimates is applied, if

$$|u_1 \cdot u_3| \leqslant \sqrt{1 - \delta^2} \quad \text{or} \quad |u_2 \cdot u_3| \leqslant \sqrt{1 - \delta^2}, \tag{6.25}$$

and otherwise the second estimate is used.

In Section 6.2.1 we shall prove that in the first case there is a constant C_1', depending only on ω, such that

$$J \leqslant \frac{\langle \ln\beta\rangle^3}{\delta(s)\lambda(s)^3} \frac{C_1'}{\prod_{j=1}^{3} |\nabla\omega(x_j)|}. \tag{6.26}$$

The other, more involved estimate, is done in Section 6.2.2. There we prove that, if we choose

$$\gamma = \frac{1}{3n_0(n_0 + 1)}, \tag{6.27}$$

then in the second case there are constants C_2' and β_0, depending only on ω, such that for all $0 < \beta \leqslant \beta_0$,

$$J \leqslant \frac{\langle \ln \beta \rangle^2}{\lambda(s)^3} \beta^{1/(n_0+1)-1} \frac{C_2'}{\prod_{j=1}^3 |\nabla \omega(x_j)|}. \tag{6.28}$$

After applying either one of the inequalities, the remaining integral over x_j can be estimated using the Young inequality,

$$\begin{aligned} &\int_{(\mathbb{T}^d)^2} \mathrm{d}x_1\, \mathrm{d}x_2 \frac{\mathbb{1}(\min_j |\nabla\omega(x_j)| \geqslant \frac{1}{2}s)}{\min_j |\nabla\omega(x_j)|} \frac{1}{\prod_{j=1}^3 |\nabla\omega(x_j)|} \\ &\quad \leqslant \sum_{j=1}^3 \int_{(\mathbb{T}^d)^2} \mathrm{d}x_1\, \mathrm{d}x_2 \frac{\mathbb{1}(|\nabla\omega(x_j)| \geqslant \frac{1}{2}s)}{|\nabla\omega(x_j)|^2} \prod_{j'=1;\ j'\neq j}^3 \frac{1}{|\nabla\omega(x_{j'})|} \\ &\quad \leqslant 3 \biggl(\int_{\mathbb{T}^d} \mathrm{d}x \frac{\mathbb{1}(|\nabla\omega(x)| \geqslant \frac{1}{2}s)}{|\nabla\omega(x)|^3} \biggr)^{2/3} \biggl(\int_{\mathbb{T}^d} \mathrm{d}x \frac{1}{|\nabla\omega(x)|^{3/2}} \biggr)^{4/3}. \end{aligned} \tag{6.29}$$

Thus by Lemma 4.1 and Assumption 2.1, there is a constant C' such that for all sufficiently small β

$$I_2(s) \leqslant C' \langle \ln \beta \rangle^{2p_0/3+3} \bigl(\lambda(s)^{-4} + \lambda(s)^{-3} \beta^{1/(n_0+1)-1} \bigr). \tag{6.30}$$

If $s \geqslant 2$, then $r_0 = 1$ and $\lambda(s)$ is equal to a non-zero constant, implying that the bound in (6.30) is independent of s, and the integral over $2 \leqslant s \leqslant M_1$ is thus easily estimated. If $0 < s \leqslant 2$, we have $r_0 = s/2$, and thus for these s,

$$\lambda(s) = \frac{\varepsilon_0}{4a_0} (\mu s/2)^{n_0} \geqslant 2^{-2-n_0} \varepsilon_0 \mu^{n_0} s^{n_0}. \tag{6.31}$$

Therefore, there is C'' such that

$$\int_{\beta^\gamma}^2 \mathrm{d}s\, I_2(s) \leqslant C'' \langle \ln \beta \rangle^{2p_0/3+3} \bigl(\beta^{-\gamma(4n_0-1)} + \beta^{-\gamma(3n_0-1)+1/(n_0+1)-1} \bigr), \tag{6.32}$$

where, by our choice of γ,

$$1 - \gamma(4n_0 - 1) = \gamma\bigl(3n_0^2 - n_0 + 1\bigr) \geqslant \gamma \quad \text{and} \quad -\gamma(3n_0 - 1) + \frac{1}{n_0+1} = \gamma. \tag{6.33}$$

Collecting all the results together, we have now proven that there are constants β_0 and C, depending only on ω, such that for all $0 < \beta \leqslant \beta_0$, $\alpha \in \mathbb{R}$, and $k_0 \in \mathbb{T}^d$,

$$I_{3\mathrm{cr}}(\alpha, k_0, \beta) \leqslant C \langle \ln \beta \rangle^{p_0+3} \beta^{\gamma-1}. \tag{6.34}$$

For $\beta \geqslant \beta_0$, we can trivially estimate $I_{3\mathrm{cr}}(\alpha, k_0, \beta) \leqslant \beta_0^{-3}$, which allows us to conclude that the dispersion relation suppresses crossings with a power of (at least) γ. Therefore, we only need to derive the estimates (6.26) and (6.28) to complete the proof of Theorem 2.3.

We proved the result for γ defined in (6.27). This value is not optimal, as shown by the one example for which the power has previously been estimated, that is, for the nearest neighbour interaction in $d = 3$. The corresponding dispersion relation is a Morse function, but there are points at which its Hessian vanishes. Thus we need to take at least $n_0 = 3$ above, which would yield $\gamma \leqslant \frac{1}{36}$. However, in [7] it has been proven that a power $\gamma = \frac{1}{4}$ can be allowed for this case.

6.2.1. Non-parallel gradients

We assume in this subsection that (6.25) holds, meaning that u_3 is not nearly parallel to one of the vectors u_1 or u_2, say to the vector u_1. This will allow us to estimate the k_3-factor in the crossing integral by integrating it out in the direction determined by the projection of u_3 orthogonal to the level set of ω at k_1. As we will show next, this yields the estimate given in (6.26).

Let us consider the integral defining J in (6.23). We change the integration variables in the following manner: if $|u_1 \cdot u_3| \leqslant \sqrt{1-\delta^2}$, we define $k_1' = k_1 - x_1$ and $k_2' = k_2 - x_2$, when $k_3 = k_1' - k_2' + x_3$. Otherwise, $|u_2 \cdot u_3| \leqslant \sqrt{1-\delta^2}$, and we define $k_1' = k_2 - x_2$ and $k_2' = k_1 - x_1$, when $k_3 = -(k_1' - k_2') + x_3$. It is thus enough derive the bound assuming $|u_1 \cdot u_3| \leqslant \sqrt{1-\delta^2}$, if we allow slightly more general dependence of k_3 on the integration variables, namely if we assume $k_3 = x_3 + \sigma(k_1' - k_2')$, with $\sigma = \pm 1$ (swapping the indices $1 \leftrightarrow 2$ in the result then produces the corresponding bound for the second case).

