Fourier–Jacobi expansion of automorphic forms on $Sp(1, q)$ generating quaternionic discrete series

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Abstract

The aim of this paper is to develop the notion of the Fourier expansion of automorphic forms on $Sp(1, q)$ generating quaternionic discrete series, which are non-holomorphic forms. There is such an expansion given by Tsuneo Arakawa, assuming the boundedness of the forms and the integrability of the discrete series. We study these automorphic forms without such assumptions. When $q > 1$ we prove the “Koecher principle” for such automorphic forms, whose validity is known for holomorphic automorphic forms except elliptic modular forms.

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0. Introduction

In this paper we study in detail the Fourier expansion of automorphic forms on the real symplectic group $G = Sp(1, q)$ of signature $(1+, q−)$ generating quaternionic discrete series representations in the sense of Gross and Wallach (cf. [12]).
The automorphic forms above were first considered by Tsuneo Arakawa in [2,3]. He defined them as automorphic forms that have reproducing kernel functions given by matrix coefficients of the quaternionic discrete series (cf. Definition 8.5). The two papers [2] and [3] deal with the dimension formulas for the spaces of such forms with respect to neat non-uniform lattice subgroups. In addition, there are several unpublished papers by Arakawa. For instance, he constructed a theta lifting from elliptic cusp forms to automorphic forms on $\text{Sp}(1,q)$ (more precisely, the restriction of a theta correspondence for $\text{SL}_2(\mathbb{R}) \times \text{SO}(4,4q)$ to $\text{SL}_2(\mathbb{R}) \times \text{Sp}(1,q)$) and proved that its images are the automorphic forms on $\text{Sp}(1,q)$ just mentioned above when $q = 1$ (recently the author has settled the case of an arbitrary $q$ in [24] by using the Fourier expansion we will discuss). Moreover, Arakawa also investigated the spinor L-functions for the forms on $\text{Sp}(1,1)$ by following the method of Andrianov [1] (see also [4]).

However, we point out that he assumed the boundedness of the automorphic forms and the integrability of the discrete series. For the advancement of the research it is quite natural to study such automorphic forms without the boundedness condition or the integrability of the discrete series. In fact, it enables us to consider non-cuspidal automorphic forms such as Eisenstein series. We provide two new definitions of the automorphic forms on $\text{Sp}(1,q)$ under such weak conditions. One of them is representation theoretic (cf. Definition 6.1) and the other one is given by using a rank one differential operator called the Schmid operator (cf. Definition 8.1).

The Fourier expansion of our automorphic forms is given as the expansion along the center of a maximal unipotent subgroup $N$ of $G$. When $q > 1$ we call this “Fourier–Jacobi expansion” after Piatetskii-Shapiro [25, Chapter 4, Section 1] since $N$ is similar to the Heisenberg group. Our result on it is stated as Theorem 6.3 for the case of $q > 1$. The Fourier expansion for the case of $q = 1$ will be described in Section 9. As an application, we obtain the following theorem.

**Theorem 0.1** (Theorem 7.1). Let $q > 1$. An automorphic form $f$ on $G$ generating a quaternionic discrete series automatically satisfies the moderate growth condition, i.e., the growth of $f$ is at most of polynomial order.

This theorem is what should be called the “Koecher principle,” which is known to be valid for holomorphic automorphic forms except elliptic modular forms (cf. [5, Theorem 10.14], [7, Corollaire de la Proposition 1], [23, Satz 1, Satz 2], [25, Chapter 4, Section 1, Lemma 1]). As far as the author knows, such a result seems new in the literature for non-holomorphic automorphic forms (D. Zagier informed the author that Hilbert–Maass forms also satisfy the Koecher principle in the above sense).

Now we explain the outline of this paper. In Section 1 we introduce the notation of the Lie groups, the Lie algebras and the associated root systems which we deal with. In Section 2 we describe representations of the maximal compact subgroup $K$. In Section 3 we review some general facts on discrete series representations and introduce quaternionic discrete series of $G$. In Section 4 we study irreducible unitary representations of the maximal unipotent subgroup $N$ in terms of the so-called “orbit method” given by A.A. Kirillov [20]. In Section 5 we deal with an explicit formula for generalized Whittaker functions for quaternionic discrete series and with the multiplicity formula for generalized Whittaker models of such discrete series (for these functions and models see Definition 5.1). This section is an important step in the study of the Fourier expansion since it provides the explicit determination of the functions appearing in the expansion. For Sections 6–8 we assume $q > 1$. In Section 6 we give our theory of the Fourier–Jacobi expansion. Our approach to study the expansion is to use the explicit formula for the generalized Whittaker functions and the spectral decomposition of $L^2$-spaces on compact nilmanifolds by Corwin and...
Greenleaf [8]. In Section 7 we prove the Koecher principle of our automorphic forms. In Section 8 we discuss the equivalence of our two definitions of the automorphic forms and Arakawa’s original definition, assuming the boundedness of the forms. In Section 9 we give remarks on the case of \( \text{Sp}(1, 1) \). Things are somewhat different for this case. Actually the moderate growth condition of the automorphic forms does not automatically hold when \( q = 1 \).

**Notation**

For a ring \( R \), \( M_{m,n}(R) \) denotes the set of matrices with their sizes \( m \times n \) and entries in \( R \). When \( m = n \) (respectively \( n = 1 \)) we also denote it by \( M_n(R) \) (respectively \( R^n \)). By \( I_n \) we mean the unit matrix of \( M_n(R) \). We denote by \( 0_{m,n} \) the zero matrix in \( M_{m,n}(R) \) and when \( n = m \) we write \( 0_n \) for it. For a pair \((i, j)\) of two positive integers \( E_{ij} \) is the matrix unit indexed by \((i, j)\) and \( \delta_{ij} \) the Kronecker’s delta. For \((a_1, a_2, \ldots, a_m) \in R^m\), \( \text{diag}(a_1, a_2, \ldots, a_m) \in M_m(R) \) denotes the diagonal matrix with its entries in \[ \{a_1, a_2, \ldots, a_m\} \].

For a real number \( a \) we set \( e(a) := \exp(2\pi \sqrt{-1}a) \). Given a measurable set \( M, \text{vol}(M) \) stands for the volume of \( M \).

For a Lie group we denote its Lie algebra by the corresponding German letter, e.g., the Lie algebra of a Lie group \( S \) is written by \( s \). For a Lie algebra \( s \), \( s \subset C \) denotes its complexification when \( s \) is real, and \( U(s) \) the universal enveloping algebra of \( s \).

The algebraic dual space of a finite-dimensional vector space \( V \) is written by \( V^* \). Given a unitary representation \((\pi, H)\) of a Lie group, \((\pi^*, H^*)\) denotes its contragredient as a unitary representation. For two representations \( \tau_1 \) and \( \tau_2 \) of one group (respectively one group or two different groups) \( \tau_1 \otimes \tau_2 \) (respectively \( \tau_1 \boxtimes \tau_2 \)) stands for the inner tensor product (respectively the outer tensor product) of them.

1. **Structure of Lie groups and Lie algebras**

Let \( \mathbb{H} := \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_4 \) be the Hamilton quaternion algebra, where we fix a basis \( \{e_1 = 1, e_2, e_3, e_4\} \) which satisfies the following defining relations:

\[
e_2^2 = e_3^2 = -e_1, \quad e_2e_3 = -e_3e_2 = e_4.
\]

Its main involution is

\[
\mathbb{H} \ni x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 \mapsto \bar{x} := x_1e_1 - x_2e_2 - x_3e_3 - x_4e_4 \in \mathbb{H}.
\]

The reduced trace \( \text{tr} \) and the reduced norm \( \nu \) of \( \mathbb{H} \) are defined by

\[
\text{tr} : \mathbb{H} \ni x \mapsto \text{tr}(x) = x + \bar{x} \in \mathbb{R}, \quad \nu : \mathbb{H} \ni x \mapsto \nu(x) := x\bar{x} \in \mathbb{R}^\geq 0,
\]

respectively. The algebra \( \mathbb{H} \) can be embedded into \( M_2(\mathbb{C}) \) by

\[
\varphi : \mathbb{H} \ni x_1 + x_2e_2 + x_3e_3 + x_4e_4 \mapsto \begin{pmatrix} x_1 + \sqrt{-1}x_2 & x_3 + \sqrt{-1}x_4 \\ -(x_3 - \sqrt{-1}x_4) & x_1 - \sqrt{-1}x_2 \end{pmatrix} \in M_2(\mathbb{C}).
\]

By \( \varphi \) we can identify \( \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \) with \( M_2(\mathbb{C}) \), and \( \text{tr} \) and \( \nu \) correspond to the trace and the determinant of \( M_2(\mathbb{C}) \), respectively.
Furthermore, we set \( X_\mathbb{R} := \{ x \in \mathbb{H} \mid \text{tr}(x) = 0 \} \) to be the space of pure quaternions. In what follows we often use \( \sqrt{\nu(\xi)} \) for \( \xi \in X_\mathbb{R} \). We define

\[
d(\xi) := \sqrt{\nu(\xi)} \quad (\xi \in X_\mathbb{R}).
\]

For an element \( X = (x_{ij})_{1 \leq i, j \leq n} \) in the matrix algebra \( M_n(\mathbb{H}) \) we set \( \bar{t} X := (\bar{x}_{ji})_{1 \leq i, j \leq n} \). For \( X \in M_{q+1}(\mathbb{H}) \) we often use a block notation

\[
X = \begin{pmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3
\end{pmatrix} \quad (q > 1) \quad \text{and} \quad X = \begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{pmatrix} \quad (q = 1),
\]

where \( a_1 \in M_{q-1}(\mathbb{H}), \bar{t}a_2, \bar{t}a_3, b_1, c_1 \in \mathbb{H}^{q-1} \), and \( b_2, c_2, b_3, c_3 \in \mathbb{H} \) when \( q > 1 \) and \( a_1, a_2, b_1, b_2 \in \mathbb{H} \) when \( q = 1 \).

Form now on we fix a definite quaternion algebra \( B \) over \( \mathbb{Q} \) contained in \( \mathbb{H} \). Let \( G \) be the algebraic group over \( \mathbb{Q} \) defined by

\[
G(\mathbb{Q}) := \{ g \in M_{q+1}(B) \mid \bar{t} \bar{g} Q g = Q \}
\]

with

\[
Q := \begin{pmatrix}
-1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} \quad (q > 1) \quad \text{and} \quad Q := \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \quad (q = 1).
\]

The group of real points of \( G \) defines the real symplectic group \( Sp(1, q) \) of signature \((1+, q-)\). This group is a real form of the complex symplectic group \( Sp(q + 1; \mathbb{C}) \) of degree \( q + 1 \) (matrix size \( 2(q + 1) \)). Throughout this paper we denote this real Lie group by \( G \).

Let \( \theta \) denote the Cartan involution of \( G \) defined by

\[
G \ni g \mapsto \bar{t} \bar{g}^{-1} \in G.
\]

Then \( K := \{ g \in G \mid \theta(g) = g \} \) forms a maximal compact subgroup, which is isomorphic to \( Sp^*(q) \times Sp^*(1) \). Here \( Sp^*(q) \) denotes the compact real form of \( Sp(q; \mathbb{C}) \) given by

\[
\begin{cases}
\{ g \in M_q(\mathbb{H}) \mid \bar{t} \bar{g} = 1_q \}.
\end{cases}
\]

In particular, \( Sp^*(1) = \{ x \in \mathbb{H} \mid \nu(x) = 1 \} \) is isomorphic to the special unitary group \( SU(2) \) of degree 2. We also denote this by \( \mathbb{H}^{(1)} \).

The quotient \( G/K \) is called the quaternion hyperbolic space (cf. [16, Chapter X, Section2]), which is realized as

\[
\mathfrak{h} := \begin{cases}
\{ z = (w, \tau) \in \mathbb{H}^{q-1} \times \mathbb{H} \mid \text{tr}(\tau) > \bar{t} \bar{w} w \} & (q > 1), \\
\{ z \in \mathbb{H} \mid \text{tr}(z) > 0 \} & (q = 1)
\end{cases}
\]

(cf. [2, Section 1], [3, (0.3)]). The group \( G \) acts transitively on this space by

\[
g \cdot z := \begin{cases}
((a_1 w + b_1 \tau + c_1) \mu(g, z)^{-1}, (a_2 w + b_2 \tau + c_2) \mu(g, z)^{-1}) & (q > 1), \\
(a_1 z + b_1) \mu(g, z)^{-1} & (q = 1),
\end{cases}
\]
where

\[\mu(g, z) = \begin{cases} 
  a_3w + b_3\tau + c_3 & (q > 1), \\
  a_2z + b_2 & (q = 1),
\end{cases}\]

for \(g = \begin{pmatrix} a_1 & b_1 \\
  a_2 & b_2 \\
  a_3 & b_3 \
\end{pmatrix}\) when \(q > 1\) and

for \(g = \begin{pmatrix} a_1 & b_1 \\
  a_2 & b_2 
\end{pmatrix}\) when \(q = 1\).

Here we note that \(\mu(g, z)\) defines an automorphy factor for \(G \times h\). When \(q > 1\) (respectively \(q = 1\)) this symmetric space has \(z_0 := (0, 0, 1)\) (respectively 1) as a base point with respect to the \(G\)-action above. The group of stabilizers of \(z_0\) in \(G\) coincides with \(K\).

The Lie algebra \(g\) of \(G\) is

\[\{ X \in M_{q+1}(\mathbb{H}) \mid {}^t\bar{X}Q + QX = 0_{q+1} \} .\]

Let \(\theta\) also denote the Cartan involution of \(g\) defined by

\[ g \ni X \mapsto -{}^t\bar{X} \in g. \]

The Lie algebra \(\mathfrak{k}\) of \(K\) is equal to \(\{ X \in g \mid \theta(X) = X \} \). We set \(p := \{ X \in g \mid \theta(X) = -X \} \). Then \(g\) admits the Cartan decomposition \(g = \mathfrak{k} \oplus p\) (for the definition see [22, Chapter VI, Section 4]).

In order to describe an Iwasawa decomposition of \(G\) and \(g\) (for the definition see [22, Chapter VI, Section 4]) we give the restricted root system of \(g\), which is of \(BC_1\)-type when \(q > 1\) (respectively \(C_1\)-type when \(q = 1\)) (cf. [22, p. 701]). Let \(H := \{\text{diag}(0, \ldots, 0, e_1, -e_1) \mid (q > 1), \text{diag}(e_1, -e_1) \mid (q = 1)\} \).

Then \(a := \mathbb{R}H\) forms a maximal abelian subalgebra of \(p\). Let \(\alpha\) be the element in \(a^*\) such that \(\alpha(H) = 1\). The root system \(\Delta(a, g)\) of \((g, a)\) is then given as

\[\{\pm\alpha, \pm 2\alpha\} \quad (q > 1) \quad \text{and} \quad \{\pm 2\alpha\} \quad (q = 1)\]

and the root spaces are

\[g_\alpha = \bigoplus_{1 \leq a \leq 4 \atop 1 \leq i \leq q-1} \mathbb{R}E_{\alpha}^{(a,i)}, \quad g_{-\alpha} = \bigoplus_{1 \leq a \leq 4 \atop 1 \leq i \leq q-1} \mathbb{R}E_{-\alpha}^{(a,i)}, \]

\[g_{2\alpha} = \bigoplus_{2 \leq a \leq 4} \mathbb{R}E_{2\alpha}^{(a)}, \quad g_{-2\alpha} = \bigoplus_{2 \leq a \leq 4} \mathbb{R}E_{-2\alpha}^{(a)} \]

with

\[E_{\alpha}^{(a,i)} := e_aE_{i,q+1} + \tilde{e}_aE_{q,i}, \quad E_{2\alpha}^{(a)} = e_aE_{q,q+1}, \quad E_{-\alpha}^{(a,i)} := {}^tE_{\alpha}^{(a,i)}, \quad E_{-2\alpha}^{(a)} := {}^tE_{2\alpha}^{(a)} .\]

When \(q = 1\) there is no \(g_{\pm \alpha}\).
Put \( n := g_\alpha \oplus g_{2\alpha} \) when \( q > 1 \) (respectively \( n = g_{2\alpha} \) when \( q = 1 \)). We can express \( n \) as follows:

\[
\begin{align*}
n &= \begin{cases}
(w, x) := \begin{pmatrix}
0_{q-1} & 0_{q-1,1} & w \\
\bar{w} & 0 & x
\end{pmatrix} & | \ w \in \mathbb{H}^{q-1}, \ x \in X_R \\
(x) := \begin{pmatrix}
0 & x \\
0 & 0
\end{pmatrix} & | \ x \in X_R
\end{cases} \\
&= \begin{cases}
(w, x) := \begin{pmatrix}
0 & 0 & w \\
1 & 0 & \bar{w}w + x
\end{pmatrix} & | \ w \in \mathbb{H}^{q-1}, \ x \in X_R \\
(x) := \begin{pmatrix}
1 & x \\
0 & 1
\end{pmatrix} & | \ x \in X_R
\end{cases}
\quad (q > 1),
\end{align*}
\]

Then \( g = n \oplus a \oplus \mathfrak{k} \) is an Iwasawa decomposition of \( g \).

Furthermore, we set \( A := \exp(a) \) and \( N := \exp(n) \). They can be expressed as

\[
\begin{align*}
A &= \begin{cases}
a = a_y := \begin{pmatrix} 1_{q-1} & \sqrt{y} \\
\sqrt{y} & \sqrt{y}^{-1}
\end{pmatrix} & | \ y \in \mathbb{R}^+ \\
a = a_y := \begin{pmatrix} \sqrt{y} & \sqrt{y}^{-1}
\end{pmatrix} & | \ y \in \mathbb{R}^+
\end{cases} \\
N &= \begin{cases}
n(w, x) := \begin{pmatrix} 1_{q-1} & 0_{q-1,1} & w \\
\bar{w} & 1 & \frac{1}{2} \bar{w}w + x
\end{pmatrix} & | \ w \in \mathbb{H}^{q-1}, \ x \in X_R \\
n(x) := \begin{pmatrix}
1 & x \\
0 & 1
\end{pmatrix} & | \ x \in X_R
\end{cases}
\quad (q > 1),
\end{align*}
\]

Then we have an Iwasawa decomposition \( G = NAK \).

Let \( dw, dx \) and \( dy \) denote the Euclidean measure on \( \mathbb{H}^{q-1}, \ X_R \) and \( \mathbb{R} \), respectively, and \( dk \) the invariant measure of \( K \) with \( \int_K dk = 1 \) (\( dw \) is omitted when \( q = 1 \)). We then choose

\[
\begin{align*}
dn &= dw \, dx, \quad (1.3) \\
dg &= y^{-2(q+1)} \, dw \, dx \, dy \, dk, \quad (1.4)
\end{align*}
\]
as our invariant measures on \( N \) and \( G \), respectively, where we note that \( dg \) is induced by the Iwasawa decomposition.

Next we introduce the absolute root system of \( g_\mathbb{C} \) with respect to the complexification of a compact Cartan subalgebra in \( g \). Let \( \mathfrak{t} \) be the compact Cartan subalgebra defined by \( \mathfrak{t} := \bigoplus_{1 \leq i \leq q+1} \mathbb{R} T_i \), where

\[
T_i := e_2 E_{ii} \quad \text{for } 1 \leq i \leq q - 1,
\]

\[
T_q := c(e_2 E_{qq}) c^{-1} = \frac{1}{2} e_2 (E_{qq} - E_{q,q+1} - E_{q+1,q} + E_{q+1,q+1}),
\]

\[
T_{q+1} := c(e_2 E_{q+1,q+1}) c^{-1} = \frac{1}{2} e_2 (E_{qq} + E_{q,q+1} + E_{q+1,q} + E_{q+1,q+1})
\]
with

\[
c := \begin{pmatrix}
1_{q-1} & 1/\sqrt{2} & 1/\sqrt{2} \\
-1/\sqrt{2} & 1/\sqrt{2} & \\
\end{pmatrix}.
\]

Let \( \beta_i \) be the element in \( t^*_C \) such that \( \beta_i(T_j) = \sqrt{-1}\delta_{ij} \). Then

\[
\Delta := \{ \pm(\beta_i \pm \beta_j), \pm 2\beta_k \mid 1 < i < j < q + 1, 1 \leq k \leq q + 1 \}
\]
is the absolute root system of \((g_C, t_C)\), which is of \( C_{q+1} \)-type (cf. [22, p. 701]). We put

\[
\Delta_c := \{ \pm(\beta_i \pm \beta_j), \pm 2\beta_k \mid 1 < i < j < q, 1 \leq k \leq q + 1 \},
\]

\[
\Delta_n := \{ \pm(\pm\beta_k + \beta_{q+1}) \mid 1 \leq k \leq q \}.
\]

Then \( \Delta_c \) (respectively \( \Delta_n \)) forms the set of compact roots (respectively non-compact roots), i.e., roots occurring in the eigenspace decomposition of \( t_C \) (respectively \( p_C \)) with respect to the \( t_C \)-action.

