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Ultra-analytic effect of Cauchy problem for a class of kinetic equations

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ABSTRACT

The smoothing effect of the Cauchy problem for a class of kinetic equations is studied. We firstly consider the spatially homogeneous nonlinear Landau equation with Maxwellian molecules and inhomogeneous linear Fokker–Planck equation to show the ultra-analytic effects of the Cauchy problem. Those smoothing effect results are optimal and similar to heat equation. In the second part, we study a model of spatially inhomogeneous linear Landau equation with Maxwellian molecules, and show the analytic effect of the Cauchy problem.

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1. Introduction

It is well known that the Cauchy problem of heat equation possesses the ultra-analytic effect phenomenon, namely, if $u(t,x)$ is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t u - \Delta_x u = 0, & x \in \mathbb{R}^d,\ t > 0, \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^d), \end{cases}$$

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then under the uniqueness hypothesis, the solution $u(t,\cdot)=e^{t\Delta_x}u_0$ is an ultra-analytic function for any $t>0$. We give now the definition of function spaces $\mathcal{A}^s(\Omega)$ where Ω is an open subset of $\mathbb{R}^d$.

Definition 1.1. For $0<s<+\infty$, we say that $f\in\mathcal{A}^s(\Omega)$, if $f\in C^\infty(\Omega)$, and there exist $C>0, N_0>0$ such that

$$\|\partial^\alpha f\|_{L^2(\Omega)}\leqslant C^{|\alpha|+1}(\alpha!)^s,\quad \forall\alpha\in\mathbb{N}^d,\ |\alpha|\geqslant N_0.$$

If the boundary of Ω is smooth, by using Sobolev embedding theorem, we have the same type estimate with L^2 norm replaced by any L^p norm for $2<p\leqslant+\infty$. On the whole space $\Omega=\mathbb{R}^d$, it is also equivalent to

$$e^{c_0(-\Delta)^{\frac{1}{2s}}}\big(\partial^{\beta_0}f\big)\in L^2\big(\mathbb{R}^d\big)$$

for some $c_0>0$ and $\beta_0\in\mathbb{N}^d$, where $e^{c_0(-\Delta)^{\frac{1}{2s}}}$ is the Fourier multiplier defined by

$$e^{c_0(-\Delta)^{\frac{1}{2s}}}u(x)=\mathcal{F}^{-1}\big(e^{c_0|\xi|^{\frac{1}{s}}}\hat{u}(\xi)\big).$$

If $s=1$, it is usual analytic function. If $s>1$, it is Gevrey class function. For $0<s<1$, it is called ultra-analytic function. Notice that all polynomial functions are ultra-analytic for any $s>0$.

It is obvious that if $u_0\in L^2(\mathbb{R}^d)$ then, for any $t>0$ and any $k\in\mathbb{N}$, we have $u(t,\cdot)=e^{-t(-\Delta_x)^k}u_0\in\mathcal{A}^{\frac{1}{2k}}(\mathbb{R}^d)$, namely, there exists $C>0$ such that for any $m\in\mathbb{N}$,

$$\begin{aligned}\big\|\big(t^m\partial_x^{2km}\big)u(t,\cdot)\big\|_{L^2(\mathbb{R}^d)}&\leqslant C^{km}\big\|\big(t(-\Delta_x)^k\big)^m u(t,\cdot)\big\|_{L^2(\mathbb{R}^d)}\\&\leqslant\|u_0\|_{L^2(\mathbb{R}^d)}C^{km}m!\leqslant\tilde{C}^{2km+1}\big((2km)!\big)^{\frac{1}{2k}},\end{aligned}$$

where $\partial_x^{2km}=\sum_{|\alpha|=2km,\,\alpha\in\mathbb{N}^d}\partial_x^\alpha$. We say that the diffusion operators $(-\Delta_x)^k$ possess the ultra-analytic effect property if $k>1/2$, the analytic effect property if $k=1/2$ and the Gevrey effect property if $0<k<1/2$.

We study the Cauchy problem for spatially homogeneous Landau equation

$$\begin{cases}f_t=Q(f,f)\equiv\nabla_v\big(\bar{a}(f)\cdot\nabla_v f-\bar{b}(f)f\big), & v\in\mathbb{R}^d,\ t>0,\\ f|_{t=0}=f_0,\end{cases}\tag{1.1}$$

where $\bar{a}(f)=(\bar{a}_{ij}(f))$ and $\bar{b}(f)=(\bar{b}_1(f),\dots,\bar{b}_d(f))$ are defined as follows (convolution is w.r.t. the variable $v\in\mathbb{R}^d$)

$$\bar{a}_{ij}(f)=a_{ij}\star f,\qquad \bar{b}_j(f)=\sum_{i=1}^d(\partial_{v_i}a_{ij})\star f,\quad i,j=1,\dots,d,$$

with

$$a_{ij}(v)=\left(\delta_{ij}-\frac{v_iv_j}{|v|^2}\right)|v|^{\gamma+2},\quad \gamma\in[-3,1].$$

We consider hereafter only the Maxwellian molecule case which corresponds to $\gamma=0$. We introduce also the notation, for $l\in\mathbb{R}$, $L^p_l(\mathbb{R}^d)=\{f;\,(1+|v|^2)^{l/2}f\in L^p(\mathbb{R}^d)\}$ is the weighted function space.

We prove the following ultra-analytic effect results for the nonlinear Cauchy problem (1.1).

Theorem 1.1. *Let $f_0 \in L^2(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d)$ and $0 < T \leqslant +\infty$. If $f(t,x) > 0$ and $f \in L^\infty(]0,T[; L^2(\mathbb{R}^d) \cap L^1_2(\mathbb{R}^d))$ is a weak solution of the Cauchy problem* (1.1)*, then for any $0 < t < T$, we have*

$$f(t,\cdot) \in \mathcal{A}^{1/2}\big(\mathbb{R}^d\big),$$

and moreover, for any $0 < T_0 < T$, there exists $c_0 > 0$ such that for any $0 < t \leqslant T_0$

$$\big\| e^{-c_0 t \Delta_v} f(t,\cdot)\big\|_{L^2(\mathbb{R}^d)} \leqslant e^{\frac{d}{2}t} \|f_0\|_{L^2(\mathbb{R}^d)}. \tag{1.2}$$

In [17], they proved the Gevrey regularity effect of the Cauchy problem for linear spatially homogeneous non-cut-off Boltzmann equation. By a careful revision for the proof of Theorem 1.2 of [17], one can also prove that the solution of the Cauchy problem (1.10) in [17] belongs to $\mathcal{A}^{\frac{1}{2\alpha}}(\mathbb{R}^d)$ for any $t > 0$, where $0 < \alpha < 1$ is the order of singularity of collision kernel of Boltzmann operator. Hence, if $\alpha \geqslant 1/2$, there is also the ultra-analytic effect phenomenon. Now the above Theorem 1.1 shows that, for Landau equation, the ultra-analytic effect phenomenon holds in nonlinear case, which is an optimal regularity result.

The ultra-analytic effect property is also true for the Cauchy problem of the following generalized Kolmogorov operators

$$\begin{cases} \partial_t u + v\cdot\nabla_x u + (-\Delta_v)^\alpha u = 0, & (x,v)\in\mathbb{R}^{2d},\ t>0,\\ u|_{t=0} = u_0 \in L^2\big(\mathbb{R}^{2d}\big), \end{cases}$$

where $0 < \alpha < \infty$, and the classical Kolmogorov operators is corresponding to $\alpha = 1$. By Fourier transformation, the explicit solution of the above Cauchy problem is given by

$$\hat{u}(t,\eta,\xi) = e^{-\int_0^t |\xi+s\eta|^{2\alpha}\,ds}\hat{u}_0(\eta,\xi+t\eta).$$

Since there exists $c_\alpha > 0$ (see Lemma 3.1 below) such that

$$c_\alpha\big(t|\xi|^{2\alpha} + t^{2\alpha+1}|\eta|^{2\alpha}\big) \leqslant \int_0^t |\xi+s\eta|^{2\alpha}\,ds, \tag{1.3}$$

we have

$$e^{c_\alpha(t(-\Delta_v)^\alpha + t^{2\alpha+1}(-\Delta_x)^\alpha)} u(t,\cdot,\cdot) \in L^2\big(\mathbb{R}^{2d}\big),$$

i.e. $u(t,\cdot,\cdot) \in \mathcal{A}^{1/(2\alpha)}(\mathbb{R}^{2d})$ for any $t > 0$.

Notice that this ultra-analytic (if $\alpha > 1/2$) effect phenomenon is similar to heat equations of (x,v) variables. That is, this means $v\cdot\nabla_x + (-\Delta_v)^\alpha$ is equivalent to $(-\Delta_x)^\alpha + (-\Delta_v)^\alpha$ by time evolution in "some sense", though the equation is only transport for x variable.