As mentioned in Section 6.2, λ is small enough that we can apply Lemma 4.9 and obtain two diffeomorphisms ψ_1 and ψ_2 such that Corollary 4.10 holds. This shows that

$$J \leqslant 2^2 \int\limits_{|y|<2\lambda} \mathrm{d}y \int\limits_{|y'|<2\lambda} \mathrm{d}y' \frac{N_d^2}{\lambda^{2d}} \frac{1}{|\alpha_1 - \omega(x_1) - |\nabla\omega(x_1)|y_1 + \mathrm{i}\beta|} \times \frac{1}{|\alpha_2 - \omega(x_2) - |\nabla\omega(x_2)|y_1' + \mathrm{i}\beta|} \frac{1}{|\alpha_3 - \omega(x_3 + \gamma(y', y)) + \mathrm{i}\beta|}, \tag{6.35}$$

where

$$\gamma(y', y) = \sigma\big(\psi_1(y) - x_1 - \big(\psi_2(y') - x_2\big)\big). \tag{6.36}$$

By Lemma 4.9, always

$$\big|\gamma(y', y)\big| \leqslant \big|\psi_1(y) - x_1\big| + \big|\psi_2(y') - x_2\big| < 8\lambda. \tag{6.37}$$

Let us denote $A = D\psi_1(0)$, which is a rotation in $\mathbb{R}^d$ with $A^T u_1 = e_1$. Let $v = Q_{u_1} u_3$ and $v' = A^T v$. By assumption, then

$$|v|^2 = 1 - (u_1 \cdot u_3)^2 > \delta^2, \tag{6.38}$$

implying that $|v'| = |v| > \delta > 0$. Also, $v_1' = 0$ since $v \cdot u_1 = 0$. Thus there is a rotation O of $\mathbb{R}^d$ for which $Oe_1 = e_1$ and $Ov' = |v'|e_2$. We change the integration variable y to $z = Oy$, yielding:

$$\begin{aligned} J &\leqslant \frac{2^2 N_d^2}{\lambda^{2d}} \int\limits_{|z|<2\lambda} \mathrm{d}z \int\limits_{|y'|<2\lambda} \mathrm{d}y' \frac{1}{|\alpha_1 - \omega(x_1) - |\nabla\omega(x_1)|z_1 + \mathrm{i}\beta|} \\ &\quad \times \frac{1}{|\alpha_2 - \omega(x_2) - |\nabla\omega(x_2)|y_1' + \mathrm{i}\beta|} \frac{1}{|\alpha_3 - \omega(x_3 + \gamma(y', O^T z)) + \mathrm{i}\beta|} \\ &\leqslant \frac{2^2 N_d^2}{\lambda^{2d}} \frac{(2\lambda)^{2d-3}}{N_{d-2}N_{d-1}} \int\limits_{|z_1|<2\lambda} \mathrm{d}z_1 \frac{1}{|\alpha_1 - \omega(x_1) - |\nabla\omega(x_1)|z_1 + \mathrm{i}\beta|} \int\limits_{|y_1'|<2\lambda} \mathrm{d}y_1' \frac{1}{|\alpha_2 - \omega(x_2) - |\nabla\omega(x_2)|y_1' + \mathrm{i}\beta|} \\ &\quad \times \sup_{\substack{|z|,|y'|<2\lambda \\ z_2=0}} \int\limits_{t^2<(2\lambda)^2-z^2} \mathrm{d}t \frac{1}{|\alpha_3 - \omega(x_3 + \gamma(y', O^T(z + te_2))) + \mathrm{i}\beta|}. \end{aligned} \tag{6.39}$$

Let us first estimate the final term, of the form

$$\int\limits_{|t|<R} \mathrm{d}t \frac{1}{|\alpha_3 - f(t) + \mathrm{i}\beta|}, \tag{6.40}$$

where

$$f(t) = \omega\big(x_3 + \Gamma(t)\big), \quad \text{with } \Gamma(t) = \gamma\big(y', O^T z + tO^T e_2\big). \tag{6.41}$$

Clearly, $|f| \leqslant M_0 < \infty$, and we shall later show that

$$\big|f'(t)\big| \geqslant \frac{1}{4}\big|\nabla\omega(x_3)\big|\delta > 0. \tag{6.42}$$

This allows applying Lemma 4.7, and proves that

$$\int\limits_{|t|<R} \mathrm{d}t\, \frac{1}{|\alpha_3 - f(t) + \mathrm{i}\beta|} \leqslant \frac{4 \cdot 6\langle \ln\langle M_0\rangle\rangle}{|\nabla\omega(x_3)|\delta} \langle \ln\beta\rangle. \tag{6.43}$$

Therefore, applying Proposition 4.6 to (6.39), yields the bound,

$$J \leqslant \frac{2^{2d+1} N_d^2}{N_{d-2}N_{d-1}} \frac{6^3\langle \ln\beta\rangle^3}{\delta\lambda^3} \frac{\langle \ln\langle M_0\rangle\rangle\langle \ln\langle M_1\rangle\rangle^2}{\prod_{j=1}^3 |\nabla\omega(x_j)|}. \tag{6.44}$$

Collecting all the constants into C_1' proves that (6.26) is valid in this case.

We still need to prove (6.42). From the definition of f,

$$f'(t) = \Gamma'(t) \cdot \nabla\omega\big(x_3 + \Gamma(t)\big). \tag{6.45}$$

Since $O^T e_2 = v'/|v'| = A^T v/|v|$,

$$\Gamma(t) = \sigma\big(\psi_1\big(O^T z + tA^T v/|v|\big) - x_1\big) - \sigma\big(\psi_2(y') - x_2\big), \tag{6.46}$$

which, together with (4.35), implies:

$$\Gamma'(t) = \frac{\sigma}{|v|} D\psi_1|_{O^T z + tv'/|v|} A^T v = \frac{\sigma}{|v|}\left(v - \frac{\nabla\omega(x)\cdot v}{\nabla\omega(x)\cdot u_1} u_1\right), \tag{6.47}$$

where $x = x(t) = \psi(O^T z + tA^T v/|v|)$, and $|\nabla\omega(x) - \nabla\omega(x_1)| \leqslant \frac{1}{2}|\nabla\omega(x_1)|$. Therefore,

$$f'(t) = \frac{\sigma}{|v|}\left(v\cdot\nabla\omega(x_3) + v\cdot\big[\nabla\omega\big(x_3+\Gamma(t)\big) - \nabla\omega(x_3)\big] - \frac{\nabla\omega(x)\cdot v}{\nabla\omega(x)\cdot u_1} u_1 \cdot \nabla\omega\big(x_3 + \Gamma(t)\big)\right). \tag{6.48}$$

Here $v\cdot\nabla\omega(x_3) = |\nabla\omega(x_3)| v\cdot u_3$, and $v\cdot u_3 = 1 - (u_1\cdot u_3)^2 = v^2$. Thus

$$\big|f'(t)\big| \geqslant \big|\nabla\omega(x_3)\big||v| - \left(\big|\nabla\omega\big(x_3+\Gamma(t)\big) - \nabla\omega(x_3)\big| + \frac{|\nabla\omega(x)\cdot v|}{|v||\nabla\omega(x)\cdot u_1|}\big|u_1\cdot\nabla\omega\big(x_3+\Gamma(t)\big)\big|\right). \tag{6.49}$$

By (6.37), $|\Gamma(t)| < 8\lambda$, and thus

$$\big|\nabla\omega\big(x_3+\Gamma(t)\big) - \nabla\omega(x_3)\big| \leqslant M_2\big|\Gamma(t)\big| < 8M_2\lambda = \lambda'. \tag{6.50}$$

In addition,

$$u_1\cdot\nabla\omega\big(x_3+\Gamma(t)\big) = \big|\nabla\omega(x_3)\big| u_1\cdot u_3 + u_1\cdot\big[\nabla\omega\big(x_3+\Gamma(t)\big) - \nabla\omega(x_3)\big], \tag{6.51}$$

yielding $|u_1\cdot\nabla\omega(x_3+\Gamma(t))| \leqslant |\nabla\omega(x_3)| + \lambda'$. As $u_1\cdot v = 0$, now $\nabla\omega(x)\cdot v = [\nabla\omega(x) - \nabla\omega(x_1)]\cdot v$, and so

$$\big|\nabla\omega(x)\cdot v\big| \leqslant |v||x - x_1|M_2 < |v|\frac{1}{2}\lambda'. \tag{6.52}$$