Let

\[
\Delta_{c,0}^+ := \{ \beta_i \pm \beta_j, 2\beta_k \mid 1 \leq i < j \leq q, 1 \leq k \leq q + 1 \},
\]

\[
\Delta_{n,0}^+ := \{ \pm\beta_k + \beta_{q+1} \mid 1 \leq k \leq q \}.
\]

We choose \( \Delta_0^+ := \Delta_{n,0}^+ \cup \Delta_{c,0}^+ \) as our standard choice of a positive system.

We can take as root vectors the following:

\[
F_{\beta_i+\beta_j} := (e_3 - \sqrt{-1}e_4)(E_{ij} + E_{ji}) \quad (1 < i < j \leq q - 1),
\]

\[
F_{\beta_i-\beta_j} := (e_1 - \sqrt{-1}e_2)E_{ij} - (e_1 + \sqrt{-1}e_2)E_{ji} \quad (1 < i < j \leq q - 1),
\]

\[
F_{\beta_k+\beta_q} := (e_3 - \sqrt{-1}e_4)(E_{k,q} - E_{k,q+1} + E_{q,k} - E_{q+1,k}) \quad (1 \leq k \leq q - 1),
\]

\[
F_{\beta_k-\beta_q} := (e_1 - \sqrt{-1}e_2)(E_{k,q} - E_{k,q+1}) - (e_1 + \sqrt{-1}e_2)(E_{q,k} - E_{q+1,k}) \quad (1 \leq k \leq q - 1),
\]

\[
F_{\beta_k+\beta_{q+1}} := (e_3 - \sqrt{-1}e_4)(E_{k,q} + E_{k,q+1}) - (e_3 + \sqrt{-1}e_2)(E_{k,q} + E_{q+1,k}) \quad (1 \leq k \leq q - 1),
\]

\[
F_{\beta_k-\beta_{q+1}} := (e_1 - \sqrt{-1}e_2)(E_{k,q} + E_{k,q+1}) + (e_1 + \sqrt{-1}e_2)(E_{k,q} + E_{q+1,k}) \quad (1 \leq k \leq q - 1),
\]

\[
F_{\beta_q+\beta_{q+1}} := -(e_3 - \sqrt{-1}e_4)E_{q,q+1} + (e_3 - \sqrt{-1}e_4)E_{q+1,q},
\]

\[
F_{\beta_{q+1}-\beta_q} := e_1(E_{qq} - E_{q+1,q+1}) - \sqrt{-1}e_2(E_{q,q+1} - E_{q+1,q}).
\]

\[
F_{2\beta_k} := (e_3 - \sqrt{-1}e_4)E_{kk} \quad (1 \leq k \leq q - 1),
\]

\[
F_{2\beta_q} := (e_3 - \sqrt{-1}e_4)(E_{qq} - E_{q,q+1} - E_{q+1,q} + E_{q+1,q+1}),
\]

\[
F_{2\beta_{q+1}} := (e_3 - \sqrt{-1}e_4)(E_{qq} + E_{q,q+1} + E_{q+1,q} + E_{q+1,q+1}).
\]

\( F_{-\beta} \) is given by the complex conjugate of \( F_\beta \) for \( \beta \in \Delta_0^+ \).
Then we have
\[ t_C = \bigoplus_{\beta \in \Delta^*_C} F_{\beta} \quad \text{and} \quad p_C := p \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{\beta \in \Delta^*_C} F_{\beta}. \]

In Section 5.2 we need Iwasawa decompositions of the generators of \( p_C \).

**Lemma 1.1.** Let \( \text{Ad}(a) \) denote the adjoint action of \( a \in A \) on \( n \). We have

\[
F_{\beta_k - \beta_q + 1} = \sqrt{y} \text{Ad}(a^{-1})(2E_{\alpha}^{(1,k)} - 2\sqrt{-1}E_{\alpha}^{(2,k)}) + F_{\beta_k - \beta_q},
\]
\[
F_{-(\beta_k - \beta_q + 1)} = \sqrt{y} \text{Ad}(a^{-1})(2E_{\alpha}^{(1,k)} + 2\sqrt{-1}E_{\alpha}^{(2,k)}) + F_{-(\beta_k - \beta_q)},
\]
\[
F_{\beta_k + \beta_q + 1} = \sqrt{y} \text{Ad}(a^{-1})(2E_{\alpha}^{(3,k)} - 2\sqrt{-1}E_{\alpha}^{(4,k)}) + F_{\beta_k + \beta_q},
\]
\[
F_{-(\beta_k + \beta_q + 1)} = \sqrt{y} \text{Ad}(a^{-1})(2E_{\alpha}^{(3,k)} + 2\sqrt{-1}E_{\alpha}^{(4,k)}) + F_{-(\beta_k + \beta_q)},
\]
\[
F_{\beta_q + \beta_q + 1} = -y \text{Ad}(a^{-1})(2E_{2\alpha}^{(3)} - 2\sqrt{-1}E_{2\alpha}^{(4)}) - \frac{1}{2}(F_{2\beta_q} - F_{2\beta_q + 1}),
\]
\[
F_{-(\beta_q + \beta_q + 1)} = -y \text{Ad}(a^{-1})(2E_{2\alpha}^{(3)} + 2\sqrt{-1}E_{2\alpha}^{(4)}) - \frac{1}{2}(F_{-2\beta_q} - F_{-2\beta_q + 1}),
\]
\[
F_{\beta_q - \beta_q + 1} = -2y \text{Ad}(a^{-1})\sqrt{-1}E_{2\alpha}^{(2)} + H - \sqrt{-1}(T_q - T_{q+1}),
\]
\[
F_{-(\beta_q - \beta_q + 1)} = 2y \text{Ad}(a^{-1})\sqrt{-1}E_{2\alpha}^{(2)} + H + \sqrt{-1}(T_q - T_{q+1}),
\]

where \( 1 \leq k \leq q - 1 \).

Here we note that only the last four formulas are necessary when \( q = 1 \).

**2. Representations of the maximal compact subgroup \( K \)**

In this section we describe the set \( \hat{K} \) of equivalence classes of irreducible finite-dimensional representations of \( K \) and provide some representations of \( K \) necessary for our study.

Recall that there is an isomorphism \( K \simeq \text{Sp}^*(q) \times \text{Sp}^*(1) \) (cf. Section 1). Hence we have a bijection

\[ \hat{K} \simeq \text{Sp}^*(q) \times \text{Sp}^*(1), \]

where \( \text{Sp}^*(q) \) (respectively \( \text{Sp}^*(1) \)) denotes the set of equivalence classes of irreducible representations of \( \text{Sp}^*(q) \) (respectively \( \text{Sp}^*(1) \)). Therefore every element of \( \hat{K} \) is expressed as an outer tensor product of irreducible representations of \( \text{Sp}^*(q) \) and \( \text{Sp}^*(1) \).

By the theorem of the highest weight (cf. [22, Theorem 5.5]) equivalence classes of irreducible representations of the \( \text{Sp}^*(q) \)-factor in \( K \) (respectively the \( \text{Sp}^*(1) \)-factor in \( K \)) are parametrized by the set of integral weights

\[ \{ n_1\beta_1 + n_2\beta_2 + \cdots + n_q\beta_q \mid (n_1, n_2, \ldots, n_q) \in \mathbb{Z}_{\geq 0}^q, n_1 \geq n_2 \geq \cdots \geq n_q \}, \]
which are dominant with respect to \( \{ \beta_i \pm \beta_j, \ 2\beta_k \mid 1 \leq i < j \leq q, \ 1 \leq k \leq q \} \) (respectively \( \{ n_q+1 \beta_{q+1} \mid n_q+1 \in \mathbb{Z}_{\geq 0} \} \), which are dominant with respect to \( \{ 2\beta_{q+1} \} \)). We define \((\tau_{n_1n_2 \ldots n_q}, V_{(n_1,n_2 \ldots n_q)})\) (respectively \((\tau_{n_{q+1}}, V_{n_{q+1}})\)) as the representation of the \(Sp^s(q)\)-factor (respectively the \(Sp^s(1)\)-factor) parametrized by the dominant integral weight \( n_1\beta_1 + n_2\beta_2 + \cdots + n_q\beta_q \) (respectively \( n_{q+1}\beta_{q+1} \)). Hence every element of \( \hat{K} \) is of the form

\[
(\tau_{n_1n_2 \ldots n_q}, V_{(n_1,n_2 \ldots n_q)\otimes V_{n_{q+1}}}) = (\tau_{n_1n_2 \ldots n_q} \otimes \tau_{n_{q+1}}, V_{(n_1,n_2 \ldots n_q) \otimes V_{n_{q+1}}})
\]

and the set \( \hat{K} \) is in bijective correspondence with the following set of integral weights dominant with respect to \( \Delta^+_c \):

\[
\{ n_1\beta_1 + n_2\beta_2 + \cdots + n_q\beta_q + n_{q+1}\beta_{q+1} \mid (n_1, n_2, \ldots, n_{q+1}) \in \mathbb{Z}_{\geq 0}^{q+1}, \ n_1 \geq n_2 \geq \cdots \geq n_q \}.
\]

Now we describe two representations in \( \hat{K} \), which we use later. For a positive integer \( \kappa \) let \( \sigma_\kappa \) be the pull-back of the \( \kappa \)th symmetric tensor representation of \( GL_2(\mathbb{C}) \) to \( \mathbb{H}^\times \) by \( \varphi \) (for \( \varphi \) see (1.2)). Consider the representation of \( K \) defined by

\[
K \ni k \mapsto \sigma_\kappa(\mu(k, z_0)).
\]

This representation is non-trivial only on the \( Sp^s(1) \)-factor and has \( \kappa\beta_{q+1} \) as its highest weight, i.e., is isomorphic to the outer tensor product of the trivial representation of \( Sp^s(q) \) and \((\tau_\kappa, V_\kappa)\). From now on we denote this representation simply by \((\tau_\kappa, V_\kappa)\). The representation space \( V_\kappa \) has a basis \( \{ v_{\kappa,i} \}_{0 \leq i \leq \kappa} \) satisfying

\[
d\tau_\kappa(T_k) v_{\kappa,i} = \begin{cases} 0 & (1 \leq k \leq q), \\ \sqrt{-1}(2i - \kappa)v_{\kappa,i} & (k = q + 1), \end{cases}
\]

\[
d\tau_\kappa(F_{\pm(\beta_i \pm \beta_j)}) = d\tau_\kappa(F_{\pm 2\beta_k}) = 0 \quad \text{for } 1 \leq i < j \leq q \text{ and } 1 \leq k \leq q,
\]

\[
d\tau_\kappa(F_{2\beta_{q+1}}) v_{\kappa,i} = 4(\kappa - i)v_{\kappa,i+1}, \quad d\tau_\kappa(F_{-2\beta_{q+1}}) v_{\kappa,i} = -4i v_{\kappa,i-1},
\]

where \( d\tau_\kappa \) denotes the differential of \( \tau_\kappa \). This formula gives the weight of \( v_{\kappa,i} \) to be \((2i - \kappa)\beta_{q+1} \). The existence of this basis is assured by the well-known formula for the infinitesimal actions of irreducible finite-dimensional representations of \( SL_2(\mathbb{C}) \). Hereafter we fix a basis \( \{ v_{\kappa,i} \}_{0 \leq i \leq \kappa} \) of \( V_\kappa \) with the above properties.

As another representation we have the adjoint representation \((Ad_K, p_\mathbb{C})\) of \( K \) on \( p_\mathbb{C} \). We can verify that, up to constant multiples, \( F_{\beta_1 + \beta_{q+1}} \) is a unique highest weight vector of the representation. In fact, the calculations of bracket products \([F, X]\) for \( F \in t_\mathbb{C} \) and \( X \in p_\mathbb{C} \) show

\[
\mathbb{C}F_{\beta_1 + \beta_{q+1}} = \{ X \in p_\mathbb{C} \mid [F_\beta, X] = 0 \ \forall \beta \in \Delta^+_c \}.
\]

Hence an isomorphism

\[
(Ad_K, p_\mathbb{C}) \cong (\tau_{(1,0,\ldots,0|1)}, V_{(1,0,\ldots,0|1)})
\]

holds.

For our subsequent argument we state the following lemma.
Lemma 2.1.

(1) The representation \( \tau_{(1,0,\ldots,0)} \in \hat{Sp}^*(q) \) is isomorphic to \( (\text{Ad}_K|_{\hat{Sp}^*(q)}, p^+) \) with \( p^+ := \sum_{1 \leq k \leq q} (\mathbb{C}F_{\beta_1+\beta_q} + \mathbb{C}F_{-\beta_1+\beta_q+1}) \).

(2) Any representation in \( \hat{K} \) is isomorphic to its contragredient, namely, it is self-dual.

Proof. For the assertion (1) it suffices to show that \( (\text{Ad}_K|_{\hat{Sp}^*(q)}, p^+) \) has a unique highest weight vector \( F_{\beta_1+\beta_q+1} \) with highest weight \( (1,0,\ldots,0) \), up to constant multiples. It is verified by a direct calculation. For the assertion (2) we need another lemma (cf. [22, Chapter II, Section 12, Problem 18, Chapter V, Section 9, Problem 1]).

Lemma 2.2.

(1) Let \( g' \) be a complex semisimple Lie algebra with a Cartan subalgebra \( \mathfrak{t}' \) and \( \Phi \) be the root system of \( (g', \mathfrak{t}') \) with a positive system \( \Phi^+ \). Then there exists a unique element \( w_0 \) in the Weyl group of \( \Phi \) such that \( w_0\Phi^+ = -\Phi^+ \).

(2) For an irreducible finite-dimensional representation of \( g' \) with highest weight \( \Lambda \), the highest weight of its contragredient is equal to \( -w_0\Lambda \).

We apply this lemma to \( \mathfrak{k}_c \), which is semisimple. Lemma 2.2(1) implies that \(-w_0\) induces an automorphism of the Dynkin diagram of \( \mathfrak{k}_c \). However, the Dynkin diagram of the Lie algebra of a complex symplectic group admits no non-trivial automorphism (cf. [18, Chapter III, Section 12, Table 1]). Thus \(-w_0 = 1\) for our case. Therefore Lemma 2.2(2) tells us that every element in \( \hat{K} \) has the same highest weight as that of its contragredient. \( \square \)

From now on we fix a \( K \)-invariant Hermitian inner product \( (\ast, \ast)_K \) of \( V_k \) with respect to \( \tau_k \). By means of this inner product we can identify \( (\tau_k, V_k) \) with \( (\tau_k^*, V_k^*) \) in view of Lemma 2.1(2).

Let us consider the representation \((\tau_k \otimes \text{Ad}_K, V_k \otimes p_c)\) of \( K \). By a direct computation we see that \((\tau_k \otimes \text{Ad}_K, V_k \otimes p_c)\) has two highest weight vectors \( v_{k,k} \otimes F_{\beta_1+\beta_q+1} \) and \( v_{k,k-1} \otimes F_{\beta_1+\beta_q+1} + v_{k,k} \otimes F_{\beta_1-\beta_q+1} \) with highest weights \( \beta_1 + (k+1)\beta_q+1 \) and \( \beta_1 + (k-1)\beta_q+1 \), respectively. Hence one obtains the decomposition

\[ V_k \otimes p_c \simeq V_{(1,0,\ldots,0|k+1)} \oplus V_{(1,0,\ldots,0|k-1)} \]

as \( K \)-modules. Let \( P_k \) be the projection given by

\[ P_k : V_k \otimes p_c \rightarrow V_{(1,0,\ldots,0|k-1)}. \quad (2.4) \]

For later use we need to describe this \( P_k \) explicitly. For that purpose we fix a basis \( \{ z_{\pm\beta_k} \}_{1 \leq k \leq q} \) of \( V_{(1,0,\ldots,0)} \) as follows:

\[ d\tau_{(1,0,\ldots,0)}(T)z_{\pm\beta_k} = \pm \beta_k(T)z_{\pm\beta_k} \quad \text{for} \quad T \in \bigoplus_{1 \leq i \leq q} \mathbb{C}T_i \quad (1 \leq k \leq q), \quad (2.5) \]

\[ d\tau_{(1,0,\ldots,0)}(F_\alpha)z_{\pm\beta_k} = \begin{cases} C_{\alpha,k} z_{\pm\beta_k + \alpha} & \text{when} \quad F_{\alpha+\beta_k+\beta_q+1} \in p^+ \quad \text{for} \quad \alpha \in \Delta_c \setminus \{ \pm 2\beta_q+1 \}, \\ 0 & \text{(otherwise)} \end{cases} \quad (2.6) \]

...
where \( d\tau(1,0,\ldots,0) \) is the differential of \( \tau(1,0,\ldots,0) \) and \( C_{\alpha,k}^\pm \) is the constant determined by \( [F_\alpha, F_{\pm \beta_k + \beta_{q+1}}] = C_{\alpha,k}^\pm F_{\alpha \pm \beta_k + \beta_{q+1}} \). This choice of the basis is justified, since \( V_{(1,0,\ldots,0)} \simeq p^+ \) as \( Sp^*(q) \)-modules (cf. Lemma 2.1(1)). This implies that \( (\tau(1,0,\ldots,0), V_{(1,0,\ldots,0)}) \) has \( \{\pm \beta_k | 1 \leq k \leq q \} \) as its set of weights.

Then we can state an explicit formula for the projection \( P_k \).

**Lemma 2.3.** \( Up to scales, P_k \) satisfies

\[
P_k(v_{k,k} \otimes F_{\pm \beta_k + \beta_{q+1}} + v_{k,0} \otimes F_{\pm \beta_k - \beta_{q+1}}) = P(v_{k,0} \otimes F_{\pm \beta_k - \beta_{q+1}}) = 0 \quad (1 \leq k \leq q),
\]

\[
P_k(v_{k,i} \otimes F_{\pm \beta_k + \beta_{q+1}}) = (k - i)v_{k-1,i} \otimes z_{\pm \beta_k} \quad (0 \leq i \leq k - 1, \ 1 \leq k \leq q),
\]

\[
P_k(v_{k,i} \otimes F_{\pm \beta_k - \beta_{q+1}}) = \begin{cases} 
\pm i v_{k-1,i-1} \otimes z_{\pm \beta_k} & (1 \leq i \leq k, \ 1 \leq k \leq q - 1), \\
\mp i v_{k-1,i-1} \otimes z_{\pm \beta_k} & (1 \leq i \leq k, \ k = q) \end{cases}
\]

**Proof.** The weights of \( v_{k,k} \otimes F_{\pm \beta_k + \beta_{q+1}} \) and \( v_{k,0} \otimes F_{\pm \beta_k - \beta_{q+1}} \) are

\[(k + 1)\beta_{q+1} \pm \beta_k \quad \text{and} \quad -(k + 1)\beta_{q+1} \pm \beta_k,
\]

respectively. Such weights never occur in \( V_{(1,0,\ldots,0,k-1)} \). This implies (2.7).

Let us deduce (2.9). First note that \( P_k(v_{k,k-1} \otimes F_{\beta_1 + \beta_{q+1}} + v_{k,k} \otimes F_{\beta_1 - \beta_{q+1}}) \) is a highest weight vector in \( V_{(1,0,\ldots,0,k-1)} \). Since the highest weight vector is unique up to constant multiples, we have

\[
P_k(v_{k,k-1} \otimes F_{\beta_1 + \beta_{q+1}} + v_{k,k} \otimes F_{\beta_1 - \beta_{q+1}}) = C v_{k-1,k-1} \otimes z_{\beta_1}
\]

(2.10)

with some constant \( C \). On the other hand, consider the action of \( F_{-2\beta_{q+1}} \) on both sides of \( P_k(v_{k,k} \otimes F_{\pm \beta_k + \beta_{q+1}}) = 0 \). Then we obtain

\[
-4k P_k(v_{k,k-1} \otimes F_{\pm \beta_k + \beta_{q+1}}) \pm 4P_k(v_{k,k} \otimes F_{\pm \beta_k - \beta_{q+1}}) = 0 \quad (1 \leq k \leq q - 1),
\]

\[
-4k P_k(v_{k,k-1} \otimes F_{\pm \beta_k + \beta_{q+1}}) \mp 4P_k(v_{k,k} \otimes F_{\pm \beta_k - \beta_{q+1}}) = 0 \quad (k = q),
\]

by the formula (2.3) and

\[
[F_{-2\beta_{q+1}}, F_{\pm \beta_k + \beta_{q+1}}] = \begin{cases} 
\pm 4F_{\pm \beta_k - \beta_{q+1}} & (1 \leq k \leq q - 1), \\
\mp 4F_{\pm \beta_k - \beta_{q+1}} & (k = q) \end{cases}
\]

Then (2.11) implies that (2.10) can be replaced by

\[
P_k(v_{k,k-1} \otimes F_{\beta_1 + \beta_{q+1}}) = C v_{k-1,k-1} \otimes z_{\beta_1}.
\]

(2.13)

where we write \( \frac{C}{k+1} \) as \( C \) again. Moreover, (2.6) and (2.13) lead to

\[
P_k(v_{k,k-1} \otimes F_{\pm \beta_k + \beta_{q+1}}) = C v_{k-1,k-1} \otimes z_{\pm \beta_k}
\]

(2.14)

for \( 1 \leq k \leq q \). Thus (2.11), (2.12) and (2.14) yield

\[
P_k(v_{k,k} \otimes F_{\pm \beta_k - \beta_{q+1}}) = \begin{cases} 
\pm \kappa (C v_{k-1,k-1} \otimes z_{\pm \beta_k}) & (1 \leq k \leq q - 1), \\
\mp \kappa (C v_{k-1,k-1} \otimes z_{\pm \beta_k}) & (k = q) \end{cases}
\]
Let $F_{-2\beta_q+1}$ act on both sides of this formula repeatedly. Then we obtain (2.9) by the formula (2.3).