We consider now a more complicate equation, the Cauchy problem for linear Fokker–Planck equation:

$$\begin{cases} f_t + v\cdot\nabla_x f = \nabla_v\cdot(\nabla_v f + vf), & (x,v)\in\mathbb{R}^{2d},\ t>0,\\ f|_{t=0} = f_0. \end{cases} \tag{1.4}$$

This equation is a natural generalization of classical Kolmogorov equation, and a simplified model of inhomogeneous Landau equation (see [20,21]). The local property of this equation is the same as classical Kolmogorov equation since the add terms $\nabla_v\cdot(vf)$ is a first order term, but for the studies of kinetic equation, v is velocity variable, and hence it is in whole space $\mathbb{R}^d_v$. Then there occurs additional difficulty for analysis of this equation.

The definition of weak solution in the function space $L^\infty(]0,T[;L^2(\mathbb{R}^{2d}_{x,v})\cap L^1_1(\mathbb{R}^{2d}_{x,v}))$ for the Cauchy problem is standard in the distribution sense, where for $1\leqslant p<+\infty$, $l\in\mathbb{R}$

$$L^p_l\big(\mathbb{R}^{2d}_{x,v}\big)=\big\{f\in\mathcal{S}'\big(\mathbb{R}^{2d}\big);\ \big(1+|v|^2\big)^{l/2}f\in L^p\big(\mathbb{R}^{2d}_{x,v}\big)\big\}.$$

The existence of weak solution is similar to full Landau equation (see [1,13]). We get also the following ultra-analytic effect result.

Theorem 1.2. *Let $f_0\in L^2(\mathbb{R}^{2d}_{x,v})\cap L^1_1(\mathbb{R}^{2d}_{x,v})$, $0<T\leqslant+\infty$. Assume that $f\in L^\infty(]0,T[;L^2(\mathbb{R}^{2d}_{x,v})\cap L^1_1(\mathbb{R}^{2d}_{x,v}))$ is a weak solution of the Cauchy problem* (1.4)*. Then, for any $0<t<T$, we have*

$$f(t,\cdot,\cdot)\in\mathcal{A}^{1/2}\big(\mathbb{R}^{2d}\big).$$

Furthermore, for any $0<T_0<T$ there exists $c_0>0$ such that for any $0<t\leqslant T_0$, we have

$$\big\|e^{-c_0(t\Delta_v+t^3\Delta_x)}f(t,\cdot,\cdot)\big\|_{L^2(\mathbb{R}^{2d})}\leqslant e^{\frac{d}{2}t}\|f_0\|_{L^2(\mathbb{R}^{2d})}. \tag{1.5}$$

Remark 1.1. The ultra-analyticity results of the above two theorems are optimal for the smoothness properties of solutions. From these results, we obtain a good understanding for the hypoellipticity of kinetic equations (see [11,14]), and also the relationship, established by Villani [19] and Desvillettes and Villani [10], between the nonlinear Landau equation (with Maxwellian molecules) and the linear Fokker–Planck equation.

We consider now the spatially inhomogeneous Landau equation

$$\begin{cases} f_t+v\cdot\nabla_x f=Q(f,f), & (x,v)\in\mathbb{R}^{2d},\ t>0,\\ f|_{t=0}=f_0(x,v). \end{cases} \tag{1.6}$$

The problem is now much more complicate since the solution f is the function of (t,x,v) variables. We consider it here only in the linearized framework around the normalized Maxwellian distribution

$$\mu(v)=(2\pi)^{-\frac{d}{2}}e^{-\frac{|v|^2}{2}},$$

which is the equilibrium state because $Q(\mu,\mu)=0$. Setting $f=\mu+g$, we consider the diffusion part of linear Landau collision operators

$$Q(\mu,g)=\nabla_v\big(\bar a(\mu)\cdot\nabla_v g-\bar b(\mu)g\big),$$

where

$$\bar a_{ij}(\mu)=a_{ij}\star\mu=\delta_{ij}\big(|v|^2+1\big)-v_iv_j,$$

$$\bar b_j(\mu)=\sum_{i=1}^d(\partial_{v_i}a_{ij})\star\mu=-v_j,\quad i,j=1,\dots,d.$$

In particular, it follows that

$$\sum_{ij=1}^d\bar a_{ij}(\mu)\xi_i\xi_j\geqslant|\xi|^2,\quad\text{for all }(v,\xi)\in\mathbb{R}^{2d}. \tag{1.7}$$

We then consider the following Cauchy problem

$$\begin{cases} g_t + v\cdot\nabla_x g = \nabla_v\big(\bar a(\mu)\cdot\nabla_v g - \bar b(\mu) g\big), & (x,v)\in\mathbb{R}^{2d},\ t>0,\\ g|_{t=0} = g_0. \end{cases} \tag{1.8}$$

We can also look this equation as a linear model of spatially inhomogeneous Landau equation, which is much more complicate than linear Fokker–Planck equation (1.4), since the coefficients of diffusion part are now variables. The existence and C^∞ regularity of weak solution for the Cauchy problem have been considered in [1]. We prove now the following:

Theorem 1.3. *Let $g_0 \in L^2(\mathbb{R}^{2d}_{x,v}) \cap L^1_2(\mathbb{R}^{2d}_{x,v})$, $0 < T \leqslant +\infty$. Assume that $g \in L^\infty(]0,T[; L^2(\mathbb{R}^{2d}_{x,v}) \cap L^1_2(\mathbb{R}^{2d}_{x,v}))$ is a weak solution of the Cauchy problem* (1.8)*. Then, for any $0<t<T$, we have*

$$g(t,\cdot,\cdot) \in \mathcal{A}^1\big(\mathbb{R}^{2d}\big).$$

Furthermore, for any $0 < T_0 < T$ there exist $C, c > 0$ such that for any $0 < t \leqslant T_0$, we have

$$\big\| e^{c(t(-\Delta_v)^{1/2} + t^2(-\Delta_x)^{1/2})} g(t,\cdot,\cdot)\big\|_{L^2(\mathbb{R}^{2d})} \leqslant e^{Ct} \|g_0\|_{L^2(\mathbb{R}^{2d})}. \tag{1.9}$$

In this theorem, we only consider the analytic effect result for the Cauchy problem (1.8), neglecting the symmetric term $Q(g,\mu)$ in the linearized operators of Landau collision operator (cf. (1.15) of [1]) because of the technical difficulty, see the remark in the end of Section 4.

There have been many results about the regularity of solutions for Boltzmann equation without angular cut-off and Landau equation, see [1–3,6,7,9,12,15,16] for the C^∞ smoothness results, and [4, 5,8,17,18] for Gevrey regularity results for Boltzmann equation and Landau equation in both cases: the spatially homogeneous and inhomogeneous cases. As for the analytic and Gevrey regularities, we remark that the propagation of Gevrey regularities of solutions is investigated in [5] for full nonlinear spatially homogeneous Landau equations, including non-Maxwellian molecule case, and the local Gevrey regularity for all variables t, x, v is considered in [4] for some semi-linear Fokker–Planck equations. Comparing those results, the ultra-analyticity for x, v variables showed in Theorem 1.1 is strong although the Maxwellian molecule case is only treated. As a related result for spatially homogeneous Boltzmann equation in the Maxwellian molecule case, we refer [8], where the propagation of Gevrey and ultra-analytic regularity is studied uniformly in time variable t. Throughout the present paper, we focus the smoothing effect of the Cauchy problem, and the uniform smoothness estimate near to $t = 0$. Concerning further details of the analytic and Gevrey regularities of solutions for Landau equations and Boltzmann equation without angular cut-off, we refer the introduction of [5] and references therein.

2. Spatially homogeneous Landau equations

We consider the Cauchy problem (1.1) and prove Theorem 1.1 in this section. We refer to the works of C. Villani [19,20] for the essential properties of homogeneous Landau equations. We suppose the existence of weak solution $f(t,v) > 0$ in $L^\infty(]0,T[; L^1_2(\mathbb{R}^d) \cap L^2(\mathbb{R}^d))$. The conservation of mass, momentum and energy reads

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} dv \equiv 0.$$

Without loss of generality, we can suppose that

$$\int_{\mathbb{R}^d} f(t,v)\,dv = 1, \quad \text{unit mass},$$

$$\int_{\mathbb{R}^d} f(t,v)v_j\,dv = 0, \quad j = 1,\dots,d, \quad \text{zero mean velocity},$$

$$\int_{\mathbb{R}^d} f(t,v)|v|^2\,dv = T_0, \quad \text{unit temperature},$$

$$\int_{\mathbb{R}^d} f(t,v)v_j v_k\,dv = T_j\delta_{jk}, \quad \sum_j^d T_j = T_0,$$

$$T_j = \int_{\mathbb{R}^d} f(t,v)v_j^2\,dv > 0, \quad j = 1,\dots,d, \quad \text{directional temperatures}.$$

Then we have

$$\bar{a}_{jk}(f) = \delta_{jk}\big(|v|^2 + T_0 - T_j\big) - v_j v_k, \tag{2.1}$$

$$\bar{b}_j(f) = -v_j, \tag{2.2}$$

$$\sum_{j,k}^d \bar{a}_{jk}(f)\xi_j\xi_k \geqslant C_1|\xi|^2, \quad \forall(v,\xi) \in \mathbb{R}^{2d}, \tag{2.3}$$

where $C_1 = \min_{1\leqslant j\leqslant d}\{T_0 - T_j\} > 0$.

