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Satgraphs and independent domination. Part 1

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Abstract

A graph G is called a *satgraph* if there exists a partition $A \cup B = V(G)$ such that

- A induces a clique [possibly, $A = \emptyset$],
- B induces a matching [i.e., $G(B)$ is a 1-regular subgraph, possibly, $B = \emptyset$], and
- there are no triangles (a, b, b'), where $a \in A$ and $b, b' \in B$.

We also introduce the hereditary closure of $\mathscr{SAT}$, denoted by $\mathscr{HSAT}$ [hereditary satgraphs]. The class $\mathscr{HSAT}$ contains split graphs. In turn, $\mathscr{HSAT}$ is contained in the class of all $(1, 2)$-split graphs [A. Gyárfás, Generalized split graphs and Ramsey numbers, J. Combin. Theory Ser. A 81 (2) (1998) 255–261], the latter being still not characterized. We characterize satgraphs in terms of forbidden induced subgraphs.

There exist close connections between satgraphs and the satisfiability problem [SAT]. In fact, SAT is linear-time equivalent to finding the independent domination number in the corresponding satgraph. It follows that the independent domination problem is NP-complete for the hereditary satgraphs. In particular, it is NP-complete for perfect graphs.

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1. Introduction

We denote by $G(X)$ the subgraph of a graph G induced by a set $X \subseteq V(G)$. An *induced matching* in a graph is a 1-regular induced subgraph.

Definition 1. A graph G is called a *satgraph* if there exists a partition $A \cup B = V(G)$ such that

(A): A induces a complete subgraph [possibly, $A = \emptyset$], and
(B): $G(B)$ is an induced matching [possibly, $B = \emptyset$], and
(AB): there are no triangles (a, b, b'), where $a \in A$ and $b, b' \in B$.

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We shall refer to the pair (A, B) as a *satpartition* of G.

The class of all satgraphs will be denoted by $\mathcal{SAT}$. If a satgraph G is considered along with a satpartition $A \cup B = V(G)$, then we shall use the notation (G, A, B). A similar convention will be adopted for all considered classes defined in terms of vertex bipartitions.

We shall show that the well-known satisfiability problem can be considered as the independent domination problem in satgraphs, and conversely. This "bridge" between a Boolean problem and a graph-theoretical problem has some interesting consequences in view of computational complexity.

Clearly, $\mathcal{SAT}$ is not a hereditary class, i.e., it is not closed under taking induced subgraphs. We introduce its *hereditary closure*, that is, a minimal hereditary class containing $\mathcal{SAT}$. As usual, $\Delta(G)$ is the maximum vertex degree of G.

Definition 2. A graph G is called a *hereditary satgraph* if there exists a partition $A \cup B = V(G)$ such that (A), (AB), and (B′) hold, where
(B′)**:** $\Delta(G(B)) \leqslant 1$ [possibly, $B = \emptyset$].
The pair (A, B) is called a *hereditary satpartition* of G.

The class of all satgraphs will be denoted by $\mathcal{HSAT}$. Clearly, $\mathcal{HSAT}$ is a hereditary closure of $\mathcal{SAT}$. We shall characterize the class $\mathcal{HSAT}$ in terms of forbidden induced subgraphs. Some close connections between satgraphs and some known classes of graphs [split graphs, generalized split graphs, Chvátal–Slater graphs, polar graphs, and perfect graphs] will be shown.

Now we recall some known definitions that will be used. Let G be a graph. A set $S \subseteq V(G)$ is called a *stable set* (or an *independent set*) if no vertices in S are adjacent. A set $D \subseteq V(G)$ is called a *dominating set* if each vertex $u \in V(G) \setminus D$ is adjacent to a vertex of D. A set $I \subseteq V(G)$ is called an *independent dominating set* if I is both independent and dominating. Equivalently, an independent dominating sets are exactly inclusion-wise maximal stable sets.

The following three decision problems are known to be NP-complete problems.

Decision Problem 1 (stability).
Instance: A *graph G and an integer k.*
Question: *Does G have a stable set S with* $|S| \geqslant k$?

Decision Problem 2 (domination).
Instance: *A graph G and an integer k.*
Question: Does G have a dominating set D with $|D| \leqslant k$?

Decision Problem 3 (independentdomination).
Instance: *A graph G* and an integer k.
Question: *Does G have an independent dominating set I with* $|I| \leqslant k$?

As usual, terms *minimal* and *maximal* are related to inclusion-wise minimal/maximal sets, while *minimum* and *maximum* refer to minimum/maximum cardinality.