The remaining formula (2.8) is settled similarly, hence we omit the proof. $\square$

3. Discrete series representations of $G$

Among the representations of $G$ we exclusively deal with quaternionic discrete series representations of $G$, which were introduced by B. Gross and N. Wallach [12]. In this section we first review several general facts on discrete series representations. After that we will restrict ourselves to quaternionic discrete series of $G$.

First of all we recall that an irreducible unitary representation of $G$ can be regarded as an irreducible $(g,K)$-module by considering its set of $K$-finite vectors (cf. [33, Theorem 3.4.11]). For the definition of a $(g,K)$-module see [33, 3.3.1]. For a $(g,K)$-module an irreducible representation of $K$ occurring in it is called a $K$-type of it.

To an irreducible $(g,K)$-module $H$ we attach an algebraic homomorphism $\chi$ from the center $Z(g_{\mathbb{C}})$ of $U(g_{\mathbb{C}})$ to $\mathbb{C}$ defined by

$$z \cdot v = \chi(z) \cdot v \quad \forall z \in Z(g_{\mathbb{C}}), \forall v \in H.$$ 

We call $\chi$ the infinitesimal character of $H$. It is known that, via the Harish-Chandra isomorphism for $Z(g_{\mathbb{C}})$, such a character is parametrized by an element $\mu$ in the dual space $s^*$ of some Cartan subalgebra $s$ of $g_{\mathbb{C}}$ (for details on this see [33, Theorem 3.2.4]). We also write $\chi_\mu$ for it.

Now we explain the discrete series representations of $G$. Note that $G$ has a compact Cartan subgroup $T$ with the Lie algebra $t$ (for $t$ see Section 1). Then we know that, by Harish-Chandra’s criterion (cf. [13, Theorem 13]), $G$ has discrete series representations, i.e., irreducible unitary representations whose matrix coefficients are square-integrable. When a discrete series representation has integrable matrix coefficients, it is said to be an integrable representation. The following proposition is well known.

**Proposition 3.1.** For an irreducible unitary representation of $G$ one of its non-zero $K$-finite matrix coefficients is square-integrable (respectively integrable) if and only if all of its matrix coefficients are square-integrable (respectively integrable).

For a proof see [21, Proposition 9.6]. It treats only the case of the square-integrability but the case of the integrability is proved similarly.

In order to review several fundamental facts on discrete series we introduce some notations. Given a positive system $\Delta^+ = \Delta^+_c \cup \Delta^+_n$ of $\Delta$ with $\Delta^+_c := \Delta^+ \cap \Delta_c$ and $\Delta^+_n := \Delta^+ \cap \Delta_n$ (for $\Delta$, $\Delta_c$ and $\Delta_n$ see Section 1), we denote by $\rho_c$ and $\rho_n$ the half sums of the positive roots in $\Delta^+_c$ and $\Delta^+_n$, respectively. Moreover, we set $\rho_G := \rho_c + \rho_n$. We call $\lambda \in t^*_C$ analytically integral if it comes from a differential of a unitary character of $T$. Let $\langle *, * \rangle$ be the non-degenerate bilinear form on $t^*_C \times t^*_C$ induced by the Killing form of $g_{\mathbb{C}}$. If $\lambda \in t^*_C$ satisfies

$$\langle \lambda, \beta \rangle \neq 0 \quad \forall \beta \in \Delta,$$

then $\lambda$ is said to be non-singular.

Every discrete series representation has the infinitesimal character of the form $\chi_\lambda$, with $\lambda \in t^*_C$ such that $\lambda + \rho_G$ is analytically integral and $\lambda$ is non-singular. Here we note that the integrity
condition on $\lambda + \rho G$ does not depend on the choice of a positive system. According to Harish-Chandra's parametrization of discrete series (cf. [13, Theorem 16] and [21, Theorems 9.20, 12.21]), two discrete series representations $\pi_\lambda$ and $\pi_{\lambda'}$ of $G$ with the infinitesimal characters $\chi_\lambda$ and $\chi_{\lambda'}$ are unitarily equivalent if and only if $\lambda$ and $\lambda'$ are conjugate under the action of the Weyl group for the compact roots. For the discrete series $\pi_\lambda$ we call $\lambda$ the Harish-Chandra parameter of it.

**Proposition 3.2.**

1. Let $\pi_\lambda$ be the discrete series representation above and $\Delta^+$ the positive system given by $\Delta^+ = \{\beta \in \Delta \mid \langle \lambda, \beta \rangle > 0\}$. Then $\pi_\lambda$ has the following distribution of $K$-types:
   - (i) the $K$-type with highest weight $\Lambda := \lambda + \rho_n - \rho_c$ occurs in $\pi_\lambda$ exactly once;
   - (ii) the highest weight of a $K$-type having non-zero multiplicity in $\pi_\lambda$ is of the form $\Lambda + \sum_{\alpha \in \Delta_n^+} n_\alpha \alpha$, $n_\alpha \in \mathbb{Z}_{\geq 0}$.

2. Any irreducible $(g, K)$-module with distribution of $K$-types as in (1) is equivalent to $\pi_\lambda$ as $(g, K)$-modules. In particular, $\pi_\lambda$ is a unique irreducible unitary representation with distribution of $K$-types as in (1), up to unitary equivalence. Namely the discrete series $\pi_\lambda$ is characterized by its $K$-module structure given in (1).

For (1) and (2) see [29, Corollary to Theorem 1, Theorem 2] and [27, Theorem 1.3]. The assertion (1) is a consequence of the “Blattner formula,” which describes the multiplicities of $K$-types in discrete series and is proved by H. Hecht and W. Schmid [14, Theorem 1.3]. A proof of (2) is given in [27, Section 4]. We call the $K$-type $(\tau_\Lambda, V_\Lambda)$ of $\pi_\lambda$ with highest weight $\Lambda$ the minimal $K$-type of $\pi_\lambda$.

In addition to the general facts above on discrete series, we give a well-known criterion of the integrability of discrete series (cf. [15, 31]).

**Proposition 3.3.** (Trombi, Varadarajan, Hecht, Schmid) A discrete series $\pi_\lambda$ is integrable if and only if

$$|\langle \lambda, \beta \rangle| > \frac{1}{2} \sum_{\alpha \in \Delta^+} |\langle \alpha, \beta \rangle| \quad \forall \beta \in \Delta_n^+ \cup (-\Delta_n^+),$$

where $\Delta^+$ is as in Proposition 3.2(1) and $\Delta_n^+$ the set of non-compact positive roots in it.

Now we take the positive systems $\Delta_{0}^+, \Delta_{c,0}^+$ and $\Delta_{n,0}^+$ given in Section 1. For a positive integer $\kappa > 2q - 1$ let $(\pi_\kappa, H_\kappa)$ be the discrete series representation of $G$ with Harish-Chandra parameter $\kappa \beta_{q+1} + \rho_c - \rho_n$. The condition $\kappa > 2q - 1$ means the non-singularity of this Harish-Chandra parameter with respect to $\Delta_{0}^+$. Furthermore,

$$\pi_\kappa \text{ is integrable if and only if } \kappa > 4q \quad (3.1)$$

by Proposition 3.3. This was also verified by Arakawa in [3, Lemma 2.10(ii)]. The minimal $K$-type of $\pi_\kappa$ is given by $(\tau_\kappa, V_\kappa)$. This $\pi_\kappa$ is a quaternionic discrete series in the sense of
Gross and Wallach [12]. For our later convenience we provide a description of $\pi_\kappa$ in terms of its distribution of $K$-types.

**Proposition 3.4.**

(1) As $K$-modules, $\pi_\kappa|_K$ is equivalent to

$$\bigoplus_{n \in \mathbb{Z}_{\geq 0}} \tau(n,0,...,0|n+\kappa).$$

(2) Up to equivalence as a $(g, K)$-module, the set of $K$-finite vectors in $H_\kappa$ is the only irreducible $(g, K)$-module with distribution of $K$-types as in (1).

**Proof.** The assertion (1) is proved in [12, Proposition 5.7] (for this see also [12, Proposition 8.4]). The assertion (2) is an immediate consequence of Proposition 3.2(2) and the assertion (1). \qed

### 4. Irreducible unitary representations of $N$

In this section we describe the unitary dual $\hat{N}$ of $N$ explicitly in terms of Kirillov’s orbit method [20].

To begin with, we describe the structure of $N$. Let

$$Z(N) := \begin{cases} \{n(0, x) \in N \mid x \in X_\mathbb{R}\} & (q > 1), \\ \{n(x) \in N \mid x \in X_\mathbb{R}\} & (q = 1), \end{cases}$$

and $N_L := \{n(w, 0) \in N \mid w \in \mathbb{H}^{q-1}\}$ when $q > 1$. When $q > 1$, $Z(N)$ is the center of $N$ and $N = N_L Z(N)$ holds. When $q = 1$, $Z(N)$ coincides with $N$, i.e., $N$ is abelian. The Lie algebra of $Z(N)$ is $g_2^\alpha$ (for $g_2^\alpha$ see Section 1).

For the subsequent argument we explicitly write the bracket product $[*,*]$ on $n$ and the products of elements of $N$. When $q > 1$ they are written as

$$[(w, x), (w', x')] = (0_{q-1,1}, {}^t \bar{w} w' - {}^t \bar{w}' w),$$

$$n(w, x)n(w', x') = n\left(w + w' \cdot \frac{1}{2}({}^t \bar{w} w' - {}^t \bar{w}' w) + x + x'\right).$$

When $q = 1$ we have $n(x)n(x') = n(x + x')$ and $[(x), (x')] = 0$, which confirm that $n$ and $N$ are abelian.

For $l \in n^*$ there exists a subalgebra $m$ of $n$ that is isotropic with respect to the anti-symmetric bilinear form $l([*,*])$ on $n$ and has the maximal isotropic dimension. We call $m$ a polarization subalgebra for $l$ (cf. [9, pp. 27–28]). We note that $m$ is not uniquely determined by $l$ in general. Let $M := \exp(m)$ and $\chi_l$ be a unitary character of $M$ defined as

$$M \ni m \mapsto e(l(\log(m))) \in \{z \in \mathbb{C} \mid z \cdot \bar{z} = 1\}.$$ 

Let $d\hat{\eta}$ denote the invariant measure on the quotient $M \backslash N$ induced by $dn$ (for $dn$ see (1.3)). From this $\chi_l$ we can construct a unitary representation $(\eta_l, H_{\eta_l})$ of $N$ defined by the right regular representation $\eta_l$ of $N$ on
\[ H_{\eta_l} := \left\{ h: \text{measurable function on } N \mid h(mn) = \chi_l(m) h(n) \text{ } \forall (m,n) \in M \times N, \int_{M \setminus N} h(n) d\hat{n} < \infty \right\}. \]

**Proposition 4.1.** (A.A. Kirillov)

1. The representation \( \eta_l \) is irreducible and does not depend on the choice of \( m \)'s, up to unitary equivalence. Any \( \eta \in \hat{N} \) is unitarily equivalent to \( \eta_l \) for some \( l \in \mathfrak{n}^* \).
2. Two representations \( \eta_l \) and \( \eta_{l'} \) are unitarily equivalent if and only if \( l = \text{Ad}^*(n) \cdot l' \) with some \( n \in N \), where \( \text{Ad}^* \) is the co-adjoint action of \( N \) on \( \mathfrak{n}^* \). That is, we have a bijection \( \hat{N} \simeq \mathfrak{n}^*/\text{Ad}^*(N) \).

For each \( n \in N \) the operator \( U_n \) with

\[ U_n: H_{\eta_l} \ni h \mapsto h\left((n^{-1})^*\right) \in H_{\eta_{\text{Ad}^* n \cdot l}} \]  

induces a unitary equivalence of \( (\eta_l, H_{\eta_l}) \) and \( (\eta_{\text{Ad}^* n \cdot l}, H_{\eta_{\text{Ad}^* n \cdot l}}) \).

For a proof of this proposition see [9, Theorems 2.2.1–2.2.4] and [20, Theorems 5.1, 5.2]. The latter assertion of (2) is verified without difficulty.

Now we describe coadjoint orbits in \( \mathfrak{n}^* \). We first note that there is an isomorphism

\[ \mathfrak{n} \simeq \begin{cases} \mathbb{H}^{q-1} \oplus X_{\mathbb{R}} & (q > 1), \\ X_{\mathbb{R}} & (q = 1), \end{cases} \]

as real vector spaces.

Let \( q > 1 \). The dual space of \( \mathbb{H}^{q-1} \) (respectively the dual space of \( X_{\mathbb{R}} \)) is identified with \( \mathbb{H}^{q-1} \) (respectively \( X_{\mathbb{R}} \)) by the non-degenerate alternating form

\[ \mathbb{H}^{q-1} \times \mathbb{H}^{q-1} \ni (w', w) \mapsto \text{tr} \xi^t \bar{w}' w \in \mathbb{R} \]  

defined for a fixed \( \xi \in X_{\mathbb{R}} \setminus \{0\} \) (respectively by the reduced trace \( \text{tr} \)). Then every \( l \in \mathfrak{n}^* \) which is non-trivial on the center, can be written as

\[ l = l_{\xi, w'} + l_{\xi} \]

with some \( w' \in \mathbb{H}^{q-1} \) and \( \xi \in X_{\mathbb{R}} \setminus \{0\} \), where \( l_{\xi, w'} \), \( l_{\xi} \in \mathfrak{n}^* \) are defined by

\[ l_{\xi, w'}((w, x)) := \text{tr} \xi^t \bar{w}' w, \quad l_{\xi}((w, x)) := \text{tr} \xi x, \]

respectively. By a direct computation we see that

\[ \text{Ad}^*(n(w, x)) \cdot (l_{\xi, w'} + l_{\xi}) = l_{\xi, w'-2w} + l_{\xi} \]
for \( \xi \in X_R \setminus \{0\} \) and that \( \text{Ad}^* (n(w, x)) \cdot l = l \) when \( l \in n^* \) is trivial on the center. These formulas imply that if \( l \in n^* \) is non-trivial on the center, its co-adjoint orbit has a representative of the form \( l_\xi \) with some \( \xi \in X_R \setminus \{0\} \) and that, otherwise, \( \text{Ad}^* N \cdot l = l \) holds.

Let \( q = 1 \). Since \( n \) is abelian, each \( l \in n^* \) is a single \( \text{Ad}^* N \)-orbit, which is of the form \( l_\xi = l \xi \) with some \( \xi \in X_R \setminus \{0\} \).

Hence any \( \eta_l \in \hat{N} \) satisfies \( \eta_l = \chi_l \).

Then the following proposition is an immediate consequence of Proposition 4.1(2).

**Proposition 4.2.** There is a bijection

\[
\hat{N} \cong \begin{cases} 
\{ l_\xi \mid \xi \in X_R \setminus \{0\} \} \cup \{ l \in n^* \mid l \text{ trivial on the center} \} & (q > 1), \\
\{ l_\xi \mid \xi \in X_R \} & (q = 1).
\end{cases}
\]

Let \( q > 1 \) again. In order to describe \( \eta \in \hat{N} \) in detail we fix our choice of a polarization subalgebra \( m \) for \( l \in n^* \). We introduce some notations for that purpose. For each \( \xi \in X_R \setminus \{0\} \) take \( \rho_\xi \in X_R \setminus \{0\} \) such that \( \rho_\xi \xi \in X_R \). Then \( \{ e_1, \xi, \rho_\xi \xi, \rho_\xi \} \) forms a basis of \( H \). Setting \( W^+_\xi := \mathbb{R} \xi + \mathbb{R} \rho_\xi \) and \( W^-_\xi := \mathbb{R} + \mathbb{R} \rho_\xi \), we let

\[
V^\pm_\xi := \{(w_1, w_2, \ldots, w_q-1) \mid w_k \in W^\pm_\xi \}. \tag{4.5}
\]

Then \( \mathbb{H}^{q-1} = V^+_\xi \oplus V^-_\xi \).

**Lemma 4.3.**

1. When \( l = l_\xi \) with \( \xi \in X_R \setminus \{0\} \)

\[
m := \{(w, x) \in n \mid w \in V^+_\xi, x \in X_R \}
\]

forms a polarization subalgebra for it. When \( l \) is trivial on the center its polarization subalgebra is \( n \).

2. Under the above choice of \( m \) we can take \( L^2(V^-_\xi) \) as a model for \( H_{\eta_l} \).

**Proof.** If the first assertion is proved, the second one follows from the right \( M = \exp(m) \)-equivariance of each element in \( H_{\eta_l} \) with respect to \( \chi_l \). It thus suffices to prove (1). First we note that the center \( g_{2\alpha} \) of \( n \) is contained in any polarization subalgebra and \( [n, n] \subset g_{2\alpha} \). If \( l \) is trivial on the center, \( l([n, n]) = 0 \) holds. Hence \( n \) is the polarization subalgebra of such \( l \). Let \( l = l_\xi \) with \( \xi \neq 0 \). A straightforward calculation and (4.1) lead to

\[
l(l[(w_1, x_1), (w_2, x_2)]) = \text{tr} \xi (l[w_1 w_2 - w_2 w_1]) = 2 \text{tr} \xi (l[w_1 w_2])
\]

for \( w_1, w_2 \in \mathbb{H}^{q-1} \) and \( x_1, x_2 \in X_R \). This is nothing but the non-degenerate alternating form on \( \mathbb{H}^{q-1} \) defined in (4.4). The subspace \( V^+_\xi \) form a maximal isotropic subspace of \( \mathbb{H}^{q-1} \) for such form. Therefore the subalgebra in the assertion forms a polarization subalgebra for \( l = l_\xi \). \( \square \)
When $q > 1$ we henceforth keep the following assumption.

**Assumption 4.4.** For $l_\xi$ with $\xi \in X_\mathbb{R} \setminus \{0\}$ we fix our choice of the polarization subalgebra $m$ as in Lemma 4.3 and realize the representation $\eta_{l_\xi}$ on $L^2(V_{l_\xi}^-)$.

Let

$$\{l_\xi^{(a)}, l_i\} \quad 1 \leq a \leq 4, \quad 1 \leq k \leq q-1, \quad 2 \leq i \leq 4$$

be the basis of $n^*$ dual to the basis

$$\{E^{(a,k)}_\alpha, E^{(i)}_{2\alpha}\} \quad 1 \leq a \leq 4, \quad 1 \leq k \leq q-1, \quad 2 \leq i \leq 4$$

of $n$. We write $l \in n^*$ as

$$l = \sum_{1 \leq a \leq 4} \xi_k^{(a)} l_k^{(a)} + \sum_{2 \leq i \leq 4} \xi_i l_i \quad \text{with } \xi_k^{(a)}, \xi_i \in \mathbb{R}.$$

Using these notations, we give a formula for the differential $d\eta$ of $\eta = \eta_{l_\xi}$ for $\xi > 0$, and a formula for the differential of a unitary character of $N$. The formula for a general $d\eta_{l_\xi}$ with a non-zero $\xi \in X_\mathbb{R}$ is essentially reduced to that of $d\eta_{l_\xi}$ above. Indeed, due to the well-known Skolem–Noether theorem, there exists $u_\xi \in H^{(1)}$ such that

$$u_\xi \{e_1, e_2, e_3, e_4\} u_\xi^{-1} = \{e_1, \xi/d(\xi), \rho_\xi \xi/d(\rho_\xi \xi), \rho_\xi /d(\rho_\xi)\} \quad (4.6)$$

(unique up to $\{\pm 1\}$). Then we see that $\eta_{l_\xi}$ is the diag($u_\xi, u_\xi, \ldots, u_\xi$)-conjugate of $\eta_{l_\xi_2 e_2}$ with some $\xi_2 > 0$ (but note that these two are not always isomorphic to each other).