Now for $N > \frac{d}{4} + 1$ and $0 < \delta < 1/N$, $c_0 > 0$, $t > 0$, set

$$G_\delta\big(t,|\xi|\big) = \frac{e^{c_0 t|\xi|^2}}{(1+\delta e^{c_0 t|\xi|^2})(1+\delta c_0 t|\xi|^2)^N}.$$

Since $G_\delta(t,\cdot) \in L^\infty(\mathbb{R}^d)$, we can use it as Fourier multiplier, denoted by

$$G_\delta(t,D_v)f(t,v) = \mathcal{F}^{-1}\big(G_\delta\big(t,|\xi|\big)\hat{f}(t,\xi)\big).$$

Then, for any $t > 0$,

$$G_\delta(t) = G_\delta(t,D_v) : L^2\big(\mathbb{R}^d\big) \to H^{2N}\big(\mathbb{R}^d\big) \subset C_b^2\big(\mathbb{R}^d\big).$$

The object of this section is to prove the uniform bound (with respect to $\delta > 0$) of

$$\big\|G_\delta(t,D_v)f(t,\cdot)\big\|_{L^2(\mathbb{R}^d)}.$$

Since $f(t,\cdot) \in L^2(\mathbb{R}^d) \cap L_2^1(\mathbb{R}^d)$ is a weak solution, we can take

$$G_\delta(t)^2 f(t,\cdot) = G_\delta(t,D_v)^2 f(t,\cdot) \in H^{2N}\big(\mathbb{R}^d\big),$$

as test function in the equation of (1.1), whence we have

$$\frac{1}{2}\frac{d}{dt}\left\|G_\delta(t)f(t,\cdot)\right\|^2_{L^2(\mathbb{R}^d)}+\sum_{j,k=1}^d\int_{\mathbb{R}^d}\bar a_{jk}(f)\big(\partial_{v_j}G_\delta(t)f(t,v)\big)\overline{\big(\partial_{v_k}G_\delta(t)f(t,v)\big)}\,dv$$

$$=\frac{1}{2}\big(\big(\partial_t G_\delta(t)\big)f,G_\delta(t)f\big)_{L^2(\mathbb{R}^d)}+\sum_{j=1}^d\int_{\mathbb{R}^d}\big(\partial_{v_j}\big(v_jf(t,v)\big)\big)\overline{G_\delta(t)^2f(t,v)}\,dv$$

$$+\sum_{j,k=1}^d\int_{\mathbb{R}^d}\big\{\bar a_{jk}(f)\big(G_\delta(t)\partial_{v_j}f(t,v)\big)-G_\delta(t)\big(\bar a_{jk}(f)\partial_{v_j}f(t,v)\big)\big\}\overline{\big(\partial_{v_k}G_\delta(t)f(t,v)\big)}\,dv.$$

To estimate the terms in the above equality, we prove the following two propositions.

Proposition 2.1. *We have*

$$C_1\left\|\nabla_vG_\delta(t)f(t)\right\|^2_{L^2(\mathbb{R}^d)}\leqslant\sum_{j,k=1}^d\int_{\mathbb{R}^d}\bar a_{jk}(f)\big(\partial_{v_j}G_\delta(t,D_v)f(t,v)\big)\overline{\big(\partial_{v_k}G_\delta(t,D_v)f(t,v)\big)}\,dv,\tag{2.4}$$

$$\big|\big(\big(\partial_tG_\delta(t)\big)f,G_\delta(t)f\big)_{L^2}\big|\leqslant c_0\left\|\nabla_vG_\delta(t)f(t)\right\|^2_{L^2},\tag{2.5}$$

$$\mathrm{Re}\sum_{j=1}^d\int_{\mathbb{R}^d}\big(\partial_{v_j}\big(v_jf(t,v)\big)\big)\overline{G_\delta(t)^2f(t,v)}\,dv\leqslant\frac{d}{2}\left\|G_\delta(t)f(t)\right\|^2_{L^2}+2c_0t\left\|\nabla_vG_\delta(t)f(t)\right\|^2_{L^2}.\tag{2.6}$$

Proof. The estimate (2.4) is exactly the elliptic condition (2.3). By using the Fourier transformation, (2.5) is deduced from the following calculus

$$\partial_tG_\delta\big(t,|\xi|\big)=c_0|\xi|^2G_\delta\big(t,|\xi|\big)\left(\frac{1}{1+\delta e^{c_0t|\xi|^2}}-\frac{N\delta}{1+\delta c_0t|\xi|^2}\right)=c_0|\xi|^2G_\delta\big(t,|\xi|\big)J_{N,\delta},$$

where

$$|J_{N,\delta}|=\left|\frac{1}{1+\delta e^{c_0t|\xi|^2}}-\frac{N\delta}{1+\delta c_0t|\xi|^2}\right|\leqslant1.$$

To treat (2.6), we use

$$\partial_{\xi_j}G_\delta\big(t,|\xi|\big)=2c_0t\xi_jG_\delta\big(t,|\xi|\big)J_{N,\delta}.\tag{2.7}$$

Then, we have

$$\mathrm{Re}\sum_{j=1}^d\int_{\mathbb{R}^d}\big(\partial_{v_j}\big(v_jf(t,v)\big)\big)\overline{G_\delta(t,D_v)^2f(t,v)}\,dv$$

$$=-\mathrm{Re}\sum_{j=1}^d\int_{\mathbb{R}^d}v_jG_\delta(t,D_v)f(t,v)\overline{\big(\partial_{v_j}G_\delta(t,D_v)f(t,v)\big)}\,dv$$

$$-\mathrm{Re}\sum_{j=1}^d\int_{\mathbb{R}^d}\big(\big[G_\delta(t,D_v),v_j\big]f(t,v)\big)\overline{\big(\partial_{v_j}G_\delta(t,D_v)f(t,v)\big)}\,dv$$

$$= \frac{d}{2}\big\|G_\delta(t)f(t,\cdot)\big\|^2_{L^2(\mathbb{R}^d)} - \mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\big(\big[G_\delta(t,D_v),v_j\big]f(t,v)\big)\overline{\big(\partial_{v_j}G_\delta(t,D_v)f(t,v)\big)}\,dv.$$

Using Fourier transformation and (2.7), we have that for $t>0$,

$$\begin{aligned}
&-\sum_{j=1}^{d}\int_{\mathbb{R}^3}\big(\big[G_\delta(t,D_v),v_j\big]f(t,v)\big)\overline{\big(\partial_{v_j}G_\delta(t,D_v)f(t,v)\big)}\,dv\\
&\quad=-\sum_{j=1}^{d}\int_{\mathbb{R}^d}\big(G_\delta(t,D_v)v_jf(t,v)-v_jG_\delta(t,D_v)f(t,v)\big)\overline{\big(\partial_{v_j}G_\delta(t,D_v)f(t,v)\big)}\,dv\\
&\quad=\sum_{j=1}^{d}\int_{\mathbb{R}^d}\big\{i\partial_{\xi_j}\big(G_\delta\big(t,|\xi|\big)\hat f(t,\xi)\big)-G_\delta\big(t,|\xi|\big)\big(i\partial_{\xi_j}\hat f(t,\xi)\big)\big\}G_\delta\big(t,|\xi|\big)\overline{i\xi_j\hat f(t,\xi)}\,d\xi\\
&\quad=\sum_{j=1}^{d}\int_{\mathbb{R}^3}\big(\partial_{\xi_j}G_\delta\big(t,|\xi|\big)\big)\hat f(t,\xi)\xi_jG_\delta\big(t,|\xi|\big)\overline{\hat f(t,\xi)}\,d\xi\\
&\quad=2c_0t\int_{\mathbb{R}^d}|\xi|^2\big|G_\delta\big(t,|\xi|\big)\hat f(t,\xi)\big|^2J_{N,\delta}\,d\xi\leqslant 2c_0t\int_{\mathbb{R}^d}|\xi|^2\big|G_\delta\big(t,|\xi|\big)\hat f(t,\xi)\big|^2\,d\xi,
\end{aligned}$$

which give (2.6). The proof of Proposition 2.1 is now complete. □

For the commutator term, the special structure of the operator implies

Proposition 2.2.

$$\sum_{j,k=1}^{d}\int_{\mathbb{R}^d}\big\{\bar a_{jk}(f)\big(G_\delta(t,D_v)\partial_{v_j}f(t,v)\big)-G_\delta(t,D_v)\big(\bar a_{jk}(f)\partial_{v_j}f(t,v)\big)\big\}\overline{\big(\partial_{v_k}G_\delta(t,D_v)f(t,v)\big)}\,dv=0.$$