Similarly, $|\nabla\omega(x)\cdot u_1| \geqslant |\nabla\omega(x_1)| - \frac{1}{2}\lambda'$. Since $\lambda' = 8M_2\lambda \leqslant \frac{\delta s}{8} \leqslant \frac{\delta}{4}|\nabla\omega(x_j)|$, we can conclude that

$$\frac{|\nabla\omega(x)\cdot v|}{|v||\nabla\omega(x)\cdot u_1|} \leqslant \frac{\lambda'}{|\nabla\omega(x_1)|} \leqslant \frac{\delta}{4}. \tag{6.53}$$

Therefore,

$$\begin{aligned}\big|f'(t)\big| &\geqslant \big|\nabla\omega(x_3)\big||v| - \frac{\delta}{4}\big|\nabla\omega(x_3)\big| - \frac{\delta}{4}\big|\nabla\omega(x_3)\big|\left(1 + \frac{\delta}{4}\right)\\ &\geqslant \big|\nabla\omega(x_3)\big|\left(|v| - \frac{3}{4}\delta\right) \geqslant \big|\nabla\omega(x_3)\big|\frac{1}{4}\delta > 0,\end{aligned} \tag{6.54}$$

and we have arrived at the estimate (6.42).

6.2.2. Nearly parallel gradients

We assume here that (6.25) is not true, i.e., that all three of the vectors u_j are nearly parallel to each other. In this case, we cannot integrate the k_3-term in the direction of its gradient. Instead, we will show that there is a direction essentially orthogonal to the gradient in which the k_3-resolvent can be integrated leading to some degree of additional decay. The additional decay will be induced by the higher order curvature of the level sets. However, we need to choose the point k_j and the direction of integration carefully, in order to make sure that the known curvature is the dominant effect. In particular, we cannot any more consider the two d-dimensional integrals independently, but we have to choose the direction in the full $2d$-dimensional space. Since we need to inspect higher order curvature effects, we will need the full machinery of the technical lemmas here.

By assumption, $|u_j \cdot u_3| > \sqrt{1-\delta^2}$ for both $j = 1, 2$. For any $u, v \in S^{d-1}$, by direct computation

$$|Q_u v| = \sqrt{1-(u \cdot v)^2}. \tag{6.55}$$

Since

$$u_1 \cdot u_2 = \big((u_1 \cdot u_3)u_3 + Q_{u_3}u_1\big) \cdot u_2 = (u_1 \cdot u_3)(u_2 \cdot u_3) + (Q_{u_3}u_1) \cdot (Q_{u_3}u_2), \tag{6.56}$$

and $|Q_{u_3}u_j| = \sqrt{1-(u_3 \cdot u_j)^2} < \delta$, we have:

$$|u_1 \cdot u_2| \geqslant |u_1 \cdot u_3||u_2 \cdot u_3| - |Q_{u_3}u_1||Q_{u_3}u_2| > 1 - 2\delta^2. \tag{6.57}$$

As $\sqrt{1-x} \geqslant 1 - 2x$ for all $0 \leqslant x \leqslant \frac{1}{4}$, we can conclude that for all $j, j' \in \{1, 2, 3\}$,

$$|u_j \cdot u_{j'}| > 1 - 2\delta^2 \geqslant \frac{1}{2}. \tag{6.58}$$

Since ω is not a constant on any affine hyperplane, the second alternative of Theorem 6.1 is valid, and we can thus find $v_3 \in S^{d-1}$ such that $v_3 \cdot u_3 = 0$ and for some $2 \leqslant \bar{n} \leqslant n_0$,

$$\frac{1}{\bar{n}!}\big|(v_3 \cdot \nabla)^{\bar{n}} \omega(x_3)\big| > \varepsilon_0 \geqslant \varepsilon\big(r_0(s), \bar{n}\big). \tag{6.59}$$

Let $\bar{v}_j = Q_{u_j} v_3$ for $j = 1, 2$. Since

$$|u_j \cdot v_3| = |Q_{u_3}u_j \cdot v_3| \leqslant |Q_{u_3}u_j| = \sqrt{1-(u_j \cdot u_3)^2} < \delta, \tag{6.60}$$

then $|\bar{v}_j| = \sqrt{1-(u_j \cdot v_3)^2} > \sqrt{1-\delta^2} > 0$. Therefore, we can define further $v_j = \bar{v}_j/|\bar{v}_j| \in S^{d-1}$, when $v_j \cdot v_3 = |\bar{v}_j| > \sqrt{1-\delta^2}$. This implies, by the same argument as for u_j, that for all $j, j' \in \{1, 2, 3\}$, $v_j \cdot v_{j'} > 1 - 2\delta^2$, and thus also:

$$|v_j - v_{j'}| = \sqrt{2(1 - v_j \cdot v_{j'})} < 2\delta. \tag{6.61}$$

We have now constructed unit vectors v_j, $j = 1, 2, 3$, such that $u_j \cdot v_j = 0$. For each j let us associate an integer n_j the smallest of integers $n \geqslant 2$ for which

$$\frac{1}{n!}\big|(v_j \cdot \nabla)^n \omega(x_j)\big| > \varepsilon(r_0(s), n), \tag{6.62}$$

if no such integer exists, let $n_j = \infty$. Let j_0 be an index which has the smallest n_j, and denote $N = n_{j_0}$. Since $n_3 \leqslant \bar{n} \leqslant n_0$, then $2 \leqslant N \leqslant n_0 < \infty$. Let $\varepsilon = 2\varepsilon(r_0, N) = 2\varepsilon_0(r_0\mu)^{n_0-N} \leqslant 1$. For any $2 \leqslant n < N$ and $j = 1, 2, 3$, we have by construction:

$$\frac{1}{n!}\big|(v_j \cdot \nabla)^n \omega(x_j)\big| \leqslant \varepsilon(r_0, n) = \frac{1}{2}\varepsilon(r_0\mu)^{N-n}, \tag{6.63}$$

and $\frac{1}{N!}|(v_{j_0} \cdot \nabla)^N \omega(x_{j_0})| > \frac{1}{2}\varepsilon$.

Let π be the unique cyclic permutation of the indices $(1, 2, 3)$ for which $\pi(3) = j_0$, and let us define $k'_j = k_{\pi(j)}$, and permute α, x, u and n similarly to yield α', x', u' and n'. We change the integration variables from (k_1, k_2) to (k'_1, k'_2). This modifies the functional dependence of k'_3 on the integration variables: for $j_0 = 3$, $k'_3 = k'_1 - k'_2 + k_0$, for $j_0 = 2$,

$k'_3 = k'_2 - k'_1 + k_0$, and for $j_0 = 1$, $k'_3 = k'_1 + k'_2 - k_0$. (The dependence of x'_3 on x'_1, x'_2, and k_0, changes accordingly with j_0.) On the other hand, since $|k_j - x_j| < \lambda$, the new integration region is contained in $|k'_1 - x'_1|, |k'_2 - x'_2| < 2\lambda$. Suppose we can find a bound for $J' = \sup_{\sigma\in\{\pm 1\}^3} J'(\sigma)$, where

$$J'(\sigma) = \frac{N_d^2}{\lambda^{2d}} \int\limits_{(\mathbb{T}^d)^2} \mathrm{d}k'_1\, \mathrm{d}k'_2 \prod_{j=1}^{2} \frac{\mathbb{1}(|k'_j - x'_j| < 2\lambda)}{|\alpha'_j - \omega(k'_j) + \mathrm{i}\beta|} \frac{1}{|\alpha'_3 - \omega(k'_3) + \mathrm{i}\beta|}, \tag{6.64}$$

assuming $j_0 = 3$, $k_3 = \sigma_1 k_1 + \sigma_2 k_2 + \sigma_3 k_0$, and $x_3 = \sigma_1 x_1 + \sigma_2 x_2 + \sigma_3 k_0$. Then a bound for J can be obtained by undoing the permutation of the indices appropriately in the bound for J'.