For $\xi = \xi_2 e_2$ with $\xi_2 > 0$ we choose $e_4$ as $\rho_\xi$. For this choice of $\xi$ and $\rho_\xi$, define $V_{\xi_2}^\pm$ as in (4.5). They lead to the polarization subalgebra $m$ for $l_{\xi_2 e_2}$, as in Lemma 4.3(1). Moreover, we write $w \in H^{q-1}$ as

$$w = (w_k^{(1)} e_1 + w_k^{(2)} e_2 + w_k^{(3)} e_3 + w_k^{(4)} e_4)_{1 \leq k \leq q-1}$$

with $(w_k^{(1)}, w_k^{(2)}, w_k^{(3)}, w_k^{(4)}) \in \mathbb{R}^4$. By a direct computation we obtain the following formula.

**Lemma 4.5.** Let $q > 1$ for the assertions (1) and (2).

1. For $\xi_2 > 0$ let $\eta := \eta_{l_{\xi_2 e_2}}$. We have

$$d\eta(E^{(1,k)}_\alpha) = \frac{\partial}{\partial w_k^{(1)}} + 4\pi \sqrt{-1} \xi_2 w_k^{(2)}, \quad d\eta(E^{(2,k)}_\alpha) = -8\pi \sqrt{-1} \xi_2 w_k^{(1)}, \quad (4.7)$$

$$d\eta(E^{(3,k)}_\alpha) = \frac{\partial}{\partial w_k^{(3)}} - 4\pi \sqrt{-1} \xi_2 w_k^{(4)}, \quad d\eta(E^{(4,k)}_\alpha) = 8\pi \sqrt{-1} \xi_2 w_k^{(3)} \quad (4.8)$$
for $1 \leq k \leq q - 1$, and
\[
d\eta(E_{2\alpha}^{(a)}) = \begin{cases} 
-4\pi \sqrt{-1} \xi_2 & (a = 2), \\
0 & (a = 3, 4).
\end{cases} \tag{4.9}
\]
(2) For
\[
l = \sum_{\substack{1 \leq k \leq q - 1 \\ 1 \leq a \leq 4 \theuler \xi_k^{(a)} l_k^{(a)}}}
\]
let $\eta := \chi_l$. We have
\[
d\eta(E_{\alpha}^{(a,k)}) = 2\pi \sqrt{-1} \xi_k^{(a)} \quad \text{for } 1 \leq a \leq 4 \text{ and } 1 \leq k \leq q - 1, \tag{4.10}
\]
\[
d\eta(E_{2\alpha}^{(a)}) = 0 \quad \text{for } 2 \leq a \leq 4. \tag{4.11}
\]
(3) For the case $q = 1$, (4.9) (respectively (4.11)) also gives the formula for the differential of $\chi_{l\xi_2^e}$ (respectively the trivial character).

5. Generalized Whittaker functions for quaternionic discrete series

In this section we study generalized Whittaker functions and generalized Whittaker models for the quaternionic discrete series representation $\pi_{\kappa}$. In Section 5.1 we state their definitions. In Section 5.2 we give an explicit formula for such functions and the multiplicity formula for such models.

5.1. Definition

To state the definition of the generalized Whittaker functions and the models we need some function spaces. For $(\eta, H_\eta) \in \hat{N}$, $H_\eta^\infty$ denotes the space of $C^\infty$-vectors in $H_\eta$. We set
\[
C_\eta^\infty(N \setminus G) := \{ F: H_\eta^\infty \text{-valued } C^\infty \text{-function on } G \mid F(ng) = \eta(n) F(g) \forall (n, g) \in N \times G \},
\]
\[
C_\eta^\infty(N \setminus G)_K := \text{the space of } K \text{-finite vectors in } C_\eta^\infty(N \setminus G).
\]
Furthermore, we put
\[
C_{\tau_\kappa}^\infty(G / K) := \{ F: V_\kappa \text{-valued } C^\infty \text{-function on } G \mid F(gk) = \tau_\kappa(k)^{-1} F(g) \forall (g, k) \in G \times K \},
\]
\[
C_{\eta,\tau_\kappa}^\infty(N \setminus G / K) := \{ W: H_\eta^\infty \otimes V_\kappa \text{-valued } C^\infty \text{-function on } G \mid W(ngk) = \eta(n) \otimes \tau_\kappa^{-1}(k) W(g), \forall (n, g, k) \in N \times G \times K \}.
\]
The space $C_\eta^\infty(N \setminus G)_K$ forms a $(g, K)$-module (for the definition of a $(g, K)$-module see [33, 3.3.1]).

Let $\iota: \tau_\kappa \rightarrow \pi_\kappa$ be the $K$-inclusion map of $\tau_\kappa$ into $\pi_\kappa$. We let $\iota^*$ be the map defined by
\[
\text{Hom}_{(g, K)}(H_\kappa, C_\eta^\infty(N \setminus G)_K) \ni \Phi \mapsto \Phi \circ \iota \in \text{Hom}_K(V_\kappa, C_\eta^\infty(N \setminus G)_K).
\]
Definition 5.1. An element in $\text{Im} \, \iota^*$ is called a generalized Whittaker function for $\pi_\kappa$ with $K$-type $\tau_\kappa$ attached to $\eta$. For $\Phi \in \text{Hom}_{(g,K)}(H_\kappa, C_\eta^\infty(N\backslash G)_K)$ we call $\Phi(H_\kappa)$ a generalized Whittaker model of $\pi_\kappa$.

Now we set
\[ S_{\eta, \tau_\kappa} (N \backslash G/K) := \{ W \in C_\infty^{\tau_\kappa}(G/K) \mid (W(ng), v)_{\kappa} \in H^\infty_{\eta}(g, v) \in G \times V_\kappa \} , \]
where we view $(W(ng), v)_{\kappa}$ as a function in $n \in N$. Then one has the following identifications:
\[ \text{Hom}_K(V_\kappa, C_\eta^\infty(N\backslash G)_K) \simeq V_\kappa^* \otimes_{\tau_\kappa} C_\eta^\infty(N\backslash G)_K \simeq V_\kappa \otimes_{\tau_\kappa} C_\eta^\infty(N\backslash G)_K \]
\[ = C_{\eta, \tau_\kappa}^\infty(N\backslash G/K) \simeq S_{\eta, \tau_\kappa} (N \backslash G/K) , \]
where
\[ V_\kappa \otimes_{\tau_\kappa} C_\eta^\infty(N\backslash G)_K := \{ v \otimes f \in V_\kappa \otimes C_\eta^\infty(N\backslash G)_K \mid v \otimes f(gk) = \tau^{-1}_\kappa(k)v \otimes f(g) , \forall (g, k) \in G \times K \} . \]

For the second identification recall that $(\tau_\kappa, V_\kappa) \in \hat{K}$ is self-dual (cf. Lemma 2.1(2)). Moreover, we note that the fourth identification above is induced by
\[ C_{\eta, \tau_\kappa}^\infty(N\backslash G/K) \ni W(g)(*) \mapsto (g \mapsto W(g)(1)) \in S_{\eta, \tau_\kappa}(N \backslash G/K) , \]
where we regard $W(g)(*)$ as a function on $N$ and $W(g)(1)$ denotes the evaluation of $W(g)(*)$ at $1 \in N$. In fact, such identification follows from the relation $W(g)(n_0n) = \eta(n_0)W(g)(n) = W(n_0g)(n)$ for any $(n_0, n) \in N \times N$.

In order to describe $\text{Im} \, \iota^*$ we introduce the Schmid operator (cf. [28, Section 7]). Let $\{X_i\}_{i \in I}$ be an orthonormal basis of $p$ with respect to the Killing form. We define the differential operator $\nabla_\kappa$ on $C_\infty^{\tau_\kappa}(G/K)$ as
\[ \nabla_\kappa \cdot W := \sum_{i \in I} dR(X_i)W \otimes X_i , \]
where $dR$ denotes the differential of the right translation $R$ of $G$. In addition to this, we remind the readers that we have defined the projection $P_\kappa : V_\kappa \otimes p_\mathbb{C} \to V_{(1, 0, \ldots, 0_{\kappa-1})}$ in (2.4).

Definition 5.2. We define the Schmid operator $D_\kappa$ on $C_\infty^{\tau_\kappa}(G/K)$ by
\[ D_\kappa := P_\kappa \circ \nabla_\kappa . \]

We note that $D_\kappa$ does not depend on the choice of $\{X_i\}_{i \in I}$.

The following proposition is due to Yamashita’s general result on embeddings of discrete series representations into induced representations (cf. [34, Proposition 2.1]).

Proposition 5.3. The map $\iota^*$ induces an injection
\[ \text{Hom}_{(g,K)}(H_\kappa, C_\eta^\infty(N\backslash G)_K) \hookrightarrow \{ W \in S_{\eta, \tau_\kappa}(N\backslash G/K) \mid D_\kappa \cdot W = 0 \} . \]
This proposition is formulated for discrete series representations of general semisimple Lie groups in [34, Proposition 2.1]. In [34, Theorem 2.4] Yamashita further asserts that this map becomes a bijection for a discrete series representation with its minimal \( K \)-type satisfying the condition “far from the walls,” which means that

\[
\kappa\beta_{q+1} - \sum_{\beta \in Q} \beta \quad \text{is} \quad \Delta_{c,0}^+ \quad \text{dominant for any subset} \quad Q \quad \text{of} \quad \Delta_{n,0}^+
\]

for our discrete series \( \pi_\kappa \) (see [34, Definition 1.7]). But we note that no quaternionic discrete series \( \pi_\kappa \) satisfies this condition. However, we can prove that the above map for \( \pi_\kappa \) is actually a bijection in the next subsection (cf. Theorem 5.11).

5.2. An explicit formula for the generalized Whittaker functions and the multiplicities of the generalized Whittaker models

Let \( W_{\kappa,l}(g) := \sum_{i=0}^\kappa w_i(g)v_{\kappa,i} \in \mathcal{S}_{\eta_l}(N \backslash G/K) \) be a generalized Whittaker function for \( \pi_\kappa \) attached to \( \eta_l \), where we fixed a basis \( \{v_{\kappa,i} \mid 0 \leq i \leq \kappa \} \) of \( V_\kappa \) satisfying (2.1)–(2.3). Since \( \eta_l(z) = \chi_l(z) \) for \( z \in \mathcal{Z}(N) \), \( W_{\kappa,l}(mg) = \chi_l(z)W_{\kappa,l}(g) \) holds for \( (z,g) \in \mathcal{Z}(N) \times G \) (for \( \mathcal{Z}(N) \) see Section 4). This and the \( K \)-equivariance of \( W_{\kappa,l} \) imply that it suffices to consider the restriction of \( W_{\kappa,l} \) to \( N_L A \) (respectively \( A \)) when \( q > 1 \) (respectively when \( q = 1 \)), in order to obtain an explicit formula for \( W_{\kappa,l} \) (for \( A \) and \( N_L \) see Sections 1 and 4, respectively). Moreover, when \( q > 1 \) and \( \eta_l = \eta_{l\xi} \) with a non-zero \( \xi \in X_{\mathfrak{g}_L} \), it suffices to deal with the restriction of \( W_{\kappa,l} \) to \( \{n(w,0) \in N \mid w \in V_\xi^- \} \times A \) since \( W_{\kappa,l}(mg) = \chi_l(m)W_{\kappa,l}(g) \) for \( m \in M = \{n(w,x) \in N \mid w \in V_\xi^+, x \in \mathcal{Z}(N) \} \), where recall that we fixed our choice of the polarization subalgebra \( m \) for \( l\xi \) in Assumption 4.4 and put \( M := \exp(m) \) (cf. Section 4).

We first write the differential equations for \( W_{\kappa,l} \) arising from the condition \( D_\kappa \cdot W_{\kappa,l} = 0 \) in Proposition 5.3. For an explicit description of the equation we choose

\[
\left\{ \begin{array}{l}
\frac{1}{4\sqrt{q+2}}X_{\beta_k-\beta_{q+1}}, \quad \frac{1}{4\sqrt{q+2}}X_{\beta_k+\beta_{q+1}}, \quad \frac{1}{2\sqrt{2(q+2)}}X_{\beta_q+\beta_{q+1}}, \quad \frac{1}{2\sqrt{2(q+2)}}X_{\beta_q-\beta_{q+1}}, \\
\frac{1}{4\sqrt{q+2}}Y_{\beta_k-\beta_{q+1}}, \quad \frac{1}{4\sqrt{q+2}}Y_{\beta_k+\beta_{q+1}}, \quad \frac{1}{2\sqrt{2(q+2)}}Y_{\beta_q+\beta_{q+1}}, \quad \frac{1}{2\sqrt{2(q+2)}}Y_{\beta_q-\beta_{q+1}}
\end{array} \right. 
\]

as an orthonormal basis of \( \mathfrak{p} \), where \( 1 \leq k \leq q - 1 \) and

\[
X_\beta := \frac{1}{2}(F_\beta + F_{-\beta}), \quad Y_\beta := \frac{1}{2\sqrt{-1}}(F_{-\beta} - F_\beta) \quad \text{for} \quad \beta \in \Delta_n^+.
\]

Using this basis, we can write \( \nabla_\kappa \cdot W_{\kappa,l} \) as follows:

\[
\nabla_\kappa \cdot W_{\kappa,l}(n(w,0)a) = \sum_{1 \leq k \leq q} (dR_{X_{\beta_k-\beta_{q+1}}} W_{\kappa,l}(n(w,0)a) \otimes X_{\beta_k-\beta_{q+1}} + dR_{X_{\beta_k+\beta_{q+1}}} W_{\kappa,l}(n(w,0)a) \otimes X_{\beta_k+\beta_{q+1}}) \\
+ \sum_{1 \leq k \leq q} (dR_{Y_{\beta_k-\beta_{q+1}}} W_{\kappa,l}(n(w,0)a) \otimes Y_{\beta_k-\beta_{q+1}} + dR_{Y_{\beta_k+\beta_{q+1}}} W_{\kappa,l}(n(w,0)a) \otimes Y_{\beta_k+\beta_{q+1}})
\]
Recall that we introduced the projection $P_\kappa$ in (2.4). From now on we normalize $P_\kappa$ so that it satisfies (2.7)–(2.9). Apply $P_\kappa$ to both sides of the equation above. By virtue of the Iwasawa decompositions for the generators of $p_\kappa$ in Lemma 1.1 and of the formula for $P_\kappa$ in Lemma 2.3, we obtain an explicit formula for $D_\kappa \cdot W_{\kappa, l} = 0$.

**Proposition 5.4.** Let $\partial := y \frac{\partial}{\partial y}$ be the Euler operator. Then the condition $D_\kappa \cdot W_{\kappa, l}(n(w, 0)a) = 0$ is equivalent to

\[
\begin{align*}
&\frac{1}{32(q+2)} \sum_{1 \leq k \leq q-1} \left\{ \sum_{0 \leq i \leq \kappa} (dR_{p_{\kappa}+\beta_{q+1}} w_i(n(w, 0)a)v_{\kappa, i} \otimes F_{-(p_{\kappa}+\beta_{q+1})}) \\
&+ dR_{-(p_{\kappa}+\beta_{q+1})} w_i(n(w, 0)a)v_{\kappa, i} \otimes F_{-p_{\kappa}+\beta_{q+1}} \right\} \\
&+ \frac{1}{32(q+2)} \sum_{1 \leq k \leq q-1} \left\{ \sum_{0 \leq i \leq \kappa} (dR_{p_{\kappa}+\beta_{q+1}} w_i(n(w, 0)a)v_{\kappa, i} \otimes F_{-(p_{\kappa}-\beta_{q+1})}) \\
&+ dR_{-(p_{\kappa}-\beta_{q+1})} w_i(n(w, 0)a)v_{\kappa, i} \otimes F_{-p_{\kappa}+\beta_{q+1}} \right\} \\
&+ \frac{1}{16(q+2)} \sum_{0 \leq i \leq \kappa} (dR_{p_{\kappa}+\beta_{q+1}} w_i(n(w, 0)a)v_{\kappa, i} \otimes F_{-p_{\kappa}+\beta_{q+1}}) \\
&+ dR_{-(p_{\kappa}+\beta_{q+1})} w_i(n(w, 0)a)v_{\kappa, i} \otimes F_{-p_{\kappa}-\beta_{q+1}} \\
&+ \frac{1}{16(q+2)} \sum_{1 \leq k \leq q-1} \sum_{i=1}^{\kappa-1} i \sqrt{y} (d\eta_l(E_{\alpha}^{3,k})) - \sqrt{1} d\eta_l(E_{\alpha}^{4,k})) w_i(n(w, 0)a)v_{\kappa-1,i-1} \otimes w_{-\beta_k} \\
&+ \frac{1}{16(q+2)} \sum_{1 \leq k \leq q-1} \sum_{i=0}^{\kappa-1} (\kappa - i) \sqrt{y} (d\eta_l(E_{\alpha}^{3,k})) + \sqrt{1} d\eta_l(E_{\alpha}^{4,k})) w_i(n(w, 0)a)v_{\kappa-1,i} \otimes w_{\beta_k} \\
&+ \frac{1}{16(q+2)} \sum_{1 \leq k \leq q-1} \sum_{i=0}^{\kappa-1} (\kappa - i) \sqrt{y} (d\eta_l(E_{\alpha}^{1,k})) - \sqrt{1} d\eta_l(E_{\alpha}^{2,k})) w_i(n(w, 0)a)v_{\kappa-1,i} \otimes w_{-\beta_k} \\
&+ \frac{1}{16(q+2)} \sum_{1 \leq k \leq q-1} \sum_{i=1}^{\kappa} i \sqrt{y} (d\eta_l(E_{\alpha}^{1,k})) + \sqrt{1} d\eta_l(E_{\alpha}^{2,k})) w_i(n(w, 0)a)v_{\kappa-1,i-1} \otimes w_{\beta_k} \\
&- \frac{1}{16(q+2)} \sum_{i=1}^{\kappa} 2i \sqrt{y} (d\eta_l(E_{\alpha}^{2,k})) - \sqrt{1} d\eta_l(E_{\alpha}^{4,k})) w_i(n(w, 0)a)v_{\kappa-1,i-1} \otimes w_{-\beta_q}
\end{align*}
\]
\[
- \frac{1}{16(q + 2)} \sum_{i=1}^{\kappa-1} 2i(\kappa - i)w_i(n(w, 0)a)v_{\kappa-1,i} \otimes w_{-\beta_q}
\]

\[
- \frac{1}{16(q + 2)} \sum_{i=1}^{\kappa-1} 2(\kappa - i)w_i(n(w, 0)a)v_{\kappa-1,i} \otimes w_{-\beta_q}
\]

\[
- \frac{1}{16(q + 2)} \sum_{i=0}^{\kappa-1} 2(\kappa - i)y(d\eta_l(E_{2a}^{(3)}) + \sqrt{-1} \eta_l(E_{2a}^{(4)})w_i(n(w, 0)a)v_{\kappa-1,i} \otimes w_{\beta_q}
\]

\[
+ \frac{1}{16(q + 1)} \sum_{i=1}^{\kappa-1} 2(\kappa - i)w_i(n(w, 0)a)v_{\kappa-1,i-1} \otimes w_{\beta_q}
\]

\[
+ \frac{1}{16(q + 1)} \sum_{i=1}^{\kappa-1} 2i w_i(n(w, 0)a)v_{\kappa-1,i-1} \otimes w_{\beta_q}
\]

\[
+ \frac{1}{16(q + 2)} \sum_{i=0}^{\kappa-1}(\kappa - i)(2\partial - 2\sqrt{-1}yd\eta_l(E_{2a}^{(2)})w_i(n(w, 0)a)v_{\kappa-1,i} \otimes w_{-\beta_q}
\]

\[
+ \frac{1}{16(q + 2)} \sum_{i=0}^{\kappa-1}(\kappa - i)(2i - (\kappa - 1))w_i(n(w, 0)a)v_{\kappa-1,i} \otimes w_{-\beta_q}
\]

\[
- \frac{1}{16(q + 2)} \sum_{i=0}^{\kappa-1}(\kappa - i)w_i(n(w, 0)a)v_{\kappa-1,i} \otimes w_{-\beta_q}
\]

\[
- \frac{1}{16(q + 2)} \sum_{i=1}^{\kappa} i(2\partial + 2\sqrt{-1}yd\eta_l(E_{2a}^{(2)})w_i(n(w, 0)a)v_{\kappa-1,i-1} \otimes w_{\beta_q}
\]

\[
+ \frac{1}{16(q + 2)} \sum_{i=1}^{\kappa} i(2i - 1 - \kappa)w_i(n(w, 0)a)v_{\kappa-1,i-1} \otimes w_{\beta_q}
\]

\[
+ \frac{1}{16(q + 2)} \sum_{i=1}^{\kappa} i w_i(n(w, 0)a)v_{\kappa-1,i-1} \otimes w_{\beta_q} = 0.
\]

As an immediate consequence of this proposition and Lemma 4.5, we have the following formula.