2. Connection with the satisfiability problem

The following satisfiability problem (or SAT) is well-known [9], see also [13].

Decision Problem 4 (SAT).
Instance: *A collection* $C = \{c_1, c_2, \dots, c_m\}$ *of clauses over a set* $X = \{x_1, x_2, \dots, x_n\}$ *of* 0–1 *variables.*
Question: *Is there a truth assignment for X that satisfies all the clauses in C*?

Recall that a *clause* over X is a conjunction of some literals, a *literal* being either a variable $x_i \in X$ or a negation of a variable $x_j \in X$, denoted by $\overline{x}_j$. A truth assignment $\mathbf{x}^0$ *satisfies* a clause c_i if $c_i(\mathbf{x}^0) = 1$. In other words, c_i involves at least one true literal l [l is a *true literal* if $l = 1$ according to the assignment $\mathbf{x}^0$].

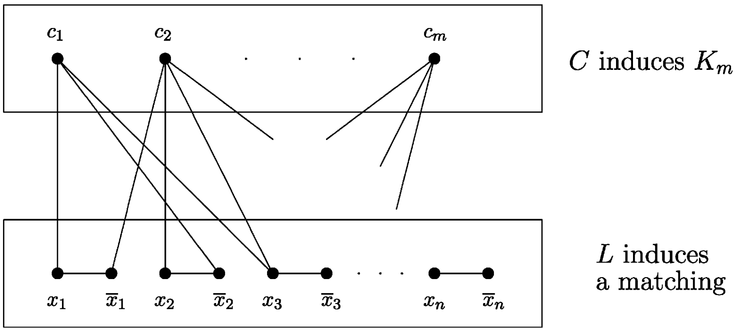

Fig. 1. An example of the Chvátal–Slater graph.

Definition 3. A Chvátal–Slater graph G associated to an instance (C, X) to SAT is defined as follows:

- The graph G has vertex-set $L \cup C$, where $L = \{x_1, \overline{x}_1, x_2, \overline{x}_2, \ldots, x_n, \overline{x}_n\}$ is the set of all literals over X.
- Edge-set of G is defined by the following rules:
  - L induces a matching with edges $x_i\overline{x}_i$, $i = 1, 2, \ldots, n$,
  - C induces a complete subgraph, and
  - a vertex $l \in L$ is adjacent to a vertex $c \in C$ if and only if the clause c involves the literal l.

Fig. 1 gives an illustration of the Chvátal–Slater graph. There $c_1 = x_1 \vee \overline{x}_2 \vee x_3$. Clearly, G can be constructed in linear time in $m = |C|$ and $n = |X|$. Chvátal and Slater [8] used this construction to prove that the not-well covered graph problem is NP-complete, see also Sankaranarayana and Stewart [25]. Recall that a graph is *well covered* if all its maximal stable sets have the same cardinality.

The class $\mathcal{CS}$ of all Chvátal–Slater graphs includes satgraphs as a proper subclass [due to condition (AB)]. However, condition (AB) is not restrictive, since we may assume without loss of generality that each instance (C, X) to SAT satisfies the following assumption.

Assumption 1 (Assumption (CL)). No clause in C involves a variable $x_i \in X$ and its negation $\overline{x}_i$ simultaneously.

Indeed, if the assumption does not hold for some clause $c_i \in C$, then every truth assignment for X satisfies c_i, and therefore c_i can be deleted from C. In terms of the Chvátal–Slater graph, Assumption (CL) corresponds to condition (AB). Given an instance (C, X) to SAT, we shall assume that it satisfies (LC), and we refer to the Chvátal–Slater graph of (C, X) as to the *corresponding satgraph* (G, A, B), where $A = C$ and $B = L$.

Proposition 1. *Let* (C, X) *be an instance to SAT satisfying Assumption* (*CL*). *Then the Chvátal–Slater graph* (G, A, B) *corresponding to* (C, X) *is a satgraph.*

Conversely, *each satgraph* (G, A, B) *is a Chvátal–Slater graph corresponding to* (C, X) *satisfying Assumption* (*CL*).

Now we establish connections between SAT and the independent domination problem within (hereditary) satgraphs.

Theorem 1. (i) *The* satisfiability problem *and the* independent domination problem *for satgraphs are linear-time equivalent*.

(ii) *The* satisfiability problem *and the* independent problem *for hereditary satgraphs are linear-time equivalent*.

Proof. It will be shown in Corollaries 4 and 5 below that given a (hereditary) satgraph, we can find a (hereditary) satpartition in polynomial time. Therefore, we may assume that each (hereditary) satgraph is given along with a (hereditary) satpartition.

