Strong Additivity, Absolute Continuity, and Compactness in Spaces of Measures

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In this paper we shall examine strong additivity, absolute continuity, and compactness (weak and strong) in the space of vector measures and discuss various relationships among these concepts. Many of the theorems cast new light on the structure of measures, even in the scalar case. Some of the results presented here have been announced in Brooks and Dinculeanu [8, 9]. Variations on the above themes are also treated in detail in Brooks [7].

In Section 1, the main concept of uniform strong additivity of a family of vector measures $\mathcal{H}$ is introduced, namely, $m(E_i) \to 0$ uniformly for $m \in \mathcal{H}$, when $(E_i)$ is a disjoint sequence of sets. This notion is inextricably tied up with weak and strong compactness of $\mathcal{H}$ in different topological settings. The existence of a positive control measure $\mu$ such that $\mathcal{H} \leq \mu$, when $\mathcal{H}$ is uniformly strongly additive is presented in Section 2. A local control measure is constructed in Section 3 by means of establishing a “synthesis theorem” which allows us to piece together locally equivalent families of positive measures. This theorem is also used to prove the existence of a local control measure for relatively weakly compact sets in the space of vector measures with local finite variation (Section 4). Criteria for weak compactness in this locally convex space (Theorem 4.2) extends the work of Dieudonne [14], who considered a special case, viz, the space of locally integrable functions on a locally compact space. Conditions concerning compactness (weak and strong) with respect to the quasivariation norm are presented in Sections 5

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and 6. The motivation for establishing criteria for weak compactness in spaces of vector measures stems from the problem of classifying weakly compact operators on function spaces, for example, $C_2(S)$, the space of $X$-valued continuous functions on a compact Hausdorff space $S$, since, by using Dinculeanu's representation theorems [16], one can show that an operator $T: C_2(S) \rightarrow Y$ is weakly compact if and only if $T^*(Y_1^*)$ is a relatively weakly compact set in the space of $X^*$-valued measures. These considerations, when $X$ is the scalar field, led Grothendieck [20] and Bartle, Dunford and Schwartz [1] to study weakly compact sets of scalar measures. For applications of compactness in the vector setting to operator theory, the reader is referred to Brooks and Lewis [12, 13].

1. Strong Additivity

Throughout this paper, $\mathcal{R}$, $\mathcal{E}$, and $\mathcal{F}$ will be respectively a ring, a δ-ring and a σ-ring of subsets of a set $T$. A δ-ring is a ring closed under countable intersections. The σ-ring (δ-ring) generated by $\mathcal{R}$ is denoted by $\sigma(\mathcal{R}) (\delta(\mathcal{R}))$. Define $\mathcal{R}_{loc} = \{A \subset T; A \cap R \in \mathcal{R}, \text{for each } R \in \mathcal{R}\}$. Let $X$ be a Banach space with norm $\| \cdot \|$ and conjugate space $X^*$. We denote by $fa(\mathcal{R}, X)$ and $ca(\mathcal{R}, X)$, respectively, the sets of finitely additive and countably additive measures $m: \mathcal{R} \rightarrow X$.

For each $E \subset T$, set $\mathcal{R} \cap E = \{A \in \mathcal{R}; A \subset E\}$; if $m \in fa(\mathcal{R}, X)$, then $m_E$ is the restriction of $m$ to $\mathcal{R} \cap E$; if $\mathcal{K} \subset fa(\mathcal{R}, X)$, set $\mathcal{K}_E = \{m_E; m \in \mathcal{K}\}$. If $P$ is a property involving a class $\mathcal{K}$ of set functions defined on $\mathcal{R}$, we say $\mathcal{K}$ has property $P$ locally providing $\mathcal{K}_E$ has property $P$ on $\mathcal{R} \cap E$ for each $E \in \mathcal{R}$.

Let $m \in fa(\mathcal{R}, X)$. We denote by $\|m\|$, $\tilde{m}$, respectively, the total variation and quasivariation functions of $m$; recall that

$$\tilde{m}(E) = \sup\{|m(A)|; A \subset E, A \in \mathcal{R}\},$$

for $E \subset T$. If $m(A_n) \rightarrow 0$ whenever $(A_n)$ is a disjoint sequence, we say $m$ is strongly additive. In this case one can show that $\sum m(A_n)$ is unconditionally convergent. A set $\mathcal{K} \subset fa(\mathcal{R}, X)$ is uniformly strongly additive if $m(A_n) \rightarrow 0$ uniformly with respect to $m \in \mathcal{K}$, whenever the $A_n$ are disjoint. Note that if $\mathcal{K} \subset ca(\mathcal{F}, X)$, then $\mathcal{K}$ is uniformly strongly additive if and only if $\mathcal{K}$ is uniformly countably additive. The concept of strong additivity was introduced by Rickart [25] (under the name "strongly bounded") and was later used in [2, 6, 10] for interchange of limit theorems and the existence of control measures. A strongly additive measure is bounded, but the converse is false. On the other hand, $m$ is strongly additive (locally strongly additive) in any of the following cases: (i) $m$ is scalar valued and bounded (locally bounded);
(ii) $m$ has bounded variation (finite variation on each set); (iii) $m$ is countably additive and $\mathcal{R}$ is a $\sigma$-ring ($\delta$-ring). Note that a countably additive measure on a ring need not be strongly additive—for example, consider a spectral measure defined on an infinite Stone algebra.

