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Some properties of Bernoulli polynomials and their generalizations

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A B S T R A C T

In this work, we investigate some well-known and new properties of the Bernoulli polynomials and their generalizations by using quasi-monomial, lowering operator and operational methods. Some of these general results can indeed be suitably specialized in order to deduce the corresponding properties and relationships involving the (generalized) Bernoulli polynomials.

1. Introduction

A polynomial $p_n(x)(n \in \mathbb{N},\ x \in \mathbb{C})$ is said to be a quasi-monomial [1] whenever two operators $\hat{M}, \hat{P}$, called the multiplicative and derivative (or lowering) operators respectively, can be defined in such a way that

$$\hat{P}p_n(x) = np_{n-1}(x), \tag{1.1}$$

$$\hat{M}p_n(x) = p_{n+1}(x), \tag{1.2}$$

which can be combined to get the identity

$$\hat{M}\hat{P}p_n(x) = np_n(x). \tag{1.3}$$

The classical Bernoulli polynomials $B_n(x)$ are defined by [2]

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} \quad (|z| < 2\pi), \tag{1.4}$$

and, consequently, the classical Bernoulli numbers $B_n := B_n(0)$ can be obtained by using the generating function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n\frac{z^n}{n!}. \tag{1.5}$$

Moreover, we have [3]

$$B_n(0) = (-1)^n B_n(1) = \frac{1}{2^{1-n} - 1}B_n\left(\frac{1}{2}\right) \quad (n \in \mathbb{N}_0). \tag{1.6}$$

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The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ are defined by [4]

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!} \quad (|z|<2\pi). \tag{1.7}$$

Clearly, the generalized Bernoulli numbers $B_n^{(\alpha)}$ are given by

$$B_n^{(\alpha)} := B_n^{(\alpha)}(0)$$

and

$$B_n(x) := B_n^{(1)}(x) \quad (n \in \mathbb{N}_0).$$

In 2005, Luo defined the Apostol–Bernoulli numbers $\mathcal{B}_n(\lambda)$ and polynomials $\mathcal{B}_n(x;\lambda)$ as

$$\frac{z}{\lambda e^z-1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\lambda)\frac{z^n}{n!}, \tag{1.8}$$

$$\frac{ze^{xz}}{\lambda e^z-1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\frac{z^n}{n!}, \tag{1.9}$$

$(|z|<2\pi$ when $\lambda=1$; $|z|<|\log\lambda|$ when $\lambda\neq 1)$.

The generalized Apostol–Bernoulli numbers $\mathcal{B}_n^{(\alpha)}(\lambda)$ and polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ are defined by [5]

$$\left(\frac{z}{\lambda e^z-1}\right)^{\alpha} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(\lambda)\frac{z^n}{n!}, \tag{1.10}$$

$$\left(\frac{z}{\lambda e^z-1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}, \tag{1.11}$$

$(|z|<2\pi$ when $\lambda=1$; $|z|<|\log\lambda|$ when $\lambda\neq 1)$.

Clearly, we have

$$B_n^{(\alpha)}(x) = \mathcal{B}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{B}_n^{(\alpha)}(\lambda) := \mathcal{E}_n^{(\alpha)}(0;\lambda), \tag{1.12}$$

$$\mathcal{B}_n(x;\lambda) := \mathcal{B}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathcal{B}_n(\lambda) := \mathcal{B}_n^{(1)}(\lambda). \tag{1.13}$$

The Appell polynomials [6] can be defined by considering the following generating function:

$$A(t)e^{xt} = \sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n, \tag{1.14}$$

where

$$A(t) = \sum_{k=0}^{\infty} \frac{R_k}{k!} t^k, \quad (A(0)\neq 0) \tag{1.15}$$

is an analytic function at $t=0$.

It is easy to see that if $A(t) = \left(\frac{t}{\lambda e^t-1}\right)^{\alpha}$, then $R_n(t) = \mathcal{B}_n^{(\alpha)}(x)$.