(i) Let (C, X) be an instance of SAT. As it was noted, we may assume without loss of generality that (C, X) satisfies Assumption (CL). The corresponding Chvátal–Slater graph G is a satgraph with a satpartition $A \cup B$ of G.

Let $|B| = 2n$. Each minimal independent dominating set in G has cardinality either n or $n+1$. If there exists a satisfying truth assignment for (C, X), then set of true literals gives an independent dominating set of cardinality n. Otherwise G has no independent dominating sets of cardinality n. Indeed, if an independent dominating set I in G contains a vertex $c \in A$, then I also contains n vertices from B as it follows from condition (AB).

Thus, there exists a satisfying truth assignment for (C, X) if and only if G has an independent dominating set of cardinality n.

(ii) We show that the independent domination problem is polynomial-time equivalent for satgraphs and hereditary satgraphs. Then we may use the result (i). Since $\mathcal{SAT} \subseteq \mathcal{HSAT}$, it is sufficient to construct a polynomial-time reduction from the independent domination problem for hereditary satgraphs to the same problem for satgraphs.

Let G be a hereditary satgraph with a hereditary satpartition (A, B). We denote by n the number of components in $G(B)$. Condition (AB) implies that for each vertex $a \in A$ all independent dominating sets that contains a have the same cardinality, say $\imath_a$. Clearly, $\imath_a$ can be easily calculated by a greedy algorithm. If $\min\{\imath_a : a \in A\} \leqslant n$, then we have found a minimum independent dominating set. Indeed, each independent dominating set that is disjoint from A contains exactly n vertices. If $\min\{\imath_a : a \in A\} \geqslant n+1$, then we construct a satgraph G' by deleting the set I_B of all isolated vertices in $G(B)$ from G. It is easy to see that condition $\min\{\imath_a : a \in A\} \geqslant n+1$ implies that I_B is contained in all independent dominating sets of G. Therefore a set I' is an independent dominating set in G' if and only if $I = I' \cup I_B$ is an independent dominating set in G. □

Corollary 1. *The* independent domination problem *is NP-complete for both* $\mathcal{SAT}$ *and* $\mathcal{HSAT}$.

3. Connections with other classes of graphs

The class $\mathcal{CS}$ of all Chvátal–Slater graphs is not hereditary. We consider its hereditary closure: the class of $(1, 2)$-polar graphs introduced by Tyshkevich and Chernyak [26]. As usual, $\overline{G}$ denotes the complement of a graph G.

Definition 4 (Tyshkevich and Chernyak [26]). Let α and β be non-negative integers. A graph G is called an (α, β)-*polar graph* if there exists a partition $A \cup B = V(G)$, called an (α, β)-*partition*, such that

- $\overline{G(A)}$ is a disjoint union of complete graphs, each having at most α vertices [possibly, $A = \emptyset$], and
- $G(B)$ is a disjoint union of complete graphs, each having at most β vertices [possibly, $B = \emptyset$].

Note that Definition 4 was also introduced for the cases where $\alpha = \infty$ and/or $\beta = \infty$. We denote by $\mathcal{POL}(\alpha, \beta)$ the class of all (α, β)-polar graphs.

Proposition 2. *The hereditary closure of the class* $\mathcal{CS}$ *is the class* $\mathcal{POL}(1, 2)$.

Proof. Straightforward. □

A general result of Zverovich and Zverovich [29] guarantees that all the classes $\mathcal{POL}(\alpha, \beta)$ are polynomial-time recognizable [for finite α and β]. Moreover, each of them has a finite forbidden induced subgraph characterization, see Zverovich [27]. Such a characterization for the class $\mathcal{POL}(1, 2)$ was found by Gagarin and Metel'skiĭ [12].

Let Z be a set of graphs. A graph G is called Z-*free* if no graph of Z is an induced subgraph of G.

Theorem 2 (*Gagarin and Metel'skiĭ [12]*). *The class of all* $(1, 2)$-*polar graphs coincides with the class of all* $\mathcal{Z}_{1,2}$-*free graphs*, *where* $\mathcal{Z}_{1,2}$ *consists of the graphs* $G_1, G_2, \ldots, G_{18}$ *shown in Fig.* 2.

Note that a graph G is a $(1, 2)$-polar graph if there exists a partition $A \cup B = V(G)$ such that A induces a complete subgraph, and B induces a (P_3, K_3)-free graph, where P_3 is the 3-path and K_3 is the complete graph of order 3.

