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Statistical inference for detrended point processes

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We consider a multivariate point process with a parametric intensity process which splits into a stochastic factor b_t and a trend function a_t of a squared polynomial form with exponents larger than $-\frac{1}{2}$. Such a process occurs in a situation where an underlying process with intensity b_t can be observed on a transformed time scale only. On the basis of the maximum likelihood estimator for the unknown parameter a detrended (or residual) process is defined by transforming the occurrence times via integrated estimated trend function. It is shown that statistics (mean intensity, periodogram estimator) based on the detrended process exhibit the same asymptotic properties as they do in the case of the underlying process (without trend function). Thus trend removal in point processes turns out to be an appropriate method to reveal properties of the (unobservable) underlying process – a concept which is well established in time series. A numerical example of an earthquake aftershock sequence illustrates the performance of the method.

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multivariate point process * intensity process * trend component * detrending * residual process * periodogram estimator

1. Introduction

In time-series analysis the method of estimating and removing a trend and of analyzing the detrended series is well established. Let a time series model

$$X_t = a_t(\alpha) + Z_t, \quad t=0, 1, \dots ,$$

be given where

$$a_t(\alpha) = \sum_{j=0}^{q} \alpha_j t^{\gamma(j)}, \quad \gamma(j) > -\tfrac{1}{2} ,$$

is a trend component and Z_t, $t=0, 1, \dots$, a stationary linear process (MA(∞) process). The detrended (or residual) time series is formed by

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$$\hat{Z}_t = X_t - a_t(\hat{\alpha}_T), \quad t=0, 1, \ldots ,$$

with $\hat{\alpha}_T$ being an estimator (e.g. least squares estimator) of α based on the time interval $[0, T]$. Properties of the (unobservable) time series Z_t are intended to be revealed by using the detrended series $\hat{Z}_t$, at least asymptotically. Some results support this intention. Take as an example the periodogram estimator

$$I_n(\omega) = \left| \sum_{t=1}^{n} Z_t e^{-i\omega t} \right|^2 / (\pi n), \quad \omega \in (0, \pi) ,$$

which fulfills under some conditions (Fuller, 1976, Theorem 7.1.2)

$$I_n(\omega)/f(\omega) \xrightarrow{\mathbb{D}} E \quad \text{(Exponential with parameter 1)} .$$

As a substitute for the unavailable estimator $I_n(\omega)$ we use

$$\hat{I}_{T,n}(\omega) = \left| \sum_{t=1}^{n} \hat{Z}_t e^{-i\omega t} \right|^2 / (\pi n), \quad \omega \in (0, \pi) ,$$

based on the detrended series $\hat{Z}_t = X_t - a_t(\hat{\alpha}_T)$. From Fuller (1976, Section 9.3) one derives

$$\hat{I}_{T,n}(\omega) - I_n(\omega) \xrightarrow{\mathbb{P}} 0 \quad \text{as } n, T \to \infty, T/n \to \infty ,$$

such that for this limit

$$\hat{I}_{T,n}(\omega)/f(\omega) \xrightarrow{\mathbb{D}} E .$$

That means, that the estimator based on the detrended series exhibits the same limit law as the estimator based on the (unobservable) stationary time series. In what follows we are concerned with related problems in multivariate point processes where – contrary to time series – the state space is discrete and the time parameter continuous. A trend in a point process is defined by a trend function multiplied to the intensity, and a trend is removed by transforming the occurrence times via the estimated integrated trend function. Multiplying a trend function a_t to the intensity amounts to a time change $t \to \sigma(t)$ (σ the inverse function of $A(t) = \int_0^t a_s \, ds$, $t \geqslant 0$). Hence it affects the distribution of the intervals between events and so the whole probability structure of the process. Thus the related time-series methods cannot be transferred to our point process context.

The standpoint of the paper can be sketched in this way: The observed point process is obtained from an (unobservable) underlying process by a transformation of the time scale. Asymptotical statistical results which apply to the underlying process (in its own time scale) are no longer valid for the observed process. The paper will show, however, that these results will also apply to the observed process after estimating the trend a_t (by $\hat{a}_t$, say) and recovering the original time scale sufficiently accurately by a time change $t \to \hat{A}(t) = \int_0^t \hat{a}_s \, ds$.

The main results of this paper are the following: Some asymptotic properties of point process statistics (mean intensity estimator, periodogram estimator) which are known to hold true for the underlying model are established for the detrended point process (see Proposition 5, Theorems 6 and 8). At least in the case of an underlying Poisson process we obtain qualitatively the same results as mentioned above for time series. In the proofs a result on the transformation of point process distributions is steadily used, which is stated in Theorem 4 below and was proved in Pruscha (1988), a paper to which we will refer by [P] in the following.

To derive our results some assumptions are to be made: (i) The evolution of the underlying process depends on the history of types (marks) of events, not on the history of occurrence times. (ii) The underlying process obeys an ergodic-type law, while stationarity is not explicitly demanded. (iii) The trend coefficients are to be calculated from a larger time interval than the statistics in consideration.

2. Estimation of a parametric trend function

We start with a canonical multivariate point process model

$$N(\beta):\quad (N_t, \mathscr{F}, \mathbb{P}_\beta) \text{ with intensity } b_t(\beta)$$

where $N_t = (N_{i,t}, i \in I)$, $t \geq 0$, I finite set, is a multivariate counting process on the canonical space $(\Omega, \mathscr{F})$ of all multivariate point process realizations,

$$b_t(\beta) = (b_{i,t}(\beta), i \in I), \quad t \geq 0, \ \beta \in \mathbb{R}^d,$$

fulfills the requirements of an intensity process (nonnegative, left-continuous, adopted to the N_t history) and $\mathbb{P}_\beta$ is the corresponding probability on $(\Omega, \mathscr{F})$ according to the existence theorem of Jacod (1975). We introduce the following assumptions on $N(\beta)$ (writing $c_+ = \Sigma_I c_i$ and denoting by (τ_n, ξ_n), $n \geq 1$, the double sequence of occurrence times and types of event):

(A0) $b_{+,t} > 0$ and $b_{i,t} = b_i^{(n)}(\xi_1, \ldots, \xi_n)$ for $\tau_n < t \leq \tau_{n+1}$.
(A1) $(1/t)\int_0^t b_{+,s}\, ds \to \mu$ ($\mathbb{P}_\beta$-a.s., $t \to \infty$), where $\mu = \mu(\beta)$ is a positive constant.

(A0) is a jump-type condition and says that on the nth interval $b_{i,t}$ depends only on the earlier types not on the earlier occurrence times. (A1) is an ergodic-type law implying $N_{+,t}/t \to \mu$ ($\mathbb{P}_\beta$ − a.s.) by Lepingle's (1978) strong convergence results.

(A2) $b_{i,t}(\beta)$ and $\log b_{i,t}(\beta)$ have continuous second-order derivatives w.r.t. $\beta \in \mathbb{R}^d$, the derivatives form left-continuous processes; the differentiations $d/d\beta$ and $d^2/d\beta^2$ are interchangeable with the integral $\Sigma_i \int_0^t b_{i,s}(\beta)\, ds$.