Proof. We introduce now polar coordinates on $\mathbb{R}^d_\xi$ by setting $r=|\xi|$ and $\omega=\xi/|\xi|\in\mathbb{S}^{d-1}$. Note that $\partial/\partial\xi_j=\omega_j\partial/\partial r+r^{-1}\Omega_j$ where Ω_j is a vector field on $\mathbb{S}^{d-1}$, and (see [14, Proposition 14.7.1])

$$\sum_{j=1}^{d}\omega_j\Omega_j=0,\qquad \sum_{j=1}^{d}\Omega_j\omega_j=d-1. \tag{2.8}$$

By using Fourier transformation, we have

$$\begin{aligned}
&-\sum_{j,k=1}^{d}\int_{\mathbb{R}^d}\big\{\bar a_{jk}(f)\big(G_\delta(t,D_v)\partial_{v_j}f(t,v)\big)-G_\delta(t,D_v)\big(\bar a_{jk}(f)\partial_{v_j}f(t,v)\big)\big\}\overline{\big(\partial_{v_k}G_\delta(t,D_v)f(t,v)\big)}\,dv\\
&\quad=\int_{\mathbb{R}^d}\left\{\sum_{j,k=1}^{d}\xi_k\big[(\delta_{jk}\Delta_\xi-\partial_{\xi_k}\partial_{\xi_j}),G_\delta\big(t,|\xi|\big)\big]\xi_j\hat f(t,\xi)\right\}\times G_\delta\big(t,|\xi|\big)\overline{\hat f(t,\xi)}\,d\xi.
\end{aligned}$$

Noting, in polar coordinates on $\mathbb{R}^d_\xi$,

$$\Delta_\xi = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\sum_{j=1}^d \Omega_j^2,$$

we have, denoting by $\tilde G(r^2) = G_\delta(t,r)$,

$$\begin{aligned}
&\sum_{j,k=1}^d \omega_k\left[\left(\delta_{jk}\left\{\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}\right\} - \left\{\left(\omega_k\partial/\partial r + r^{-1}\Omega_k\right)\left(\omega_j\partial/\partial r + r^{-1}\Omega_j\right)\right\}\right), \tilde G\left(r^2\right)\right]\omega_j \\
&\quad = \left[\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}, \tilde G\left(r^2\right)\right] - \left[\left(\sum_{k=1}^d\left(\omega_k^2\partial/\partial r + r^{-1}\omega_k\Omega_k\right)\sum_{j=1}^d\left(\omega_j^2\partial/\partial r + r^{-1}\Omega_j\omega_j\right)\right), \tilde G\left(r^2\right)\right] \\
&\quad = \left[\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}, \tilde G\left(r^2\right)\right] - \left[\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\frac{d-1}{r}, \tilde G\left(r^2\right)\right] = 0,
\end{aligned}$$

where we have used (2.8). Then we finish the proof of Proposition 2.2. □

Remark 2.1. In the above proof of Proposition 2.2, we have used the polar coordinates in the dual variable of v, which is essentially related to a form of the Landau operator with Maxwellian molecules. We notice that the same relation (in v variable) was described by Villani [19] and Desvillettes and Villani [10].

End of proof of Theorem 1.1. From Propositions 2.1 and 2.2, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|G_\delta(t)f(t,\cdot)\right\|^2_{L^2(\mathbb{R}^d)} + \left(C_1 - \frac{1}{2}c_0 - 2c_0 t\right)\left\|\nabla_v G_\delta(t)f(t,\cdot)\right\|^2_{L^2(\mathbb{R}^d)} \\
&\quad \leqslant \frac{d}{2}\left\|G_\delta(t)f(t,\cdot)\right\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

For any $0 < T_0 < T$, choose c_0 small enough such that $C_1 - \frac{1}{2}c_0 - 2c_0T_0 \geqslant 0$. Then we get

$$\frac{d}{dt}\left\|G_\delta(t)f(t,\cdot)\right\|_{L^2(\mathbb{R}^d)} \leqslant \frac{d}{2}\left\|G_\delta(t)f(t,\cdot)\right\|_{L^2(\mathbb{R}^d)}. \tag{2.9}$$

Integrating the inequality (2.9) on $]0,t[$, we obtain

$$\left\|G_\delta(t)f(t,\cdot)\right\|_{L^2(\mathbb{R}^d)} \leqslant e^{\frac{d}{2}t}\|f_0\|_{L^2(\mathbb{R}^d)}. \tag{2.10}$$

Take limit $\delta \to 0$ in (2.10). Then we get

$$\left\|e^{-c_0 t\Delta_v}f(t,\cdot)\right\|_{L^2(\mathbb{R}^d)} \leqslant e^{\frac{d}{2}t}\|f_0\|_{L^2(\mathbb{R}^d)} \tag{2.11}$$

for any $0 < t \leqslant T_0$. We have now proved $f(t,\cdot) \in \mathcal{A}^{1/2}(\mathbb{R}^d)$ and Theorem 1.1. □

3. Linear Fokker–Planck equations

In the paper [19], there is an exact solution for spatially homogeneous linear Fokker–Planck equation. In the inhomogeneous case we can also obtain an exact solution of the Cauchy problem (1.4). Denote by

$$\hat{f}(t,\eta,\xi) = \mathcal{F}_{x,v}\big(f(t,x,v)\big)$$

the partial Fourier transformation of f with respect to (x,v) variable. Then, by Fourier transformation for (x,v) variables, the linear Fokker–Planck equation (1.4) becomes

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{f}(t,\eta,\xi) - \eta\cdot\nabla_\xi \hat{f}(t,\eta,\xi) + \xi\cdot\nabla_\xi \hat{f}(t,\eta,\xi) = -|\xi|^2\hat{f}(t,\eta,\xi), \\ \hat{f}|_{t=0} = \mathcal{F}(f_0)(\eta,\xi). \end{cases}$$

Therefore we obtain the exact solution

$$\hat{f}(t,\xi,\eta) = \hat{f}\big(0,\xi e^{-t} + \eta\big(1-e^{-t}\big),\eta\big)\exp\left(-\int_0^t \big|\xi e^{\tau-t} + \eta\big(1-e^{\tau-t}\big)\big|^2\,d\tau\right).$$

Note that

$$\begin{aligned} &\int_0^t \big|\xi e^{-\tau} + \eta\big(1-e^{-\tau}\big)\big|^2\,d\tau \\ &\quad = \frac{1-e^{-2t}}{2}|\xi|^2 + \big(1-e^{-t}\big)^2\xi\cdot\eta + \left(t - \frac{3+e^{-2t}}{2} + 2e^{-t}\right)|\eta|^2 \\ &\quad = \left(X - \frac{X^2}{2}\right)|\xi|^2 + X^2\xi\cdot\eta + \left(-\log(1-X) - X - \frac{X^2}{2}\right)|\eta|^2, \end{aligned}$$

where $X = 1-e^{-t} \sim t$. We have for $0 < K < 2/3$

$$\int_0^t \big|\xi e^{-\tau} + \eta\big(1-e^{-\tau}\big)\big|^2\,d\tau \geqslant X\big(1 - 1/(2K) - X/2\big)|\xi|^2 + (1/3 - K/2)X^3|\eta|^2.$$

Hence for $t \sim X < 2 - 1/K$, we get

$$f(t,\cdot,\cdot) \in \mathcal{A}^{1/2}\big(\mathbb{R}^{2d}\big),$$

so that the ultra-analytic effect holds for any $t > 0$ by means of the semi-group property. But we cannot get the uniform estimate (1.5).

We present now the proof of (1.5) which implies the ultra-analytic effect, by commutator estimates similarly as for homogeneous Landau equation. Set

$$w(t,\eta,\xi) = \hat{f}(t,\eta,\xi - t\eta).$$

Then the Cauchy problem (1.4) is equivalent to

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,\eta,\xi) = -|\xi - t\eta|^2 w(t,\eta,\xi) - (\xi - t\eta)\cdot\nabla_\xi w(t,\eta,\xi), \\ w|_{t=0} = \mathcal{F}(f_0)(\eta,\xi). \end{cases} \tag{3.1}$$

Since we need to study the function $\int_0^t |\xi - s\eta|^2\,ds$, we prove the following estimate.

Lemma 3.1. *For any $\alpha > 0$, there exists a constant $c_\alpha > 0$ such that*

$$\int_0^t |\xi - s\eta|^\alpha\,ds \geqslant c_\alpha\big(t|\xi|^\alpha + t^{\alpha+1}|\eta|^\alpha\big). \tag{3.2}$$

Remark 3.1. If $\alpha = 2$, we can get the above estimate by direct calculation. The following simple proof is due to Seiji Ukai.