Since $2\lambda \leqslant |\nabla\omega(x_j)|/a_0$, for all $j = 1, 2, 3$, we can apply Lemma 4.9 and Corollary 4.10 to both of the k-integrals. We denote the corresponding diffeomorphisms by ψ_1 and ψ_2, and obtain the bound:

$$\begin{aligned} J'(\sigma) \leqslant{}& \frac{2^2 N_d^2}{\lambda^{2d}} \int\limits_{|y|<4\lambda} \mathrm{d}y \int\limits_{|y'|<4\lambda} \mathrm{d}y' \frac{1}{|\alpha_1 - \omega(x_1) - |\nabla\omega(x_1)| y_1 + \mathrm{i}\beta|} \\ &\times \frac{1}{|\alpha_2 - \omega(x_2) - |\nabla\omega(x_2)| y'_1 + \mathrm{i}\beta|} \frac{1}{|\alpha_3 - \omega(x_3 + \gamma(y', y)) + \mathrm{i}\beta|}, \end{aligned} \tag{6.65}$$

where

$$\gamma(y', y) = \sigma_1\big(\psi_1(y) - x_1\big) + \sigma_2\big(\psi_2(y') - x_2\big), \tag{6.66}$$

and by Lemma 4.9, always

$$\big|\psi_1(y) - x_1\big|, \big|\psi_2(y') - x_2\big| < 2^3\lambda \quad \text{and} \quad \big|\gamma(y', y)\big| < 2^4\lambda. \tag{6.67}$$

For $j = 1, 2$, let us denote the rotation $D\psi_j(0)$ by A_j and define $\tilde{v}_j = A_j^T v_j$. Then $\tilde{v}_j \cdot e_1 = v_j \cdot u_j = 0$, and there is a rotation O_j of $\mathbb{R}^d$ for which $O_j e_1 = e_1$ and $O_j \tilde{v}_j = e_2$. We change variables to $z = O_1 y$ and $z' = O_2 y'$, and evaluate first the z_2 and z'_2 integrals. This yields:

$$\begin{aligned} J'(\sigma) \leqslant{}& \frac{2^2 N_d^2}{\lambda^{2d}} \int\limits_{|z|<4\lambda} \mathrm{d}z \int\limits_{|z'|<4\lambda} \mathrm{d}z' \frac{1}{|\alpha_1 - \omega(x_1) - |\nabla\omega(x_1)| z_1 + \mathrm{i}\beta|} \\ &\times \frac{1}{|\alpha_2 - \omega(x_2) - |\nabla\omega(x_2)| z'_1 + \mathrm{i}\beta|} \frac{1}{|\alpha_3 - \omega(x_3 + \gamma(O_2^T z', O_1^T z)) + \mathrm{i}\beta|} \\ \leqslant{}& \frac{2^2 N_d^2}{\lambda^{2d}} \frac{(4\lambda)^{2(d-2)}}{N_{d-2}^2} \int\limits_{|z_1|<4\lambda} \mathrm{d}z_1 \frac{1}{|\alpha_1 - \omega(x_1) - |\nabla\omega(x_1)| z_1 + \mathrm{i}\beta|} \\ &\times \int\limits_{|z'_1|<4\lambda} \mathrm{d}z'_1 \frac{1}{|\alpha_2 - \omega(x_2) - |\nabla\omega(x_2)| z'_1 + \mathrm{i}\beta|} \\ &\times \sup_{\substack{|z|,|z'|<2\lambda \\ z_2, z'_2 = 0}} \int\limits_{t_1^2 < (2\lambda)^2 - z^2} \mathrm{d}t_1 \int\limits_{t_2^2 < (2\lambda)^2 - (z')^2} \mathrm{d}t_2 \frac{1}{|\alpha_3 - f(t_1, t_2; z', z) + \mathrm{i}\beta|}, \end{aligned} \tag{6.68}$$

where

$$f(t_1, t_2; z', z) = \omega\big(x_3 + \gamma\big(O_2^T(z' + t_2 e_2), O_1^T(z + t_1 e_1)\big)\big) = \omega\big(x_3 + \gamma(\tilde{z}_2 + t_2\tilde{v}_2, \tilde{z}_1 + t_1\tilde{v}_1)\big), \tag{6.69}$$

with $\tilde{z}_2 = O_2^T z'$ and $\tilde{z}_1 = O_1^T z$. Let us denote the final supremum by J''. Applying Proposition 4.7, we find the bound:

$$J'(\sigma) \leqslant \frac{2^{4d-6} N_d^2}{N_{d-2}^2} \frac{6^2 \langle \ln\langle M_1\rangle\rangle^2}{|\nabla\omega(x_1)||\nabla\omega(x_2)|} \frac{\langle \ln\beta\rangle^2}{\lambda^4} J''. \tag{6.70}$$

We still need to estimate J''. To do this we need to make a diversion and first prove the following lemma:

Lemma 6.6. *Let an integer $n \geqslant 2$ and $a, b, c \in \mathbb{R}$ be given and suppose $|c| \geqslant \varepsilon' > 0$. Then there is $\nu \in \mathbb{R}$, with $|\nu|, |1-\nu| \leqslant 2$, for which*

$$\left|\nu^n a + (1-\nu)^n b + c\right| \geqslant \frac{\varepsilon'}{2}. \tag{6.71}$$

Proof. Let $f(\nu) = \nu^n a + (1-\nu)^n b + c$, and let us first assume that $a \geqslant |b|$. Then, if $a \leqslant |c| - \varepsilon'/2$, we have $|f(1)| \geqslant |c| - |a| \geqslant \varepsilon'/2$, and choosing $\nu = 1$ suffices. Alternatively, if $a > |c| - \varepsilon'/2$, we have $a > \varepsilon'/2$ and thus

$$f(2) \geqslant 2^n a - |b| - |c| > a(2^n - 2) - \frac{\varepsilon'}{2} \geqslant \frac{\varepsilon'}{2}(2^n - 3) \geqslant \frac{\varepsilon'}{2}. \tag{6.72}$$

Therefore, the estimate holds then for either $\nu = 1$ or $\nu = 2$. If $a \leqslant -|b|$, we have $-a \geqslant |b|$, and after swapping the signs of a, b, c, we can apply the above result to conclude that again $|f(\nu)| \geqslant \varepsilon'/2$ at either $\nu = 1$ or $\nu = 2$. We have then proven the result for $|a| \geqslant |b|$. Finally, if $|a| < |b|$, we apply the above result for $\nu' = 1 - \nu$, and conclude that in this case either $|f(0)|$ or $|f(-1)|$ is greater than or equal to $\varepsilon'/2$. Thus the bound is attained at one of the points $\nu \in \{-1, 0, 1, 2\}$, which implies $|\nu|, |1-\nu| \leqslant 2$. □

For both $j = 1, 2$, let $\tilde{g}_{j,n} = \tilde{g}_n(x_j, v_j)$ be defined as in item (2) of Lemma 4.12. Then we employ Lemma 6.6 with $n = N$, and

$$\begin{aligned} a &= -\sigma_1^{N+1} \tilde{g}_{1,N} u_1 \cdot \nabla\omega(x_3), \\ b &= -\sigma_2^{N+1} \tilde{g}_{2,N} u_2 \cdot \nabla\omega(x_3), \\ c &= \frac{1}{N!} (v_3 \cdot \nabla)^N \omega(x_3). \end{aligned} \tag{6.73}$$