Let a function (trend function) $a_t(\alpha) = (p_t(\alpha))^2$, $t \geq 0$, $\alpha \in \mathbb{R}^{q+1}$, be given with

$$p_t(\alpha)=\sum_{j=0}^{q}\alpha_j t^{\gamma(j)}, \quad -\tfrac{1}{2}<\gamma(0)<\cdots<\gamma(q),\ \alpha_q>0\,,$$

and a canonical multivariate point process model

$$N(\alpha,\beta):\quad (N_t,\mathscr{F},\mathbb{P}_{\alpha,\beta}) \text{ with intensity } \lambda_t(\alpha,\beta)\,,$$

where

$$\lambda_t(\alpha,\beta)=(\lambda_{i,t}(\alpha,\beta),\, i\in I),\quad t\geqslant 0,\ \lambda_{i,t}(\alpha,\beta)=a_t(\alpha)b_{i,t}(\beta)\,.$$

$N(\beta)$ and $N(\alpha,\beta)$ play the roles of the underlying and of the trend affected point process model, respectively. In the following we will write $\vartheta=(\alpha,\beta)\in\mathbb{R}^{q+1+d}$. (A2) allows to define a log likelihood function $l_t(\vartheta)$, the score vector $J_t(\vartheta)=\mathrm{d}l_t(\vartheta)/\mathrm{d}\vartheta$ of dimension $q+1+d$ and the $(q+1+d)\times(q+1+d)$ matrix $K_t(\vartheta)=\mathrm{d}^2 l_t(\vartheta)/\mathrm{d}\vartheta^2$ of second order derivatives.

As a consequence of (A0) and (A1) we have in the $N(\alpha,\beta)$ model

$$\lim_t N_{+,t}/t^{2r+1}=\lim_n n/\tau_n^{2r+1}=h^2,\quad \mathbb{P}_\vartheta\text{-a.s.},\quad h^2=\alpha_q^2\mu/(2r+1)\,, \tag{1}$$

cf. [P, Lemma 2.1] where $r=\gamma(q)$ is the largest exponent in $p_t(\alpha)$. We will call an N_t-adapted process $\hat{\vartheta}_t\in\mathbb{R}^{q+1+d}$ a consistent m.l. estimator for ϑ, if for each $\varepsilon>0$,

$$\mathbb{P}_\vartheta(\,|\hat{\vartheta}_t-\vartheta|\leqslant\varepsilon,\ J_t(\hat{\vartheta}_t)=0)\to 1 \tag{2}$$

as $t\to\infty$. Defining the $(q+1)\times(q+1)$ and $d\times d$ matrices

$$D_t^{(\alpha)}=\mathrm{Diag}(t^{\gamma(j)+1/2}, j=0,\ldots,q)\,,\qquad D_t^{(\beta)}=\mathrm{Diag}(t^{r+1/2})$$

as well as the $(q+1+d)\times(q+1+d)$ matrix

$$D_t=\mathrm{Diag}(D_t^{(\alpha)},D_t^{(\beta)})$$

we can formulate the following assumption:

(A3) There exists a consistent m.l. estimator $\hat{\vartheta}_T=(\hat{\alpha}_T,\hat{\beta}_T)$ for $\vartheta=(\alpha,\beta)$ in the sense of (2) and asymptotic normality holds in the form

$$D_T(\hat{\vartheta}_T-\vartheta)\overset{\mathbb{D}_\vartheta}{\to}\mathrm{N}(0,V) \tag{3}$$

as $T\to\infty$, with some positive-definite matrix $V=V(\vartheta)$.

Remarks. 1. Examples of (underlying) $N(\beta)$ models where (A0)–(A3) are fulfilled are
– irreducible positive-recurrent Markov processes with finite state space;
– certain Markov branching processes with random immigration rate;
– certain linear OM (learning) processes;
cf. [P, Section 5]. Markov renewal processes, however, are excluded, since (A0) requires a piecewise constant intensity function.

2. By using the square of the trend polynomial $p_t(\alpha)$, the intensity λ_t remains nonnegative and the asymptotic analysis seems to become somewhat easier.

3. Three lemmas on the estimated trend

For the integrated trend function

$$A(t)=\int_0^t a_s(\alpha)\ \mathrm{d}s=\sum_{0\leqslant j\leqslant k\leqslant q} c_{jk}\alpha_j\alpha_k t^{\gamma(j)+\gamma(k)+1}$$

we introduce the estimator

$$\hat{A}_T(t)=\int_0^t a_s(\hat{\alpha}_T)\ \mathrm{d}s\ .$$

Lemma 1. *Let* (A0)–(A3) *be fulfilled. We have for* $T=T_n$ *with* $T_n/\tau_n \xrightarrow{\mathbb{P}_\vartheta} \infty$,

$$(\hat{A}_T(\tau_n)-A(\tau_n))/\sqrt{n} \xrightarrow{\mathbb{P}_\vartheta} 0 \quad (n\to\infty)\ .$$

Proof. One calculates by using the abbreviation $\hat{\Delta}_{j,k,T}=\hat{\alpha}_{j,T}\hat{\alpha}_{k,T}-\alpha_j\alpha_k$,

$$(\hat{A}_T(\tau_n)-A(\tau_n))/\sqrt{n}=\sum_{0\leqslant j\leqslant k\leqslant q} c_{jk}\hat{\Delta}_{j,k,T}T^{\gamma(j)+1/2}f_{j,k}(n)$$

with

$$f_{j,k}(n)=\tau_n^{\gamma(j)+\gamma(k)+1}/(T^{\gamma(j)+1/2}n^{1/2})\ .$$

Since from (1) and the assumption on T_n/τ_n,

$$f_{j,k}(n) \xrightarrow{\mathbb{P}_\vartheta} 0 \quad (n\to\infty)\ ,$$

one derives by using $\hat{\Delta}_{j,k,T}=(\hat{\alpha}_{j,T}-\alpha_j)\hat{\alpha}_{k,T}+(\hat{\alpha}_{k,T}-\alpha_k)\alpha_j$ and (3) the assertion. □

To derive results on bounded expectations we have to stipulate further conditions. Let $t_0\geqslant 1$, $\beta_0>0$, $c_0<\infty$ and for $t\to\infty$:

(A4) (i) $(1/t)\int_0^t \mathbb{E}_\beta b_{+,s}(\beta)\ \mathrm{d}s\to\mu$;
(ii) $b_{+,s}(\beta_t^*)\geqslant\beta_0$ for all $t_0\leqslant s\leqslant t$, $\mathbb{P}_\beta$-a.s.;
(iii) $(1/t)\int_0^t \mathbb{E}_\beta[\sqrt{t}(b_{+,s}(\beta_t^*)-b_{+,s}(\beta)]^2\ \mathrm{d}s\to c_0$;

where in (ii) and (iii) β_t^*, $t\geqslant 0$, is a d-dimensional process with $\beta_t^*=\beta^*(N_s, 0\leqslant s\leqslant t)$ and with $\sqrt{t}(\beta_t^*-\beta)$, $t\geqslant 0$, stochastically bounded w.r.t. $\mathbb{P}_\beta$.