Proof of Lemma 3.1. Setting $s = t\tau$ and $\tilde\eta = t\eta$, we see that the estimate is equivalent to

$$\int_0^1 |\xi - \tau\tilde\eta|^\alpha\,d\tau \geqslant c_\alpha\big(|\xi|^\alpha + |\tilde\eta|^\alpha\big).$$

Since this is trivial when $\tilde\eta = 0$, we may assume $\tilde\eta \neq 0$. If $|\xi| < |\tilde\eta|$ then

$$\begin{aligned} \int_0^1 |\xi - \tau\tilde\eta|^\alpha\,d\tau &\geqslant |\tilde\eta|^\alpha \int_0^1 \left|\tau - \frac{|\xi|}{|\tilde\eta|}\right|^\alpha d\tau \\ &= |\tilde\eta|^\alpha\left\{\int_0^{|\xi|/|\tilde\eta|}\left(\frac{|\xi|}{|\tilde\eta|} - \tau\right)^\alpha d\tau + \int_{|\xi|/|\tilde\eta|}^1 \left(\tau - \frac{|\xi|}{|\tilde\eta|}\right)^\alpha d\tau\right\} \\ &\geqslant \frac{|\tilde\eta|^\alpha}{\alpha+1}\min_{0\leqslant\theta\leqslant1}\big(\theta^{\alpha+1} + (1-\theta)^{\alpha+1}\big) = \frac{|\tilde\eta|^\alpha}{2^\alpha(\alpha+1)} \\ &\geqslant \frac{1}{2^{\alpha+1}(\alpha+1)}\big(|\xi|^\alpha + |\tilde\eta|^\alpha\big). \end{aligned}$$

If $|\xi| \geqslant |\tilde\eta|$ then

$$\begin{aligned} \int_0^1 |\xi - \tau\tilde\eta|^\alpha\,d\tau \geqslant |\xi|^\alpha \int_0^1 \left(1 - \tau\frac{|\tilde\eta|}{|\xi|}\right)^\alpha d\tau \geqslant |\xi|^\alpha \int_0^1 (1-\tau)^\alpha\,d\tau \\ = \frac{|\xi|^\alpha}{\alpha+1} \geqslant \frac{1}{2(\alpha+1)}\big(|\xi|^\alpha + |\tilde\eta|^\alpha\big). \end{aligned}$$

Hence we obtain (3.2). □

Set now

$$\phi(t,\eta,\xi)=c_0\left(\int_0^t|\xi-s\eta|^2\,ds-\frac{c_2}{2}t^3|\eta|^2\right),$$

where $c_0>0$ is a small constant to choose later, and c_2 is the constant in (3.2) with $\alpha=2$. Then (3.2) implies

$$\phi(t,\eta,\xi)\geqslant c_0\frac{c_2}{2}\left(t|\xi|^2+t^3|\eta|^2\right). \tag{3.3}$$

Let $N=(2d+1)/4$. For $0<\delta<1/4N^2$ and $t>0$, set

$$G_\delta=G_\delta(t,\eta,\xi)=\frac{e^{\phi(t,\eta,\xi)}}{(1+\delta e^{\phi(t,\eta,\xi)})(1+\delta(|\eta|^2+|\xi|^2))^N}. \tag{3.4}$$

Since $G_\delta(t,\cdot,\cdot)\in L^\infty(\mathbb{R}^{2d})$, we can use it as Fourier multiplier, denoted by

$$\left(G_\delta(t,D_x,D_v)u\right)(t,x,v)=\mathcal{F}^{-1}_{\eta,\xi}\left(G_\delta(t,\eta,\xi)\hat{u}(t,\eta,\xi)\right).$$

Lemma 3.2. *Assume that $f(t,\cdot)\in L^2(\mathbb{R}^{2d}_{x,v})\cap L^1_1(\mathbb{R}^{2d}_{x,v})$ for any $t\in\,]0,T[$. Then $\nabla_\xi w(t,\eta,\xi)\in L^\infty(\mathbb{R}^{2d}_{\eta,\xi})$, and*

$$|\xi-t\eta|G_\delta(t,\eta,\xi)^2\bar{w}(t,\eta,\xi),\ |\eta|G_\delta(t,\eta,\xi)^2\bar{w}(t,\eta,\xi),\ \nabla_\xi\left(G_\delta(t,\eta,\xi)^2\bar{w}(t,\eta,\xi)\right) \tag{3.5}$$

belong to $L^2(\mathbb{R}^{2d}_{\eta,\xi})$ for any $t\in\,]0,T[$.

Proof. Since $\partial_{\xi_j}w=-i\mathcal{F}(v_jf)$, it follows from $f\in L^1_1(\mathbb{R}^{2d}_{x,v})$ that $\nabla_\xi w(t,\eta,\xi)\in L^\infty(\mathbb{R}^{2d}_{\eta,\xi})$. Noting

$$|\xi-t\eta|G_\delta(t,\eta,\xi)^2,\,|\eta|G_\delta(t,\eta,\xi)^2\in L^\infty\left(\mathbb{R}^{2d}_{\eta,\xi}\right),$$

we see that the first two terms of (3.5) are obvious. To check the last term in (3.5), note

$$\begin{aligned}\partial_{\xi_j}G_\delta(t,\eta,\xi)&=2c_0t\left(\xi_j-\frac{t}{2}\eta_j\right)G_\delta(t,\eta,\xi)\frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})}\\&\quad-\frac{2N\delta\xi_j}{(1+\delta(|\eta|^2+|\xi|^2))}G_\delta(t,\eta,\xi).\end{aligned} \tag{3.6}$$

Then, we have

$$\begin{aligned}\nabla_\xi\left(G_\delta(t,\eta,\xi)^2\bar{w}(t,\eta,\xi)\right)&=G_\delta(t,\eta,\xi)^2\nabla_\xi\bar{w}(t,\eta,\xi)+\nabla_\xi\left(G_\delta(t,\eta,\xi)^2\right)\bar{w}(t,\eta,\xi)\\&=G_\delta(t,\eta,\xi)^2\nabla_\xi\bar{w}(t,\eta,\xi)\\&\quad+4c_0t\left(\xi-\frac{t}{2}\eta\right)\frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})}G_\delta(t,\eta,\xi)^2\bar{w}(t,\eta,\xi)\\&\quad-\frac{4N\delta\xi}{(1+\delta(|\eta|^2+|\xi|^2))}G_\delta(t,\eta,\xi)^2\bar{w}(t,\eta,\xi).\end{aligned}$$

Since $G_\delta(t,\eta,\xi)^2 \in L^2(\mathbb{R}^{2d}_{x,v})$ we have

$$G_\delta(t,\eta,\xi)^2 \nabla_\xi \bar{w}(t,\eta,\xi) \in L^2\big(\mathbb{R}^{2d}\big).$$

Using

$$\left|\frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})}\right| \leqslant 1, \qquad \left|\frac{2N\delta\xi}{(1+\delta(|\eta|^2+|\xi|^2))}\right| \leqslant 1,$$

and

$$\begin{aligned}
&\left|\left(\xi - \frac{t}{2}\eta\right) G_\delta(t,\eta,\xi)^2 \frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})} \bar{w}(t,\eta,\xi)\right| \\
&\quad \leqslant \left|\xi - \frac{t}{2}\eta\right| G_\delta(t,\eta,\xi)^2 \big|\bar{w}(t,\eta,\xi)\big| \\
&\quad \leqslant |\xi - t\eta| G_\delta(t,\eta,\xi)^2 \big|\bar{w}(t,\eta,\xi)\big| + \frac{t}{2}|\eta| G_\delta(t,\eta,\xi)^2 \big|\bar{w}(t,\eta,\xi)\big| \in L^2\big(\mathbb{R}^{2d}\big).
\end{aligned}$$

We have proved Lemma 3.2. □

We take now $G_\delta(t,\eta,\xi)^2 \bar{w}(t,\eta,\xi)$ as test function in the equation of (3.1). Then we have

$$\begin{aligned}
&\frac{d}{dt}\big\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\big\|^2_{L^2(\mathbb{R}^{2d})} + 2\int_{\mathbb{R}^{2d}} \big|(\xi - t\eta)G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big|^2\, d\eta\, d\xi \\
&\quad = 2\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} w(t,\eta,\xi)\overline{\big(\partial_{\xi_j}(\xi_j - t\eta_j)G_\delta(t,\eta,\xi)^2 w(t,\eta,\xi)\big)}\, d\eta\, d\xi \\
&\qquad + \big(\big(\partial_t G_\delta(t,\cdot,\cdot)\big)w(t,\cdot,\cdot), G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\big)_{L^2(\mathbb{R}^{2d})}.
\end{aligned} \tag{3.7}$$

We prove now the following:

Proposition 3.1. *We have*

$$\begin{aligned}
&\big(\big(\partial_t G_\delta(t,\cdot,\cdot)\big)w, G_\delta(t,\cdot,\cdot)w\big)_{L^2(\mathbb{R}^{2d})} \\
&\quad = c_0\int_{\mathbb{R}^{2d}} \big|(\xi - t\eta)G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big|^2\, d\eta\, d\xi \\
&\qquad - \frac{3}{2}c_0c_2t^2\int_{\mathbb{R}^{2d}} |\eta|^2\big|G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big|^2 \frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})}\, d\eta\, d\xi.
\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
&\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} w(t,\eta,\xi)\overline{\partial_{\xi_j}\big((\xi_j-t\eta_j)G_\delta(t,\eta,\xi)^2 w(t,\eta,\xi)\big)}\,d\eta\,d\xi\\
&\leqslant\left(2c_0t+\frac{c_0t^2}{3c_2}+c_0\right)\int_{\mathbb{R}^{2d}}\big|(\xi-t\eta)G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big|^2\,d\eta\,d\xi\\
&\quad+\frac{d+2N^2\delta/c_0}{2}\big\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\big\|^2_{L^2(\mathbb{R}^{2d})}\\
&\quad+\frac{3}{4}c_0c_2t^2\int_{\mathbb{R}^{2d}}|\eta|^2\big|G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big|^2\frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})}\,d\eta\,d\xi.
\end{aligned}
\tag{3.9}
$$