From [7], we know that the multiplicative and derivative operators of $R_n(x)$ are

$$\hat{M} = (x+\alpha_0) + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{n-k}, \tag{1.16}$$

$$\hat{P} = D_x, \tag{1.17}$$

where

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}. \tag{1.18}$$

By using (1.3), we have the following lemma.

Lemma 1.1 ([7]). *The Appell polynomials $R_n(x)$ defined by (1.14) satisfy the differential equation*

$$\frac{\alpha_{n-1}}{(n-1)!}y^{(n)}+\frac{\alpha_{n-2}}{(n-2)!}y^{(n-1)}+\cdots+\frac{\alpha_1}{1!}y''+(x+\alpha_0)y'-ny=0, \tag{1.19}$$

where the numerical coefficients $\alpha_k,\ k=1,2,\ldots,n-1$, are defined in (1.19), and are linked to the values R_k by the following relations:

$$R_{k+1}=\sum_{h=0}^{k}\binom{k}{h}R_h\alpha_{k-h}.$$

Let $\mathscr{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$, and the polynomial sequence $\{P_n\}_{n\geq 0}$ be a polynomial set. $\{P_n\}_{n\geq 0}$ is called a σ-Appell polynomial set of the transfer power series A generated by

$$G(x,t)=A(t)G_0(x,t)=\sum_{n=0}^{\infty}\frac{P_n(x)}{n!}t^n, \tag{1.20}$$

where $G_0(x,t)$ is a solution of the system

$$\sigma G_0(x,t)=tG_0(x,t),$$
$$G_0(x,0)=1.$$

In [8], the authors investigated the coefficients of connection between two polynomials. And there is a result concerning coefficients of connection between two σ-Appell polynomial sets.

Lemma 1.2 ([8]). *Let $\sigma\in\Lambda^{(-1)}$. Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be two σ-Appell polynomial sets of transfer power series A_1 and A_2, respectively. Then*

$$Q_n(x)=\sum_{m=0}^{n}\frac{n!}{m!}\alpha_{n-m}P_m(x), \tag{1.21}$$

where

$$\frac{A_2(t)}{A_1(t)}=\sum_{k=0}^{\infty}\alpha_k t^k.$$

Recently, several interesting properties and relationships involving the classical as well as the generalized Bernoulli polynomials and the Apostol–Bernoulli polynomials were investigated [3,9,5,10–15]. In this work, we want to investigate some well-known and new properties of these polynomials from different angles.

In Section 2, we propose to prove some relationships and differential equations involving the generalized Bernoulli polynomials.

In Section 3, we want to consider the problem of connection of the generalized Apostol–Bernoulli polynomials with some other polynomials.

In Section 4, we shall extend the generalized Apostol–Bernoulli polynomials to Hermite-based generalized Apostol–Bernoulli polynomials by an operational method. And some of their properties are given.

2. Recursion formulas and differential equations

From Section 1, we know that the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ are Appell polynomials with $A(t)=\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}$. And by using (1.4), we obtain

$$\frac{A'(t)}{A(t)}=-\alpha\sum_{n=0}^{\infty}\frac{B_{n+1}(1)}{n+1}\frac{t^n}{n!}.$$

By using (1.16)–(1.18), we can obtain the multiplicative and derivative operators of the generalized Bernoulli polynomials:

$$\hat{M}=\left(x-\frac{1}{2}\alpha\right)-\alpha\sum_{k=0}^{n-1}\frac{B_{n-k+1}(1)}{(n-k+1)!}D_x^{n-k},\qquad \hat{P}=D_x. \tag{2.1}$$

From the generating function (1.8), we can easily obtain

$$\frac{\partial^p}{\partial x^p}B_n^{(\alpha)}(x)=\frac{n!}{(n-p)!}B_{n-p}^{(\alpha)}(x). \tag{2.2}$$

By using (1.2), (2.1) and (2.2), we obtain the following result.