**Lemma 1.1.** Let $m: \mathcal{R} \to \mathcal{X}$ be strongly additive. Then there exists a sequence of sets $A_n \in \mathcal{R}$ such that $m$ vanishes outside $\bigcup A_n$. In particular, if $m$ is defined on a $\sigma$-ring $\mathcal{S}$, then $m$ vanishes outside a set belonging to $\mathcal{S}$.

**Proof.** Apply Zorn's lemma to the set of all families $(E_n)$ of pairwise disjoint sets from $\mathcal{R}$ such that $m(E_n) \neq 0$, and obtain a maximal family $(E_n)_{n \in \Delta}$. Since $m$ is strongly additive, $\Delta$ is at most countable. The lemma then follows.

**Lemma 1.2.** Let $\mathcal{X} \subseteq ca(\mathcal{R}, \mathcal{X})$ be a family of locally uniformly strongly additive measures. Then for every monotone sequence of sets $A_n \to A$, with $A_n, A \in \mathcal{R}$, we have $\tilde{m}(A_n) \to \tilde{m}(A)$ uniformly with respect to $m \in \mathcal{X}$.

**Proof.** Consider the special case when $A_n \not\subseteq \phi$. Suppose we deny the conclusion. There exists an $\epsilon > 0$ and $m \in \mathcal{X}$ such that $m_n(A_n) \geq \epsilon$, $n = 1, 2, \ldots$. Let $A_{n_1} = A_1$ and choose a $B_1 \in \mathcal{R} \cap A_{n_1}$ satisfying $|m_n(B_1)| > \epsilon$. Since $m_1(B_1 \cap A_n) \to 0$, we may choose an $A_{n_2}$ such that

$$|m_1(B_1 \cap A_{n_2})| < \epsilon/2.$$ 

Let $C_1 = B_1 \cap (A_{n_1} - A_{n_2})$. Then $|m_1(C_1)| \geq \epsilon/2$. By induction, obtain $(n_p)$ and $C_p \in \mathcal{R}$ such that $C_p \subseteq A_{n_p} - A_{n_{p+1}}$ and $|m_p(C_p)| \geq \epsilon/2$, which contradicts the assumption of local uniform strong additivity, since $(C_p)$ is a disjoint sequence of sets contained in $A_1$. In general, if $A_n$ is a monotone sequence with limit in $\mathcal{R}$, use the above case and the subadditivity of $\tilde{m}$.

### 2. Absolute Continuity and Control Measures

Absolute continuity ($\epsilon, \delta$ definition) of $m: \mathcal{R} \to \mathcal{X}$ with respect to a positive (not necessarily finite) finitely additive measure $\mu$ is denoted by $m \ll \mu$. Thus, $m \ll \mu$ locally means that for every set $E \subseteq \mathcal{R}$, we have

$$\lim_{\mu(A) \to 0} \frac{m(A)}{\mu(A)} = 0,$$

where $A \subseteq E$. If $\mathcal{X} \subseteq ca(\mathcal{R}, \mathcal{X})$, then $m \ll \mu$ (uniformly) means that $m \ll \mu$ (uniformly) for $m \in \mathcal{X}$. A set $E \subseteq \mathcal{R}$ is a $\mathcal{X}$-null set if

$$\sup \mathcal{X}(A) = \sup \{\tilde{m}(A): m \in \mathcal{X}\} = 0.$$
We say that \( \mu \) is a \((local) \) control measure for \( \mathcal{X} \) if \( \mathcal{X} \ll \mu \) (locally) and \( \mathcal{X} \) and \( \mu \) have the same null sets. The main theorem in Brooks [2] states that \( m : \mathcal{R} \rightarrow \mathcal{X} \) is strongly additive if and only if there exists a bounded control measure \( \mu \) for \( m \) such that \( \mu(A) \ll m\ll \mu(A) \); if \( m \) is countably additive, then \( \mu \) is also countable additive.

The following theorem from Brooks [5], which we include for the reader's convenience, shows the relationship between uniform strong additivity and absolute continuity.

**Theorem 2.1.** Let \( \mathcal{X} \subset \text{fa}(\mathcal{R}, \mathcal{X}) \) and let \( \mu \) be a positive (possibly infinite) finitely additive measure. If \( \mathcal{X} \) is uniformly strongly additive and \( \mathcal{X} \ll \mu \), then \( \mathcal{X} \ll \mu \) uniformly.