Proof. The estimate (3.8) is deduced from

$$
\partial_t G_\delta(t,\eta,\xi)=c_0\left(|\xi-t\eta|^2-\frac{3}{2}c_2t^2|\eta|^2\right)G_\delta(t,\eta,\xi)\frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})}.
$$

Since it follows from (3.6) that

$$
\begin{aligned}
\mathcal{I}&=\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} w(t,\eta,\xi)\overline{\partial_{\xi_j}\big((\xi_j-t\eta_j)G_\delta(t,\eta,\xi)^2 w(t,\eta,\xi)\big)}\,d\eta\,d\xi\\
&=\mathrm{Re}\,2c_0t\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}(\xi_j-t\eta_j)\left(\xi_j-\frac{t}{2}\eta_j\right)\big|G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big|^2\frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})}\,d\eta\,d\xi\\
&\quad-\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}\frac{2N\delta\xi_j(\xi_j-t\eta_j)}{(1+\delta(|\eta|^2+|\xi|^2))}\big|G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big|^2\,d\eta\,d\xi\\
&\quad-\mathrm{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}(\xi_j-t\eta_j)\big(\partial_{\xi_j}G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big)\overline{G_\delta(t,\eta,\xi)w(t,\eta,\xi)}\,d\eta\,d\xi,
\end{aligned}
$$

we get

$$
\begin{aligned}
\mathcal{I}&=2c_0t\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}(\xi_j-t\eta_j)\left(\xi_j-\frac{t}{2}\eta_j\right)\big|G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big|^2\frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})}\,d\eta\,d\xi\\
&\quad-\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}\frac{2N\delta\xi_j(\xi_j-t\eta_j)}{(1+\delta(|\eta|^2+|\xi|^2))}\big|G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big|^2\,d\eta\,d\xi+\frac{d}{2}\big\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\big\|^2_{L^2(\mathbb{R}^{2d})}\\
&=2c_0t\int_{\mathbb{R}^{2d}}|\xi-t\eta|^2\big|G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big|^2\frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})}\,d\eta\,d\xi\\
&\quad+c_0t^2\int_{\mathbb{R}^{2d}}(\xi-t\eta)\cdot\eta\big|G_\delta(t,\eta,\xi)w(t,\eta,\xi)\big|^2\frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})}\,d\eta\,d\xi
\end{aligned}
$$

$$- \sum_{j=1}^{d} \int_{\mathbb{R}^{2d}} \frac{2N\delta\xi_j(\xi_j - t\eta_j)}{(1+\delta(|\eta|^2+|\xi|^2))} \left|G_\delta(t,\eta,\xi)w(t,\eta,\xi)\right|^2 d\eta\, d\xi$$

$$+ \frac{d}{2} \left\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\right\|^2_{L^2(\mathbb{R}^{2d})}.$$

For the last term, noting

$$\sum_{j=1}^{d} \frac{2N\delta\xi_j(\xi_j - t\eta_j)}{(1+\delta(|\eta|^2+|\xi|^2))} \leqslant \frac{(N^2/c_0)\delta^2|\xi|^2 + c_0|\xi - t\eta|^2}{(1+\delta(|\eta|^2+|\xi|^2))} \leqslant N^2\delta/c_0 + c_0|\xi - t\eta|^2,$$

we finally obtain

$$\mathcal{I} \leqslant \left(2c_0 t + \frac{c_0 t^2}{3c_2} + c_0\right) \int_{\mathbb{R}^{2d}} \left|(\xi - t\eta)G_\delta(t,\eta,\xi)w(t,\eta,\xi)\right|^2 d\eta\, d\xi$$

$$+ \frac{d + 2N^2\delta/c_0}{2} \left\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\right\|^2_{L^2(\mathbb{R}^{2d})}$$

$$+ \frac{3}{4} c_0 c_2 t^2 \int_{\mathbb{R}^{2d}} |\eta|^2 \left|G_\delta(t,\eta,\xi)w(t,\eta,\xi)\right|^2 \frac{1}{(1+\delta e^{\phi(t,\eta,\xi)})}\, d\eta\, d\xi.$$

Thus we have proved Proposition 3.1. □

End of proof of Theorem 1.2. Now Eq. (3.7), the estimate (3.8) and (3.9) deduce

$$\frac{d}{dt} \left\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\right\|^2_{L^2(\mathbb{R}^{2d})} + \left(2 - 3c_0 - 4c_0 t - \frac{2c_0 t^2}{3c_2}\right) \int_{\mathbb{R}^{2d}} \left|(\xi - t\eta)G_\delta(t,\eta,\xi)w(t,\eta,\xi)\right|^2 d\eta\, d\xi$$

$$\leqslant \left(d + 2N^2\delta/c_0\right) \left\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\right\|^2_{L^2(\mathbb{R}^{2d})}.$$

Then for any $0 < T_0 < T$ choose $c_0 > 0$ (depends on T_0) small enough such that

$$2 - 3c_0 - 4c_0 T_0 - \frac{2c_0 T_0^2}{3c_2} \geqslant 0,$$

then for any $0 < t \leqslant T_0$,

$$\frac{d}{dt} \left\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\right\|_{L^2(\mathbb{R}^{2d})} \leqslant \frac{d + 2N^2\delta/c_0}{2} \left\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\right\|_{L^2(\mathbb{R}^{2d})},$$

which gives

$$\left\|G_\delta(t,\cdot,\cdot)w(t,\cdot,\cdot)\right\|_{L^2(\mathbb{R}^{2d})} \leqslant e^{\frac{d+2N^2\delta/c_0}{2}t} \|f_0\|_{L^2(\mathbb{R}^{2d})}.$$

Take $\delta \to 0$, we have

$$\int_{\mathbb{R}^{2d}} e^{c_0\int_0^t|\xi-s\eta|^2\,ds-c_1t^3|\eta|^2}\left|\hat f(t,\eta,\xi-t\eta)\right|^2 d\eta\,d\xi$$
$$=\int_{\mathbb{R}^{2d}} e^{c_0\int_0^t|\xi+(t-s)\eta|^2\,ds-c_1t^3|\eta|^2}\left|\hat f(t,\eta,\xi)\right|^2 d\eta\,d\xi\leqslant e^{dt}\|f_0\|^2_{L^2(\mathbb{R}^{2d})}.$$

By using (3.3), we get finally

$$\left\|e^{-\tilde c_0(t\Delta_v+t^3\Delta_x)}f(t,\cdot,\cdot)\right\|_{L^2(\mathbb{R}^{2d})}\leqslant e^{\frac{d}{2}t}\|f_0\|_{L^2(\mathbb{R}^{2d})}$$

for any $0<t\leqslant T_0$, where $\tilde c_0=\frac{c_0c_2}{2}>0$. This is the desired estimate (1.5), which implies

$$f(t,\cdot,\cdot)\in\mathcal{A}^{1/2}\big(\mathbb{R}^{2d}\big).$$

We have thus proved Theorem 1.2. □

4. Linear model of inhomogeneous Landau equations

We prove now Theorem 1.3 in this section. By the change of variables $(t,x,v)\to(t,x+vt,v)$, the Cauchy problem (1.8) is reduced to

$$\begin{cases} f_t=(\nabla_v-t\nabla_x)\big(\bar a(\mu)\cdot(\nabla_v-t\nabla_x)f-\bar b(\mu)f\big),\\ f|_{t=0}=g_0(x,v),\end{cases}\tag{4.1}$$

where $f(t,x,v)=g(t,x+vt,v)$. Recall that

$$\bar a_{ij}(\mu)=a_{ij}\star\mu=\delta_{ij}\big(|v|^2+1\big)-v_iv_j,$$
$$\bar b_j(\mu)=\sum_{i=1}^d(\partial_{v_i}a_{ij})\star\mu=-v_j,\quad i,j=1,\dots,d,$$

and

$$\sum_{ij=1}^d\bar a_{ij}(\mu)\xi_i\xi_j\geqslant|\xi|^2,\quad\text{for all }(v,\xi)\in\mathbb{R}^{2d}.$$

In view of this Cauchy problem, we set

$$\Psi(t,\eta,\xi)=c_0\int_0^t|\xi-s\eta|\,ds,$$

for a sufficiently small $c_0>0$ which will be chosen later on. Then we can use (3.2) with $\alpha=1$ to estimate Ψ. Set

$$F_\delta(t,\eta,\xi)=\frac{e^\Psi}{(1+\delta e^\Psi)(1+\delta\Psi)^N}$$

for $N = d+1$, $0 < \delta \leqslant \frac{1}{N}$. If A is a first order differential operator of (t, η, ξ) variables then we have

$$AF_\delta = \left(\frac{1}{1+\delta e^{\Psi}} - \frac{N\delta}{1+\delta\Psi}\right)(A\Psi)F_\delta, \tag{4.2}$$

and

$$\left|\frac{1}{1+\delta e^{\Psi}} - \frac{N\delta}{1+\delta\Psi}\right| \leqslant 1.$$