**Proposition 5.5.** For the assertions (1) and (2) below we assume that \( q > 1 \).

(1) Let \( \eta_l \) be a character of \( N \), which is trivial on the center. Then \( D_\kappa \cdot W_{\kappa,i}(n(w, 0)a) = 0 \) is equivalent to the following system of differential equations:

\[
-(i + 1)(2\pi \sqrt{-1}\xi_k^{(3)} + 2\pi \xi_k^{(4)})w_{i+1}(n(w, 0)a)
+(\kappa - i)(2\pi \sqrt{-1}\xi_k^{(1)} + 2\pi \xi_k^{(2)})w_i(n(w, 0)a) = 0,
\]

(5.1)
For \( \eta_l = \eta_{l_2} \) with \( \xi_2 > 0 \), \( D_\kappa \cdot W_{k,l}(n(w^-_\xi, 0)a) = 0 \) for \( w^-_\xi \in V^-_\xi \) is equivalent to the following system of differential equations:

\[
-(i + 1) \left( \frac{\partial}{\partial w^{(3)}_k} + 8\pi \xi_2 w^{(3)}_k \right) w_{i+1}(n(w^-_\xi, 0)a) \\
+ (\kappa - i) \left( \frac{\partial}{\partial w^{(1)}_k} - 8\pi \xi_2 w^{(1)}_k \right) w_i(n(w^-_\xi, 0)a) = 0,
\]

\[
(i + 1) \left( \frac{\partial}{\partial w^{(1)}_k} + 8\pi \xi_2 w^{(1)}_k \right) w_{i+1}(n(w^-_\xi, 0)a) \\
+ (\kappa - i) \left( \frac{\partial}{\partial w^{(3)}_k} - 8\pi \xi_2 w^{(3)}_k \right) w_i(n(w^-_\xi, 0)a) = 0,
\]

for \( 1 \leq k \leq q - 1 \) and \( 0 \leq i \leq \kappa - 1 \), and

\[
2\partial w_i(n(w^-_\xi, 0)a) - 8\pi \xi_2 w_i(n(w^-_\xi, 0)a) - (\kappa + 2) w_i(n(w^-_\xi, 0)a) = 0,
\]

\[
2\partial w_{i+1}(n(w^-_\xi, 0)a) + 8\pi \xi_2 w_{i+1}(n(w^-_\xi, 0)a) - (\kappa + 2) w_{i+1}(n(w^-_\xi, 0)a) = 0,
\]

for \( 0 \leq i \leq \kappa - 1 \).

(3) Let \( q = 1 \), and replace \( w_i(n(w, 0)a) \) and \( w_i(n(w^-_\xi, 0)a) \) by \( w_i(a) \) in (1) and (2) for \( 0 \leq i \leq \kappa \). Then, if \( \eta_l \) is the trivial character (respectively \( \chi_{l_2} \) with \( \xi_2 > 0 \)), \( D_\kappa \cdot W_{k,l}(a) = 0 \) is equivalent to Eqs. (5.3) and (5.4) in (1) (respectively (5.7) and (5.8) in (2)).

We solve the differential equations above. First we consider the case where \( \eta_l \) is a character of \( N \) trivial on the center.

**Theorem 5.6.** Let \( \eta_l = \chi_l \) be as above. The solutions of \( D_\kappa \cdot W_{k,l}(n(w, 0)a) = 0 \) or \( D_\kappa \cdot W_{k,l}(a) = 0 \) are

\[
\begin{align*}
\{ & y^{q/2+1}(C_0 v_{k,0} + C_1 v_{k,1} + \cdots + C_\kappa v_{k,\kappa}) \\
& 0
\end{align*}
\]

\( q: \) arbitrary, \( \chi_l: \) the trivial character, \( q > 1, \chi_l: \) a non-trivial character,

where \( C_i \) is a constant depending only on \( i \) for \( 0 \leq i \leq \kappa \).
We omit the proof of this theorem. Actually it is obtained by the differential equations in Proposition 5.5(1), which are not difficult to solve.

Next we deal with the remaining case, i.e., the case where \( q > 1 \) and \( \eta_l \neq \chi_l \) or where \( q = 1 \) and \( \eta_l \) is a non-trivial character. For the former case we let \( l = l_\xi \) with \( \xi \in X_{\mathbb{R}} \setminus \{0\} \).

**Theorem 5.7.** Let \( \xi \in X_{\mathbb{R}} \setminus \{0\} \) and let us fix \( u_\xi \in \mathbb{H}^{(1)} \) (respectively \( \rho_\xi \in X_{\mathbb{R}} \setminus \{0\} \)) satisfying (4.6) (respectively the condition just before Lemma 4.3).

(1) When \( q = 1 \) and \( \eta_l = \chi_l \xi \), the solutions of the equation \( D_\kappa \cdot W_{\kappa, l}(a) = 0 \) are
\[
C_0 y^{k/2+1} \exp(4\pi d(\xi)y) \cdot \sigma_\kappa(u_\xi)v_{\kappa,0} + C_\kappa y^{k/2+1} \exp(-4\pi d(\xi)y) \cdot \sigma_\kappa(u_\xi)v_{\kappa,k},
\]
where \( C_0 \) and \( C_\kappa \) are two arbitrary constants.

(2) When \( q > 1 \) and \( \eta_l = \eta_l \xi \), the unique solution of \( D_\kappa \cdot W_{\kappa, l}(n(w, 0)a) = 0 \) is given as
\[
y^{k/2+1} \exp(-4\pi d(\xi)y) \exp(-4\pi d(\xi)(t w_\xi - \xi w_\xi - \xi)) \cdot \sigma_\kappa(u_\xi)v_{\kappa,k},
\]
up to constant multiples, where \( w = w^+_\xi + w^-_\xi \) with \( w^\pm_\xi \in V^\pm_\xi \).

Furthermore, these expressions of the solutions in (1) and (2) do not depend on the choice of \( u_\xi \)'s and \( \rho_\xi \)'s, up to constant multiples. In other words, we can replace \( u_\xi \) by any \( u \in \mathbb{H}^{(1)} \) such that \( u e_2 u^{-1} = \xi / d(\xi) \) in the formulas above.

**Proof.** By (4.2) we see that
\[
n(w, 0) = n \left( w^+_\xi, \frac{1}{2}(t w^-_\xi w^+_\xi - t w^+_\xi w^-_\xi) \right)n(w^-_\xi, 0).
\]
Thereby we have
\[
W_{\kappa, l}(n(w, 0)a) = \chi_l \left( n \left( w^+_\xi, \frac{1}{2}(t w^-_\xi w^+_\xi - t w^+_\xi w^-_\xi) \right) \right) W_{\kappa, l}(n(w^-_\xi, 0)a)
\]
\[
= \exp(\text{tr} \xi (t w^-_\xi w^+_\xi)) W_{\kappa, l}(n(w^-_\xi, 0)a)
\]
when \( q > 1 \) and \( \eta_l = \eta_l \xi \).

Hence we concentrate on solving the differential equations for this \( W_{\kappa, l}(n(w^-_\xi, 0)a) \), assuming \( q = 1 \). This also provides a proof for the case of \( q = 1 \). We first consider the case where \( \xi = \xi_2 e_2 \) with \( \xi_2 > 0 \) and \( \rho_\xi = e_4 \). Around the end of this proof we explain how the case of a general \( \xi \in X_{\mathbb{R}} \setminus \{0\} \) is reduced to this case.

We begin with solving (5.7) and (5.8). In (5.7) (respectively (5.8)) replace \( w_i(n(w^-_\xi, 0)a) \) (respectively \( w_{i+1}(n(w^-_\xi, 0)a) \)) by \( \exp(4\pi \xi_2 y)w'_i(n(w^-_\xi, 0)a) \) (respectively \( \exp(-4\pi \xi_2 y) \times w'_{i+1}(n(w^-_\xi, 0)a) \)). Then \( w'_i(n(w^-_\xi, 0)a) \) and \( w''_{i+1}(n(w^-_\xi, 0)a) \) satisfy (5.3) and (5.4), respectively. We thus obtain
\( W_{k,l}(n(w^{-}_ξ,0)a) \)
\[
= C_0(w^{-}_ξ)^{y^{\nu}/2+1} \exp(4\pi \xi_2 y) v_{\kappa,0} + C_κ(w^{-}_ξ)^{y^{\nu}/2+1} \exp(-4\pi \xi_2 y) v_{\kappa,κ},
\]

where \( C_0(w^{-}_ξ) \) and \( C_κ(w^{-}_ξ) \) are not dependent on \( y \).

Now assume \( q = 1 \) and replace \( W_{k,l}(n(w^{-}_ξ,0)a) \) by \( W_{k,l}(a) \) in this equation. Furthermore, exchange \( C_0(w^{-}_ξ) \) and \( C_κ(w^{-}_ξ) \) with two arbitrary constants \( C_0 \) and \( C_κ \), respectively. Then we see that the formula (5.9) proves the case of \( q = 1 \) for \( \xi = \xi_2 e_2 \) with \( \xi_2 > 0 \).

Next we solve the differential equations (5.5) and (5.6). By virtue of (5.9) Eqs. (5.5) and (5.6) are reduced to
\[
\left( \frac{∂}{∂w_k^{(1)}} + 8\pi \xi_2 w_k^{(1)} \right) C_κ(w^{-}_ξ) = 0,
\]
\[
\left( \frac{∂}{∂w_k^{(3)}} - 8\pi \xi_2 w_k^{(3)} \right) C_0(w^{-}_ξ) = 0,
\]
for \( 1 \leq k \leq q - 1 \). Solving these equations, we have
\[
C_κ(w^{-}_ξ) = C_κ \exp(-4\pi \xi_2 (\xi^{-}_w w^{-}_ξ)), \quad C_0(w^{-}_ξ) = C_0 \exp(4\pi \xi_2 (\xi^{-}_w w^{-}_ξ)),
\]
where \( C_0 \) and \( C_κ \) are constants not dependent on \( w^{-}_ξ \). Since the second solution is not square-integrable with respect to \( w^{-}_ξ \), we see \( w_0(n(w^{-}_ξ,0)a) = 0 \) in view of Lemma 4.3(2) or Assumption 4.4. As a result we obtain
\[
W_{k,l}(n(w^{-}_ξ,0)a) = C_κ y^{\nu}/2+1 \exp(-4\pi \xi_2 y) \exp(-4\pi \xi_2 (\xi^{-}_w w^{-}_ξ)) \cdot v_{\kappa,κ},
\]
which leads to the solution in the second assertion for the case \( \xi = \xi_2 e_2 \) with \( \xi_2 > 0 \).

For a general \( \xi \) we choose \( \{e_1, \xi/d(\xi), \rho_ξ \xi/d(\rho_ξ \xi), \rho_ξ/d(\rho ξ ξ)\} \) as a basis of \( \mathbb{H} \) instead of \( \{e_1, e_2, e_3, e_4\} \). These two bases are related to each other via the inner automorphism of \( \mathbb{H} \) by \( u_ξ \). Due to this change of bases, each root vector for the restricted root system and the absolute root system is replaced by its diag\( (u_ξ, u_ξ, \ldots, u_ξ) \)-conjugate. Then we realize that the problem for a general \( \xi \) is reduced to the case where \( \xi = \xi_2 e_2 \) and \( ρ_ξ = e_4 \) with \( \xi_2 > 0 \), whence we obtain the solutions in the assertion.

Our remaining task is to verify the independence of the expressions of the solutions with respect to the choice of \( u_ξ \)-s and \( ρ_ξ \)-s. We first discuss such independence for the vector part \( σ_κ(u_ξ)v_{κ,κ} \). Let us take \( u'_ξ \in \mathbb{H}^{(1)}_1 \) and \( ρ'_ξ \in X_\mathbb{R} \setminus \{0\} \) with the same property as \( u_ξ \) and \( ρ_ξ \), respectively. Then \( u'_ξ = u_ξ u \) holds with some \( u \in \mathbb{H}^{(1)}_1 \) such that \( u e_2^{-1} u^{-1} = e_2 \). Now we note that \( \{b \in \mathbb{H} \mid b e_2 = e_2 b\} = \mathbb{R}(e_2) := \mathbb{R} + \mathbb{R} e_2 \) and that every element of \( \{b \in \mathbb{R}(e_2) \mid ν(b) = 1\} \) is in the image of \( \{re_2 \mid r \in \mathbb{R}\} \) by the exponential map. Then we see by the formula (2.1) that \( σ_κ(u)v_{κ,κ} \) is a constant multiple of \( v_{κ,κ} \), which implies our desired independence on \( σ_κ(u_ξ)v_{κ,κ} \).

As for the scalar parts of the solutions it is helpful to note that we have \( u'_ξ = u'u_ξ \) with some \( u' \in \mathbb{H}^{(1)}_1 \) such that \( u'_ξ u^{-1}_ξ = ξ \). This enables us to check that such parts remain the same under any choice of \( ξ \)-s and \( ρ_ξ \)-s. This completes the proof. \( \Box \)
Proposition 5.3, Theorems 5.6 and 5.7 give an inequality

$$\dim \text{Hom}_{(g,K)}(H_\kappa, C_\eta^\infty(N \setminus G)_K) \leq \begin{cases} 
\kappa + 1 & (\eta; \text{trivial character}), \\
2 & (q = 1 \text{ and } \eta; \text{non-trivial character}), \\
1 & (q > 1 \text{ and } \eta; \text{not character}), \\
0 & (q > 1 \text{ and } \eta; \text{non-trivial character}).
\end{cases}$$

Here note that Theorems 5.6 and 5.7 essentially exhaust the generalized Whittaker functions for all of $\eta \in \hat{N}$ in view of Propositions 4.1(2) and 4.2.

Let $\eta_l$ be as in Theorems 5.6 and 5.7 and further assume that $\eta_l$ is an element in $\hat{N}$ such that $D_\kappa \cdot W_{\kappa,l} = 0$ has a non-zero solution. We note that Theorems 5.6 and 5.7 provide a basis of the solution space of $D_\kappa \cdot W_{\kappa,l} = 0$ for such $\eta_l$. We let $W_{\kappa,l}$ be any fixed element of such basis. For such $W_{\kappa,l}$ we denote by $\pi(W_{\kappa,l})$ the $C$-span of $$\{(W_{\kappa,l}(g), v) \mid g \in G, v \in V_\kappa\}.$$ To prove that the formula above actually becomes an equality, we verify the following.

**Proposition 5.8.** For $\eta_l$ and $W_{\kappa,l}$ above we have an isomorphism $\pi(W_{\kappa,l}) \simeq H_\kappa$ as $(g,K)$-modules.

**Proof.** Proposition 3.4(2) implies that this proposition is a consequence of the following lemma.

**Lemma 5.9.** As a $(g,K)$-module, $\pi(W_{\kappa,l})$ is irreducible. As a $K$-module, it decomposes into

$$\pi(W_{\kappa,l}) \mid_K \simeq \bigoplus_{n \geq 0} (\tau(n,0,\ldots,0|n+\kappa), V(n,0,\ldots,0|n+\kappa)).$$

First we study the structure of $\pi(W_{\kappa,l})$ as a $K$-module. Let $$\{F^a_1, F^a_2, \ldots, F^a_s\}$$ be bases of $p_C$ and $$\{F^b_1, F^b_2, \ldots, F^b_t\}$$ be bases of $k_C$, respectively. Due to the Poincaré–Birkhoff–Witt theorem (cf. [22, Chapter III, Section 2, Theorem 3.8]) the set

$$\{(F^a_1 F^a_2 \ldots F^a_s) \cdot (F^b_1 F^b_2 \ldots F^b_t) \mid a_i, b_j \in \mathbb{Z}_{\geq 0}\}$$

forms a basis of the universal enveloping algebra $U(g_C)$ of $g_C$. For a non-negative integer $n$ we put

$$H_n := \mathbb{C}\text{-span of } \left\{ dR(F^{a_1} F^{a_2} \ldots F^{a_s})(W_{\kappa,l}, v) \mid v \in V_\kappa, \sum_{1 \leq i \leq s} a_i \leq n \right\},$$

$$W_n := H_n / H_{n-1} \quad (n \geq 1), \quad W_0 := H_0 \simeq V^*_\kappa \simeq V_\kappa.$$

Both of $H_n$ and $W_n$ are $K$-stable.

In order to have more detailed information on $W_n$, the following lemma is necessary.

**Lemma 5.10.** Let $\{v_{\kappa,i}^n\}_{0 \leq i \leq \kappa}$ be the basis of $V_\kappa$ dual to $\{v_{\kappa,i}\}_{0 \leq i \leq \kappa}$. Then $W_{\kappa,l}$ satisfies

$$(\kappa - i) dR(F_{\pm \beta_k - \beta_q + 1})(W_{\kappa,l}, v_{\kappa,i}^n)_\kappa = \pm (i + 1) dR(F_{\pm \beta_k + \beta_q + 1})(W_{\kappa,l}, v_{\kappa,i+1}^n)_\kappa,$$

$$(\kappa - i) dR(F_{\pm \beta_q - \beta_q + 1})(W_{\kappa,l}, v_{\kappa,i}^n)_\kappa = \mp (i + 1) dR(F_{\pm \beta_q + \beta_q + 1})(W_{\kappa,l}, v_{\kappa,i+1}^n)_\kappa,$$

for $0 \leq i \leq \kappa - 1$. For $i = \kappa$ we have $W_{\kappa,l} \mid_{(g,K)} = 0$.
where \( k \) ranges over \( 1 \leq k \leq q - 1 \) for the first formula and \( i \) over \( 0 \leq i \leq \kappa - 1 \) for both formulas.

This follows from Lemma 2.3 and the condition \( D_k \cdot W_{k,l} \equiv 0 \). For this it is convenient to review the explicit description of \( \nabla_K \cdot W_{k,l} \) just before Proposition 5.4.

Once we prove \( W_n \simeq V_{(n,0,...,0)}(\kappa+\kappa) \), then the \( K \)-module structures of \( H_n \) and \( \pi(W_{k,l}) \) will be also understood. By Lemma 5.10 we see that \( W_n \) is spanned by

\[
dR \left( \prod_{1 \leq j \leq n} F_{\alpha_j} \right) (W_{k,l}, v^\ast_{k,l})_k \mod H_{n-1}(\alpha_j \in \Delta^+_n, 0) \tag{5.10}
\]

for \( 0 \leq i \leq \kappa - 1 \), and

\[
dR \left( \prod_{1 \leq j \leq a} F_{\alpha_j} \prod_{1 \leq k \leq n-a} F_{\alpha_k} \right) (W_{k,l}, v^\ast_{k,l})_k \mod H_{n-1}(\alpha_j \in \Delta^+_n, 0, \alpha_k \in -\Delta^+_n, 0) \tag{5.11}
\]

for \( 0 \leq a \leq n \), where we take, as the order of the above products, the standard order of \( \Delta^+_n \) and \( -\Delta^+_n, 0 \) (for \( \Delta^+_n, 0 \) see Section 1), and where we allow the overlap of \( \alpha_j \)'s or \( \alpha_k \)'s for the products.

Let us verify that these vectors form a basis of \( W_n \) and \( W_n \simeq V_{(n,0,...,0)}(\kappa+\kappa) \). For that purpose we first assume that each of them is non-zero. Then we see that they form a basis of \( \Pi \) with its eigenvalue \( \kappa \) (for \( \kappa \) see Section 1), and we can check by a direct computation that the eigenvalues of them with respect to the \( t_{\mathbb{C}} \)-actions are distinct. In addition, we can verify under such an assumption that \( W_n \) has a unique highest weight vector \( dR((F_{\beta_1+\beta_{q+1}})^n)(W_{k,l}, v^\ast_{k,0})_k \) with highest weight \( n\beta_1 + (n + \kappa)\beta_{q+1} \), up to constant multiples, whence \( W_n \) is an irreducible \( K \)-module isomorphic to \( V_{(n,0,...,0)}(\kappa+\kappa) \).

Next we realize that it suffices to prove \( W_n \not\equiv \{0\} \) for the claim just above. With the help of the explicit formula of \( W_{k,l} \) we can show that the \( \mathbb{C} \)-span of \( \{dR(F^n_{\beta})(W_{k,l}(\tau_k^\ast(k)v)_k | k \in K, \beta \in \{\pm(\beta_q - \beta_{q+1})\}) \} \) with each fixed non-zero \( v \in V_k \) contains a non-zero eigenvector with its eigenvalue \( -(n\beta_2 + (n + \kappa)\beta_{q+1})(T) \) or \( (n\beta_2 - (n + \kappa)\beta_{q+1})(T) \) for \( T \in t_{\mathbb{C}} \) (for \( t \) see Section 1). Such a vector never occurs in \( H_{n-1} \), whence \( W_n \not\equiv \{0\} \). Therefore we conclude that \( W_n \simeq V_{(n,0,...,0)(\kappa+\kappa)} \) and the generators above actually form a basis of this.