It is interesting to compare this class with the following. A graph G is *almost bipartite* if there exists a partition $A \cup B = V(G)$ such that A is a stable set, and B induces a (P_3, K_3)-free graph. Recognizing almost bipartite graphs is an NP-complete problem, see Chernyak and Chernyak [6]. In particular, this class cannot be characterized by a finite number of forbidden induced subgraphs.

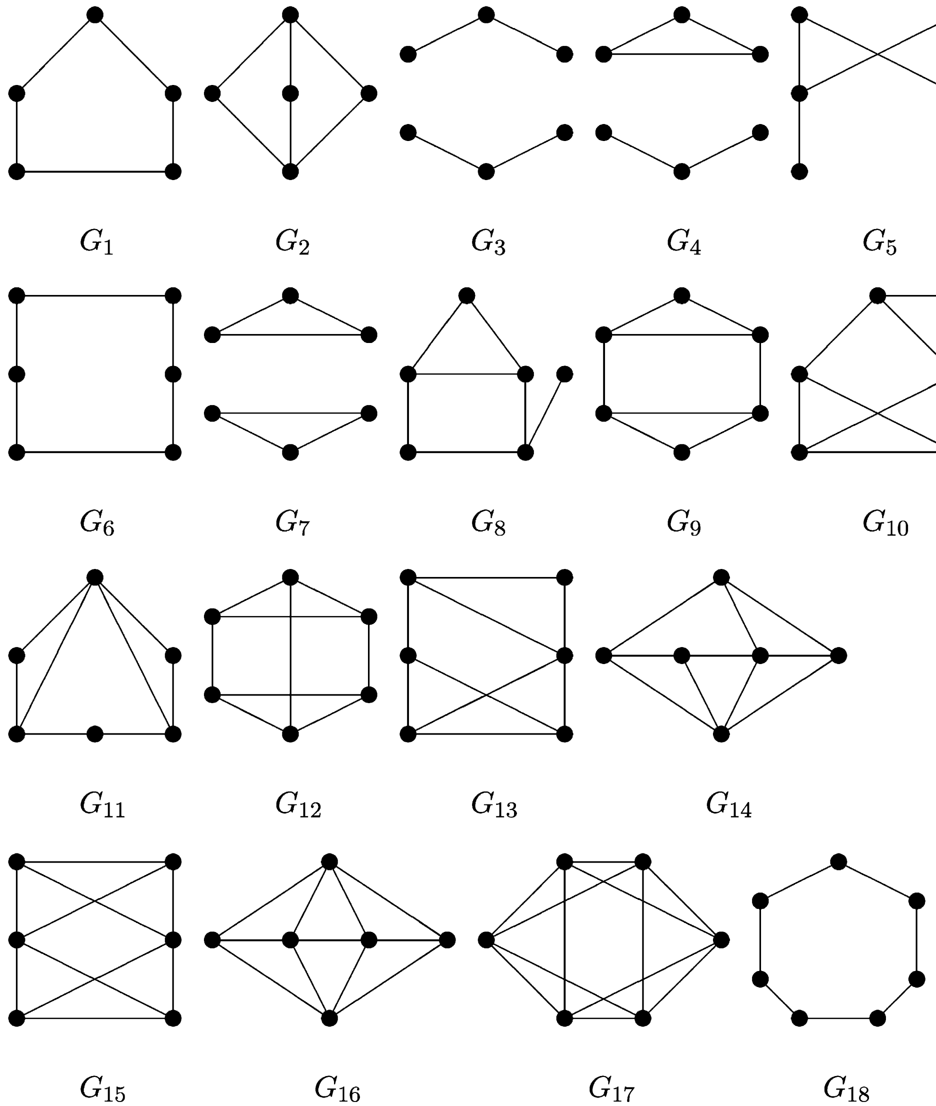

Fig. 2. Minimal forbidden induced subgraphs for the class $\mathcal{POL}(1, 2)$.

The class $\mathcal{HSAT}$ of all hereditary satgraphs includes all split graphs that constitutes a subclass of perfect graphs, see Ramírez Alfonsín and Reed [24].

Definition 5. A graph G is a *split graph* if there exists a partition $A \cup B = V(G)$ such that A induces a complete subgraph, and B is a stable set [*each of* A, B *may be empty*].

Theorem 3 (*Foldes and Hammer [11]*). *The class of all split graphs is characterized by the following three minimal forbidden induced subgraph*: $2K_2$ [the graph of order 4 with two disjoint edges], C_4 [the 4-cycle] *and* C_5 [the 5-cycle].