Note that condition (A4) is formulated in terms of the underlying $N(\beta)$ model according to the general standpoint adopted in this paper. With the same technique as in the proof of [P, Lemma 2.2] – which makes use heavily of Theorem 4 below and a Cesaro convergence argument [P, Appendix A] – one deduces from (A4) for the $N(\vartheta)$ model (note that the $\mathbb{P}_\beta$-boundedness of $t^{1/2}(\beta_t^* - \beta)$ carries over to the $\mathbb{P}_\vartheta$-boundedness of $t^{r+1/2}(\beta_t^* - \beta)$, $t \geq 0$, such that $\hat{\beta}_t$ can be chosen as β_t^*)

$$\frac{1}{t}\int_0^t \mathbb{E}_\vartheta b_{+,s}(\beta)\ \mathrm{d}s \to \mu\,, \tag{4}$$

$$b_{+,x}(\hat{\beta}_t) \geq \beta_0 \quad \text{for all } t_0 \leq s \leq t, \quad \mathbb{P}_\vartheta\text{-a.s.}$$

$$(t_0 \text{ possibly larger than in (A4)})\,, \tag{5}$$

$$\frac{1}{t}\int_0^t \mathbb{E}_\vartheta B_{s,t}^2\ \mathrm{d}s \to c_0', \quad \text{where } B_{s,t} = t^{r+1/2}(b_{+,s}(\hat{\beta}_t) - b_{+,s}(\beta))\,. \tag{6}$$

For convenience our last condition will be formulated directly in the $N(\vartheta)$ model. The superscript (α) will refer to the components $(\alpha_0, \ldots, \alpha_q)$ of ϑ.

(A5) $\mathbb{E}_\vartheta(J_{j,t}^{(\alpha)}(\hat{\vartheta}_t))^2 \leq C\,\mathbb{E}_\vartheta(J_{j,t}^{(\alpha)}(\alpha, \hat{\beta}_t))^2$ for all $t \geq t_0, j = 0, \ldots, q$ (C finite constant).

Condition (A5) is fulfilled, if

$$J_t^{(\alpha)}(\vartheta) = \left(2\int_0^t s^{\gamma(j)}/p_s(\alpha)(\mathrm{d}N_{+,s} - p_s^2(\alpha)b_{+,s}(\beta)\ \mathrm{d}s)\right)_j$$

vanishes at $\hat{\vartheta}_t$ $\mathbb{P}_\vartheta$–a.s. Using (5) and a quadratic form argument we conclude the a.s. invertibility of the matrix

$$K_t^{(\alpha,\alpha)}(\vartheta) = \left(-2\int_0^t s^{\gamma(j)+\gamma(k)}(\mathrm{d}N_{+,s}/a_s(\alpha) + b_{+,s}(\beta)\ \mathrm{d}s)\right)_{jk}$$

for $t \geq t_0$. The stochastic boundedness expressed by (3) can now be sharpened w.r.t. the α components.

Lemma 2. *Under* (A0)–(A5) *we have a finite constant C_0 such that for all $T \geq t_0$,*

$$\mathbb{E}_\vartheta[\,|\hat{\alpha}_{j,T} - \alpha_j|\,T^{\gamma(j)+1/2}]^2 \leq C_0\,.$$

Proof. (i) Letting (within this proof only) $\vartheta = (\alpha, \hat{\beta}_T)$, $\vartheta(u) = (\alpha(u), \hat{\beta}_T)$, $\alpha(u) = \alpha + u(\hat{\alpha}_T - \alpha)$, and $\hat{\vartheta}_T = (\hat{\alpha}_T, \hat{\beta}_T)$ as usually. Then we have from the fundamental theorem

$$J_T^{(\alpha)}(\hat{\vartheta}_T) = J_T^{(\alpha)}(\vartheta) + W_T^{(\alpha,\alpha)}(\hat{\vartheta}_T, \vartheta)(\hat{\alpha}_T - \alpha) \tag{7}$$

where $W_T^{(\alpha,\alpha)}(\hat{\vartheta}_T, \vartheta) = \int_0^1 K_T^{(\alpha,\alpha)}(\vartheta(u))\ \mathrm{d}u$ is a symmetric, invertible matrix (read the integral component-wise). Hence from (7) $\mathbb{P}_\vartheta$-a.s.

$$T^{\gamma(i)+1/2}(\hat{\alpha}_{i,T}-\alpha_i)=\sum_{j=0}^{q} C_{ij,T}(D_{j,T}-\hat{D}_{j,T}),\quad i=0,\ldots,q\,, \tag{8}$$

where

$$C_{ij,T}=-T^{\gamma(i)+\gamma(j)+1}[W_T^{(\alpha,\alpha)}(\hat{\vartheta}_T,\vartheta)]_{ij}^{-1}\,,$$
$$D_{j,T}=T^{-(\gamma(j)+1/2)}(J_T^{(\alpha)}(\vartheta))_j\,,$$

$\hat{D}_{j,T}$ as $D_{j,T}$ with ϑ replaced by $\hat{\vartheta}_T$.

(ii) For the term $D_{j,T}$ we calculate

$$\begin{aligned}\mathbb{E}_\vartheta D_{j,T}^2&=4T^{-2\gamma(j)-1}\mathbb{E}_\vartheta\left\{\int_0^T (s^{\gamma(j)}/p_s(\alpha))(\mathrm{d}N_{+,s}-p_s^2 b_{+,s}(\hat{\beta}_T)\,\mathrm{d}s)\right\}^2\\&\leqslant 8T^{-2\gamma(j)-1}\mathbb{E}_\vartheta\left\{\int_0^T (s^{\gamma(j)}/p_s(\alpha))(\mathrm{d}N_{+,s}-p_s^2 b_{+,s}(\beta)\,\mathrm{d}s)\right\}^2\\&\quad+8T^{-2\gamma(j)-1}\mathbb{E}_\vartheta\left\{\int_0^T s^{\gamma(j)}p_s(\alpha)(b_{+,s}(\beta)-b_{+,s}(\hat{\beta}_T))\,\mathrm{d}s\right\}^2\\&=I_T+II_T,\quad\text{say}\,.\end{aligned}$$

By a well-known property of squared point process integrals we have

$$I_T=8T^{-2\gamma(j)-1}\int_0^T s^{2\gamma(j)}\mathbb{E}_\vartheta b_{+,s}(\beta)\,\mathrm{d}s$$

and thus $I_T\to$ const by using (4) and a Cesaro convergence argument. By using Jensen's inequality, the quantity $B_{s,T}$ of (6) and $\gamma=8/(\gamma(j)+\frac{1}{2})$ we can write

$$\begin{aligned}II_T&\leqslant\gamma\mathbb{E}_\vartheta T^{-\gamma(j)-1/2}\int_0^T s^{\gamma(j)-1/2}\{s^{1/2}p_s(\alpha)(b_{+,s}(\beta)-b_{+,s}(\hat{\beta}_T))\}^2\,\mathrm{d}s\\&\leqslant\gamma T^{-\gamma(j)-1/2}\int_0^T s^{\gamma(j)-1/2}\{s^{-r}p_s(\alpha)\}^2\mathbb{E}_\vartheta B_{s,T}^2\,\mathrm{d}s\end{aligned}$$

the right-hand side converging towards some constant (use (6), $p_s(\alpha)/s^r\to\alpha_q$, and once again a Cesaro convergence argument).