As $|c| > \frac{1}{2}\varepsilon$, this yields a $\nu \in \mathbb{R}$ such that $|\nu|, |1-\nu| \leqslant 2$, and $|\tilde{c}| \geqslant \frac{1}{4}\varepsilon$ with

$$\tilde{c} = \left(-\nu^N \sigma_1^{N+1} \tilde{g}_{1,N} u_1 - (1-\nu)^N \sigma_2^{N+1} \tilde{g}_{2,N} u_2\right) \cdot \nabla\omega(x_3) + \frac{1}{N!}(v_3 \cdot \nabla)^N \omega(x_3). \tag{6.74}$$

Now $|\tilde{z}_1| = |z| < 2\lambda$, $|\tilde{z}_2| = |z'| < 2\lambda$, and also $\tilde{z}_j \cdot \tilde{v}_j = 0$, with the t_j-integration going over values with $|t_j|^2 < (2\lambda)^2 - |\tilde{z}_j|^2$. We make the final change of variables $(t_1, t_2) \to (t, t')$, given by:

$$t_1 = \sigma_1(-t' + \nu t) \quad \text{and} \quad t_2 = \sigma_2\big(t' + (1-\nu)t\big), \tag{6.75}$$

where ν is the constant found above. The Jacobian of the transformation is always 1, and it has the inverse:

$$t = \sigma_1 t_1 + \sigma_2 t_2 \quad \text{and} \quad t' = \nu\sigma_2 t_2 - (1-\nu)\sigma_1 t_1, \tag{6.76}$$

and thus inside the new integration region,

$$|t| \leqslant |t_1| + |t_2| < 2^3\lambda \quad \text{and} \quad |t'| < \big(|\nu| + |1-\nu|\big)4\lambda \leqslant 2^4\lambda. \tag{6.77}$$

Therefore,

$$\begin{aligned} \int\limits_{t_1^2 < (2\lambda)^2 - z^2} \mathrm{d}t_1 \int\limits_{t_2^2 < (2\lambda)^2 - (z')^2} \mathrm{d}t_2 \frac{1}{|\alpha_3 - f(t_1, t_2; z', z) + \mathrm{i}\beta|} &\leqslant \int \mathrm{d}t' \int\limits_{I(t')} \mathrm{d}t \frac{2}{|\alpha_3 - F(t; t') + \mathrm{i}\beta|} \\ &\leqslant 2^6\lambda \sup_{t'} \int\limits_{I(t')} \mathrm{d}t \frac{1}{|\alpha_3 - F(t; t') + \mathrm{i}\beta|}, \end{aligned} \tag{6.78}$$

where the integration region over t, that is $I(t')$, depends on t', but it always is an interval of a length less than $2^4\lambda$. The final integral contains the function,

$$\begin{aligned} F(t; t') &= f(t_1, t_2; z', z) = \omega\big(x_3 + \gamma(\tilde{z}_2 + t_2\tilde{v}_2, \tilde{z}_1 + t_1\tilde{v}_1)\big) \\ &= \omega\big(\tilde{x}_3 + \sigma_1\big(\psi_1(\tilde{y}_1 + \sigma_1\nu t\tilde{v}_1) - \psi_1(\tilde{y}_1)\big) + \sigma_2\big(\psi_2\big(\tilde{y}_2 + \sigma_2(1-\nu)t\tilde{v}_2\big) - \psi_2(\tilde{y}_2)\big)\big), \end{aligned} \tag{6.79}$$

where

$$\tilde{y}_1 = \tilde{y}_1(t') = \tilde{z}_1 - t'\sigma_1\tilde{v}_1 \quad \text{and} \quad \tilde{y}_2 = \tilde{z}_2 + t'\sigma_2\tilde{v}_2, \tag{6.80}$$

and

$$\tilde{x}_3 = \tilde{x}_3(t') = x_3 + \sigma_1\big(\psi_1(\tilde{y}_1) - x_1\big) + \sigma_2\big(\psi_2(\tilde{y}_2) - x_2\big). \tag{6.81}$$

Let us then define, as in Proposition 4.11,

$$\gamma_j(\tau) = \psi_j(\tilde{y}_j + \tau\tilde{v}_j) \quad \text{and} \quad \Gamma_j(\tau) = \gamma_j(\tau) - \tau v_j - \psi_j(\tilde{y}_j). \tag{6.82}$$

As $\sigma_1 v_1(\sigma_1\nu t) + \sigma_2 v_2(\sigma_2(1-\nu)t) = (\nu v_1 + (1-\nu)v_2)t$, then $F(t) = \omega(\Gamma(t))$ with

$$\Gamma(t) = \tilde{x}_3 + t v_0 + \sigma_1\Gamma_1(\sigma_1\nu t) + \sigma_2\Gamma_2\big(\sigma_2(1-\nu)t\big). \tag{6.83}$$

Here $v_0 = \nu v_1 + (1-\nu)v_2$, and it thus satisfies:

$$|v_0 - v_3| \leqslant 2\big(|v_1 - v_3| + |v_2 - v_3|\big) < 4\delta. \tag{6.84}$$

By Lemma A.1,

$$\begin{aligned}\frac{1}{N!}\left|\frac{\mathrm{d}^N}{\mathrm{d}t^N}F(t)\right| \geqslant{}& \frac{1}{N!}\left|\Gamma^{(N)}(t)\cdot\nabla\omega\big(\Gamma(t)\big) + \big(\Gamma^{(1)}(t)\cdot\nabla\big)^N\omega\big(\Gamma(t)\big)\right| \\ &- \sum_{k=2}^{N-1}\sum_{m\in\mathbb{N}_+^k}\mathbb{1}\left(\sum_{j=1}^k m_j = N\right)M_k\prod_{j=1}^k\left[\frac{1}{m_j!}\left|\Gamma^{(m_j)}(t)\right|\right].\end{aligned} \tag{6.85}$$

Here $\Gamma^{(1)}(t) = v_0 + \nu\Gamma_1^{(1)}(\sigma_1\nu t) + (1-\nu)\Gamma_2^{(1)}(\sigma_2(1-\nu)t)$, and, by (6.84) and Proposition 4.11, it satisfies the bound:

$$\left|\Gamma^{(1)}(t) - v_3\right| \leqslant 4\delta + 4\varepsilon\mu^N \leqslant 1, \tag{6.86}$$

which implies in particular that $|\Gamma^{(1)}(t)| \leqslant 2$. Note that we can apply the Proposition, since ε, μ, and N are clearly in the right range, and also the expansion radius satisfies $2\lambda \leqslant \frac{1}{2}a_0^{-1}\varepsilon_0(r_0\mu)^{n_0} = a_0^{-1}\varepsilon(r_0\mu)^N$. For all $n \geqslant 2$, we similarly get:

$$\Gamma^{(n)}(t) = \sigma_1^{n+1}\nu^n\Gamma_1^{(n)}(\sigma_1\nu t) + \sigma_2^{n+1}(1-\nu)^n\Gamma_2^{(n)}\big(\sigma_2(1-\nu)t\big), \tag{6.87}$$

satisfying, with $\widetilde{C} = 1 + M_{n_0} \geqslant 1 + \frac{M_N}{N!}$, the bounds

$$\left|\frac{1}{n!}\Gamma^{(n)}(t)\right| \leqslant \begin{cases} 2^{n+2}\varepsilon\mu^{N-n}r_0^{N-1-n}, & \text{for } 2 \leqslant n < N, \\ 2^{N+2}\widetilde{C}r_0^{-1}, & \text{for } n = N, \\ 2^{N+3}\widetilde{C}\mu^{-1}r_0^{-2}, & \text{for } n = N+1. \end{cases} \tag{6.88}$$