**Proof.** If we deny the conclusion of the theorem, there exists an \( \varepsilon > 0 \) and sequences \((m_j), (\delta_k)\) and \((E_j)\) such that \( |m_{k+1}(E_{k+1})| \gg \varepsilon, \mu(E_{k+1}) \ll \delta_{k+1} \) and \( |m_i(E)| \ll \varepsilon/2^{k+3} \) for \( i = 1, \ldots, k \). Let \( F_1 = E_0 \); assume there exists an \( i_2 > 2 \) such that \( m_{i_2}(F_1 \cap E_{i_2}) \gg \varepsilon/4 \). Let \( F_2 = F_1 - E_{i_2} \). In general, assume \( F_1, \ldots, F_k; i_1, \ldots, i_k \) have been chosen and that there exists an \( i_{k+1} > i_k \) such that \( |m_{i_k+1}(F_k \cap E_{i_{k+1}})| \gg \varepsilon/4 \). Let \( F_{k+1} = F_k - E_{i_{k+1}} \). If this process did not terminate, we would obtain a disjoint sequence of sets \((F_k - F_{k+1})\) such that \( |m_{i_k+1}(F_k \cap E_{i_{k+1}})| \gg \varepsilon/4 \). This contradicts the uniform strong additivity of \( \mathcal{X} \). Hence, we can find an \( F_k \) and an \( i_k \) so that \( |m_j(F_k \cap E_j)| < \varepsilon/4 \) for \( j > i_k \). Let

\[
p_1 = i_k, \quad H_1 = F_k, \quad m_i^{(1)} = m_{p_1+i}, \quad E_i^{(1)} = E_{p_1+i} - H_1.
\]

Observe that
\[
|m_2(H_1)| \geq |m_2(E_2)| - \left\{ \sum_{j=1}^{k-1} |m_2(F_j - F_{j+1})| \right\} > \varepsilon - \sum_{j=1}^{k-1} \varepsilon/2^{j+2} > \varepsilon - \varepsilon/4,
\]

since
\[
|m_2(F_j - F_{j+1})| = |m_2(F_j \cap E_{i_{j+1}})| < \varepsilon/2^{j+2}.
\]

By a similar process obtain an \( F_k^{(1)} \) and an \( i_{k'} \), such that
\[
|m_j^{(1)}(F_k^{(1)} \cap E_{i_{k'}})| < \varepsilon/8 \quad \text{for all } j > i_{k'}.
\]

Note that \( |m_i^{(1)}(H_2)| \geq \varepsilon - \varepsilon/4 - \varepsilon/8 \) and \( |m_i^{(1)}(E_i^{(1)})| > \varepsilon - \varepsilon/4 \), where \( H_2 = F_k^{(1)} \). Let

\[
p_2 = i_{k'}, \quad m_i^{(2)} = m_{p_2+i}^{(1)}, \quad \text{and} \quad E_i^{(2)} = E_{p_2+i} - H_2.
\]

At the \( k \)th stage, assume that for \( j = 1, 2, \ldots, k, H_j \) and \( p_j \) are defined so that
\[
E_i^{(j)} = E_{p_{j+1}} - H_j, \quad m_i^{(j)} = m_{p_{j+1}}^{(j-1)}, \quad |m_i^{(j)}(E_i^{(j)})| > \varepsilon - \varepsilon/4 - \cdots - \varepsilon/2^{j+1}
\]
and \( H_1, \ldots, H_k \) are disjoint (set \( m_i^{(0)} = m_{i+1}^{(0)} = E_i^{(0)} = E_i \)). In addition,
\[
| m_i^{(j-1)}(H_j) | > \varepsilon - \varepsilon/4 - \cdots - \varepsilon/2^{j+1}.
\]
As before, if we let \( F_j^{(k)} = E_j^{(k)} \), there exists an \( i_t \) such that
\[
| m_i^{(k)}(F_j^{(k)} \cap E_j^{(k)}) | < \varepsilon/2^{k+3} \quad \text{for all } j > i_t.
\]
Let
\[
p_{k+1} = i_t, \quad H_{k+1} = F_{i_t}^{(k)}, \quad m_{i_t}^{(k+1)} = m_{p_{k+1}}^{(k)} - H_{k+1}.
\]
In this fashion, we obtain a disjoint sequence \( (H_k) \) and \( m_1^{(k)} \) such that
\[
| m_1^{(k)}(H_{k+1}) | > \varepsilon/2, \quad \text{which contradicts the uniform strong additivity of } \mathcal{H}.
\]
Using the above result, we can now establish the following theorem.

**Theorem 2.2.** Suppose that \( \mathcal{H} \subseteq \text{ca}(\mathcal{R}, \mathcal{X}) \) and \( \mathcal{H} \) is uniformly strongly additive. Then each element in \( \mathcal{H} \) can be extended to a unique countably additive measure on \( \sigma(\mathcal{R}) \) and the extensions are uniformly countably additive on \( \sigma(\mathcal{R}) \).