Taking

$$F_\delta(t, D_x, D_v)^2 f = F_\delta(t)^2 f \in H^{2N}\big(\mathbb{R}^{2d}\big)$$

as a test function in the weak solution formula of (4.1), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|F_\delta(t)f\right\|^2_{L^2(\mathbb{R}^{2d})} + \big(\bar{a}(\mu)\big((\nabla_v - t\nabla_x)F_\delta(t)f\big), \big((\nabla_v - t\nabla_x)F_\delta(t)f\big)\big)_{L^2(\mathbb{R}^{2d})} \\
&\quad = -\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} v_j f\overline{\big((\partial_{v_j} - t\partial_{x_j})F_\delta(t)^2 f\big)}\,dx\,dv + \frac{1}{2}\big((\partial_t F_\delta)f, F_\delta(t)f\big)_{L^2(\mathbb{R}^{2d})} \\
&\qquad + \sum_{j,k=1}^{d}\int_{\mathbb{R}^{2d}}\big\{\bar{a}_{jk}(\mu)\big(F_\delta(t)(\partial_{v_j} - t\partial_{x_j})\big)f - F_\delta(t)\big(\bar{a}_{jk}(\mu)(\partial_{v_j} - t\partial_{x_j})f\big)\big\}\overline{\big((\partial_{v_k} - t\partial_{x_k})F_\delta(t)f\big)}\,dx\,dv.
\end{aligned}$$

We prove now the following results.

Proposition 4.1. *We have*

$$\left\|(\nabla_v - t\nabla_x)F_\delta(t)f\right\|^2_{L^2(\mathbb{R}^{2d})} \leqslant \big(\bar{a}(\mu)\big((\nabla_v - t\nabla_x)F_\delta(t)f\big), \big((\nabla_v - t\nabla_x)F_\delta(t)f\big)\big)_{L^2(\mathbb{R}^{2d})}, \tag{4.3}$$

$$\left|\big(\big(\partial_t F_\delta(t)\big)f, F_\delta(t)f\big)_{L^2}\right| \leqslant c_0\left\|(\nabla_v - t\nabla_x)F_\delta(t)f\right\|_{L^2}\left\|F_\delta(t)f\right\|_{L^2}, \tag{4.4}$$

$$-\operatorname{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}} v_j f\overline{\big((\partial_{v_j} - t\partial_{x_j})F_\delta(t)^2 f\big)} \leqslant \frac{d}{2}\left\|F_\delta(t)f\right\|^2_{L^2} + c_0 t\left\|(\nabla_v - t\nabla_x)F_\delta f(t)\right\|_{L^2}\left\|F_\delta f(t)\right\|_{L^2}. \tag{4.5}$$

Proof. The estimate (4.3) is a direct consequence of the elliptic condition (1.7). Using the Fourier transformation and noting (4.2), we see that (4.4) is derived from

$$\partial_t F_\delta(t, \eta, \xi) = \left(\frac{1}{1+\delta e^{\Psi}} - \frac{N\delta}{1+\delta\Psi}\right)(\partial_t\Psi)F_\delta, \quad \partial_t\Psi = c_0|\xi - t\eta|.$$

For (4.5), we have firstly

$$-\operatorname{Re}\sum_{j=1}^{d}\int_{\mathbb{R}^6} v_j F_\delta(t)f\overline{\big((\partial_{v_j} - t\partial_{x_j})F_\delta(t)f\big)} = \frac{d}{2}\left\|F_\delta(t)f\right\|^2_{L^2}.$$

For the commutators $[v_j, F_\delta(t)]$, using Fourier transformation, we have that for $t > 0$ and $\hat{f} = \hat{f}(t, \eta, \xi)$

$$
\begin{aligned}
&-\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}\big(\big[F_\delta(t,D_x,D_v),v_j\big]f(t,x,v)\big)\overline{\big((\partial_{v_j}-t\partial_{x_j})F_\delta(t,D_x,D_v)f(t,x,v)\big)}\,dx\,dv\\
&\quad=-\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}\big(F_\delta(t,D_x,D_v)v_jf(t)-v_jF_\delta(t,D_x,D_v)f(t)\big)\overline{\big((\partial_{v_j}-t\partial_{x_j})F_\delta(t,D_v)f(t)\big)}\,dx\,dv\\
&\quad=\sum_{j=1}^{3}\int_{\mathbb{R}^{2d}}\big\{i\partial_{\xi_j}\big(F_\delta(t,\eta,\xi)\hat f(t)\big)-F_\delta(t,\eta,\xi)\big(i\partial_{\xi_j}\hat f(t)\big)\big\}F_\delta(t,\eta,\xi)\overline{i(\xi_j-t\eta_j)\hat f(t)}\,d\eta\,d\xi\\
&\quad=\sum_{j=1}^{d}\int_{\mathbb{R}^{2d}}\big(\partial_{\xi_j}F_\delta(t,\eta,\xi)\big)\hat f(t)(\xi_j-t\eta_j)F_\delta(t,\eta,\xi)\overline{\hat f(t)}\,d\eta\,d\xi\\
&\quad\leqslant c_0t\int_{\mathbb{R}^{2d}}|\xi-t\eta|\big|F_\delta(t,\eta,\xi)\hat f(t)\big|^2\,d\eta\,d\xi\leqslant c_0t\big\|(\nabla_v-t\nabla_x)F_\delta f(t)\big\|_{L^2}\big\|F_\delta f(t)\big\|_{L^2},
\end{aligned}
$$

where, in view of (4.2), we have used the fact that

$$
\left|\sum_{j=1}^{d}(\partial_{\xi_j}\Psi)(t,\eta,\xi)\times(\xi_j-t\eta_j)\right|\leqslant c_0\int_0^1\left|\sum_{j=1}^{3}\frac{\xi_j-s\eta_j}{|\xi-s\eta|}(\xi_j-t\eta_j)\right|ds\leqslant c_0t|\xi-t\eta|.
$$

Thus (4.5) has been proved. □

For the commutator terms, we have

Proposition 4.2. *There exists a constant $C_1>0$ independent of $\delta>0$ such that*

$$
\begin{aligned}
&\left|\sum_{j,k=1}^{d}\int_{\mathbb{R}^{2d}}\big\{\bar a_{jk}(\mu)\big(F_\delta(t)(\partial_{v_j}-t\partial_{x_j})\big)f-F_\delta(t)\big(\bar a_{jk}(\mu)(\partial_{v_j}-t\partial_{x_j})f\big)\big\}\overline{\big((\partial_{v_k}-t\partial_{x_k})F_\delta(t)f\big)}\right|\\
&\quad\leqslant C_1\big\{(c_0t)^2\big\|(\nabla_v-t\nabla_x)F_\delta(t)f\big\|_{L^2}^2+\big\|F_\delta(t)f\big\|_{L^2}^2\big\}.
\end{aligned}\tag{4.6}
$$

Proof. In order to prove (4.6), we introduce the polar coordinates of ξ centered at $t\eta$, that is,

$$
r=|\xi-t\eta|\quad\text{and}\quad\omega=\frac{\xi-t\eta}{|\xi-t\eta|}\in\mathbb{S}^{d-1}.
$$

Note again that $\partial/\partial\xi_j=\omega_j\partial/\partial r+r^{-1}\Omega_j$ where Ω_j is a vector field on $\mathbb{S}^{d-1}$. We have again

$$
\sum_{j=1}^{d}\omega_j\Omega_j=0,\qquad\sum_{j=1}^{d}\Omega_j\omega_j=d-1.
$$

By means of Plancherel formula, we have

$$
\sum_{j,k=1}^{d}\int_{\mathbb{R}^{2d}}\big\{\bar a_{jk}(\mu)\big(F_\delta(t)(\partial_{v_j}-t\partial_{x_j})\big)-F_\delta(t)\big(\bar a_{jk}(\mu)(\partial_{v_j}-t\partial_{x_j})f\big)\big\}\overline{\big((\partial_{v_k}-t\partial_{x_k})F_\delta(t)f\big)}
$$

$$= -\int_{\mathbb{R}^{2d}} \left\{ \sum_{j,k=1}^{d} (\xi_k - t\eta_k)\big[(\delta_{jk}\Delta_\xi - \partial_{\xi_k}\partial_{\xi_j}), F_\delta(t,\eta,\xi)\big](\xi_j - t\eta_j)\hat{f}(t) \right\} \overline{F_\delta(t,\eta,\xi)\hat{f}(t)}\, d\xi\, d\eta$$
$$= J.$$