The reducibility of finite-dimensional representations of compact topological groups yields \( H_n \simeq \bigoplus_{0 \leq i \leq n} (\tau_{(i,0,...,0)(i+\kappa)}), V_{(i,0,...,0)(i+\kappa)} \). As a result we deduce from the Poincaré–Birkhoff–Witt theorem that \( \pi(W_{k,l})|_K \) has the decomposition as a \( K \)-module in the assertion.

It remains to show the irreducibility of \( \pi(W_{k,l}) \) as a \((g, K)\)-module. Let \( \Pi \) be any \((g, K)\)-submodule of \( \pi(W_{k,l}) \) containing \((\tau_K, V_K)\). Then \((\tau_{(n,0,...,0)(n+\kappa)}, V_{(n,0,...,0)(n+\kappa)}) \) occurs in \( \Pi \) for all non-negative integer \( n \). In fact, once \((\tau_K, V_K) \) occurs in \( \Pi \), the \( U(g_{\mathbb{C}}) \)-stability of \( \Pi \) implies that \( \Pi \) contains a vector \( dR((F_{\beta_{q+1}})^n)(W_{k,l}, v^\ast_{k,0})_k \) for each \( n \), which gives a highest weight vector of the \( (n,0,...,0)(n+\kappa) \)-factor in \( \pi(W_{k,l}) \). This yields \( \Pi = \pi(W_{k,l}) \) as \((g, K)\)-modules. Therefore \( \pi(W_{k,l}) \) is an irreducible \((g, K)\)-module. \( \square \)

**Theorem 5.11.** We have a bijection

\[
\text{Hom}_{(g, K)}(H_K, C_n^\infty(N \setminus G)_K) \simeq \{W \in S_\kappa, \tau_K (N \setminus G/K) \mid D_K \cdot W = 0\},
\]
hence

\[
\dim_{\mathbb{C}} \text{Hom}_{(g,\kappa)}(H_\kappa, C^\infty_\eta(N\backslash G)_K) = \begin{cases} 
\kappa + 1 & (\eta: \text{trivial character}), \\
2 & (q = 1 \text{ and } \eta: \text{non-trivial character}), \\
1 & (q > 1 \text{ and } \eta: \text{not character}), \\
0 & (q > 1 \text{ and } \eta: \text{non-trivial character}).
\end{cases}
\]

**Proof.** This is an immediate consequence of Propositions 5.3 and 5.8. \qed

**Remark 5.12.** Recall that \(\pi_\kappa\) is a discrete series representation without the condition “far from the walls” on its minimal \(K\)-type (see the end of Section 5.1). There is an expectation that the map \(t^*\) in Proposition 5.3 would be a bijection for any discrete series representations of general semisimple Lie groups. Theorem 5.11 tells us that generalized Whittaker functions for \(\pi_\kappa\) give evidence of such an expectation. Some other evidence exists. For instance, the result by Taniguchi [30] on discrete series Whittaker functions of \(SU(n, 1)\) and \(Spin(2n, 1)\) provides such examples. In spite of that, the expectation does not seem to be generally settled yet.

### 6. Fourier–Jacobi expansion

For Sections 6–8 we assume \(q > 1\). This section consists of three subsections. In Section 6.1 we define an automorphic form on \(G\) generating the quaternionic discrete series \(\pi_\kappa\) and state our theorem of the Fourier–Jacobi expansion. In Section 6.2 we study certain theta functions, which are proved to appear in the Fourier expansion. Section 6.3 proves the theorem. The proof is based on our results of the generalized Whittaker functions and the theory of spectral decomposition of \(L^2\)-spaces on compact nilmanifolds by Corwin and Greenleaf [8]. Actually we need the former to describe the functions appearing in the Fourier series and the latter to know what irreducible unitary representations of \(N\) contribute to the expansion.

#### 6.1. Definition of the automorphic forms and the statement of the Fourier expansion

Let \(\Gamma\) be an arithmetic subgroup of \(G\) contained in \(\mathcal{G}(\mathbb{Q})\) (for \(\mathcal{G}\) see Section 1). The automorphic forms which we consider are defined as follows.

**Definition 6.1.** A \(V_\kappa\)-valued \(C^\infty\)-function \(f\) on \(G\) is called an automorphic form generating \(\pi_\kappa\) with respect to \(\Gamma\) if \(f\) satisfies (i) and (ii) as follows:

(i) \(f(\gamma gk) = \tau_\kappa(k)^{-1} f(g) \forall (\gamma, g, k) \in \Gamma \times G \times K\),

(ii) \(\mathbb{C}\)-span of \(\{(f(*g), v)_\kappa | g \in G, v \in V_\kappa\}\) is isomorphic to \(\pi_\kappa\) as \((g, K)\)-modules.

For this definition we note that a definition of an automorphic form needs the “moderate growth condition” in general, i.e., it is at most of polynomial order. However, we will see in Section 7 that this automorphic form \(f\) automatically satisfies such condition. This property of \(f\) should be called the “Koecher principle” (cf. [5, Theorem 10.14], [7, Corollaire de la Proposition 1], [23, Satz 1, Satz 2], [25, Chapter 4, Section 1, Lemma 1]).

Let \(P\) be the standard proper \(\mathbb{Q}\)-parabolic subgroup of \(\mathcal{G}\), which is uniquely determined since the \(\mathbb{Q}\)-algebraic group \(\mathcal{G}\) is of \(\mathbb{Q}\)-rank one. We call the quotient \(\Gamma\backslash \mathcal{G}(\mathbb{Q})/P(\mathbb{Q})\), the set of \(\Gamma\)-cusps. As is well known, this set is finite. From now on we fix a complete set \(\Xi\) of representatives of it. For each \(c \in \Xi\) we set
\[ N_{\Gamma,c} := N \cap c^{-1} \Gamma c, \]
\[ X_{\Gamma,c} := \{ x \in X^R \mid n(0,x) \in N_{\Gamma,c} \}, \]
\[ X^*_{\Gamma,c} := \text{the dual lattice of } X_{\Gamma,c} \text{ with respect to the reduced trace } \text{tr}. \]

For a fixed \( g \in G \) we regard \( f(cn(w,x)g) \) as a function in \( n(w,x) \in N \). Since this is left \( N_{\Gamma,c} \)-invariant we see that
\[ f(cn(w,x+x')g) = f(cn(w,x)g) \quad \forall x' \in X_{\Gamma,c}. \]

Therefore we can decompose \( f(cn(w,x)g) \) into
\[ f(cn(w,x)g) = \sum_{\xi \in X^*_{\Gamma,c}} f_\xi(w,g) e(\text{tr} \xi x) \quad (6.1) \]

with
\[ f_\xi(w,g) := \frac{1}{\text{vol}(X^R/X_{\Gamma,c})} \int_{X^R/X_{\Gamma,c}} f(cn(w,x)g)e(\text{tr} \xi x) \, dx. \]

This is said to be the Fourier–Jacobi expansion at a cusp \( c \). In order to make \( f_\xi(w,g) \) more explicit we need a certain space \( \Theta_{\xi,c} \) of theta functions for each \( c \in \Sigma \) and \( \xi \in X^*_{\Gamma,c} \setminus \{0\} \). To define \( \Theta_{\xi,c} \) we introduce
\[ \Lambda_c := \{ \lambda \in B^{q-1} \mid n(\lambda,x_\lambda) \in N_{\Gamma,c}, \exists x_\lambda \in X^R \cap B \}, \]
\[ k_\xi(w',w) := 2^{4(q-1)}\nu(\xi)^{q-1} \exp(-2\pi d(\xi)^t(w-w')(w-w'))e(-\text{tr} \xi^t \bar{w}'w), \]
where we note that \( x_\lambda \in X^R \cap B \) above is unique modulo \( X_{\Gamma,c} \) for each \( \lambda \in L_c \).

**Definition 6.2.** For each \( \xi \in X^*_{\Gamma,c} \setminus \{0\} \) we set
\[ \Theta_{\xi,c} := \{ \vartheta \in C(\mathbb{H}^{q-1}) \mid \vartheta(w+\lambda) = e(\text{tr} \xi^t \bar{w}\lambda - x_\lambda)\vartheta(w), \forall \lambda \in \Lambda_c, \]
\[ \int_{\mathbb{H}^{q-1}} k_\xi(w',w)\vartheta(w') \, dw' = \vartheta(w) \}, \]

where \( C(\mathbb{H}^{q-1}) \) is the space of continuous functions on \( \mathbb{H}^{q-1} \).

Here we note that this notion of theta functions was introduced by Arakawa in one of his unpublished notes.

For each \( \xi \in X_{\Gamma,c} \setminus \{0\} \) let us fix \( u_\xi \in \mathbb{H}^{(1)} \) such that \( u_\xi e_2 u_\xi^{-1} = \xi/d(\xi) \). With this \( \Theta_{\xi,c} \) we state our result on the Fourier expansion.
Theorem 6.3. At each cusp \( c \in \Xi \) an automorphic form \( f \) generating \( \pi_\kappa \) has a Fourier expansion

\[
 f(cn(w, x)ay) = \sum_{i=0}^{\kappa} C_i^f y^{\kappa/2+1} v_{\kappa, i} + \sum_{\xi \in X_{\Gamma,c}^* \backslash \{0\}} a^f_\xi (w) y^{\kappa/2+1} \exp(-4\pi d(\xi)y) e(\tr \xi x) \cdot \sigma_\kappa(u_\xi)v_{\kappa, \kappa},
\]

where \( a^f_\xi (w) \in \Theta_{\xi, c} \) and \( C_i^f \) is a constant depending only on \( f \) and \( i \).

Remark 6.4.

1. We will see in Section 6.3 that the expansion above does not depend on the choice of \( u_\xi \)'s by virtue of Theorem 5.7.
2. This theorem is a generalization of Arakawa’s Fourier expansion in [3, Theorem 6.1], which deals with the case where \( f \) is bounded and \( \pi_\kappa \) is integrable. In one of his unpublished notes Arakawa had already shown that theta functions in \( \Theta_{\xi, c} \) appear in the Fourier expansion for that case. However, we note that his proof depends on the condition

\[
 c_\kappa \int_G \omega_\kappa (g^{-1}h) f(g) \, dg = f(h)
\]

in his definition of the automorphic forms (cf. Definition 8.5), where \( c_\kappa \) is some constant dependent only on \( \kappa \), and \( \omega_\kappa \) is the \( \tau_\kappa \)-spherical function coming from matrix coefficients of \( \pi_\kappa \) (cf. Section 8). But this condition is not well defined when \( f \) is unbounded. Hence we need another approach to a proof of the generalized theorem above, which we will consider soon.

6.2. Spectral decomposition of \( L^2(N_{\Gamma,c}\backslash N) \) and an explicit structure of \( \Theta_{\xi,c} \)

In this subsection we study \( \Theta_{\xi,c} \) and the space \( L^2(N_{\Gamma,c}\backslash N) \) of \( L^2 \)-functions on \( N_{\Gamma,c}\backslash N \) in detail. The invariant measure of \( N_{\Gamma,c}\backslash N \) induced by \( dn \) (for \( dn \) see (1.3)) defines the \( L^2 \)-norm of \( L^2(N_{\Gamma,c}\backslash N) \). With respect to such norm \( L^2(N_{\Gamma,c}\backslash N) \) forms a unitary representation by the right regular representation of \( N \).

To begin with, we consider a decomposition of \( L^2(N_{\Gamma,c}\backslash N) \) into a sum of irreducible unitary representations of \( N \). For \( (\eta, H_\eta) \in \hat{N} \) let \( m(\eta) := \dim_C \text{Hom}_N(H_\eta, L^2(N_{\Gamma,c}\backslash N)) \), i.e., the multiplicity of \( \eta \) in \( L^2(N_{\Gamma,c}\backslash N) \). We note that \( N_{\Gamma,c}\backslash N \) is compact. Then we have the following lemma (cf. [10, Chapter I, Sections 2, 3]).

Proposition 6.5. (Gel’fand, Graev, Piatetskii-Shapiro) The space \( L^2(N_{\Gamma,c}\backslash N) \) decomposes discretely into

\[
 L^2(N_{\Gamma,c}\backslash N) \simeq \bigoplus_{\eta \in \hat{N}} \bigoplus_{1 \leq i \leq m(\eta)} H_{\eta}^{(i)} \simeq \bigoplus_{\eta \in \hat{N}} \text{Hom}_N(H_\eta, L^2(N_{\Gamma,c}\backslash N)) \otimes H_\eta
\]
and each multiplicity \( m(\eta) \) is finite, where \( \oplus \) denotes the Hilbert space direct sum and each \( H^{(i)}_\eta \) is equivalent to \( H_\eta \) as unitary \( N \)-modules.

Now we decompose \( L^2(N_{\Gamma,c}\backslash N) \) along the center of \( N \). Then

\[
L^2(N_{\Gamma,c}\backslash N) \simeq \bigoplus_{\xi \in X^*_{\Gamma,c}} L^2_\xi(N_{\Gamma,c}\backslash N)
\]

with

\[
L^2_\xi(N_{\Gamma,c}\backslash N) := \{ \Phi \in L^2(N_{\Gamma,c}\backslash N) \mid \Phi(n(w,x)) = e(\text{tr}\xi x)\Phi(n(w,0)) \forall n(w,x) \in N \}.
\]

Recall that, up to unitary equivalence, an irreducible unitary representation of \( N \) is uniquely determined by its central character when it is non-trivial on the center (cf. Proposition 4.2). Hence Proposition 6.5 implies that \( L^2_\xi(N_{\Gamma,c}\backslash N) \) for \( \xi \in X^*_{\Gamma,c} \{0\} \) is unitarily equivalent to a finite copy of \( \eta_{l_\xi} \) (for \( \eta_{l_\xi} \) see Section 4). For a proof of Theorem 6.3 we need to know \( \text{Hom}_N(H_{\eta_{l_\xi}}, L^2(N_{\Gamma,c}\backslash N)) \) or the \( \eta_{l_\xi} \)-isotypic component of \( L^2(N_{\Gamma,c}\backslash N) \) explicitly. To this end we review the results of Corwin and Greenleaf [8] on spectral decomposition of \( L^2 \)-spaces on compact nilmanifolds.

In order to state their results let \( N \) be a generally connected nilpotent Lie group with Lie algebra \( \mathfrak{n} \) for the moment. Given a co-compact discrete subgroup \( \Gamma_N \) of \( N \), we set \( n_\mathbb{Q} := \mathbb{Q}\text{-span of } \log(\Gamma_N) \). A linear form \( l \in \mathfrak{n}^* \) is said to be \( \mathbb{Q} \)-rational if \( l(n_\mathbb{Q}) \subset \mathbb{Q} \). For a \( \mathbb{Q} \)-rational linear form \( l \) a polarization subalgebra \( m \) (for the definition see Section 4) is called \( \mathbb{Q} \)-rational if \( m \cap n_\mathbb{Q} \) provides a \( \mathbb{Q} \)-structure of \( m \). Then \( M = \exp(m) \) also has a \( \mathbb{Q} \)-structure. The existence of such \( m \) is known (cf. [9, Proposition 5.2.6]). For \( l \in \mathfrak{n}^* \) we can define a character \( \chi_l \) of \( M \) and an element \((\eta_l, H_{\eta_l})\) of \( \hat{N} \) similarly as in Section 4.

**Proposition 6.6.** Let \( \Gamma_N \) be as above and define a \( \mathbb{Q} \)-structure of \( \mathfrak{n} \) by this \( \Gamma_N \). Let \( l \in \mathfrak{n}^* \) be \( \mathbb{Q} \)-rational and \( M := \exp(m) \) with a \( \mathbb{Q} \)-rational polarization algebra \( m \) for \( l \).

1. Put

\[
\mathcal{O}(l)_{\Gamma_N} := \{ n \in (\Gamma_N\backslash N/M)^* \mid \chi_{\text{Ad}^* (n^{-1})}(n M n^{-1} \cap \Gamma_N) \equiv 1 \},
\]

where \((\Gamma_N\backslash N/M)^*\) stands for the totality of the double cosets meeting \( N_\mathbb{Q} := \exp(n_\mathbb{Q}) \). Then

\[
\dim \mathbb{C} \text{Hom}_N(H_{\eta_l}, L^2(\Gamma_N\backslash N)) = \sharp \mathcal{O}(l)_{\Gamma_N}.
\]

2. For any representative \( n \) of \( \mathcal{O}(l)_{\Gamma_N} \) we define \( \Theta_{l,n} \in \text{Hom}_N(\eta_{l_\xi}, L^2(\Gamma_N\backslash N)) \) by

\[
\Theta_{l,n}(h)(n') := \sum_{\gamma \in n M n^{-1} \cap \Gamma_N \backslash \Gamma_N} U_n(h)(\gamma n') \quad \text{for } h \in H_{\eta_l} \text{ and } n' \in N,
\]

where see (4.3) for \( U_n \). Then \( \bigoplus_{n \in \mathcal{O}(l)_{\Gamma_N}} \Theta_{l,n}(H_{\eta_l}) \) forms the \( \eta_l \)-isotypic component of \( L^2(\Gamma_N\backslash N) \).
For (1), see [8, Theorem 5.1] and for (2), see [8, Sections 5, 6]. This kind of result was also considered by Howe [17] and Richardson [26].

Let us return to our nilpotent Lie group \( N \). In order to apply this proposition to \( L^2(N_{\Gamma,c} \backslash \mathbb{N}) \) and \( L^2(\xi, N_{\Gamma,c} \backslash \mathbb{N}) \) we introduce several notations.

For \( \xi \in X_{\mathbb{R}} \setminus \{0\} \) we henceforth denote \((\eta_{\xi}, H_{\eta_{\xi}})\) simply by \((\eta_{\xi}, H_{\xi})\).

For \( \xi \in X_{\mathbb{R}} \cap B^\times \) so that \( \xi \rho_{\xi} \in X_{\mathbb{R}} \cap B^\times \), and define \( V^\pm_{\xi} \) under this choice of \( \xi \) and \( \rho_{\xi} \) (for \( V^\pm_\xi \) see Section 4). Furthermore, we set

\[
\hat{\Lambda}_{\xi,c} := \left\{ \lambda \in B^q-1 \mid \text{tr}(\xi(t^\lambda \mu + x_{\mu})) \in \mathbb{Z} \forall n(\mu, x_{\mu}) \in M \cap N_{\Gamma,c} \right\},
\]

\[
\hat{\Lambda}^\pm_{\xi,c} := \text{Pr}^\pm(\hat{\Lambda}_{\xi,c}), \quad \Lambda^\pm_{\xi,c} := \text{Pr}^\pm(\Lambda_c),
\]

where \( \text{Pr}^\pm \) denotes the projection of \( \mathbb{H}^{q-1} \) onto \( V^\pm_\xi \).

For a representative \( \lambda^- \) of \( \hat{\Lambda}^-_{\xi,c}/\Lambda^-_{\xi,c} \) we define \( \Theta^\lambda^-_{\xi}(H_{\eta_{\xi}}) \) forms the \( \eta_{\xi} \)-isotypic component of \( L^2(\xi, N_{\Gamma,c} \backslash \mathbb{N}) \). Namely

\[
L^2(\xi, N_{\Gamma,c} \backslash \mathbb{N}) = \bigoplus_{\lambda^- \in \hat{\Lambda}^-_{\xi,c}/\Lambda^-_{\xi,c}} \Theta^\lambda^-_{\xi}(H_{\eta_{\xi}}).
\]

Next, in order to consider \( \Theta_{\xi,c} \) in detail, we introduce

\[
\Theta'_{\xi,c} := \{ \theta: \text{measurable function on } \mathbb{H}^{q-1} \mid \theta(w + \lambda) = e^{\text{tr} \xi(t^\lambda \bar{\omega} \lambda - x_{\lambda})} \theta(w) \forall \lambda \in \Lambda_c, \| \theta \|_{\xi,c} < \infty \},
\]

where \( \| * \|_{\xi,c} \) denotes the norm induced by the inner product

\[
(\theta_1, \theta_2)_{\xi,c} := \int_{\mathbb{H}^{q-1}/\Lambda_c} \theta_1(w)\overline{\theta_2(w)} \, dw \quad \text{for } \theta_1, \theta_2 \in \Theta'_{\xi,c}.
\]
This space forms a Hilbert space with respect to \((\cdot, \cdot)_{\xi,c}\).