We show that all hereditary satgraphs are perfect. Recall that a graph G is called *perfect* if $\omega(H) = \chi(H)$ for each induced subgraph H of G, where $\omega(H)$ is the clique number of H [the size of the largest complete subgraph in H], and $\chi(H)$ is the chromatic number of H [the minimum number of colors in proper vertex colorings of H].

The following result can be proved directly, but it immediately follows from the Perfect Graph Theorem of Chudnovsky et al. [7] and Theorem 2.

Proposition 3. *All* $(1, 2)$*-polar graphs are perfect. In particular, all hereditary satgraphs are perfect graphs.*

Let us compare complexity of some optimization on split graphs and on hereditary satgraphs. The domination problem is NP-complete for the split graphs [2], and therefore it is hard for the hereditary satgraphs. The stability problem [even its weighted version] can be solved in polynomial time for both split graphs and hereditary satgraphs, since it holds for the whole class of perfect graphs, see [14]. It is easy to see that the independent domination problem is polynomial-time solvable on split graphs. However, we shall show that it is NP-complete in the class $\mathcal{HSAT}$.

A generalization of split graphs was given by Gyárfás [15]. We restrict ourselves with (1, 2)-split graphs only.

Definition 6. A graph G is called a $(1, 2)$-*split graph* if there exists a partition $A \cup B = V(G)$ such that

(A1): A induces a complete subgraph, and
(B1): B induces a triangle-free subgraph [i.e., a K_3-free graph].

It follows from the definitions that $\mathcal{POL}(1, 2)$ is a subclass of the class $\mathcal{SPLIT}(1, 2)$ of all (1, 2)-split graphs. A result of Gyárfás [15] and a more general result of Zverovich [27] imply that the class $\mathcal{SPLIT}(1, 2)$ has a finite forbidden induced subgraph characterization. However, it is hard to find such a characterization. Currently we know 340 minimal forbidden induced subgraphs for $\mathcal{SPLIT}(1, 2)$; they were found using computer search by Vadim Zverovich (Western England University, UK).

Gyárfás et al. [16] propose a common generalization of the mentioned Gyárfás' result and an interesting theorem of Kézdy et al. [18] on cocolorings of perfect graphs.

Open Problem 1. *Find a finite forbidden induced subgraph characterization of the class* $\mathcal{SPLIT}(1, 2)$.

There is another interesting connection between hereditary satgraphs and split graphs. A graph G is a *locally split graph* if the neighborhood $N(u)$ of each vertex $u \in V(G)$ induces a split graph.

Proposition 4. *All hereditary satgraphs are locally split graphs.*

Proof. Straightforward. □

4. A characterization of hereditary satgraphs

The following theorem gives a forbidden induced subgraph characterization of the class $\mathcal{HSAT}$ of all hereditary satgraphs.

Theorem 4. *The class of all hereditary satgraphs coincides with the class of all* $\mathcal{Z}_{SAT}$*-free graphs, where the set* $\mathcal{Z}_{SAT}$ *consists of the graphs* $F_1, F_2, \dots, F_{21}$ *shown in Fig.* 3.

Proof. *Necessity*. It is easy to check that each of the graphs $F_1, F_2, \dots, F_{21}$ in Fig. 3 is not a hereditary satgraph. Therefore none of them can be an induced subgraph of a hereditary satgraph.

Sufficiency. Let G be a minimal forbidden induced subgraph for the class $\mathcal{HSAT}$. Suppose that the statement does not hold, i.e., G is not isomorphic to any of $F_1, F_2, \dots, F_{21}$. By minimality of G, none of $F_1, F_2, \dots, F_{21}$ is an induced subgraph of G. □

Claim 1. *G is a* $(1, 2)$*-polar graph.*

Proof. By Theorem 2, it is enough to show that G is a $\mathcal{Z}_{1,2}$-free graph, where $\mathcal{Z}_{1,2}$ consists of the graphs $G_1, G_2, \dots, G_{18}$ shown in Fig. 2. The graphs G_{11}, G_{13} and G_{15} contain an induced F_3. The graph G_{17} contains an induced F_4. The other graphs in $\mathcal{Z}_{1,2}$ are contained in $\mathcal{Z}_{SAT}$, see Fig. 3. Therefore G is a $\mathcal{Z}_{1,2}$-free graph. □