(iii) Denoting by c_1, c_2, c_3 finite, positive constants, by $D_T^{(\alpha)}$ the diagonal matrix introduced above, and letting

$$C_T=-D_T^{(\alpha)}[W_T^{(\alpha,\alpha)}(\hat{\vartheta}_T,\vartheta)]^{-1}D_T^{(\alpha)}\,,$$
$$\mathrm{d}R_s(u)=(1/a_s(\alpha(u)))\,\mathrm{d}N_{+,s}+b_{+,s}(\hat{\beta}_T)\,\mathrm{d}s\,,$$

one calculates by using (5) and assuming $\frac{1}{2}T\geqslant t_0$, that $\mathbb{P}_\vartheta$-a.s.

$$(C_{ij,T})^2\leqslant c_1\lambda_{\max}^2(C_T)=c_1\lambda_{\min}^{-2}(C_T^{-1})=c_1\left[\min_{|a|=1}a^T C_T^{-1}a\right]^{-2}$$

$$=c_1\left[2\min\frac{1}{T}\sum_j\sum_k a_j a_k T^{-\gamma(j)-\gamma(k)}\int_0^1\int_0^T s^{\gamma(j)+\gamma(k)}\,\mathrm{d}R_s(u)\,\mathrm{d}u\right]^{-2}$$

$$=c_1\left[2\min\frac{1}{T}\int_0^1\int_0^T\left(\sum_j a_j T^{-\gamma(j)}s^{\gamma(j)}\right)^2\mathrm{d}R_s(u)\,\mathrm{d}u\right]^{-2}$$

$$\leqslant c_2\left[\min\frac{1}{T}\int_{T/2}^T\left(\sum_j a_j(s/T)^{\gamma(j)}\right)^2 b_{+,s}(\hat{\beta}_T)\,\mathrm{d}s\right]^{-2}$$

$$\leqslant c_3\left[\min\int_{1/2}^1\left(\sum_j a_j u^{\gamma(j)}\right)^2\mathrm{d}u\right]^{-2}\leqslant c_4\,.$$

(iv) The assertion follows now from (8) by using (ii), (iii) as well as (A5). □

Lemma 3. *Under* (A0)–(A5) *we have for* $T\geqslant t_0$ *and* $\tau_n'=\max(\tau_n,1)$,

$$\mathbb{E}_\vartheta(T^{\gamma(0)+1/2}/\tau_n'^{2r+1})\,|\hat{A}_T(\tau_n)-A(\tau_n)|\leqslant C_1 \tag{9}$$

where C_1 *is some finite constant.*

Proof. Writing I_ϑ for the left-hand side of (9), we deduce from Lemma 2 by making use of

$$\hat{\Delta}_{j,k,T}=\hat{\alpha}_{j,T}\hat{\alpha}_{k,T}-\alpha_j\alpha_k=(\hat{\alpha}_{j,T}-\alpha_j)\hat{\alpha}_{k,T}+(\hat{\alpha}_{k,T}-\alpha_k)\alpha_j$$

that

$$I_\vartheta\leqslant\sum_{j\leqslant k}\sum c_{jk}\mathbb{E}_\vartheta|\hat{\Delta}_{j,k,T}|\,T^{\gamma(j)+1/2}f_{j,k}\leqslant\sum_{j\leqslant k}\sum c_{jk}C_2 f_{j,k}$$

where C_2 is some finite constant and

$$f_{j,k}=\tau_n'^{\gamma(j)+\gamma(k)+1}T^{\gamma(0)-\gamma(j)}/\tau_n'^{2r+1}\leqslant 1\,. \quad \square$$

4. Analysis of the detrended process

As suggested by Lewis (1972) in the case of inhomogeneous Poisson processes (i.e. the case $b_t=1$), we define a detrended (or residual) process by the double sequence $(\hat{\tau}_n,\hat{\xi}_n)$, $n\geqslant 1$ where $\hat{\xi}_n=\xi_n$ and

$$\hat{\tau}_n=\hat{A}_T(\tau_n)\,. \tag{10}$$

The counting process belonging to (10) is $\hat{N}_{+,t}=N_{+,\hat{\sigma}(t)}$, where $\hat{\sigma}(t)=\hat{\sigma}_T(t)$, $t\geqslant 0$, is the inverse function of $\hat{A}_T(t)$, $t\geqslant 0$.

For the following analysis we need a result on the transformation of point process distributions. If $\mathbb{D}_\beta$ and $\mathbb{D}_\vartheta$ denote as above distributions w.r.t. $\mathbb{P}_\beta$ and $\mathbb{P}_\vartheta$, respectively, and $\sigma(t)$ the inverse function of $A(t)$, $t \geqslant 0$, then we can prove (cf. [P, Appendix, Corollary B. 3]) the following two equivalent statements:

Theorem 4. *Under* (A0) *we have*

$$\mathbb{D}_\beta(\tau_n, n \geqslant 1) = \mathbb{D}_\vartheta(A(\tau_n), n \geqslant 1) ,$$

$$\mathbb{D}_\beta(N_{+,t}, t \geqslant 0) = \mathbb{D}_\vartheta(N_{+,\sigma(t)}, t \geqslant 0) .$$

Remarks. 1. Note that the detrended occurrence times (10) are intended to estimate $A(\tau_n)$.

2. In this theorem $a_t(\alpha)$, $t \geqslant 0$, is allowed to be a random process; thus this theorem generalizes a well-known result on transforming a univariate point process to a Poisson process.