We define a map \(I_{\xi}\) between \(L^2_{\xi}(N_{\Gamma,c}\setminus N)\) and \(\Theta'_{\xi,c}\) as follows:

\[
I_{\xi}: L^2_{\xi}(N_{\Gamma,c}\setminus N) \ni \Phi \mapsto \left( \mathbb{H}^{q-1} \ni w \mapsto \Phi(n(w,0)) \right) \in \Theta'_{\xi,c}.
\]

The \(L^2\)-norm of \(L^2(N_{\Gamma,c}\setminus N)\) also defines a norm of \(L^2_{\xi}(N_{\Gamma,c}\setminus N)\). Multiplying the norm by \(\mathrm{vol}(X_{\mathbb{R}}/X_{\Gamma,c})^{-1}\), we see that \(I_{\xi}\) preserves the norms of \(L^2_{\xi}(N_{\Gamma,c}\setminus N)\) and \(\Theta'_{\xi,c}\). Then one obtains the following lemma without difficulty.

**Lemma 6.8.** The map \(I_{\xi}\) gives an isomorphism between \(L^2_{\xi}(N_{\Gamma,c}\setminus N)\) and \(\Theta'_{\xi,c}\) as Hilbert spaces.

We provide a complete orthogonal basis of \(\Theta'_{\xi,c}\) now. In view of Assumption 4.4, Proposition 6.7(2) and Lemma 6.8, we understand that an orthogonal basis of \(L^2(V_{\xi}^-)\) is useful to give one of \(\Theta'_{\xi,c}\). One can identify \(V_{\xi}^-\) with \(\mathbb{R}^{2(q-1)}\) by

\[
V_{\xi}^- = W_{\xi}^- \oplus \cdots \oplus W_{\xi}^- \ni \left( a_i + b_i \rho_{\xi} \xi / d(\rho_{\xi} \xi) \right)_{1 \leq i \leq q-1} \mapsto (a_1, b_1, \ldots, a_{q-1}, b_{q-1}) \in \mathbb{R}^{2(q-1)}
\]

and can regard \(L^2(V_{\xi}^-)\) as \(L^2(\mathbb{R}^{2(q-1)})\). Let

\[
h_n(t) := e^{4\pi d(\xi)t^2} \frac{d^n}{dt^n} e^{-8\pi d(\xi)t^2}
\]

be the Hermite function on \(\mathbb{R}\) indexed by \(n \in \mathbb{Z}_{\geq 0}\) and put

\[
h_n(t) := \prod_{1 \leq i \leq 2(q-1)} h_{n_i}(t_i)
\]

for \(n := (n_1, n_2, \ldots, n_{2(q-1)}) \in \mathbb{Z}_{\geq 0}^{2(q-1)}\) and \(t := (t_1, t_2, \ldots, t_{2(q-1)}) \in \mathbb{R}^{2(q-1)}\). By the isomorphism \(V_{\xi}^- \simeq \mathbb{R}^{2(q-1)}\) we view \(h_n\) as a function in \(w_{\xi}^- \in V_{\xi}^-\) and denote its evaluation at \(w_{\xi}^-\) by \(h_n(w_{\xi}^-)\). It is known that \(\{h_n\}_{n \in \mathbb{Z}_{\geq 0}^{2(q-1)}}\) forms a complete orthogonal basis of \(L^2(\mathbb{R}^{2(q-1)})\) (cf. [32, Chapter 12, Section 12.1.5]). Thus we can also regard it as a complete orthogonal basis of \(L^2(V_{\xi}^-)\).

For \((n, \lambda^-) \in \mathbb{Z}_{\geq 0}^{2(q-1)} \times \hat{\Lambda}_{\xi,c}^- / \Lambda_{\xi,c}^-\) we extend \(h_n(w_{\xi}^-) \in L^2(V_{\xi}^-)\) to an element

\[
H_n: n(w, x) \mapsto h_n(w_{\xi}^-) e(\operatorname{tr}_{\xi}(t_{w_{\xi}^-} w_{\xi}^+ + x))
\]

in \(H_{\xi,c}\), where \(w = w_{\xi}^+ + w_{\xi}^-\) with \(w_{\xi}^\pm \in V_{\xi}^\pm\), and set

\[
\theta_{\xi,n}^\lambda(w) := I_{\xi}(\theta_{\xi}^\lambda(H_n))(w).
\]

Then Lemma 6.8 implies the following.

**Lemma 6.9.** The space \(\Theta'_{\xi,c}\) has \(\{\theta_{\xi,n}^\lambda(w) \mid n \in \mathbb{Z}_{\geq 0}^{2(q-1)}\} \subseteq \hat{\Lambda}_{\xi,c}^- / \Lambda_{\xi,c}^-\) as a complete orthogonal basis.
Now we are ready to give a detailed description of $\Theta_{\xi,c}$. We set
\[\theta^{\lambda,-}_{\xi}(w) := \theta^{\lambda,-}_{\xi,n}(w) \quad \text{with} \quad n = (0, 0, \ldots, 0) \in \mathbb{Z}^{q-1}_{\geq 0} \quad (6.2)\]
and denote by $K_{\xi}(w', w)$
\[2^{4(q-1)}\nu(q)^{-1} \sum_{\lambda \in \Lambda_c} \exp\left(-2\pi d(\xi)'(w-w'-\lambda)(w-w'-\lambda)\right) \times \exp\left(-\text{tr} \xi\left(t' (w' + \lambda)(w - \lambda) + x_\lambda\right)\right).\]
In addition, we note that the second defining condition of $\Theta_{\xi,c}$ can be rewritten as
\[\int_{\mathbb{H}^{q-1}/\Lambda_c} K_{\xi}(w', w) \theta(w') \, dw' = \theta(w) \quad (6.3)\]
and that $K_{\xi}$ is bounded.

Lemma 6.10. With a fixed $w' \in \mathbb{H}^{q-1}$, $K_{\xi}(w', w) \in \Theta_{\xi,c}$ as a function in $w \in \mathbb{H}^{q-1}$.

Proof. The first defining condition of $\Theta_{\xi,c}$ for $K_{\xi}(*, w)$ is confirmed by a straightforward calculation. As for the second defining condition of $\Theta_{\xi,c}$ we recall a well-known formula
\[\int_{\mathbb{R}} \exp(-ax^2 + 2bx) \, dx = \left(\frac{\pi}{a}\right)^{1/2} \exp\left(\frac{b^2}{a}\right), \quad (6.4)\]
where $a, b \in \mathbb{C}$ such that $\text{Re}(a) > 0$ (cf. [19, Lemma 1, p. 5]). Using this, we verify the condition for this function by a formal computation. □

Now we state our result on $\Theta_{\xi,c}$.

Theorem 6.11. The set of theta functions $\left\{\theta^{\lambda,-}_{\xi}\right\}_{\lambda,- \in \hat{\Lambda}_{\xi,c} / \Lambda_{\xi,c}^-}$ forms a basis of $\Theta_{\xi,c}$, and we have a dimension formula for $\Theta_{\xi,c}$ as follows:
\[\dim_{\mathbb{C}} \Theta_{\xi,c} = (\hat{\Lambda}_{\xi,c}^- / \Lambda_{\xi,c}^-) = 2^{4(q-1)}\nu(q)^{-1} \text{vol}(\mathbb{H}^{q-1}/\Lambda_c).\]

Proof. First we prove
\[\dim_{\mathbb{C}} \Theta_{\xi,c} = 2^{4(q-1)}\nu(q)^{-1} \text{vol}(\mathbb{H}^{q-1}/\Lambda_c).\]
For this formula we follow the proof which Arakawa gave in one of his unpublished notes. The proof starts from
\[\dim_{\mathbb{C}} \Theta_{\xi,c} = \int_{\mathbb{H}^{q-1}/\Lambda_c} K_{\xi}(w, w) \, dw.\]
Noting Lemma 6.10, (6.3) and \( \overline{K_\xi (w', w)} = K_\xi (w, w') \), we can deduce this by an argument similar to the standard proof of “Godement’s formula” [11, Théorème 8] for the dimension of modular forms (see also the proof of [2, Theorem 1]).

By putting the explicit form of \( K_\xi (w, w) \) into the right-hand side, we obtain

\[
\int_{\mathbb{H}^{q-1}/\Lambda_c} K_\xi (w, w) \, dx = 2^4(q-1) v(\xi)^{q-1} \int_{\mathbb{H}^{q-1}/\Lambda_c} \sum_{\lambda \in \Lambda_c} \exp(-2\pi d(\xi)^\ell \bar{\lambda}\lambda) \, e(-\text{tr} \xi (2^\ell \bar{\lambda} w + x_\lambda)) \, dw
\]

\[
= 2^4(q-1) v(\xi)^{q-1} \sum_{\lambda \in \Lambda_c} \exp(-2\pi d(\xi)^\ell \bar{\lambda}\lambda) \int_{\mathbb{H}^{q-1}/\Lambda_c} e(-\text{tr} \xi (2^\ell \bar{\lambda} w + x_\lambda)) \, dw
\]

\[
= 2^4(q-1) v(\xi)^{q-1} \text{vol}(\mathbb{H}^{q-1}/\Lambda_c),
\]

where we note that

\[
\int_{\mathbb{H}^{q-1}/\Lambda_c} e(-2 \text{tr} \xi^\ell \bar{\lambda} w) \, dw
\]

is non-vanishing only if \( \lambda = 0 \) since \( e(-2 \text{tr} \xi^\ell \bar{\lambda} w) \) defines an additive character on \( \mathbb{H}^{q-1}/\Lambda_c \).

Next we show that \( \dim_{\mathbb{C}} \Theta_{\xi, c} = \#(\hat{\Lambda}_{\xi, c}/\Lambda_{\xi, c}) \). Then the proof of the theorem is complete. Now we need a formula

\[
\int_{\mathbb{H}^{q-1}} k_\xi (w', w) \theta_{\xi, n} (w) \, dw' = \begin{cases} \theta_{\xi, n} (w) = \theta_{\xi, n}^{-} (w) & (n = (0, 0, \ldots, 0)), \\ 0 & (\text{otherwise}). \end{cases}
\]

This is verified by the formula (6.4) and the orthogonality of the Hermite functions. This formula and Lemma 6.9 imply \( \Theta_{\xi, c} = \bigoplus_{\lambda \in \Lambda_{\xi, c}/\Lambda_{\xi, c}} \theta_{\xi, n}^{-} \). Thus our proof of this theorem is finished. \( \square \)

6.3. Proof of Theorem 6.3

Throughout this subsection we denote by \( f \) an automorphic form on \( G \) generating \( \pi_{\kappa} \) defined in Definition 6.1. The following proposition is the first step for the proof of Theorem 6.3.

Proposition 6.12. An automorphic form \( f \) generating \( \pi_{\kappa} \) satisfies \( D_{\kappa} \cdot f = 0 \), where recall that \( D_{\kappa} \) denotes the Schmid operator (cf. Section 5.1).

Proof. We denote by \( H_f \) the \( (g, K) \)-module in Definition 6.1(ii). By definition this is isomorphic to \( H_{\kappa} \) as \( (g, K) \)-modules, where \( H_{\kappa} \) denotes the representation space of \( \pi_{\kappa} \) (cf. Section 3). By the definition of \( D_{\kappa} \) we see that \( D_{\kappa} \cdot f \) is \( V_{(1,0,\ldots,0|\kappa-1)} \)-valued and that the coefficient functions of \( D_{\kappa} \cdot f \) belong to \( U(g)H_f = H_f \). The \( K \)-module generated by such coefficients is isomorphic to \( V^*_{(1,0,\ldots,0|\kappa-1)} \simeq V_{(1,0,\ldots,0|\kappa-1)} \) (cf. Lemma 2.1(2)) if it is non-trivial. However, the discrete series representation \( (\pi_{\kappa}, H_{\kappa}) \) does not admit \( V_{(1,0,\ldots,0|\kappa-1)} \) since the highest weight of \( V_{(1,0,\ldots,0|\kappa-1)} \).
is lower than that of the minimal $K$-type $(\tau_\kappa, V_\kappa)$ of $(\pi_\kappa, H_\kappa)$. This implies that $D_\kappa \cdot f = 0$ has to hold.

Recall that $f_\xi(w, g)e(tr\xi x)$ appears in the $\xi$-component of the Fourier–Jacobi expansion of $f$ in (6.1). This belongs to $L^2_\xi(N_{\Gamma, c}\setminus \mathcal{N}) \otimes V_\kappa$ for each fixed $g \in G$. Proposition 6.7(2) implies

$$f_\xi(w, g)e(tr\xi x) = \sum_{i=0}^{\kappa} e^{\lambda_\xi i} (W^{f, i}_{\eta_\xi}(g)) (n(w, x)) \cdot v_{\kappa, i},$$

where $W^{f, i}_{\eta_\xi}$ is an $H_{\eta_\xi}^\infty$-valued $C^\infty$-function on $G$.

We see that $W^{f}_{\eta_\xi}(g) := \sum_{i=0}^{\kappa} W^{f, i}_{\eta_\xi}(g) \cdot v_{\kappa, i}$ is a generalized Whittaker function defined in Section 5.1. In fact, the right $K$-equivariance of $f$ with respect to $\tau_\kappa$ is also valid for $W^{f}_{\eta_\xi}$ and its left $\mathcal{N}$-equivariance with respect to $\eta_\xi$ is verified in a formal way. Furthermore, we note that the condition $D_\kappa \cdot f = 0$ implies $D_\kappa \cdot W^{f}_{\eta_\xi} = 0$. Hence, due to Theorem 5.11, $W^{f}_{\eta_\xi}$ is nothing but a generalized Whittaker function for $\pi_\kappa$ attached to $\eta_\xi$.

Now, for the choice of $\rho_\xi$ given in the setting of Proposition 6.7, let us take $u_\xi \in H^1(1)$ so that it satisfies (4.6). By virtue of the explicit formulas for the generalized Whittaker functions in Theorems 5.6 and 5.7, we see that

$$f_\xi(w, a_y)e(tr\xi x) = \begin{cases} \sum_{i=0}^{\kappa} C^f_i y^{\kappa/2+1} v_{\kappa, i} & (\xi = 0), \\
\sum_{\lambda^- \in \Lambda^-_{\xi, c}/\Lambda^-_{\xi, c}} C^f_{\xi, \lambda^-} \theta^{\lambda^-}_{\xi}(w) y^{\kappa/2+1} \exp(-4\pi d(\xi)y) e^{(tr\xi x) \cdot \sigma_\kappa(u_\xi) v_{\kappa, \kappa}} & (\xi \neq 0), \end{cases}$$

where $C^f_i$ (respectively $C^f_{\xi, \lambda^-}$) denotes a constant dependent only on $(i, f)$ (respectively $(\xi, \lambda^-, f)$). Here we remark that we can replace $u_\xi$ by any $u \in \mathbb{H}^{(1)}$ such that $ueu^{-1} = \xi/d(\xi)$ in view of Theorem 5.7, i.e., the Fourier expansion does not depend on the choice of $u_\xi$’s. According to Theorem 6.11, $\theta^{\lambda^-}_{\xi}(w) \in \Theta_{\xi, c}$. We can therefore take $\sum_{\lambda^- \in \Lambda^-_{\xi, c}/\Lambda^-_{\xi, c}} C^f_{\xi, \lambda^-} \theta^{\lambda^-}_{\xi}(w)$ as $a^f_\xi(w)$ in Theorem 6.3. Thus we conclude that the Fourier–Jacobi expansion of $f$ at a cusp $c$ is of the form in Theorem 6.3.

7. The Koecher principle of the automorphic forms

Let $\{\theta^\mu_{\xi}\}_{1 \leq \mu \leq m(\xi)}$ be an orthonormal basis of $\Theta_{\xi, c}$ with $m(\xi) = \dim_{\mathbb{C}} \Theta_{\xi, c}$. With this basis the Fourier–Jacobi expansion at a cusp $c$ can be written as

$$f(cn(w, x)a) = \sum_{i=1}^{\kappa} C^f_i y^{\kappa/2+1} v_{\kappa, i}$$

$$+ \sum_{\xi \in \mathcal{X}^\kappa_{\Gamma, c} \setminus \{0\}} \sum_{\mu=1}^{m(\xi)} C^f_{\xi, \mu} \theta^\mu_{\xi}(w) y^{\kappa/2+1} e^{-4\pi d(\xi)y} e^{(tr\xi x) \cdot \sigma_\kappa(u_\xi) v_{\kappa, \kappa}},$$

where $C^f_i$ and $C^f_{\xi, \mu}$ are the coefficients dependent only on $(i, f)$ and $(\xi, \mu, f)$, respectively.
Now we state another main theorem of this paper.

**Theorem 7.1 (Koecher principle).** An automorphic form \( f \) generating \( \pi_{\kappa} \) with respect to \( \Gamma \) automatically satisfies the moderate growth condition, i.e., the growth of \( f \) is at most of polynomial order.

**Proof.** In order to prove this theorem we provide a lemma. To state it we introduce a Siegel set \( \mathcal{S}_{\omega,\epsilon} := \omega \cdot A[\epsilon] \cdot K \) with \( A[\epsilon] := \{ a \in A \mid y > \epsilon \} \) for a compact neighborhood \( \omega \) of \( N \) at the identity and a positive number \( \epsilon \).

**Lemma 7.2.**

1. (Reduction theory) With a suitable choice of \( \omega \) and \( \epsilon \) we have \( G = \bigcup_{c \in \Xi} \Gamma \cdot \mathcal{S}_{\omega,\epsilon} \).
2. The theta function \( \theta_{\xi}^n \) (respectively \( K_{\xi} \)) has a uniform estimate
   \[
   \left| \theta_{\xi}^n(w) \right| < C_{\Gamma,c} \sqrt{\text{vol}(\mathbb{H}^{q-1}/\Lambda_c)} \nu(\xi)^{q-1} \quad \text{(respectively } \left| K_{\xi}(w,w') \right| < C_{\Gamma,c} \nu(\xi)^{q-1})
   \]
   on \( \mathbb{H}^{q-1} \) (respectively on \( \mathbb{H}^{q-1} \times \mathbb{H}^{q-1} \)), where \( C_{\Gamma,c} \) is a constant depending only on \( \Gamma \) and \( c \).
3. For \( \epsilon > 0 \) set \( G_{\epsilon} := \{ g = n(w,x)a_yk \in G \mid y \geq \epsilon \} \). On \( G_{\epsilon} \) we have an estimate of the coefficient \( C_{f,\xi,\mu} \):
   \[
   \left| C_{f,\xi,\mu} \right| < C_{f,c,\epsilon_0} \exp(4\pi d(\xi)\epsilon_0)
   \]
   with any fixed positive number \( \epsilon_0 < \epsilon \), where \( C_{f,c,\epsilon_0} \) is a constant dependent only on \( f \), \( c \in \Xi \) and \( \epsilon_0 \).

**Proof.** The assertion (1) is well known and see Borel [6, Proposition 15.6]. First we prove the assertion (2). For that purpose we need another lemma.