**Proof.** Let \( m \in \mathcal{H} \). By Theorem 2 in [2], there is a bounded countably additive control measure \( \mu \) for \( m \) such that \( \mu \leq m \). Regard \( (\mathcal{R}, \rho) \) as a pseudo metric space, where \( \rho(A, B) = \mu(A \triangle B) \). Note that \( \mu \) can be extended to \( \mu_1 \) on \( \sigma(\mathcal{R}) \) and \( (\mathcal{R}, \rho) \) is dense in \( (\sigma(\mathcal{R}), \rho_1) \). Since \( m \) is uniformly continuous on \( (\mathcal{R}, \rho) \), there exists a unique extension, say \( m_1 \) of \( m \) to \( (\sigma(\mathcal{R}), \rho_1) \). Also, \( \mu_1 \leq m_1 \). For notational convenience, \( \mathcal{H} \) will denote the family of extensions to \( \sigma(\mathcal{R}) \). If \( \mathcal{H} \) is not uniformly countably additive on \( \sigma(\mathcal{R}) \), there exists a sequence of sets \( E_n \in \sigma(\mathcal{R}) \), measures \( m_n \in \mathcal{H} \) and an \( \varepsilon > 0 \) such that
\[
| m_n(E_n) | > \varepsilon, \quad n = 1, 2, \ldots \quad (*)
\]
Let \( \mu = \sum (2^n B_n)^{-1} \mu_n \), where \( B_n \) is a positive bound for \( \mu_n \), the control measure of \( m_n \). By Theorem 2.1, \( \{m_n : n = 1, 2, \ldots\} \leq \mu \) uniformly on \( \mathcal{R} \). We now show uniform absolute continuity on \( \sigma(\mathcal{R}) \). Let
\[
\mathcal{H} = \{x^{*} m_n : n = 1, 2, \ldots, | x^{*} | \leq 1, x^{*} \in \mathcal{X}^{*}\}.
\]
Let \( \varepsilon > 0 \) be given. There exists a \( \delta > 0 \) such that \( | \sigma | \leq \varepsilon \) whenever \( E \in \mathcal{R}, \mu(E) < \delta \) and \( \sigma \in \mathcal{H} \). Let \( A \in \sigma(\mathcal{R}) \) such that \( \mu(A) < \delta \); choose disjoint sets \( R_n \in \mathcal{R} \) covering \( A \) such that \( \mu(\bigcup R_n) < \delta \). Then \( | \sigma | (\bigcup R_n) < \varepsilon \) for all \( n \) and \( \sigma \in \mathcal{H} \). This implies that \( | \sigma | (A) \leq | \sigma | (\bigcup R_n) \leq \varepsilon \), for all \( \sigma \in \mathcal{H} \). Thus, \( \mathcal{H} \leq \mu \) uniformly on \( \sigma(\mathcal{R}) \), which in turn contradicts \( (*) \).

The following remark is deduced from Theorems 2.1 and 2.2.
Remark 2.2. (i) If \( m \in ca(\mathcal{R}, X) \) and \( m \) is strongly additive, then there exists a unique countably additive extension of \( m \) to \( \sigma(\mathcal{R}) \). If \( m \in ca(\mathcal{R}, X) \) and \( m \) is locally strongly additive, then \( m \) can be uniquely extended to a countably additive measure on \( \delta(\mathcal{R}) \). In either case, the quasivariation (variation) of the extension is the extension of the quasivariation (variation).

(ii) Suppose that \( m \in ca(\mathcal{R}, X) \) is locally strongly additive and \( m \) vanishes outside a set \( R \in \mathcal{R}_{loc} \), where \( R = \bigcup R_n \), \( R_n \in \mathcal{R} \). Then one can show that the extension of \( m \) to \( \delta(\mathcal{R}) \) also vanishes outside \( R \). If \( m \in ca(\mathcal{R}, X) \) is locally strongly additive and has local \( \sigma \)-finite variation, then one can show that the extension of \( m \) to \( \delta(\mathcal{R}) \) has local \( \sigma \)-finite variation.

(iii) Suppose \( \mathcal{H} \subset ca(\sigma(\mathcal{R}), X) \) and \( \mu \) is a positive countably additive measure on \( \sigma(\mathcal{R}) \). If \( \mathcal{H} \leq \mu \) uniformly on \( \mathcal{R} \), then \( \mathcal{H} \leq \mu \) uniformly on \( \sigma(\mathcal{R}) \).

The following result extends a theorem in Brooks and Walker [11].

**Theorem 2.3.** Let \( \mathcal{H} \subset f\alpha(\mathcal{R}, X) \). The following two assertions are equivalent:

(a) \( \mathcal{H} \) is uniformly strongly additive;

(b) There exists a bounded control measure \( \mu \) such that \( \mathcal{H} \leq \mu \) uniformly.

Moreover, in (b) we can choose \( \mu \) such that \( \mu(E) \leq \sup \mathcal{H}(E) \). If \( \mathcal{H} \subset ca(\mathcal{R}, X) \), then \( \mu \) in (b) can be chosen to be countably additive. Finally, if \( \mathcal{H} \leq \lambda \), for some measure \( \lambda \), then \( \mathcal{H} \leq \lambda \) uniformly and \( \mu \leq \lambda \).