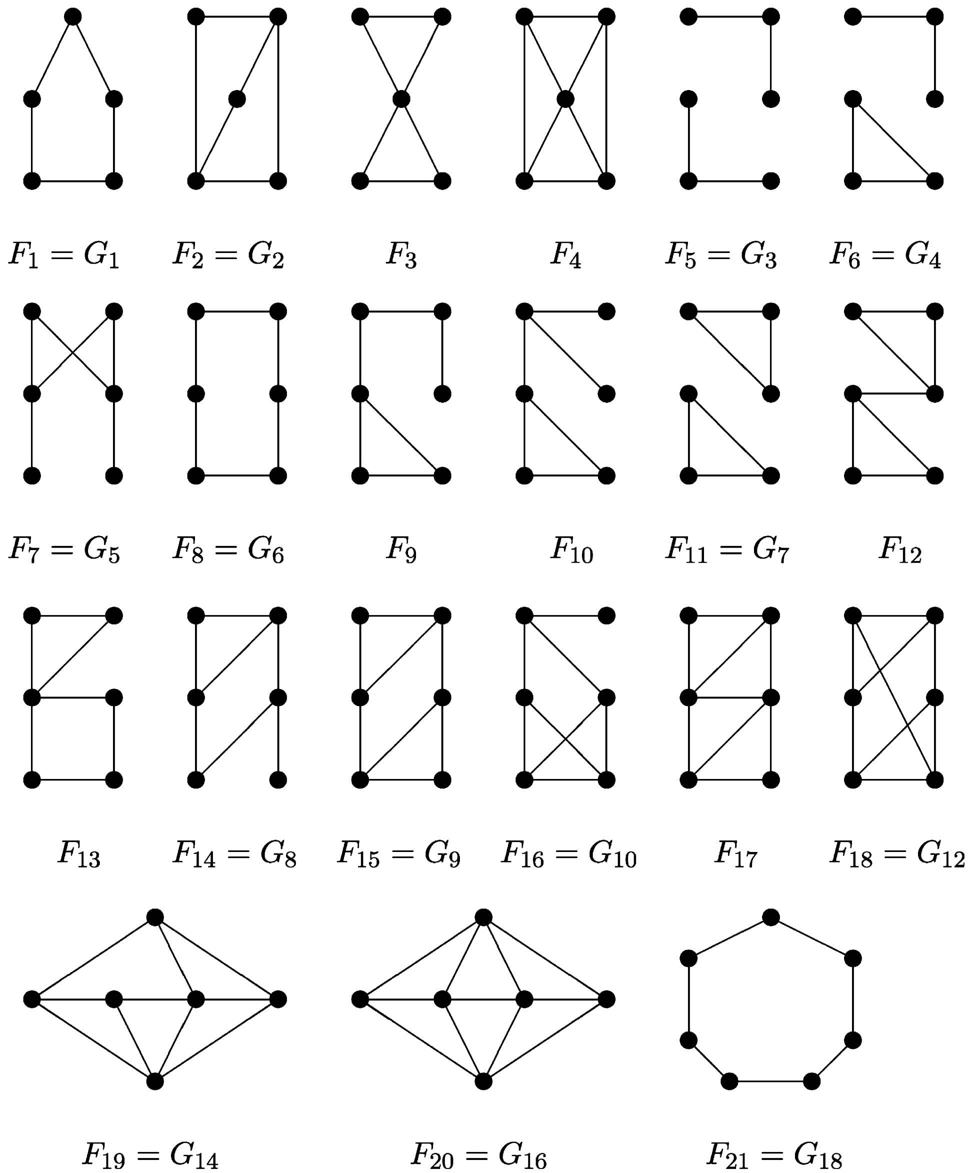

Fig. 3. Minimal forbidden induced subgraphs for the class $\mathcal{HSAT}$.

Given a (1, 2)-polar partition $A \cup B$, we define a *forbidden triangle* in $A \cup B$ as a set $T = \{a, b_1, b_2\}$, where the vertices $a \in A$ and $b_1, b_2 \in B$ induce a triangle. The edge $b_1 b_2$ is called the *base* of T. Clearly, a (1, 2)-polar partition without forbidden triangles is a hereditary satpartition.

By Claim 1, there exists a (1, 2)-polar partition of G. We choose a (1, 2)-polar partition $A \cup B = V(G)$ of G.

Claim 2. *All forbidden triangles in $A \cup B$ have the same base.*

Proof. Suppose that the statement does not hold, i.e., there are two forbidden triangles in $A \cup B$, namely (a, b_1, b_2) and (a', b'_1, b'_2) with different bases $b_1 b_2$ and $b'_1 b'_2$.

Case 1: $a = a'$.

In this case the set $\{a, b_1, b_2, b'_1, b'_2\}$ induces F_3, a contradiction.