Consider then the sum over k in (6.85). Since $k \geqslant 2$, inside the sum always $m_j \leqslant N-1$. Denoting, as before, $\ell = |\{j \mid m_j = 1\}| \leqslant k-1$, we thus have

$$\prod_{j=1}^k\left[\frac{1}{m_j!}\left|\Gamma^{(m_j)}(t)\right|\right] \leqslant 2^{\ell+3(k-\ell)+\sum_j(m_j-1)}\big(\varepsilon\mu^{N-1}\big)^{k-\ell}\mu^{\sum_j(1-m_j)} \leqslant 2^{2k+N}\varepsilon\mu^{N-1+k-N} \leqslant 2^{3N-2}\varepsilon\mu. \tag{6.89}$$

Therefore, the sum over k is bounded by:

$$2^{3N-2}\varepsilon\mu\sum_{k=2}^{N-1}\binom{N-1}{k-1}M_{N-1} \leqslant 2^{4N-3}\varepsilon\mu M_{N-1} \leqslant \frac{1}{2^4}\varepsilon. \tag{6.90}$$

To estimate the first term in (6.85), we use the estimates in item (3) of Lemma 4.12, with $b = \mu^{-1} \geqslant 1 + 2^N + M_{N+1}2^{2N+1}$. Firstly,

$$\begin{aligned}
&\left|\frac{1}{N!}\Gamma^{(N)}(t)\cdot\nabla\omega\big(\Gamma(t)\big)-\big(-\nu^N\sigma_1^{N+1}\tilde g_{1,N}u_1-(1-\nu)^N\sigma_2^{N+1}\tilde g_{2,N}u_2\big)\cdot\nabla\omega(x_3)\right|\\
&\quad\leqslant\left|\frac{1}{N!}\Gamma^{(N)}(t)\right|M_2\big|\Gamma(t)-x_3\big|+\big|\nabla\omega(x_3)\big|2^N\big(\big|g_{1,N}(\sigma_1\nu t)-\tilde g_{1,N}\big|+\big|g_{2,N}\big(\sigma_2(1-\nu)t\big)-\tilde g_{2,N}\big|\big)\\
&\quad\leqslant 2^4\lambda M_2 2^{N+2}\widetilde C r_0^{-1}+M_1 2^{N+1}a_0\mu^{1-N}\lambda r_0^{-N}\leqslant 2^{N+3}\varepsilon(r_0\mu)^N\widetilde C r_0^{-1}+M_1 2^{N+1}\varepsilon\mu\\
&\quad\leqslant\varepsilon\mu\big(\mu 2^{N+3}\widetilde C+M_1 2^{N+1}\big)\leqslant\varepsilon\mu\big(1+M_1 2^{n_0+1}\big)\leqslant 2^{-n_0-2}\varepsilon\leqslant 2^{-4}\varepsilon.
\end{aligned}\tag{6.91}$$

Secondly, by (6.86),

$$\begin{aligned}
\big|\big(\Gamma^{(1)}(t)\cdot\nabla\big)^N\omega\big(\Gamma(t)\big)-(v_3\cdot\nabla)^N\omega(x_3)\big|&\leqslant\sum_{k=1}^{N}\binom{N}{k}\big|\Gamma^{(1)}(t)-v_3\big|^k M_N+M_{N+1}2^4\lambda\\
&\leqslant 4\big(\lambda+\varepsilon\mu^N\big)M_N 2^N+M_{N+1}2^4\lambda\leqslant\varepsilon\mu^N M_{N+1}\big(2^{N+3}+2^4\big)\\
&\leqslant\varepsilon\mu^2 M_{N+1}2^{N+4}\leqslant 2^{-4}\varepsilon.
\end{aligned}\tag{6.92}$$

Therefore,

$$\left|\frac{1}{N!}\Gamma^{(N)}(t)\cdot\nabla\omega\big(\Gamma(t)\big)+\frac{1}{N!}\big(\Gamma^{(1)}(t)\cdot\nabla\big)^N\omega\big(\Gamma(t)\big)-\tilde c\right|\leqslant 2^{-3}\varepsilon,\tag{6.93}$$

which can be combined with the previous estimate for the sum over k in (6.85) to prove that for all allowed t,

$$\frac{1}{N!}\left|\frac{\mathrm{d}^N}{\mathrm{d}t^N}F(t)\right|\geqslant|\tilde c|-\frac{3}{2^4}\varepsilon\geqslant\frac{1}{2^4}\varepsilon.\tag{6.94}$$

On the other hand, applying Lemma A.1 once more proves that

$$\begin{aligned}
\frac{1}{(N+1)!}\big|F^{(N+1)}(t)\big|&\leqslant\sum_{k=1}^{N+1}\sum_{m\in\mathbb{N}_+^k}\mathbb{1}\left(\sum_{j=1}^{k}m_j=N+1\right)\prod_{j=1}^{k-1}\frac{m_j}{\sum_{j'=j}^{k}m_{j'}}\prod_{j=1}^{k}\Big[\frac{1}{m_j!}\big|\Gamma^{(m_j)}(t)\big|\Big]M_k\\
&\leqslant\frac{|\Gamma^{(N+1)}(t)|}{(N+1)!}+\widetilde C r_0^{-1}\sum_{k=2}^{N+1}\sum_{m\in\mathbb{N}_+^k}\mathbb{1}\left(\sum_{j=1}^{k}m_j=N+1\right)2^{\sum_j(m_j+2)}M_{N+1}\\
&\leqslant 2^{N+3}\widetilde C\mu^{-1}r_0^{-2}+\widetilde C r_0^{-1}\sum_{k=2}^{N+1}\binom{N}{k-1}2^{N+1+2k}M_{N+1}\\
&\leqslant 2^{N+3}\widetilde C\mu^{-1}r_0^{-2}\big(1+\mu 2^{3N}M_{N+1}\big)\leqslant 2^{N+4}\widetilde C\mu^{-1}r_0^{-2},
\end{aligned}\tag{6.95}$$

where we have used the fact that $m_j=N$ can appear only once in the product over j. Since $r_0\geqslant\frac12\beta^\gamma$, we have also:

$$\begin{aligned}
\varepsilon'=\frac{\varepsilon}{2^4}\frac{1}{2^{N+1}(N+1)^N}\frac{r_0^2\mu}{2^{N+4}\widetilde C}&\geqslant\varepsilon_0 r_0^{n_0-N+2}\frac{\mu^{n_0-N+1}}{\widetilde C}\frac{1}{2^{2n_0+8}(n_0+1)^{n_0}}\\
&\geqslant\beta^{\gamma n_0}\varepsilon_0\frac{\mu^{n_0-1}}{\widetilde C}\frac{1}{2^{3n_0+8}(n_0+1)^{n_0}}.
\end{aligned}\tag{6.96}$$

If this is raised to the power $N+1$, the result is bounded from below by an (n_0-dependent) constant times $\beta^{\gamma n_0(n_0+1)}$. Therefore, as long as $\gamma^{-1}>n_0(n_0+1)$, there is $\beta_0>0$, such that we can apply the conclusion of Proposition 4.8 for all $0<\beta\leqslant\beta_0$. For such values of β and all allowed t', we have:

$$\begin{aligned}
&\int_{I(t')}\mathrm{d}t\,\frac{1}{|\alpha_3-F(t;t')+\mathrm{i}\beta|}\leqslant 2^{N+1}(N+1)^N\left(\frac{2^4\lambda}{\varepsilon 2^{-4}}\beta^{1/(N+1)-1}+2^{N+4}\widetilde C\mu^{-1}r_0^{-2}2^{4/N}\varepsilon^{-1/N}\beta^{1/N-1}\right)\\
&\quad\leqslant 2^{n_0+1}(n_0+1)^{n_0}\left(2^8\frac{\varepsilon_0}{4a_0}(r_0\mu)^{n_0}\frac{1}{2\varepsilon_0}(r_0\mu)^{N-n_0}\beta^{1/(N+1)-1}+2^{n_0+6}\widetilde C\mu^{-1}r_0^{-2}\big(2\varepsilon_0(r_0\mu)^{n_0-N}\big)^{-1/N}\beta^{1/N-1}\right)\\
&\quad\leqslant 2^{3n_0+8}(n_0+1)^{n_0}\widetilde C\mu^{-n_0/N}\varepsilon_0^{-1/N}\big(\beta^{1/(N+1)-1}+\beta^{-\gamma(1+n_0/N)+1/N-1}\big).
\end{aligned}\tag{6.97}$$