**Proof.** Obviously (b) implies (a). Assume now that \( \mathcal{H} \) is uniformly strongly additive. Let \( \mathcal{H} = \{x^*m: m \in \mathcal{H}, |x^*| \leq 1\} \). By passing to the Stone ring, we may assume that \( \mathcal{H} \subset ca(\mathcal{R}, C) \); the set of extensions of elements from \( \mathcal{H} \) to \( \sigma(\mathcal{R}) \) will also be denoted by \( \mathcal{H} \). We shall now modify the technique of Bartle, Dunford, and Schwartz [19, p. 307] and prove that for any \( \epsilon > 0 \) there exists a \( \delta(\epsilon) > 0 \) and a finite set \( \{\sigma_1,...,\sigma_n\} \subset \mathcal{H} \) such that \( |\sigma_i| E < \delta \), for all \( 1 \leq i \leq n \), implies that \( |\sigma(E)| < \epsilon \) for all \( \sigma \in \mathcal{H} \). If we deny this, there exists a sequence of sets \( E_i \subset \sigma(\mathcal{R}) \) and \( (\sigma_i) \subset \mathcal{H} \) such that \( |\sigma_i| E_j < \epsilon/2^{j+1} \), for \( 1 \leq i \leq j \), and \( |\sigma_j(E_j)| \geq \epsilon \), \( j = 1, 2,... \). If \( B_n = \bigcup_{i \geq n} E_i \), then \( |\sigma_i| (B_n) \leq \epsilon/2^j \) for \( i < n \), while \( |\sigma_{n+1}(B_n)| \geq \epsilon/2 \). Let \( B = \bigcap B_n \); then for each \( i \), \( \sigma_i(B) = 0 \). Hence, \( (B_n - B) \not\leq \phi \), but \( |\sigma_{n+1}(B_n - B)| \geq \epsilon/2 \), \( n = 1, 2,... \). This contradicts the uniform countable additivity of \( \mathcal{H} \). Let \( \delta_n = \delta(2^{-n}) \), and let

\[
v_n = \sum_{i \leq k(n)} 2^{-i} |\sigma_i^n|,
\]

where \( (\sigma_i^n)_{i=1}^{k(n)} \) is the set corresponding to \( \epsilon = 2^{-n} \). Define \( \nu = \sum (2^n Q_n)^{-1} v_n \), where \( Q_n \) is a positive bound for \( v_n \). Note that \( \mathcal{H} \leq \nu \) uniformly on \( \sigma(\mathcal{R}) \) and \( \nu \leq \sup \mathcal{H} \). The restriction \( \mu \) of \( \nu/4 \) to \( \mathcal{R} \) is the required measure.
Suppose that $\mathcal{N} \subseteq \text{ca}(\mathcal{R}, \mathcal{X})$. If $A_i \not\subseteq \phi$, by Lemma 1.2 we have $\sup \mathcal{N}(A_i) \to 0$; hence, $\mu(A_i) \to 0$. Thus, $\mu$ is countable additive. The last conclusion follows from Theorem 2.1.

3. LOCAL STRONG ADDITIVITY AND LOCAL CONTROL MEASURES

As was mentioned in Section 2, a strongly additive measure on a ring has a bounded control measure. We now consider the setting in which the vector measure is locally strongly additive. We know that locally there is a bounded control measure. The question is: Does there exist a global control measure? To solve this, a synthesis theorem is established which allows us to piece together a family of locally equivalent scalar measures to form a global measure. An example is given to show that in general the global measure is not bounded. A positive measure $\mu$ will be called locally $\sigma$-finite on $\mathcal{R}$ if every element of $\mathcal{R}$ is the countable union of sets in $\mathcal{R}$ of finite $\mu$-measure.

SYNTHESIS THEOREM 3.1. Let $\mathcal{R}$ be a $\delta$-ring. Suppose that for each $R \in \mathcal{R}$ there exists a positive, finite, countably additive measure $\mu_R$ defined on $\mathcal{R} \cap R$ such that for $R \subseteq R'$, $\mu_R$ and $\mu_{R'}$ are mutually absolutely continuous on $\mathcal{R} \cap R$.

Then there exists a positive, locally $\sigma$-finite, countably additive measure $\nu$ on $\mathcal{R}$ such that $\mu_R \ll \nu$ on $\mathcal{R} \cap R$, for every $R \in \mathcal{R}$, and $\nu(E) = 0$ if and only if $\mu_R(E \cap R) = 0$ for every $R \in \mathcal{R}$.