4.1. Estimation of the model parameter μ

The reciprocal $\nu = 1/\mu$ of the final intensity μ, introduced by (A1), is usually estimated by $\nu_n = \tau_n / n$. For many point processes $N(\beta)$ we have

$$\sqrt{n}(\nu_n - \nu) \overset{\mathbb{D}_\beta}{\rightarrow} \mathrm{N}(0, \sigma_\nu^2), \quad \sigma_\nu^2 > 0 , \tag{11}$$

cf. Brillinger (1975, p. 73). As an estimator of ν, based on the detrended process (10), we introduce

$$\hat{\nu}_{T,n} = \hat{\tau}_n / n = \hat{A}_T(\tau_n) / n .$$

Proposition 5. *Let* (A0)–(A3) *as well as* (11) *be fulfilled. Then we have for* $T = T_n$ *with* $T_n/\tau_n \overset{\mathbb{P}_\vartheta}{\rightarrow} \infty$ *as* $n \rightarrow \infty$,

$$\sqrt{n}(\hat{\nu}_{T,n} - \nu) \overset{\mathbb{D}_\vartheta}{\rightarrow} \mathrm{N}(0, \sigma_\nu^2) . \tag{12}$$

Proof. From (11) we obtain by using Theorem 4,

$$\sqrt{n}(A(\tau_n)/n - \nu) \overset{\mathbb{D}_\vartheta}{\rightarrow} \mathrm{N}(0, \sigma_\nu^2) .$$

Assertion (12) is then an immediate consequence of Lemma 1. □

4.2. Periodogram estimator

The periodogram estimator for point processes is usually defined by

$$I_t(\omega) = \left| \int_0^t \mathrm{e}^{-\mathrm{i}\omega s} \, \mathrm{d}N_s \right|^2 / (\pi t) = \left| \sum_{k=1}^{N_t} \mathrm{e}^{-\mathrm{i}\omega\tau_k} \right|^2 / (\pi t)$$

or

$$I_{(n)}(\omega)=\left|\sum_{k=1}^{n} e^{-i\omega\tau_k}\right|^2/(\pi n)=I_{\tau_n}(\omega)\tau_n/n\,,$$

where from now on N_t stands for $N_{+,t}$ and $\hat{N}_t$ for $\hat{N}_{+,t}$. For many stationary point processes $N(\beta)$ with a spectral density $f(\omega)>0$ we have cf. Brillinger (1975, p. 83) or Cox and Lewis (1966, p. 127) as $t\to\infty$ and $n\to\infty$, respectively

$$(I_t(\omega)/f(\omega),\,I_t(\omega')/f(\omega'))\overset{\mathbb{D}_\beta}{\to}(E,E')\quad(\omega\neq\omega')\,,\tag{13}$$

where E and E' are two independent exponentials, as well as

$$\text{law (13) with } I_t(\omega) \text{ replaced by } \mu I_{(n)}(\omega)\,.\tag{14}$$

We introduce the periodogram estimator based on the detrended process (10),

$$\hat{I}_{T,t}(\omega)=\left|\sum_{k=1}^{\hat{N}_t} e^{-i\omega\hat{\tau}_k}\right|^2/(\pi t),\qquad \hat{I}_{T,(n)}(\omega)=\left|\sum_{k=1}^{n} e^{-i\omega\hat{\tau}_k}\right|^2/(\pi n)\,.$$

Theorem 6. *Let* (A0)–(A5) *as well as* (14) *be fulfilled and* $d>3/(2\gamma(0)+1)$. *Then we have for* $T=n^d$ *as* $n\to\infty$,

$$(\hat{I}_{T,(n)}(\omega)/f(\omega),\,\hat{I}_{T,(n)}(\omega')/f(\omega'))\overset{\mathbb{D}_\vartheta}{\to}(E,E')/\mu\quad(\omega\neq\omega')\,.\tag{15}$$

Proof. As a consequence of Theorem 4 we have

$$\mathbb{D}_\beta\left(\sum_{k=1}^{n} e^{-i\omega\tau_k},\ \omega>0\right)=\mathbb{D}_\vartheta\left(\sum_{k=1}^{n} e^{-i\omega\tilde{\tau}_k},\ \omega>0\right),$$

where we have set $\tilde{\tau}_k=A(\tau_k)$. Hence, with respect to the inequality $|a|^2-|b|^2\leqslant|a-b|(|a|+|b|)$ it suffices to show that

$$\sum_{k=1}^{n}|e^{-i\omega\hat{\tau}_k}-e^{-i\tilde{\tau}_k}|/\sqrt{n}\overset{\mathbb{P}_\vartheta}{\to}0$$

or that $\hat{D}_{T,n}\overset{\mathbb{P}_\vartheta}{\to}0$ where

$$\hat{D}_{T,n}=\sum_{k=1}^{n}|e^{-i\omega\tilde{\tau}_k}-e^{-i\omega\hat{\tau}_k}|k/(\sqrt{n}\,\tau_k'^{2r+1}),\quad \tau_k'=\max(1,\tau_k)$$

(note that $\tau_k'^{2r+1}/k\to1/h^2$ $\mathbb{P}_\vartheta$-a.s. due to (1)). Now using $|e^{ix}-e^{iy}|\leqslant|x-y|$ and Lemma 3 we get

$$\mathbb{E}_\vartheta \hat{D}_{T,n} \leqslant \omega \sum_{k=1}^{n} \mathbb{E}_\vartheta |A(\tau_k) - \hat{A}_T(\tau_k)| T^{\gamma(0)+1/2} f_{k,n} / (\sqrt{n}\, \tau_k^{\prime 2r+1})$$

$$\leqslant \omega C_1 \frac{1}{\sqrt{n}} \sum_{k=1}^{n} f_{k,n} \to 0 \quad (n \to \infty)$$

where we have set $f_{k,n} = k/n^{d(\gamma(0)+1/2)}$. □

Remarks. 1. Periodogram analysis in detrended point processes should not be used for the search for hidden periodicities (note that the time scale is transformed) but to test the observed process against hypothetical point process models $N(\beta)$.

2. Writing $a_n \sim b_n$ if $a_n/b_n \to c_0$ $(0 < c_0 < \infty)$ we have from (1) $\mathbb{P}_\vartheta$-a.s.

$$T = n^d \sim \tau_n^c, \quad c = d(2r+1) .$$

Since $c > 3$ under the assumptions of Theorem 6 the time interval $[0, T]$ for estimating α runs far ahead of the time interval $[0, \tau_n]$ for calculating the periodogram. In time-series analysis (see Section 1) we have $c > 1$ and thus a much better result. Note, however, that in Section 1 a linear time-series model was assumed, while in Theorem 6 a rather general point process model is allowed.

3. For Poisson processes the results are much better (in comparison with Theorem 6) as the next theorem will show (namely $c > f + 1/(2r+1)$ with $f = 1$ or 2 if $\gamma(0) = \frac{1}{2}$ or 0, respectively). Note that the proof of Theorem 8 leans heavily on the structural properties of the Poisson process and allows for the sake of simplicity only the case $r = \gamma(q) > 0$.

4. The difference between Theorems 6 and 8 reflects the fact that there is a lack of an intermediate model which the linear model represents in time-series analysis.

4.3. Periodogram estimator (Poissonian case)

Now we consider as underlying model $N(\beta)$ the model NP(1) of a homogeneous Poisson process with rate 1. The notation $\mathbb{E}_\beta$, $\mathbb{D}_\beta$ etc. now refers to the NP(1) model, while $\mathbb{E}_\vartheta$, $\mathbb{D}_\vartheta$ etc. pertain to the trend-affected NP(1) model, i.e. an inhomogeneous Poisson process with rate function $a_t(\alpha)$.