Theorem 2.1. *For any integral $n \geq 1$, the following linear recurrence relation for the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ holds true:*

$$B_{n+1}^{(\alpha)}(x) = \left(x - \frac{1}{2}\alpha\right) B_n^{(\alpha)}(x) - \alpha \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_{n-k+1}(1)}{n-k+1} B_k^{(\alpha)}(x). \tag{2.3}$$

Furthermore, applying Lemma 1.1 to $B_n^{(\alpha)}(x)$, we obtain

Theorem 2.2. *The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ satisfy the differential equation*

$$\frac{B_n(1)}{(n)!} y^{(n)} + \frac{B_{n-1}(1)}{(n-1)!} y^{(n-1)} + \cdots + \frac{B_2(1)}{2!} y'' - \left(\frac{x}{\alpha} - \frac{1}{2}\right) y' + \frac{n}{\alpha} y = 0. \tag{2.4}$$

In the special case of (2.3) and (2.4) when $\alpha = 1$, we obtain the following results.

Corollary 2.1. *For any integral $n \geq 1$, the following linear recurrence relation for the Bernoulli polynomials $B_n(x)$ holds true:*

$$B_{n+1}(x) = \left(x - \frac{1}{2}\right) B_n(x) - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_{n-k+1}(1)}{n-k+1} B_k(x). \tag{2.5}$$

Corollary 2.2. *The Bernoulli polynomials $B_n(x)$ satisfy the differential equation*

$$\frac{B_n(1)}{(n)!} y^{(n)} + \frac{B_{n-1}(1)}{(n-1)!} y^{(n-1)} + \cdots + \frac{B_2(1)}{2!} y'' - \left(x - \frac{1}{2}\right) y' + ny = 0. \tag{2.6}$$

3. Connection problems

From the generating function (1.11), we can easily obtain

$$\frac{\partial}{\partial x} \mathcal{B}_n^{(\alpha)}(x;\lambda) = n\mathcal{B}_{n-1}^{(\alpha)}(x;\lambda).$$

So by the definition of the σ-Appell polynomial (1.20), we know that the generalized Apostol–Bernoulli polynomials are a D-Appell polynomial set, D being the derivative operator.

From Table 1 in [8], we know that the lowering operators of monomials x^n and the Gould–Hopper polynomials [16] $g_n^m(x,h)$ are all D. And their transfer power series $A(t)$ are 1 and e^{ht^m} respectively.

Applying Lemma 1.2 to $P_n(x) = x^n$, $Q_n(x) = \mathcal{B}_n^{(\alpha)}(x;\lambda)$, we get the well-known identity [5]

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{B}_{n-m}^{(\alpha)}(\lambda) x^m. \tag{3.1}$$

Applying Lemma 1.2 to $P_n(x) = \mathcal{B}_n(x;\lambda)$, $Q_n(x) = \mathcal{B}_n^{(\alpha)}(x;\lambda)$, we get the well-known identity [5]

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{B}_{n-m}^{(\alpha-1)}(\lambda) \mathcal{B}_m(x;\lambda). \tag{3.2}$$

Applying Lemma 1.2 to $P_n(x) = g_n^m(x,h)$, $Q_n(x) = \mathcal{B}_n^{(\alpha)}(x;\lambda)$, we get a new identity:

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = \sum_{r=0}^{n} \frac{n!}{r!} \left[\sum_{k=0}^{[n-r/m]} (-1)^k \frac{h^k}{k!(n-r-mk)!} \mathcal{B}_{n-r-mk}^{(\alpha)}(\lambda) \right] g_r^m(x,h). \tag{3.3}$$

In particular, for Hermite polynomials, since $H_n(x) = g_n^2(2x,-1)$, we have

$$\mathcal{B}_n^{(\alpha)}(2x;\lambda) = \sum_{r=0}^{n} \frac{n!}{r!} \left[\sum_{k=0}^{[n-r/2]} \frac{1}{k!(n-r-2k)!} \mathcal{B}_{n-r-2k}^{(\alpha)}(\lambda) \right] H_r(x). \tag{3.4}$$