Proof. Let $\mathcal{N} = \{E \in \mathcal{R}: \mu_E(E) = 0\}$. Then $\mathcal{N}$ is an ideal of $\mathcal{R}$. Let $\mathcal{F}$ be the set of all families $\mathcal{A} = \{D_\alpha: \alpha \in \Delta\}$, where each $D_\alpha \in \mathcal{R} - \mathcal{N}$ and if $\alpha_1 \neq \alpha_2$, then $D_{\alpha_1} \cap D_{\alpha_2} \in \mathcal{N}$. Order $\mathcal{F}$ by set inclusion. Observe that $\mathcal{F}$ is inductively ordered. Let $\mathcal{A} = \{D_\alpha: \alpha \in \Delta\}$ be a maximal element in $\mathcal{F}$. Note that, by the maximality of $\mathcal{A}$, we have $A \in \mathcal{N}$ if and only if $A \cap D_\alpha \in \mathcal{N}$ for every $\alpha \in \Delta$, which in turn implies that $\mu_\alpha(A \cap D_\alpha) = 0$ for all $\alpha$ (where $\mu_\alpha = \mu_{D_\alpha}$). For $A \in \mathcal{R} - \mathcal{N}$, the set $\{\alpha: A \cap D_\alpha \notin \mathcal{N}\}$ is nonempty and at most countable. To see the last statement, note that

$$\sum_{\alpha \in I} \mu_\alpha(A \cap D_\alpha) \leq \mu_\alpha(A) < \infty$$

for every finite set $I \subseteq \Delta$; hence,

$$\sum_{\alpha \in I} \mu_\alpha(A \cap D_\alpha) < \infty.$$

Now define $\nu$ on $\mathcal{R}$ by $\nu(A) = \sum_\alpha \mu_\alpha(A \cap D_\alpha)$. To prove that $\nu$ is locally $\sigma$-finite, observe that $\nu(D_\alpha) = \mu_\alpha(D_\alpha) < \infty$. Since there exist at most a countable number of sets, say $D_{\alpha_1}$ such that $D_{\alpha_1} \cap A \notin \mathcal{N}$, we have
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A = \bigcup \{A \cap D_\alpha\} \cup N, \text{ where } N = A - \bigcup D_\alpha. \text{ Since } \mathcal{D} \text{ is a } \delta\text{-ring, } N \in \mathcal{D}. \text{ Also } D_\alpha \cap N \in \mathcal{N} \text{ for every } \alpha; \text{ hence, } \nu(N) = 0. \text{ This shows that } A \text{ is } \sigma\text{-finite. Next we show that } \nu \text{ is countably additive. Let } A \in \mathcal{D}. \text{ Suppose } (D_\alpha) \text{ is the sequence of elements of } \mathcal{A} \text{ such that } A \cap D_\alpha \notin \mathcal{N}. \text{ Consequently, for } E \in \mathcal{D} \cap A, \text{ we have } \nu(E) = \sum \mu_\alpha(D_\alpha \cap E). \text{ It then follows that } \nu|_{\mathcal{D} \cap A} \text{ is the pointwise limit of a countable number of countably additive measures; hence, } \nu \text{ is countably additive on } \mathcal{D} \cap A, \text{ thus, countably additive on } \mathcal{D}. \text{ Next we assert that } \mu_R \ll \nu \text{ on } \mathcal{D} \cap R. \text{ In fact, if } \nu(A) = 0, \text{ for } A \in \mathcal{D} \cap R, \text{ then } \sum \mu_\alpha(A \cap D_\alpha) = 0; \text{ hence, } A \in \mathcal{N}. \text{ Therefore, } \mu_\alpha(A) = 0; \text{ this in turn implies that } \mu_R(A) = 0. \text{ Since } \mu_R \text{ is finite, it follows that } \mu_R \ll \nu \text{ on } \mathcal{D} \cap R. \text{ Finally, suppose that } \mu_E(E \cap A) = 0 \text{ for every } E \in \mathcal{D}. \text{ Then } E \in \mathcal{N}; \text{ hence, } \nu(A) = 0. \text{ I}

Remark 3.1. If \mathcal{R} \text{ is a ring and the } \mu_R \text{ are only finitely additive, then we are still able to obtain a positive finitely additive measure } \nu \text{ on } \mathcal{R} \text{ such that } \mu_R \ll \nu | \mathcal{R} \cap R \text{ for every } R \in \mathcal{R}, \text{ and } \nu(A) = 0 \text{ if and only if } \nu_\alpha(A) = 0 (\text{for example, the trivial measure } \nu(A) = 0 \text{ if } A \in \mathcal{N} \text{ and } \nu(A) = \infty \text{ otherwise}), \text{ but we do not know whether we can choose } \nu \text{ to be locally } \sigma\text{-finite. If the } \mu_R \text{ are countably additive, then } \nu \text{ can be chosen to be a countably additive measure such that it has a (not necessarily unique) locally } \sigma\text{-finite extension to the } \delta\text{-ring generated by } \mathcal{R}.

Theorem 3.2. Let \mathcal{D} be a } \delta\text{-ring and let } \mathcal{N} \subset \text{ca}(\mathcal{D}, \mathcal{X}) \text{ be locally uniformly strongly additive. Then there exists a positive, locally } \sigma\text{-finite, countably additive measure } \mu \text{ defined on } \mathcal{D} \text{ such that } \mathcal{N} \ll \mu \text{ locally uniformly and the } \mathcal{N}\text{-null sets and the } \mu\text{-null sets coincide.}

In particular, a locally strongly additive measure has a control measure of the type described above.