Define the random element $F(t, T)$ from $\bigcup_n \mathbb{R}^n$ by

$$F(t, T) = (\tau_{N_t+1}, \ldots, \tau_{N_T}) \quad (t < T) .$$

Lemma 7. *For the* NP(1) *model we have*

$$\mathbb{E}_\beta \sum_{k=1}^{N_t} \Psi(\tau_k, F(t, T)) = \mathbb{E}_\beta \int_0^t \Psi(u, F(t, T))\, du ,$$

where $t < T$ and Ψ is a nonnegative measurable function on $\mathbb{R} \times \bigcup_n \mathbb{R}^n$.

Proof. Writing $\hat{\Psi}(u)$ instead of $\Psi(u, f)$, $f \in \bigcup_n \mathbb{R}^n$ and making use of the independence of $F(0, t)$ and $F(t, T)$, of Corollary 4.38 in Breiman (1968) and of the uniform conditional property we get

$$\mathbb{E}_\beta \sum_{k=1}^{N_t} \Psi(\tau_k, F) = \mathbb{E}_F \mathbb{E}_\beta\left(\sum_{k=1}^{N_t} \hat{\Psi}(\tau_k)\right)$$

$$= \mathbb{E}_F \sum_{n=0}^{\infty} \mathbb{P}_\beta(N_t = n) \mathbb{E}_G\left(\sum_{k=1}^{N_t} \hat{\Psi}(\tau_k) \;\middle|\; N_t = n\right)$$

$$= \mathbb{E}_F \sum_{n=0}^{\infty} \mathbb{P}_\beta(N_t = n) \sum_{k=1}^{n} \mathbb{E}_U \hat{\Psi}(U_k t)$$

$$= \mathbb{E}_F \sum_{n=0}^{\infty} \mathbb{P}_\beta(N_t = n) n \int_0^1 \hat{\Psi}(ut)\, \mathrm{d}u$$

$$= \mathbb{E}_F t \int_0^1 \hat{\Psi}(ut)\, \mathrm{d}u = \mathbb{E}_\beta \int_0^t \Psi(u, F)\, \mathrm{d}u$$

where $\mathbb{E}_U$, $\mathbb{E}_F$, $\mathbb{E}_G$ pertains to the expectation w.r.t. the uniform distribution on $[0, 1]$, the distribution of $F = F(t, T)$ and of $G = F(0, t)$, respectively. □

Theorem 8. *Let* $T = t^c$, $c > \max\{1, 1/(\gamma(0) + \frac{1}{2}) + 1/(2r+1)\}$. *Then for the inhomogeneous Poisson process with rate* $a_t(\alpha)$ *and* $r = \gamma(q) > 0$ *we have as* $t \to \infty$,

$$\pi(\hat{I}_{T,t}(\omega), \hat{I}_{T,t}(\omega')) \overset{\mathbb{D}_\vartheta}{\to} (E, E') \quad (\omega \neq \omega')$$

provided that the estimation $\hat{\alpha}_T$ *of* α *is based on* $F(t, T)$.

Proof. (i) For the NP(1) model the asymptotic result (13) with $f(\omega) = 1/\pi$ holds true, cf. Cox and Lewis (1966, p. 127). Further we have according to Theorem 4,

$$\mathbb{D}_\beta\left(\sum_{k=1}^{N_t} \mathrm{e}^{-\mathrm{i}\omega\tau_k}, \omega > 0\right) = \mathbb{D}_\vartheta\left(\sum_{k=1}^{\tilde{N}_t} \mathrm{e}^{-\mathrm{i}\omega\tilde{\tau}_k}, \omega > 0\right),$$

where $\tilde{N}_t = N_{\sigma(t)}$, $\sigma(t)$ inverse function of $A(t)$, and $\tilde{\tau}_k = A(\tau_k)$. Hence, putting

$$\tilde{H}_t = \sum_{k=1}^{\tilde{N}_t} \mathrm{e}^{-\mathrm{i}\omega\tilde{\tau}_k}/\sqrt{t}, \qquad \hat{H}_t = \sum_{k=1}^{\hat{N}_t} \mathrm{e}^{-\mathrm{i}\omega\hat{\tau}_k}/\sqrt{t}, \qquad \overset{*}{H}_t = \sum_{k=1}^{\tilde{N}_t} \mathrm{e}^{-\mathrm{i}\omega\hat{\tau}_k}/\sqrt{t}$$

it suffices to show that

$$\tilde{H}_t - \overset{*}{H}_t \overset{\mathbb{P}_\vartheta}{\to} 0, \quad \overset{*}{H}_t - \hat{H}_t \overset{\mathbb{P}_\vartheta}{\to} 0.$$

(ii) Letting $\hat{f}(u) = \hat{A}_T(\sigma(u))$ we have by using Lemma 7 and Theorem 4,

$$\mathbb{E}_\vartheta\left|\sum_{k=1}^{\tilde{N}_t}(\mathrm{e}^{-\mathrm{i}\omega A(\tau_k)}-\mathrm{e}^{-\mathrm{i}\omega\hat{A}_T(\tau_k)})\right|/\sqrt{t}\leqslant\omega\mathbb{E}_\vartheta\sum_{k=1}^{\tilde{N}_t}\left|A(\tau_k)-\hat{A}_T(\tau_k)\right|/\sqrt{t}$$

$$=\omega\mathbb{E}_\vartheta\sum_{k=1}^{\tilde{N}_t}|\tilde{\tau}_k-\hat{f}(\tilde{\tau}_k)|/\sqrt{t}$$

$$=\omega\mathbb{E}_F\mathbb{E}_G\sum_{k=1}^{N_t}|\tau_k-\hat{f}(\tau_k)|/\sqrt{t}$$

$$=\omega\mathbb{E}_\vartheta\int_0^t|u-\hat{f}(u)|\,\mathrm{d}u/\sqrt{t}\equiv I_t,\quad\text{say},$$

where $\mathbb{E}_F=\mathbb{E}_{F,\vartheta}$, $\mathbb{E}_G=\mathbb{E}_{G,\beta}$ is similarly used as in the proof of Lemma 7 (note that $\sigma(t)\leqslant t$ for large t due to $\gamma(q)>0$, such that $(\tau_1,\ldots,\tau_{\tilde{N}_t},\tilde{N}_t)$ and $F(t,T)$ are independent). Writing $A(t)=A(t,\alpha)$ we now apply the mean value theorem to $A(\sigma(u),\hat{\alpha}_T)-A(\sigma(u),\alpha)=\hat{f}(u)-u$. With α^* lying between α and $\hat{\alpha}_T$ we have, taking $\partial/\partial\alpha_j A(t)=2\int_0^t p_s(\alpha)s^{\gamma(j)}\,\mathrm{d}s$ into regard,

$$|\hat{f}(u)-u|=2\left|\sum_{j=0}^{q}(\hat{\alpha}_{j,T}-\alpha_j)\int_0^{\sigma(u)}p_s(\alpha^*)s^{\gamma(j)}\,\mathrm{d}s\right|$$

$$\leqslant 2c_1\sum_{j=0}^{q}\hat{Z}_{j,T}\sigma(u)^{\gamma(q)+\gamma(j)+1}T^{-\gamma(j)-1/2}$$