Since we have not aimed at optimal estimates here, we do not try to optimise the extra decay arising from the crossing. Instead, let us prove that the choice given in (6.27) is sufficient. Then we can also choose explicitly,

$$\beta_0 = \left(\varepsilon_0 \frac{\mu^{n_0-1}}{\widetilde{C}} \frac{1}{2^{3n_0+8}(n_0+1)^{n_0}}\right)^{3(n_0+1)/2}, \tag{6.98}$$

since, for all $0 < \beta \leqslant \beta_0$, then $\beta \leqslant \beta^{1/3}\beta_0^{2/3} \leqslant (\varepsilon')^{N+1}$.

With these choices, the power of β in the second term in (6.97) is:

$$-\gamma\left(1+\frac{n_0}{N}\right)+\frac{1}{N}-1 = \frac{1}{3n_0(n_0+1)N}\left(-N-n_0+3n_0+3n_0^2\right)-1 \geqslant \frac{n_0}{(n_0+1)N}-1 \geqslant \frac{1}{n_0+1}-1. \tag{6.99}$$

Therefore, by (6.78), we have proved:

$$J'' \leqslant \lambda 2^{3n_0+15}(n_0+1)^{n_0}\widetilde{C}\mu^{-n_0/2}\varepsilon_0^{-1/2}\beta^{1/(n_0+1)-1}. \tag{6.100}$$

Combining this with (6.70) proves the validity of (6.28) for

$$C_2' = 2^{3n_0+9+4d}(n_0+1)^{n_0}\widetilde{C}\mu^{-\frac{n_0}{2}}\varepsilon_0^{-1/2}\frac{N_d^2}{N_{d-2}^2}6^2\langle\ln\langle M_1\rangle\rangle^2 M_1 \tag{6.101}$$

when γ is chosen as in (6.27) and β is sufficiently small. For notational simplicity, we have added the missing gradient factor $|\nabla\omega(x_3)|$ to the denominator: this makes the estimate invariant under permutations of the indices, and thus allows to use it directly for the original integral. This completes the proof of Theorem 2.3.

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Appendix A. Differentials of composite functions

Lemma A.1. *Let $d, n \in \mathbb{N}_+$, an open interval I, $\Gamma \in C^{(n)}(I, \mathbb{R}^d)$, and $f \in C^{(n)}(\mathbb{R}^d, \mathbb{R})$ be given. Then for all $t \in I$,*

$$\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n} f\big(\Gamma(t)\big) = \sum_{k=1}^{n}\sum_{m\in\mathbb{N}_+^k} \mathbb{1}\left(\sum_{j=1}^{k} m_j = n\right)\prod_{j=1}^{k-1}\frac{m_j}{\sum_{j'=j}^{k} m_{j'}}\prod_{j=1}^{k}\left[\frac{1}{m_j!}\Gamma^{(m_j)}(t)\cdot\nabla\right] f\bigg|_{\Gamma(t)}. \tag{A.1}$$

Proof. The result holds for $n = 1$. For the induction step, let us assume it holds for values up to n. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n} f\big(\Gamma(t)\big)\right] &= \sum_{k=1}^{n}\sum_{m\in\mathbb{N}_+^k}\mathbb{1}\left(\sum_{j=1}^{k} m_j = n\right)\prod_{j=1}^{k-1}\frac{m_j}{\sum_{j'=j}^{k} m_{j'}} \\ &\quad\times\left\{\sum_{\ell=1}^{k}\prod_{j=1,\ j\neq\ell}^{k}\left[\frac{1}{m_j!}\Gamma^{(m_j)}(t)\cdot\nabla\right]\left[\frac{1}{m_\ell!}\Gamma^{(m_\ell+1)}(t)\cdot\nabla\right] f\bigg|_{\Gamma(t)}\right. \\ &\quad\left.+\prod_{j=1}^{k}\left[\frac{1}{m_j!}\Gamma^{(m_j)}(t)\cdot\nabla\right]\left[\Gamma^{(1)}(t)\cdot\nabla\right] f\bigg|_{\Gamma(t)}\right\}.\end{aligned} \tag{A.2}$$

In the first term, we take out the sum over ℓ, and then change variables from m to M so that $M_\ell = m_\ell + 1$ and otherwise $M_j = m_j$. This yields a term:

$$\sum_{k=1}^{n}\sum_{\ell=1}^{k}\sum_{M\in\mathbb{N}_+^k}\mathbb{1}\left(\sum_{j=1}^{k}M_j=n+1\right)\mathbb{1}(M_\ell\geqslant 2)\prod_{j=\ell+1}^{k-1}\frac{M_j}{\sum_{j'=j}^{k}M_{j'}}$$
$$\times M_\ell\frac{M_\ell-1}{\sum_{j'=\ell}^{k}M_{j'}-1}\prod_{j=1}^{\ell-1}\frac{M_j}{\sum_{j'=j}^{k}M_{j'}-1}\prod_{j=1}^{k}\left[\frac{1}{M_j!}\Gamma^{(M_j)}(t)\cdot\nabla\right]f\bigg|_{\Gamma(t)}. \tag{A.3}$$

For the second term, we add one more sum over $m=1$, and then shift the k sum accordingly. This yields:

$$\sum_{k=2}^{n+1}\sum_{m\in\mathbb{N}_+^k}\mathbb{1}\left(\sum_{j=1}^{k}m_j=n+1\right)\mathbb{1}(m_k=1)\prod_{j=1}^{k-1}\frac{m_j}{\sum_{j'=j}^{k}m_{j'}-1}\prod_{j=1}^{k}\left[\frac{1}{m_j!}\Gamma^{(m_j)}(t)\cdot\nabla\right]f\bigg|_{\Gamma(t)}. \tag{A.4}$$

It is then an explicit computation to check that the $k=1$ term in (A.3) is equal to the $k=1$ term in (A.1) times $n+1$ (after setting $n\to n+1$), and that the same holds for $k=n+1$ term in (A.4).