Case 2: $a \neq a'$.

In this case the sets $\{b_1, b_2\}$ and $\{b'_1, b'_2\}$ are disjoint. The vertex a is adjacent to at most one of b'_1, b'_2 [otherwise we have essentially Case 1]. Similarly, the vertex a' is adjacent to at most one of b_1, b_2

- If a is non-adjacent to both b'_1 and b'_2, and a' is non-adjacent to both b_1 and b_2, then the set $\{a, a', b_1, b_2, b'_1, b'_2\}$ induces F_{12}, a contradiction.
- If a is non-adjacent to both b'_1 and b'_2, and a' is adjacent to exactly one of b_1, b_2 [we may assume that a' is adjacent to b_1], then the set $\{a, a', b_1, b'_1, b'_2\}$ induces F_3, a contradiction.
- If a is adjacent to exactly one of b'_1, b'_2, and a' is adjacent to exactly one of b_1, b_2, then the set $\{a, a', b_1, b_2, b'_1, b'_2\}$ induces F_{17}, a contradiction.

Thus the both cases are impossible. □

Now we fix a forbidden triangle $T = (a_1, b_1, b_2)$ in $A \cup B$.

Claim 3. *There exist vertices* $a \in A$ *and* $b \in \{b_1, b_2\}$ *such that* $A' \cup B'$ *is a* $(1, 2)$*-polar partition of* G, *where* $A' = (A \setminus \{a\}) \cup \{b\}$ *and* $B' = (B \setminus \{b\}) \cup \{a\}$.

Proof. Let $\overline{N}_i$, $i = 1, 2$, denote the set of all vertices in A that are non-adjacent to the vertex b_i. We show that either $\overline{N}_1 \subseteq \overline{N}_2$ or $\overline{N}_2 \subseteq \overline{N}_1$. If it does not hold, there exist vertices $a'_1 \in \overline{N}_1 \backslash \overline{N}_2$ and $a'_2 \in \overline{N}_2 \backslash \overline{N}_1$. Then the set $\{a_1, a'_1, a'_2, b_1, b_2\}$ induces F_4, a contradiction. By symmetry, we may assume that $\overline{N}_1 \subseteq \overline{N}_2$. We put $b = b_1$.

If a'_1 and a'_2 are two distinct vertices in $\overline{N}_1$, then the set $\{a_1, a'_1, a'_2, b_1, b_2\}$ induces F_3, a contradiction. Therefore $|\overline{N}_1| \leqslant 1$.

If $\overline{N}_1 = \emptyset$, then we can construct a new $(1, 2)$-polar partition $(A \cup \{b_1\}) \cup (B \backslash \{b_1\})$ of G. Claim 2 implies that it is a hereditary satpartition of G, a contradiction. It follows that $\overline{N}_1 = \{a\}$, and the set $A' = (A \backslash \{a\}) \cup \{b\}$ induces a complete subgraph.

It remains to show that B' induces a (P_3, K_3)-free graph. Suppose it does not hold. Clearly, each induced P_3 or K_3 in $G(B')$ must involve the vertex a and two vertices of $B \setminus \{b\}$. Claim 2 implies that $G(B')$ cannot contain K_3. Indeed, otherwise there is a forbidden triangle (a, b_3, b_4) in $A \cup B$ with base $b_3b_4 \neq b_1b_2$.

Since $\overline{N}_1 \subseteq \overline{N}_2$ and $a \in \overline{N}_1$, we have $a \in \overline{N}_2$, i.e., a is non-adjacent to b_2. Hence we have two possibilities for an induced P_3 in $G(B')$:

(a) $P_3 = (a, b_3, b_4)$ [with edges ab_3 and b_3b_4], or
(b) $P_3 = (b_3, a, b_4)$ [with edges b_3a and ab_4].

In both cases, $\{b_3, b_4\} \subseteq B \backslash \{b_1, b_2\}$. By Claim 2, the vertex a_1 can be adjacent at most one of b_3, b_4. It is easy to check that $G(a, a_1, b_1, b_2, b_3, b_4)$ either is isomorphic to one of F_9, F_{10}, F_{13}, or it contains an induced F_3, a contradiction.

To obtain a final contradiction, we show that the pair (A', B') of Claim 3 is a hereditary satpartition of G. To see that, suppose that condition (AB) fails for some vertices $a_2 \in A'$ and $b_3, b_4 \in B'$. In other words, the set $\{a_2, b_3, b_4\}$ induces a triangle T. Clearly, $a_2 \neq b_1$. Claim 2 implies that $a \in T$. Therefore $T = \{a_2, a, b_3\}$, where $b_3 \in B \setminus \{b_1, b_2\}$ [since a is not adjacent to b_2]. Recall that b_1 is adjacent to a.