Proof. Let } R \in \mathcal{D}. \text{ Since } \mathcal{N}_R \text{ is uniformly strongly additive on } \mathcal{D} \cap R, \text{ by Theorem 2.3, there exists a } \mu_R \text{ defined on } \mathcal{D} \cap R \text{ such that } \mathcal{N}_R \ll \mu_R \text{ uniformly and } \mu_R \ll \sup \mathcal{N} \text{ on } \mathcal{D} \cap R. \text{ Since the family } \{\mu_R: R \in \mathcal{D}\} \text{ satisfies the hypothesis of the synthesis theorem, the conclusion follows.}

Remark 3.2. (i) We do not know whether the measure } \nu \text{ in the Synthesis Theorem and Theorem 3.2 can be chosen to be finite.

(ii) In general, it is not possible to choose } \nu \text{ to be bounded in the synthesis theorem, or to satisfy } \mu \ll \sup \mathcal{N} \text{ in Theorem 3.2, as the following example shows.

Example 3.2. We give an example of a vector measure } \mathbf{m} \text{ defined on a } \delta\text{-ring which is countably additive, locally strongly additive and bounded; however, } \mathbf{m} \text{ has no bounded control measure. Since the family of local control
measures satisfies the hypothesis of the synthesis theorem, the global measure cannot be bounded. Let \((A_\alpha)\) be an uncountable number of nonempty disjoint subsets of a set \(S\). Let \(\mathcal{D}\) be the \(\delta\)-ring consisting of the empty set and all finite unions of sets \(A_\alpha\). Let \(\mathcal{X}\) be the Banach space of bounded functions on \(S\) with the supremum norm. Define \(m: \mathcal{D} \to \mathcal{X}\) by \(m(E) = \xi_E\), \(E \in \mathcal{D}\). If \(\mu\) is a control measure for \(m\), then it follows that \(\mu(A_\alpha) > 0\) for every \(\alpha\). Hence, \(\mu\) is not bounded; in particular, \(\mu\) is not dominated by \(\tilde{m}\).

However, as the next theorem shows, in special cases we can find a control measure dominated by \(\sup \mathcal{K}\).

**Theorem 3.3.** Suppose that \(\mathcal{K} \subseteq \text{ca}(\mathcal{R}, \mathcal{X})\) and assume that there exists a set \(R \in \mathcal{R}_{\text{loc}}\) such that:

(i) Every measure in \(\mathcal{K}\) vanishes outside \(R\);

(ii) \(R = \bigcup_{n=1}^{\infty} R_n\), where \(R_n \in \mathcal{R}\). If \(\mathcal{K}\) is uniformly locally strongly additive, then there exists a positive, bounded, countably additive measure \(\nu\) defined on \(\mathcal{R}\) such that \(\mathcal{K} \subseteq \nu\) locally uniformly and \(\nu \leq \sup \mathcal{K}\).

**Remark.** If \(\mathcal{R}\) is a \(\delta\)-ring, then (ii) implies that \(R \in \mathcal{R}_{\text{loc}}\).

**Proof.** We may assume that the \(R_n\) are disjoint; let \(\mathcal{D} = \delta(\mathcal{R})\). Each \(\mathcal{D}_n = \mathcal{D} \cap R_n\) is the \(\sigma\)-algebra generated by \(\mathcal{R} \cap R_n\). Every \(m \in \mathcal{K}\) can be extended to a countably additive measure \(m'\) on \(\mathcal{D}\), which vanishes outside \(R\), by virtue of Remark 2.2(ii). In view of Theorem 2.2, the set \(\mathcal{K}' = \{m': m \in \mathcal{K}\}\) is locally uniformly strongly additive on \(\mathcal{D}\). Let \(\mathcal{K}'_n = \{m' | \mathcal{D}_n: m' \in \mathcal{K}'\}\). By Theorem 2.3, there exists a positive, bounded, countably additive measure \(\mu_n\) on \(\mathcal{D}_n\) such that \(\mathcal{K}'_n \ll \mu_n\) uniformly and \(\mu_n \leq \sup \mathcal{K}'_n\). If we set \(\mu'(A) = \sum 2^{-n}(1 + \mu_n(R_n))^{-1} \mu_n(A \cap R_n)\), for \(A \in \mathcal{D}\), then \(\mu'\) is a positive, countably additive measure satisfying \(\mathcal{K}'_n \ll \mu'(D_n)\) uniformly and \(\mu' \leq \sup \mathcal{K}'\). Next we prove that \(\mathcal{K}' \ll \mu'\) locally uniformly. Let \(A \in \mathcal{D}\) and let \(\epsilon > 0\). If we set \(B_n = \bigcup_{i>n}(A \cap R_i)\), then \((B_n)\) is decreasing and \(B = \bigcap B_n\) is disjoint from \(R\); thus, \(\mathcal{K}'\) vanishes on \(B\). By Lemma 1.2, we have \(\tilde{m}'(B_n) \to \tilde{m}'(B) = 0\) uniformly for \(m' \in \mathcal{K}'\). Choose \(N\) such that \(\tilde{m}'(B_N) < \epsilon/2\) for all \(m' \in \mathcal{K}'\). Choose \(\delta > 0\) such that \(\tilde{m}'(E \cap R_i) < \epsilon/(2(N - 1))\), for all \(1 \leq i \leq N\), whenever \(E \in \mathcal{D} \cap A\) and \(\mu'(E) < \delta\). Hence, \(\tilde{m}'(E) < \epsilon\) for all \(m' \in \mathcal{K}'\), whenever \(\mu'(E) < \delta\) and \(E \in \mathcal{D} \cap A\). That is, \(\mathcal{K}' \ll \mu'\) locally uniformly. The restriction \(\nu\) of \(\mu'\) to \(\mathcal{R}\) satisfies the conclusion of the theorem.