for $\sigma(u)\geqslant 1$ where $\hat{Z}_{j,T}=|\hat{\alpha}_{j,T}-\alpha_j|T^{\gamma(j)+1/2}$. Hence by Lemma 2 ((A0)–(A5) are fulfilled for the NP(1) model) for large t with finite constants c_j,

$$I_t\leqslant c_2\sum_{j=0}^{q}T^{-\gamma(j)-1/2}\int_0^t\sigma(u)^{\gamma(q)+\gamma(j)+1}\,\mathrm{d}u/\sqrt{t}$$

$$\leqslant c_3\sum_{j=0}^{q}T^{-\gamma(j)-1/2}\sigma(t)^{\gamma(q)+\gamma(j)+1+2\gamma(q)+1}/\sqrt{t}$$

$$\leqslant c_4\sum_{j=0}^{q}t^{-c(\gamma(j)+1/2)}t^{(\gamma(q)+\gamma(j)+1)/(2\gamma(q)+1)+1/2}=c_4\sum_{j=0}^{q}t^{\kappa(j)}$$

taking $\sigma(t)^{2\gamma(q)+1}/t\rightarrow$ const into regard. Since $\kappa(j)<0$ by our assumption on c, we arrive at $\mathbb{E}_\vartheta|\tilde{H}_t-\overset{*}{H}_t|\rightarrow 0$.

(iii) In order to show $\overset{*}{H}_t-\hat{H}_t\overset{\mathbb{P}_\vartheta}{\rightarrow}0$ we begin with a proof of

$$(t-A(\hat{\sigma}(t)))/\sqrt{t}\overset{\mathbb{P}_\vartheta}{\rightarrow}0.\tag{16}$$

In fact, applying the mean value theorem to $t-A(\hat{\sigma}(t))=A(\sigma(t),\alpha)-A(\hat{\sigma}(t),\alpha)$ and then to $\sigma(t)-\hat{\sigma}(t)=\sigma(t,\alpha)-\sigma(t,\hat{\alpha})$ we obtain

$$|t-A(\hat{\sigma}(t))|/\sqrt{t} \leqslant e^*(t) \sum_{j=0}^{q} \hat{Z}_{j,T} t^{\kappa(j)}$$

with $\kappa(j)<0$, $\hat{Z}_{j,t}$ as in part (ii), and $e^*(t)=a_{\sigma^*}(\alpha)/a_{\sigma(t,\alpha^*)}(\alpha^*)$, where σ^* lies between $\sigma(t)$ and $\hat{\sigma}(t)$ and α^* between α and $\hat{\alpha}_T$. Assertion (16) now follows from $e^*(t) \xrightarrow{\mathbb{P}_\vartheta} 1$ $(t\to\infty)$ and the stochastic boundedness of $\hat{Z}_{j,T}$.

Next we want to show that

$$\text{the process } \overset{*}{H}_t-\hat{H}_t,\ t\geqslant 0, \text{ is stochastically bounded w.r.t. } \mathbb{P}_\vartheta. \tag{17}$$

In fact, writing

$$\frac{N_{\sigma(t)}-N_{\hat{\sigma}(t)}}{\sqrt{t}}=\frac{N_{\sigma(t)}-t}{\sqrt{t}}+\frac{t-A(\hat{\sigma}(t))}{\sqrt{t}}+\frac{A(\hat{\sigma}(t))-N_{\hat{\sigma}(t)}}{\sqrt{t}},$$

we know that the first and the third term converges in $\mathbb{P}_\vartheta$-law towards the normal distribution (apply the random time change theorem 17.1 of Billingsley (1968) along the line of arguments used in Aalen (1976), p. 66)), while (16) completes the proof of (17).

To complete the proof of $\overset{*}{H}_t-\hat{H}_t \xrightarrow{\mathbb{P}} >0$ we can restrict – as a consequence of (17) – the estimators $\hat{\alpha}_j(j<q)$, $\hat{\alpha}_q$ to the intervals $[\alpha_j-1,\ \alpha_j+1]$ and $[\frac{1}{2}\alpha_q,\ 2\alpha_q]$, respectively. Now using $\mathbb{E}_F=\mathbb{E}_{F,\vartheta}$ and $\mathbb{E}_G=\mathbb{E}_{G,\beta}$ as in part (ii), we have for large t,

$$\begin{aligned}\mathbb{E}_\vartheta|\overset{*}{H}_t-\hat{H}_t| &\leqslant \mathbb{E}_\vartheta|\tilde{N}_t-\hat{N}_t|/\sqrt{t}=\mathbb{E}_\vartheta|\tilde{N}_t-\tilde{N}_{A(\hat{\sigma}(t))}|/\sqrt{t}\\ &=\mathbb{E}_F\mathbb{E}_G|N_t-N_{A(\hat{\sigma}(t))}|/\sqrt{t}=\mathbb{E}_\vartheta|t-A(\hat{\sigma}(t))|/\sqrt{t}\\ &\to 0 \quad (t\to\infty)\end{aligned}$$

due to (16) and the fact that the convergence (16) is (by the restrictions of the $\hat{\alpha}_j$'s) bounded. □

5. Application

We will now apply the method to the sequence (τ_n, ξ_n), $n=1, \ldots, 355$, of aftershocks of the Friuli earthquake (May–Sept. 1976) where τ_n refers to the occurrence time and ξ_n to the type of the nth shock. We will only distinguish two types 1 and 2 according to the magnitude of the shock (<2.5 or $\geqslant 2.5$ ML, respectively). The time-series plot of the number of shocks per 2 days, i.e. $X_t=N_{th}-N_{(t-1)h}$, $t=1, 2, \ldots$ ($h=2$ days), shows a decreasing tendency (Figure 1). As underlying model we suppose a certain generalization of the two state Markov process, namely a two state linear OM process (see Pruscha, 1983, 1986), while the function

$$a_t(\alpha)=(\alpha_2 t^{-0.2}+\alpha_1 t^{-0.3}+\alpha_0 t^{-0.4})^2, \quad t\geqslant 0,$$

should fit the reciprocal trend. The inclusion of further terms t^γ does not improve the likelihood significantly. Different to Theorems 6 and 8 above we base here the calculation

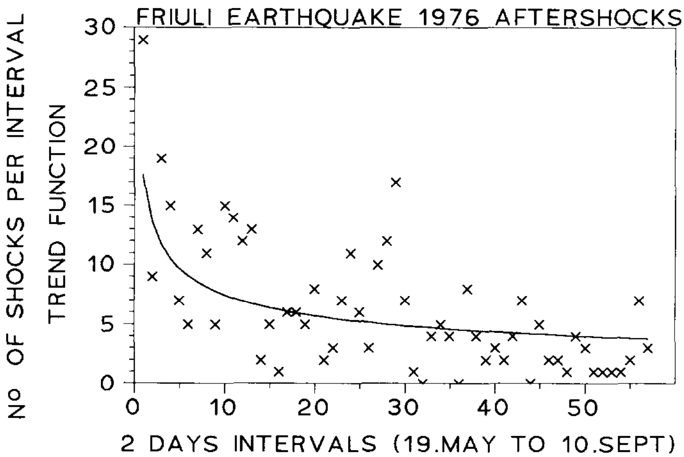

Fig. 1. Number of shocks plotted over the 57 consecutive 2 day intervals of the period 19 May to 10 Sept. 1976. A trend function $(\hat{\alpha}_2 t^{-0.2} + \hat{\alpha}_1 t^{-0.3} + \hat{\alpha}_0 t^{-0.4})^2 \bar{b}$ is fitted with $\hat{\alpha}_2 = 1 \cdot 48^{-0.2} = 0.461$ (preassigned), $\hat{\alpha}_1 = 0.115$, $\hat{\alpha}_0 = -0.105$, $\bar{b} = 79.3$ and t in [2 days].