Noting again

$$\Delta_\xi = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\sum_{l=1}^{d} \Omega_l^2,$$

we have with $\tilde{F}_\delta(t,\eta,r,\omega) = F_\delta(t,\eta,r\cdot\omega + t\eta) = F_\delta(t,\eta,\xi)$

$$-\sum_{j,k=1}^{d} \omega_k \left[\left(\delta_{jk} \left\{ \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\sum_{l=1}^{d}\Omega_l^2 \right\} - \left\{ \left(\omega_k \frac{\partial}{\partial r} + r^{-1}\Omega_k\right)\left(\omega_j\frac{\partial}{\partial r} + r^{-1}\Omega_j\right) \right\} \right), \tilde{F}_\delta \right] \omega_j$$
$$= -\left[\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}, \tilde{F}_\delta\right] + \left[\left(\sum_{k=1}^{d} \left(\omega_k^2 \frac{\partial}{\partial r} + r^{-1}\omega_k\Omega_k\right) \sum_{j=1}^{d}\left(\omega_j^2\frac{\partial}{\partial r} + r^{-1}\Omega_j\omega_j\right)\right), \tilde{F}_\delta \right]$$
$$- \frac{1}{r^2}\sum_{j=1}^{d}\omega_j \left[\sum_{l=1}^{d}\Omega_l^2, \tilde{F}_\delta\right]\omega_j = A_1 + A_2 + A_3.$$

Note again that

$$A_1 + A_2 = -\left[\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}, \tilde{F}_\delta\right] + \left[\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\frac{d-1}{r}, \tilde{F}_\delta\right] = 0.$$

On the other hand, we have in view of (4.2)

$$A_3 = -\frac{1}{r^2}\sum_{j,l=1}^{d} \omega_j\big(2\Omega_l[\Omega_l, \tilde{F}_\delta] - \big[\Omega_l, [\Omega_l, \tilde{F}_\delta]\big]\big)\omega_j$$
$$= -\frac{1}{r^2}\sum_{j,l=1}^{d}\omega_j\Bigg(2\Omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi} - \frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)\tilde{F}_\delta$$
$$- \left(\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi} - \frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)^2 + \left(\Omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi} - \frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)\right)\right)\tilde{F}_\delta\Bigg)\omega_j.$$

Putting $w_j = \omega_j \tilde{F}_\delta w$ with $w(t,\eta,r,\omega) = \hat{f}(t,\eta,r\cdot\omega + t\eta)$, we have

$$J = \operatorname{Re} J = \operatorname{Re}\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}} r^2(A_3 w)\overline{\tilde{F}_\delta w} r^{d-1}\, dr\, d\omega\, d\eta$$
$$= -\sum_{j,l=1}^{d}\operatorname{Re}\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}} \left\{2\Omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^\Psi} - \frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)w_j\right\}\overline{w_j} r^{d-1}\, dr\, d\omega\, d\eta$$

$$+\sum_{j,l=1}^{d}\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}\left(\left(\frac{(\Omega_l\Psi)}{1+\delta e^{\Psi}}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)^2+\left(\Omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^{\Psi}}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)\right)\right)|w_j|^2r^{d-1}\,dr\,d\omega\,d\eta$$
$$=J_1+J_2.$$

Since $\Omega_l^*=-\Omega_l+(d-1)\omega_l$, the integration by parts gives

$$J_1=-\sum_{j,l=1}^{d}\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}\left\{\left(\Omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^{\Psi}}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)\right)\right.$$
$$\left.+(d-1)\omega_l\left(\frac{(\Omega_l\Psi)}{1+\delta e^{\Psi}}-\frac{N\delta(\Omega_l\Psi)}{1+\delta\Psi}\right)\right\}|w_j|^2r^{d-1}\,dr\,d\omega\,d\eta.$$

Hence we obtain

$$J=\sum_{j,l=1}^{d}\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}\left\{\left(\frac{1}{1+\delta e^{\Psi}}-\frac{N\delta}{1+\delta\Psi}\right)^2(\Omega_l\Psi)^2\right.$$
$$\left.-(d-1)\omega_l\left(\frac{1}{1+\delta e^{\Psi}}-\frac{N\delta}{1+\delta\Psi}\right)(\Omega_l\Psi)\right\}|w_j|^2r^{d-1}\,dr\,d\omega\,d\eta$$
$$=\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}\left\{\left(\frac{1}{1+\delta e^{\Psi}}-\frac{N\delta}{1+\delta\Psi}\right)^2\left(\sum_{l=1}^{d}(\Omega_l\Psi)^2\right)\right.$$
$$\left.-(d-1)\left(\frac{1}{1+\delta e^{\Psi}}-\frac{N\delta}{1+\delta\Psi}\right)\left(\sum_{l=1}^{d}\omega_l(\Omega_l\Psi)\right)\right\}|\tilde F_\delta w|^2r^{d-1}\,dr\,d\omega\,d\eta. \tag{4.7}$$

Since there exists a constant $C_d>0$ such that

$$|\Omega_l\Psi|=c_0r\left|\sum_{j=1}^{d}\int_0^t\frac{\xi_j-s\eta_j}{|\xi-s\eta|}\,ds(\Omega_l\omega_j)\right|\leqslant c_0C_dtr, \tag{4.8}$$

we have

$$|J|\leqslant C_d'\left\{(c_0t)^2\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}r^2|\tilde F_\delta w|^2r^{d-1}\,dr\,d\omega\,d\eta+\int_{\mathbb{R}^d_\eta}\int_0^\infty\int_{S^{d-1}}|\tilde F_\delta w|^2r^{d-1}\,dr\,d\omega\,d\eta\right\},$$

which yields (4.6). The proof of Proposition 4.2 is now complete. □

End of proof of Theorem 1.3. From Propositions 4.1 and 4.2, there exist constants $C_2, C_3>0$ independent of $\delta>0$ and $t>0$ such that

$$\frac{1}{2}\frac{d}{dt}\left\|(F_\delta f)(t)\right\|^2_{L^2(\mathbb{R}^{2d})}+\left(\frac{1}{2}-(c_0t)^2C_2\right)\left\|(\nabla_v-t\nabla_x)(F_\delta f)(t)\right\|^2_{L^2(\mathbb{R}^{2d})}$$
$$\leqslant C_3\left\|(F_\delta f)(t)\right\|^2_{L^2(\mathbb{R}^{2d})}.$$

So that if $\frac{1}{2} - (c_0 t)^2 C_2 \geqslant 0$, we have

$$\frac{d}{dt}\big\|(F_\delta f)(t)\big\|_{L^2(\mathbb{R}^{2d})} \leqslant C_3\big\|(F_\delta f)(t)\big\|_{L^2(\mathbb{R}^{2d})}. \tag{4.9}$$

Using the fact $(F_\delta f)(0) = \frac{1}{1+\delta} g_0$, we get

$$\big\|(F_\delta f)(t)\big\|_{L^2(\mathbb{R}^{2d})} \leqslant e^{C_3 t}\|g_0\|_{L^2(\mathbb{R}^{2d})}.$$

Take the limit $\delta \to 0$. Then we have

$$\int_{\mathbb{R}^{2d}} e^{2\Psi(t,\eta,\xi)}\big|\hat{f}(t,\eta,\xi)\big|^2\,d\eta\,d\xi \leqslant e^{2C_3 t}\|g_0\|^2_{L^2(\mathbb{R}^{2d})}. \tag{4.10}$$

On the other hand, by Lemma 3.1, there exists a $c_1 > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^{2d}} e^{2\Psi(t,\eta,\xi)}\big|\hat{f}(t,\eta,\xi)\big|^2\,d\eta\,d\xi &= \int_{\mathbb{R}^{2d}} e^{2c_0\int_0^t|\xi-s\eta|\,ds}\big|\hat{g}(t,\eta,\xi-t\eta)\big|^2\,d\eta\,d\xi\\ &= \int_{\mathbb{R}^{2d}} e^{2c_0\int_0^t|\xi+(t-s)\eta|\,ds}\big|\hat{g}(t,\eta,\xi)\big|^2\,d\eta\,d\xi\\ &\geqslant \int_{\mathbb{R}^{2d}} e^{2c_0c_1(t|\xi|+t^2|\eta|)}\big|\hat{g}(t,\eta,\xi)\big|^2\,d\eta\,d\xi.\end{aligned}$$

Finally, for any $0 < T_0 < T$, choosing $c_0 > 0$ small enough such that $\frac{1}{2} - (c_0 T_0)^2 C_2 \geqslant 0$, we have proved

$$\int_{\mathbb{R}^{2d}} \big|e^{c_0c_1(t(-\Delta_v)^{1/2}+t^2(-\Delta_x)^{1/2})}g(t,x,v)\big|^2\,dx\,dv \leqslant e^{2C_3 t}\|g_0\|^2_{L^2(\mathbb{R}^{2d})} \quad \text{for any } 0 < t \leqslant T_0,$$

which completes the proof of Theorem 1.3 with $C = 2C_3$ depending only on d. □

Remark 4.1. The formulas (4.7) and (4.8) show that we cannot get the ultra-analytic effect of order 1/2 as in Theorem 1.2. It is the same reason why we do not consider the symmetric term $Q(g,\mu)$ in Eq. (1.8) as in [1].

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