**Lemma 7.3.**

1. With a fixed positive real number \( a \) we have an estimate
   \[
   \sum_{n \in \mathbb{Z}} \exp(-a(s+n)^2) \leq 2 \left( \sqrt{\frac{\pi}{a}} + 1 \right)
   \]
   for any \( s \in \mathbb{R} \).
2. Let \( \{ \alpha_i \}_{1 \leq i \leq 4(q-1)} \) be a \( \mathbb{Z} \)-basis of \( \Lambda_c \) and write \( w \in \mathbb{H}^{q-1} \) as \( \sum_{i=1}^{4(q-1)} x_i \alpha_i \) with \( x_i \in \mathbb{R} \) for \( 1 \leq i \leq 4(q-1) \), where we recall that the lattice \( \Lambda_c \) of \( \mathbb{H}^{q-1} \) is given just before Definition 6.2. Then there exists a positive constant \( C_{\Lambda_c} \) depending only on \( \Lambda_c \) such that
   \[
   \langle t \bar{w}w \rangle \geq C_{\Lambda_c} \left( \sum_{i=1}^{4(q-1)} x_i^2 \right).
   \]
Proof. The first assertion is settled by the following elementary calculation:

\[
\sum_{n \in \mathbb{Z}} \exp(-a(s + n)^2) \leq \sum_{n \in \mathbb{Z}} \exp(-an^2) + 1 < 2 \sum_{n \in \mathbb{Z}} \exp(-an^2) < 2 \left( \int_{\mathbb{R}} \exp(-at^2) \, dt + 1 \right) = 2 \left( \sqrt{\frac{\pi}{a}} + 1 \right).
\]

Here we use (6.4) to obtain the last equality. For the second assertion it suffices to verify it on the unit sphere

\[
\left\{ (x_i)_{1 \leq i \leq 4(q-1)} \in \mathbb{R}^{4(q-1)} \mid \sum_{i=1}^{4(q-1)} x_i^2 = 1 \right\} \quad \text{in } \mathbb{R}^{4(q-1)}.
\]

The assertion then follows immediately. □

Noting this lemma and the inequalities given in its proof, one has an estimate

\[
|K_\xi (w', w)| < 2^{4(q-1)} v(\xi)^{q-1} \sum_{\lambda \in \Lambda_c} \exp(-2\pi d(\xi)^t (w - w' - \lambda)(w - w' - \lambda))
\]

\[
< 2^{8(q-1)} v(\xi)^{q-1} \sum_{(n_i)_{1 \leq i \leq 4(q-1)} \in \mathbb{Z}^{4(q-1)}} \exp\left(-2\pi d(\xi)C_{\Lambda_c}\left(\sum_{i=1}^{4(q-1)} n_i^2\right)\right)
\]

\[
< 2^{8(q-1)} v(\xi)^{q-1} \prod_{i=1}^{4(q-1)} \left( \int_{\mathbb{R}} \exp(-2\pi d(\xi)C_{\Lambda_c} x_i^2) \, dx_i + 1 \right)
\]

\[
= 2^{8(q-1)} v(\xi)^{q-1} (2^{-1/2} v(\xi)^{-1/4} C_{\Lambda_c}^{-1/2} + 1)^{4(q-1)}.
\]

The maximal value of \(v(\xi)^{-1/4}\) on \(X_{\Gamma,c}^*\) and the constant \(C_{\Lambda_c}\) are dependent only on \(\Gamma\) and \(c\). Hence we see that there is a suitable constant \(C_{\Gamma,c}\) depending only on \(\Gamma\) and \(c\) such that

\[
|K_\xi (w', w)| < C_{\Gamma,c} v(\xi)^{q-1}.
\]

For the estimate of \(|\theta^\mu_\xi (w)|\) recall that \(\theta^\mu_\xi\) satisfies

\[
\theta^\mu_\xi (w) = \int_{\mathbb{H}^{q-1}/\Lambda_c} K_\xi (w', w) \theta^\mu_\xi (w') \, dw'
\]

(cf. (6.3)). By the Schwartz inequality we have

\[
|\theta^\mu_\xi (w)|^2 = \left| \int_{\mathbb{H}^{q-1}/\Lambda_c} K_\xi (w, w') \theta^\mu_\xi (w') \, dw' \right|^2
\]
\[
\begin{align*}
\leq & \int_{\mathbb{H}^{q-1}/\Lambda_c} |K_{\xi}(w, w')|^2 \, dw' \int_{\mathbb{H}^{q-1}/\Lambda_c} |\theta_{\xi}^\mu(w')|^2 \, dw' \\
& < \text{vol}(\mathbb{H}^{q-1}/\Lambda_c) C_{T, c}^2 v(\xi)^{2(q-1)}.
\end{align*}
\]

This proves the estimate of \(|\theta_{\xi}^\mu(w)|\) in the assertion (2).

Finally we prove the assertion (3). With \(\epsilon_0\) in the assertion set

\[
a_0 = \left( \frac{1}{q-1}, \frac{\sqrt{\epsilon_0}}{\sqrt{\epsilon_0} - 1} \right).
\]

Then we get

\[
\exp(-4\pi d(\xi)\epsilon_0)\epsilon_0^{K/2+1} C_{f, \mu}^f \int_{N_{\Gamma, c}\setminus N} |\theta_{\xi}^\mu(w)|^2 \, dw \, dx \cdot \sigma_k(u_{\xi}) v_{\kappa, \kappa}
\]

\[
= \int_{N_{\Gamma, c}\setminus N} f(cn(w, x)a_0) \overline{\theta_{\xi}^\mu(w)} e(\text{tr}_{\xi} x) \, dw \, dx
\]

by the Fourier–Jacobi expansion in (7.1). Since the norm of \(\theta_{\xi}^\mu\) is one, we have

\[
\int_{N_{\Gamma, c}\setminus N} |\theta_{\xi}^\mu(w)|^2 \, dw \, dx = \text{vol}(X_\mathbb{R} / X_{\Gamma, c}).
\]

From this

\[
\left\| \exp(-4\pi d(\xi)\epsilon_0)\epsilon_0^{K/2+1} C_{f, \mu}^f \int_{N_{\Gamma, c}\setminus N} |\theta_{\xi}^\mu(w)|^2 \, dw \, dx \cdot \sigma_k(u_{\xi}) v_{\kappa, \kappa} \right\|_\kappa
\]

\[
= \exp(-4\pi d(\xi)\epsilon_0)\epsilon_0^{K/2+1} C_{f, \mu}^f \text{vol}(X_\mathbb{R} / X_{\Gamma, c}) \| v_{\kappa, \kappa} \|_\kappa
\]

follows, where \(\| \cdot \|_\kappa\) denotes the norm induced by the fixed inner product \((\cdot, \cdot)_\kappa\) of \(V_\kappa\) (for \((\cdot, \cdot)_\kappa\) see Section 2).

On the other hand, the boundedness of \(f(cn(w, x)a_0)\) as a function in \(n(w, x) \in N\) and the Schwartz inequality imply

\[
\left\| \int_{N_{\Gamma, c}\setminus N} f(cn(w, x)a_0) \overline{\theta_{\xi}^\mu(w)} e(\text{tr}_{\xi} x) \, dw \, dx \right\|_\kappa
\]

\[
< C_{f, c, \epsilon_0} \text{vol}(X_\mathbb{R} / X_{\Gamma, c}),
\]

where \(C_{f, c, \epsilon_0}\) is a constant depending only on \(f\), \(c\) and \(\epsilon_0\). Thus (7.2)–(7.4) yield

\[
e^{-4\pi d(\xi)\epsilon_0} C_{f, \mu}^f \leq C_{f, c, \epsilon_0},
\]
where we denote $\epsilon_0^{-k/2-1} \|v_{k,\kappa}\|_k^{-1} C_{f,c,\epsilon_0} C_{f,c,\epsilon_0}$ by $C_{f,c,\epsilon_0}$ again. As a result we get our desired formula in the assertion (3).

Now we start proving the theorem. Lemma 7.2 and the Fourier expansion (7.1) lead to

$$\|f(cn(w, x)a)\|_k < \frac{y^{k/2+1}}{2} \left\{ C_{f,c} + C_{f,c,\epsilon_0} C_{\Gamma,c} \sqrt{\text{vol}(\mathbb{H}^{q-1}/\Lambda_c)} \right. \times \sum_{\xi \in \mathcal{X}_{\Gamma,c}\setminus\{0\}} \sum_{\mu=1}^{m(\xi)} n(\xi)^{q-1} \exp(-4\pi d(\xi)(y - \epsilon_0)) \|v_{k,\kappa}\|_k \left\{ \right\}$$

on $G_\epsilon$, with any fixed positive number $\epsilon_0 < \epsilon$ and a suitable constant $C_{f,c}$ dependent only on $f$ and $c$. The summation over $\xi \in \mathcal{X}_{\Gamma,c}\setminus\{0\}$ and $1 \leq \mu \leq m(\xi)$ is bounded by its evaluation at $y = \epsilon$. Furthermore, recall that $m(\xi) = 2^4(q-1) n(\xi)^{q-1} \text{vol}(\mathbb{H}^{q-1}/\Lambda_c)$ in Theorem 6.11. From these we deduce

$$\|f(cn(w, x)a)\|_k < \frac{y^{k/2+1}}{2} \left\{ C_{f,c} + C_{f,c,\epsilon_0} C_{\Gamma,c} 2^4(q-1) \text{vol}(\mathbb{H}^{q-1}/\Lambda_c)^{3/2} \times \sum_{\xi \in \mathcal{X}_{\Gamma,c}\setminus\{0\}} n(\xi)^{2(q-1)} \exp(-4\pi d(\xi)(\epsilon - \epsilon_0)) \|v_{k,\kappa}\|_k \left\{ \right\} \right.$$

$$< C_{f,\Gamma,c,\epsilon_0} y^{k/2+1}$$

on $G_\epsilon$ with a suitable constant $C_{f,\Gamma,c,\epsilon_0}$ depending only on $f$, $\Gamma$, $c$ and $\epsilon_0$. As a result, we verify the moderate growth condition of $f$ on $G_\epsilon$. By Lemma 7.2(1) we have thus proved that the moderate growth condition holds on the whole group $G$. \(\square\)

When our automorphic form is bounded, its Fourier expansion has no term corresponding to the trivial character of $N$, which we want to call the constant term. From the proof of the theorem above we deduce the following corollary.

**Corollary 7.4.** For a positive number $\epsilon$ let $G_\epsilon$ be as in Lemma 7.2(3). The Fourier expansion of a bounded automorphic form generating $\pi_\kappa$ converges absolutely and uniformly on $G_\epsilon$ for any fixed $\epsilon \in \mathbb{R}_{>0}$.

**Remark 7.5.** As Theorems 5.6 and 5.7 indicate, there exist no generalized Whittaker functions of non-moderate growth for the case of $q > 1$. This leads to an essential reason why the Koecher principle holds. Actually the proofs of the Koecher principle for holomorphic automorphic forms start from showing non-appearance of terms of non-moderate growth in their Fourier series, and such non-appearance property enables us to deduce a good estimate of the norm of the Fourier series to yield the principle (cf. [7, Théorème 1, Corollaire de la Proposition 1], [23, Satz 1, Satz 2] and [25, Chapter 4, Section 1, Lemma 1]). However, due to the non-existence of such Whittaker functions mentioned above, all we have to do to prove the principle for our case is just to obtain such a good estimate of the Fourier series.
8. Comparison with another two definitions of automorphic forms generating $\pi_\kappa$

In this section we provide another two definitions of automorphic forms on $G$ generating $\pi_\kappa$. One of them is defined by means of the Schmid operator $D_\kappa$ and the other is given by using some reproducing kernel function. The latter one was introduced by Arakawa in [2,3]. For this section we remind the readers that we have fixed the invariant measure $dg$ of $G$ in (1.4).

First we think of the former definition.

**Definition 8.1.** We also call a $V_\kappa$-valued $C^\infty$-function $f$ on $G$ an automorphic form generating $\pi_\kappa$ with respect to $\Gamma$ if it satisfies:

(i) $f(\gamma g k) = \tau_\kappa(k)^{-1} f(g) \forall (\gamma, G, K) \in \Gamma \times G \times K$,

(ii) $D_\kappa \cdot f = 0$.

**Theorem 8.2.** Definitions 6.1 and 8.1 give the same automorphic forms.

**Proof.** According to Proposition 6.12, an automorphic form $f$ defined in Definition 6.1 satisfies $D_\kappa \cdot f = 0$. Conversely let $f$ be an automorphic form defined in Definition 8.1. We note that (6.5) is deduced from Proposition 6.7(2) and the condition $D_\kappa \cdot f = 0$. We then see that such $f$ has the Fourier expansion of the form in Theorem 6.3, which turned out to be a sum of generalized Whittaker functions for $\pi_\kappa$ in Section 6.3. Therefore, by an argument similar to the proof of Proposition 5.8, we verify that $f$ generates $\pi_\kappa$ as a $(g, K)$-module. \(\square\)

Next we consider Arakawa’s definition of the automorphic forms associated with $\pi_\kappa$ given in [2,3]. Before reviewing it we have to discuss the $\tau_\kappa$-spherical function $\omega_\kappa : G \to \text{End}(V_\kappa)$ explicitly given by Arakawa in [2,3] as follows:

$$\omega_\kappa(g) := \sigma_\kappa\left(D(g)^{-1} \nu(D(g))^{-1}\right) \quad (g \in G),$$

where $D(g) := \frac{1}{2}(\tau(g \cdot z_0) + 1)\mu(g, z_0)$ with

$$\tau(g \cdot z_0) := \begin{cases} 
\text{the second entry of } g \cdot z_0 \in \mathfrak{h} & (q > 1), \\
\text{ } & (q = 1)
\end{cases}$$

(see [2, Section 1] for the case $q = 1$ and [3, Section 3.3, (3.5)] for the case $q > 1$). In [3, Section 2.6] Arakawa remarked that the coefficients of $\omega_\kappa$ are regarded as matrix coefficients of $\pi_\kappa$. We now state the following.

**Lemma 8.3.** For $v \in V_\kappa$, $D_\kappa \cdot (\omega_\kappa(g) \cdot v) = 0$.

**Proof.** For the proof we need the following lemma, which is given in [34, Theorem 1.5].

**Lemma 8.4.** Let $\chi_\kappa$ be the infinitesimal character of the quaternionic discrete series $\pi_\kappa$ and $\Omega$ the Casimir element of $U(g)$. Then the two spaces

$$\left\{ F \in C^{\infty}_c(G/K) \mid \Omega \cdot F = \chi_\kappa(\Omega)F, \int_G \left\| F(g) \right\|_k^2 dg < \infty \right\},$$

(8.1)
\[
\left\{ F \in C^{\infty}_{\tau \kappa} (G/K) \mid \mathcal{D}_{\kappa} \cdot F = 0, \quad \int_{G} \| F(g) \|_{\kappa}^2 \, dg < \infty \right\}
\] (8.2)

coincide and the left regular representation of \(G\) on them realizes \((\pi_{\kappa}, H_{\kappa})\), where recall that \(\| * \|_{\kappa}\) denotes the norm of \(V_{\kappa}\) induced by the inner product \((*,*)_{\kappa}\) (for \((*,*)_{\kappa}\) see Section 2). Here see Section 5.1 for \(C^{\infty}_{\tau \kappa} (G/K)\).

According to [3, Section 2.6], \(\omega_{\kappa}(g) \cdot v\) belongs to the space in (8.1) for each \(v \in V_{\kappa}\). Then the assertion is an immediate consequence of this lemma. \(\square\)

We further introduce a constant
\[
c_{\kappa} := 2^{-2(q+1)} \pi^{-2q} (\kappa + 1 - 2q)_{2q} \quad \text{with} \quad (\kappa + 1 - 2q)_{2q} := (\kappa + 1 - 2q)(\kappa + 2 - 2q) \ldots \kappa.
\]
This constant \(c_{\kappa}\) is the quotient of a formal degree of \(\pi_{\kappa}\) by \(\dim V_{\kappa}\) (cf. [3, Section 2.6]).

Assume \(\kappa > 4q\), which means that \(\pi_{\kappa}\) and \(\omega_{\kappa}\) are integrable (cf. (3.1)). Under this assumption we state Arakawa’s definition of the automorphic forms.

**Definition 8.5.** For \(\kappa\) as above let \(\mathcal{A}_{0}(1 \setminus G, \omega_{\kappa})\) be the space of \(V_{\kappa}\)-valued continuous functions \(f\) on \(G\) satisfying:

(i) \(f\) is bounded on \(G\),
(ii) \(f(\gamma g k) = \tau_{\kappa}(k)^{-1} f(g) \) \(\forall (\gamma, g, k) \in \Gamma \times G \times K\),
(iii) \(c_{\kappa} \int_{G} \omega_{\kappa}(g^{-1} h) f(g) \, dg = f(h) \) \(\forall h \in G\).

Here we remark that the automorphic forms under this definition are cuspidal (cf. [3, Proposition 3.1]).

Before coming to another theorem of this section we need one more step. For a \(V_{\kappa}\)-valued continuous function \(\theta\) on \(\mathbb{H}^{q-1}\) satisfying the first defining condition of \(\Theta_{\xi, c}\), let \(W_{\theta} : G \to V_{\kappa}\) be defined by
\[
W_{\theta}(g) = \tau_{\kappa}(k)^{-1} \theta(w) y^{\kappa/2} \exp(-4\pi d(\xi) y) e((\text{tr} \xi) x) \quad (g = n(w, x) a k).
\]
This is bounded on \(G\) (when \(\kappa/2 + 1 > 0\)). Moreover, let \(\theta^{(i)}(w)\) be the coefficient of \(\theta(w)\) for \(\sigma_{\kappa}(u_{\xi}) v_{\kappa, i}\) with \(0 \leq i \leq \kappa\), where \(u_{\xi}\) denotes any fixed element in \(H^{(1)}\) such that \(u_{\xi} e_{2} u_{\xi}^{-1} = \xi / d(\xi)\). With the help of the Fourier transformation formula of \(\omega_{\kappa}\) in [2, Lemma 1.2] we obtain the following lemma by a direct computation.

**Lemma 8.6.** Let \(\kappa > 4q\). The condition \(c_{\kappa} \int_{G} \omega_{\kappa}(g^{-1} h) W_{\theta}(g) \, dg = W_{\theta}(h)\) holds if and only if
\[
\theta^{(i)}(w) = 0 \quad (0 \leq i \leq \kappa - 1), \quad \theta^{(\kappa)}(w) \in \Theta_{\xi, c}.
\]

Now we are ready to state the second theorem of this section.
Theorem 8.7. Let $\pi_\kappa$ be integrable. We denote by $A_0(\Gamma \backslash G, \pi_\kappa)$ (respectively $A_0(\Gamma \backslash G, D_\kappa)$) the space of bounded automorphic forms under Definition 6.1 (respectively Definition 8.1). Then we have

$$A_0(\Gamma \backslash G, \pi_\kappa) = A_0(\Gamma \backslash G, D_\kappa) = A_0(\Gamma \backslash G, \omega_\kappa).$$

Proof. Theorem 8.2 reduces the problem to showing only the second equation. Let $f \in A_0(\Gamma \backslash G, \omega_\kappa)$. Then $f$ is a $C^\infty$-function. In fact, $\omega_\kappa$ remains integrable under any infinitesimal actions by elements in $U(\mathfrak{g})$ since the integrability of one matrix coefficient of $\pi_\kappa$ implies such property of every matrix coefficient of $\pi_\kappa$ (cf. Proposition 3.1). Thereby the condition (iii) in Definition 8.5 admits exchanging of the integral over $G$ and any infinitesimal actions of $U(\mathfrak{g})$ with respect to $h$. Hence Lemma 8.3 implies $D_\kappa \cdot f = 0$. On the other hand, let $f \in A_0(\Gamma \backslash G, D_\kappa)$. Due to Corollary 7.4 the integral of the Fourier expansion of $f$ against $\omega_\kappa$ over $G$ admits exchanging of the sum and the integration. Therefore we see that $f$ satisfies the third condition in Definition 8.5 by virtue of its Fourier expansion and Lemma 8.6.

9. Remarks on the case of $Sp(1, 1)$

For the case of $G = Sp(1, 1)$ we can also define an automorphic form $f$ generating $\pi_\kappa$ by Definitions 6.1 or 8.1. However, only the defining conditions in Definitions 6.1 or 8.1 do not ensure the Koecher principle for the case of $q = 1$. This is due to two reasons as follows.

First we note that there is a non-zero Whittaker function rapidly increasing with respect to $A$ besides the non-zero rapidly decreasing one, as Theorem 5.7(1) indicates. If there appears such a function in the Fourier expansion the Koecher principle is not valid.

The second reason is the following. To deny the appearance of the rapidly increasing terms in the expansion we need a subgroup $\Gamma_0$ of $c^{-1} \Gamma_c$ such that:

- it is of infinite order modulo the center of $N_{\Gamma, c}$ ($= N_{\Gamma, c}$ when $q = 1$),
- it stabilizes the index set $X_{\Gamma, c}^*$ of the Fourier expansion, and each $\Gamma_0$-orbit in $X_{\Gamma, c}^* \setminus \{0\}$ has infinite order.

In fact, if there exists such $\Gamma_0$, we would be able to prove the non-appearance of rapidly increasing Whittaker functions in the expansion by following the proofs of the Koecher principle in [7, Théorème 1], [23, Satz 1, Satz 2] and [25, Chapter 4, Section 1, Lemma 1], etc. However, $Sp(1, 1)$ does not have such $\Gamma_0$. For this it is helpful to note that the structure of $Sp(1, 1)$ is similar to that of $SL_2(\mathbb{R})$ in that they have the same restricted root system of type $C_1 = A_1$. Moreover, we note that the definition of holomorphic modular forms on $SL_2(\mathbb{R})$ needs the moderate growth condition because of the non-existence of such $\Gamma_0$. Namely the lack of the Koecher principle for our present case is just like that for holomorphic forms on $SL_2(\mathbb{R})$.

Thus we have to impose the moderate growth condition on Definitions 6.1 and 8.1 for the case of $Sp(1, 1)$. Using the same notations as in Theorem 6.3, the Fourier expansion at a cusp $c \in \Xi$ of an automorphic form $f$ under such definition can be expressed as

$$f(cn(x)a) = \sum_{i=0}^\kappa C^f_i y^{k/2+1} v_{k,i} + \sum_{\xi \in X_{\Gamma, c}^* \setminus \{0\}} C^f_\xi y^{k/2+1} \exp(-4\pi d(\xi)y) e(tr(\xi x)) \sigma_k(u_\xi) v_{k,\kappa},$$

where the constant $C^f_\xi$ depends only on $(\xi, f)$. 

When we assume the boundedness of $f$ and the integrability of $\pi_\kappa$ we can also define our automorphic forms by means of the reproducing kernel function $\omega_\kappa$, as in Definition 8.5. Under such assumptions we can similarly prove Corollary 7.4. Then, by the same reasoning as in the case of $q > 1$, we see that Theorem 8.7 is also valid for the case of $q = 1$.

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