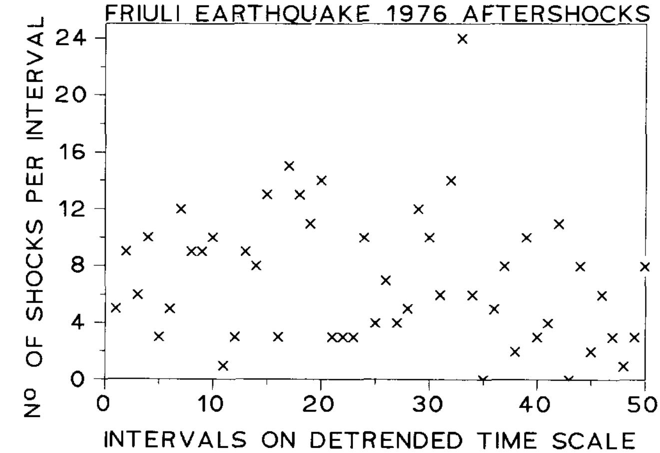

Fig. 2. Number of shocks per interval plotted over the consecutive intervals $[h(t-1),ht]$, $t=1, \ldots, 50$, $h = \hat{A}(\tau_n)/50$, on the detrended time scale $(n=355)$.

of both the parameter estimations and the statistics on the maximal interval $[0, \tau_{355}]$. Detrending according to (10) produces detrended occurrence times $\hat{\tau}_n$, $n=1, \ldots, 355$, and a corresponding time-series plot $\hat{X}_t = \hat{N}_{th} - \hat{N}_{(t-1)h}$, $t=1, 2, \ldots$ (Figure 2), which no longer reveals an obvious trend (but still one outlier).

The periodogram based on the original (trend affected) process exhibits high values for small cycle numbers which is typical for trend affected processes. They are removed in the periodogram based on the detrended process. But there are still values outside the simultaneous bound testing the hypothesis of an homogeneous Poisson process, such that the Poisson model for the underlying process can be rejected. (See Figure 3.)

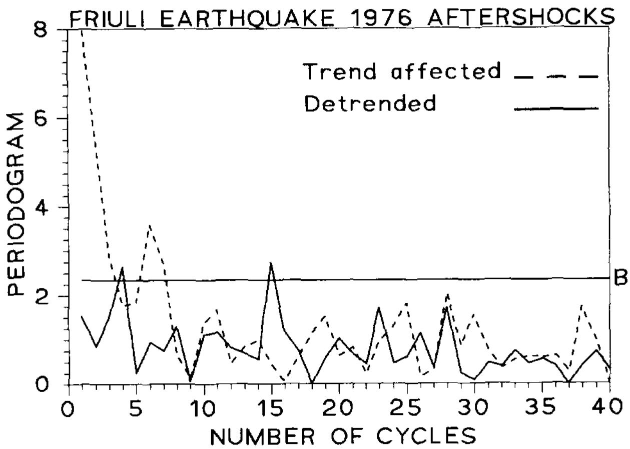

Fig. 3. Periodogram estimators $I_{(n)}(\omega)$ and $\hat{I}_{(n)}(\omega)$ based on the original and on the detrended occurrence times, resp.; evaluated at $\omega=\omega_k=2\pi k/\tau_n$ and $\omega_k=2\pi k/\hat{A}(\tau_n)$, resp., and plotted over $k=1, \ldots, 40$ $(n=355)$. $B=B_{40}=-\ln(0.025/40)/\pi$ gives a simultaneous bound for the 40 periodogram values under the hypothesis of a homogeneous Poisson process, cf. Cox and Lewis (1966, p. 99).

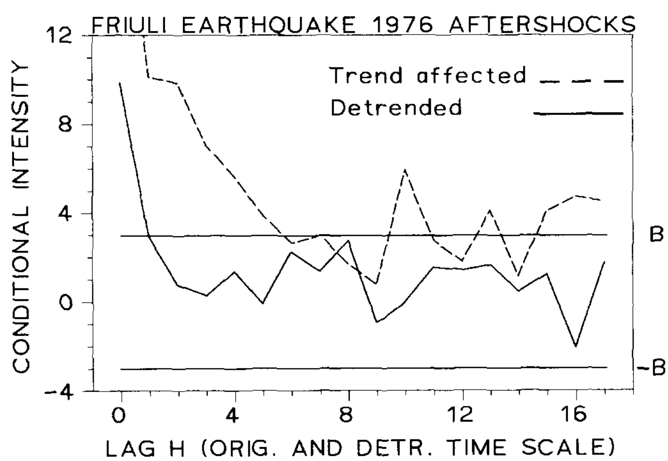

Fig. 4. Conditional (or Palm) intensity function estimators $\nu_n(h)$ and $\hat{\nu}_n(h)$, based on the original and detrended occurrence times, resp.; evaluated at time lags $h=L\cdot k$, $L=\tau_n/100$ and $L=\hat{A}(\tau_n)/100$, resp. $(n=355)$, and plotted in standardized form over $k=0, \ldots, 17$. $B=B_{18}=u(1-0.025/18)$ gives a simultaneous bound for the 18 values under the hypothesis of a homogeneous Poisson process, cf. Cox and Lewis (1966, p. 123).

Qualitatively the same features are revealed by the plots of the estimated conditional intensity function (L a small bandwidth, $h=$ the time lag)

$$\nu_n(h)=\int_0^{\tau_n}(N_{s+h+L}-N_{s+h})\ \mathrm{d}N_s/[Lnf(h)]$$

$$=\sum_{0<j<k\leqslant n} 1(h<\tau_k-\tau_j\leqslant h+L)/[Lnf(h)]$$

for the original process and for the detrended process (where $\tau_k - \tau_j$ is replaced by $\hat{\tau}_k - \hat{\tau}_j$), see Cox and Lewis (1966, p. 123). Here, $f(h) = (\tau_n - h - \frac{1}{2}L)/\tau_n$ is a correction factor. Even after trend removal significantly high values for small time lags h (compared with the Poissonian model) indicate a cluster effect in the sequence of aftershocks (see Ogata, 1988, for related results). (See Figure 4.)

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