For $2\leqslant k\leqslant n$ we need to sum the corresponding terms in (A.3) and (A.4). Their sum can be written as

$$\sum_{m\in\mathbb{N}_+^k}\mathbb{1}\left(\sum_{j=1}^{k}m_j=n+1\right)\prod_{j=1}^{k-1}\frac{m_j}{\sum_{j'=j}^{k}m_{j'}}\prod_{j=1}^{k}\left[\frac{1}{m_j!}\Gamma^{(m_j)}(t)\cdot\nabla\right]f\bigg|_{\Gamma(t)}$$
$$\times\left\{\sum_{\ell=1}^{k}\mathbb{1}(m_\ell\geqslant 2)(m_\ell-1)\prod_{j=1}^{\ell}\frac{\sum_{j'=j}^{k}m_{j'}}{\sum_{j'=j}^{k}m_{j'}-1}+\mathbb{1}(m_k=1)\prod_{j=1}^{k-1}\frac{\sum_{j'=j}^{k}m_{j'}}{\sum_{j'=j}^{k}m_{j'}-1}\right\}. \tag{A.5}$$

For the computation of the term in the curly brackets, let us separate $\ell=k$ term. When $\ell<k$, all the terms in the denominator are non-zero, as for $j<k$, we have $\sum_{j'=j}^{k}m_{j'}-1>0$ due to $m_\ell\geqslant 1$. Therefore, we can apply the property:

$$(m_\ell-1)\mathbb{1}(m_\ell\geqslant 2)=m_\ell-1=\sum_{j'=\ell}^{k}m_{j'}-1-\sum_{j'=\ell+1}^{k}m_{j'}, \tag{A.6}$$

which shows that the sum over $\ell<k$ is equal to,

$$\sum_{\ell=1}^{k-1}\frac{\prod_{j=1}^{\ell}\sum_{j'=j}^{k}m_{j'}}{\prod_{j=1}^{\ell-1}(\sum_{j'=j}^{k}m_{j'}-1)}-\sum_{\ell=1}^{k-1}\frac{\prod_{j=1}^{\ell+1}\sum_{j'=j}^{k}m_{j'}}{\prod_{j=1}^{\ell}(\sum_{j'=j}^{k}m_{j'}-1)}=\sum_{j'=1}^{k}m_{j'}-m_k\prod_{j=1}^{k-1}\frac{\sum_{j'=j}^{k}m_{j'}}{\sum_{j'=j}^{k}m_{j'}-1}. \tag{A.7}$$

The second term here is cancelled by the remaining terms in the curly brackets. (If $m_k>1$, the $\ell=k$ term in the sum cancels it and the last term in (A.5) is zero; if $m_k=1$, the opposite happens.) Therefore, the term in the curly brackets is equal to $\sum_{j=1}^{k}m_j=n+1$. This proves (A.1). □

Appendix B. Properties of $\langle x\rangle$

Proposition B.1. *Let $\langle x\rangle=\sqrt{1+x^2}$. Then for all $x,y\in\mathbb{R}$,*

(1) $|x|<\langle x\rangle$.
(2) *If* $|x|\leqslant|y|$, *then* $\langle x\rangle\leqslant\langle y\rangle$ *and* $\langle\ln\langle x\rangle\rangle\leqslant\langle\ln\langle y\rangle\rangle$.
(3) $\langle x+y\rangle<\langle x\rangle+\langle y\rangle\leqslant 2\langle x\rangle\langle y\rangle$.
(4) $\langle xy\rangle\leqslant\langle x\rangle\langle y\rangle$, *and, if* $|x|\geqslant 1$, $\langle xy\rangle\leqslant|x|\langle y\rangle$.

Proof. Items (1) and (2) are obvious. The first inequality of item (3) is proven by

$$\langle x\rangle+\langle y\rangle-\langle x+y\rangle=\frac{(\langle x\rangle+\langle y\rangle)^2-\langle x+y\rangle^2}{\langle x\rangle+\langle y\rangle+\langle x+y\rangle}=\frac{2\langle x\rangle\langle y\rangle+1+x^2+1+y^2-(1+x^2+y^2+2xy)}{\langle x\rangle+\langle y\rangle+\langle x+y\rangle}$$
$$=\frac{1+2\langle x\rangle\langle y\rangle(1-\frac{x}{\langle x\rangle}\frac{y}{\langle y\rangle})}{\langle x\rangle+\langle y\rangle+\langle x+y\rangle}>0. \tag{B.1}$$

The proofs of the remaining inequalities in (3) and (4) are very similar, and we will skip them. □

Appendix C. Morse functions

We prove here the following result which shows that Morse functions are covered by the main results given in the text.

Proposition C.1. *Let $d \geqslant 3$, and assume ω is a real-analytic and $\mathbb{Z}^d$-periodic Morse function on $\mathbb{R}^d$. Then ω satisfies Assumption* 2.1*, and we can take $p_0 = 0$ for $d \geqslant 4$ and $p_0 = 1$ for $d = 3$.*

Proof. Define f_ω by (2.5). Let $X = [-1/2, 1/2]^d$, and let x_j, $j = 1, \ldots, n$, enumerate the critical points of ω in X (as ω is a Morse function, there can be no accumulation of its critical points, and thus $n < \infty$). Let also $A_j = D^2\omega(x_j)$ be the Hessian of ω at x_j, let $\lambda_j^{(i)}$ denote its eigenvalues, and define $a_j = \min_i |\lambda_j^{(i)}|$ and $b_j = \max_i |\lambda_j^{(i)}|$. By assumption, A_j is invertible, and thus we have $0 < a_j \leqslant b_j < \infty$.

By Taylor's formula, now for any $x \in \mathbb{R}^d$ and j,

$$\nabla\omega(x) = \nabla\omega(x) - \nabla\omega(x_j) = A_j(x - x_j) + R_j(x), \tag{C.1}$$

where $|R_j(x)| \leqslant \frac{1}{2}\|\omega\|_3'|x - x_j|^2$. Here, by using an orthogonal transformation which diagonalises the Hermitian matrix A_j, we find:

$$a_j|x - x_j| \leqslant \big|A_j(x - x_j)\big| \leqslant b_j|x - x_j|. \tag{C.2}$$

Let $r_j = a_j/\|\omega\|_3'$ which is non-zero, as $\|\omega\|_3'$ is finite. Then we can conclude, by using the triangle inequality, that whenever $|x - x_j| \leqslant r_j$,

$$\frac{a_j}{2}|x - x_j| \leqslant \big|\nabla\omega(x)\big| \leqslant \frac{3b_j}{2}|x - x_j|. \tag{C.3}$$

Let $U_j = \{x \mid |x - x_j| < r_j\}$, $j = 1, \ldots, n$, and denote $K = X \setminus (\bigcup_j U_j)$. Then K is compact, and contains no critical points of ω. Therefore, by continuity of $\nabla\omega$, we have $c = \min_{x\in K} |\nabla\omega(x)| > 0$. We split the integration region into parts by removing the balls U_j, which yields for all $0 < s \leqslant c$,

$$\begin{aligned} f_\omega(s) &= \int_X \mathrm{d}x\, \frac{1}{|\nabla\omega(x)|^3}\mathbb{1}\big(\big|\nabla\omega(x)\big| \geqslant s\big) \leqslant \int_K \mathrm{d}x\, \frac{1}{c^3} + \sum_{j=1}^n \int_{U_j} \mathrm{d}x\, \frac{1}{|\nabla\omega(x)|^3}\mathbb{1}\big(\big|\nabla\omega(x)\big| \geqslant s\big) \\ &\leqslant \frac{1}{c^3} + \sum_{j=1}^n 2^{-3}a_j^3\big|S^{d-1}\big| \int_{2s/(3b_j)}^{r_j} \mathrm{d}r\, r^{d-1-3}. \end{aligned} \tag{C.4}$$

If $d > 3$, the final integral over r is less than $\int_0^{r_j} \mathrm{d}r\, r^{d-1-3} = \frac{1}{d-3} r_j^{d-3}$. Therefore, we can conclude that then $\inf_{s>0} f_\omega(s) < \infty$, as claimed in the proposition. Otherwise, $d = 3$, and

$$\int_{2s/(3b_j)}^{r_j} \mathrm{d}r\, r^{d-1-3} = \ln\left(\frac{3b_j r_j}{2s}\right) = \ln\left(\frac{3b_j r_j}{2}\right) + \ln s^{-1}. \tag{C.5}$$

Then (C.4) implies that $f_\omega(s) \leqslant c_0\langle \ln s\rangle$ for some finite constant c_0, proving the validity of Assumption 2.1 with $p_0 = 1$. □

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