Here are all possible variants:

- If $a_2 = a_1$, the set $\{a, a_1, b_1, b_2, b_3\}$ induces F_3, a contradiction. Therefore we may assume that a_1 is non-adjacent to b_3, since a_2 is adjacent to a.
- If $a_2 \neq a_1$ and a_2 is non-adjacent to b_2, the set $\{a, a_1, a_2b_1, b_2, b_3\}$ induces F_{12}.
- If $a_2 \neq a_1$ and a_2 is adjacent to b_2, the set $\{a, a_2, b_1, b_2, b_3\}$ induces F_3.

Thus, each of the variants produces a contradiction. □

Corollary 2. *The* independent domination problem *is NP-complete within the class of all* $(F_1, F_2, \ldots, F_{21})$*-free graphs.*

A *linear graph* is an induced subgraph of a path. Complexity of the stability problem and the independent domination problem is unknown for H-free graphs, where H is a linear graph with at least five non-isolated vertices. A linear graph consisting of two disjoint components P_3 is denoted by $2P_3$, see F_5 in Fig. 3.

Corollary 3. *The* independent domination problem *is NP-complete within* $2P_3$*-free graphs.*

In this connection, we can mention an interesting result of Korobitsyn [19]: if a linear graph H is not obtained from an induced subgraph of P_4 by adding isolated vertices [possibly, none], then the domination problem for H-free graphs is NP-complete. In particular, this problem is NP-complete for $2P_3$-free graphs.

Open Problem 2. *What is complexity of the* stability problem *for* $2P_3$*-free graphs*?

Corollary 4. *The class* $\mathcal{HSAT}$ *is polynomial-time recognizable. Moreover, it is possible to construct a hereditary satpartition for any hereditary satgraph in polynomial time.*

Proof. The result of Zverovich and Zverovich [29] mentioned above implies that it is possible to recognize $(1, 2)$-polar graphs in polynomial time. Moreover, a $(1, 2)$-polar partition of a $(1, 2)$-polar graph can be constructed in polynomial time.

Let G be a graph tested on membership in $\mathcal{HSAT}$. Since $\mathcal{HSAT} \subseteq \mathcal{POL}(1, 2)$, we either

- reject G as being not a hereditary satgraph in case of $G \notin \mathcal{POL}(1, 2)$, or
- construct a $(1, 2)$-polar partition (A, B) of G in polynomial time.

Then we apply the Proof of Theorem 4 to (A, B). As a result, we either construct a hereditary satpartition of G, or we find a forbidden induced subgraph F_i in G in polynomial time. □

Corollary 5. *The class* $\mathcal{SAT}$ *is polynomial-time recognizable. Moreover, it is possible to construct a satpartition for any satgraph in polynomial time.*

Proof. Let G be a graph tested on membership in $\mathcal{SAT}$. We apply Corollary 4 to G. If $G \notin \mathcal{HSAT}$, then $G \notin \mathcal{SAT}$, since $\mathcal{SAT} \subseteq \mathcal{HSAT}$. Otherwise we can construct a hereditary satpartition (A, B) of G in polynomial time.

Suppose that there exists a satpartition (A', B') of G. Each of the sets $A \cap B'$ and $A' \cap B$ induces a complete graph, since they are subsets of A and A', respectively. Conditions $\Delta(G(B)) \leqslant 1$ and $\Delta(G(B')) \leqslant 1$ imply that $|A \cap B'| \leqslant 2$ and $|A' \cap B| \leqslant 2$.

Thus, to construct (A', B') or to find out that G has no satpartitions, it is sufficient to consider all variants $((A \backslash X) \cup Y, (B \backslash Y) \cup X)$, where $X \subseteq A$, $|X| \leqslant 2$, $Y \subseteq B$, and $|Y| \leqslant 2$. Clearly, there exist polynomially many such variants. For each of them we can check whether it is a satpartition in polynomial time. □

Corollary 6. *Given a* (*hereditary*) *satgraph*, *it is possible to construct all its* (*hereditary*) *satpartitions in polynomial time*.

Proof. By Corollaries 4 and 5, we can construct one (hereditary) satpartition in polynomial time. Then we proceed as in the Proof of Corollary 5. □

In the accompanying paper we propose some classes, where SAT and/or the independent domination problem can be solved in polynomial time.

5. Uncited reference

[28].

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