In the special case of (3.1) when $\alpha = 1$ and $\lambda = 1$, we have

$$B_n(x) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m} x^m. \tag{3.5}$$

And if we apply Lemma 1.2 to $P_n(x) = B_n(x)$, $Q_n(x) = x^n$, we can obtain the following familiar expansion [4]:

$$x^n = \frac{1}{n+1} \sum_{m=0}^{n} \binom{n+1}{m} B_m(x). \tag{3.6}$$

4. Hermite-based generalized Apostol–Bernoulli polynomials

The two-variable Hermite–Kampé de Fériet polynomials (2VHKdFP) $H_n(x, y)$ are defined by the series [17]

$$H_n(x, y) = n! \sum_{r=0}^{[n/2]} \frac{x^{n-2r} y^r}{r!(n-2r)!} \tag{4.1}$$

with the following generating function:

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y). \tag{4.2}$$

And the 2VHKdFP $H_n(x, y)$ are also defined through the operational identity

$$\exp\left(y \frac{\partial^2}{\partial x^2}\right) \{x^n\} = H_n(x, y). \tag{4.3}$$

Acting with the operator $\exp\left(y \frac{\partial^2}{\partial x^2}\right)$ on (1.11), and by the identity [18]

$$\exp\left(y \frac{\partial^2}{\partial x^2}\right) \{\exp(-ax^2 + bx)\} = \frac{1}{\sqrt{1+4ay}} \exp\left(-\frac{ax^2 - bx - b^2 y}{1+4ay}\right), \tag{4.4}$$

we define the Hermite-based generalized Apostol–Bernoulli polynomials ${}_H\mathcal{B}_n^{(\alpha)}(x, y; \lambda)$ through the generating function

$$\left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} e^{xz+yz^2} = \sum_{n=0}^{\infty} {}_H\mathcal{B}_n^{(\alpha)}(x, y; \lambda) \frac{z^n}{n!}. \tag{4.5}$$

Clearly, we have

$${}_H\mathcal{B}_n(x, y, \lambda) := {}_H\mathcal{B}_n^{(1)}(x, y, \lambda), \; {}_H\mathcal{B}_n(x, y) = {}_H\mathcal{B}_n^{(1)}(x, y, 1).$$

From the generating function (4.5), we easily obtain

$$\frac{\partial}{\partial x} {}_H\mathcal{B}_n^{(\alpha)}(x, y; \lambda) = n {}_H\mathcal{B}_{n-1}^{(\alpha)}(x, y; \lambda) \tag{4.6}$$

and

$$\frac{\partial}{\partial y} {}_H\mathcal{B}_n^{(\alpha)}(x, y; \lambda) = n(n-1) {}_H\mathcal{B}_{n-2}^{(\alpha)}(x, y; \lambda), \tag{4.7}$$

which can be combined to get the identity

$$\frac{\partial^2}{\partial x^2} {}_H\mathcal{B}_n^{(\alpha)}(x, y; \lambda) = \frac{\partial}{\partial y} {}_H\mathcal{B}_n^{(\alpha)}(x, y; \lambda). \tag{4.8}$$

Acting with the operator $\exp\left(y \frac{\partial^2}{\partial x^2}\right)$ on both sides of (3.1), (3.2) and (3.6), and by using (4.3), we obtain

$${}_H\mathcal{B}_n^{(\alpha)}(x, y; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{B}_{n-m}^{(\alpha)}(\lambda) H_m(x, y), \tag{4.9}$$

$${}_H\mathcal{B}_n^{(\alpha)}(x, y; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{B}_{n-m}^{(\alpha-1)}(\lambda) {}_H\mathcal{B}_m(x, y, \lambda), \tag{4.10}$$

$$H_n(x, y) = \frac{1}{n+1} \sum_{m=0}^{n} \binom{n+1}{m} {}_H B_n(x, y). \tag{4.11}$$

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