Restricting Theorems 3.2 and 3.3 to the case of a single measure, we can obtain a local control measure for a countable set of measures which are locally strongly additive (not necessarily uniformly and not necessarily vanishing outside the union of a sequence of sets).

**Theorem 3.4.** Let \(\mathcal{D}\) be a \(\delta\)-ring and suppose that \((m_n)\) is a sequence of
locally strongly additive measures belonging to \(ca(D, X)\). Then there exists a positive, locally \(\sigma\)-finite, countably additive measure \(\mu\) on \(D\) such that \(m_n \leq \mu\) locally for each \(n\), and \(\mu(A) = 0\) if and only if \(\sup_n m_n(A) = 0\). If, in addition, each \(m_n\) vanishes outside a countable union of sets in \(D\) (in particular, if each \(m_n\) is strongly additive), then we can choose \(\mu\) to be bounded and \(\mu \leq \sup_n m_n\).

Proof. For the second part of the theorem, use Theorem 3.3 to obtain a bounded control measure \(\mu_n \leq m_n\). If we take \(\mu = \sum 2^{-n}(1 + \mu_n(T))^{-1} \mu_n\), then \(\mu\) satisfies the final conclusion of the theorem. Now using the second part, we can obtain, for each \(R \in D\), a positive, bounded, countably additive measure \(\mu_R\) on \(D \cap R\) such that \(m_n \leq \mu_R\) and \(\mu_R \leq \sup m_n\) on \(D \cap R\). The first conclusion then follows by using the synthesis theorem.

4. Weak Compactness with Respect to the Variation Topology

In this section we shall consider weakly compact sets in the Banach space \(favo(R, X)\) of finitely additive measures \(m: R \rightarrow X\) with bounded variation, endowed with the variation norm, that is, the norm of \(m\) is \(|m| (T)\). Strongly compact sets in this space have been characterized by Brooks [4].

The following property of Banach spaces plays an important role in the study of weak and strong compactness in the space of vector measures. The Banach space \(X\) has the Radon-Nikodym property (property R-N) if every countably additive \(X\)-valued measure \(m\) of bounded variation on a \(\sigma\)-ring \(\mathcal{S}\), which is absolutely continuous with respect to a positive, finite, countably additive measure \(\mu\) on \(\mathcal{S}\), can be expressed as the indefinite integral \(m(A) = \int_A g d\mu\), for \(A \in \mathcal{S}\), of a Bochner integrable function \(g: T \rightarrow X\).

Recall that \(X\) has property R-N if it is reflexive [23] or a separable dual [18].

A set in a topological space is relatively compact if its closure is compact. The following theorem improves a criterion of weak compactness given by Brooks [3] for reflexive spaces.

Theorem 4.1. If \(\mathcal{H} \subseteq favo(R, X)\) is relatively weakly compact, then

1. \(\mathcal{H}\) is bounded in \(favo(R, X)\);
2. \(\mathcal{H}(A) = \{m(A) : m \in \mathcal{H}\}\) is relatively weakly compact in \(X\), for every \(A \in R\);
3. \(|\mathcal{H}| = \{|m| : m \in \mathcal{H}\}\) is uniformly strongly additive.

Conversely, assume that both \(X\) and \(X^*\) have property R-N. If \(\mathcal{H} \subseteq favo(R, X)\) satisfies conditions (1)–(3) above, then \(\mathcal{H}\) is relatively weakly compact. If \(X\) is reflexive, condition (2) is superfluous.
Proof. The theorem has been proved by Brooks [3] in the case \( \mathcal{X} \) is reflexive. The same proof, with minor modifications, can be used if \( \mathcal{X} \) and \( \mathcal{X}^* \) have property R-N, providing that \( \mathcal{H} \) is a \( \sigma \)-ring and \( \mathcal{H} \) consists of countably additive measures. We shall reduce the general case to this one.

Assume, therefore, that \( \mathcal{X} \) and \( \mathcal{X}^* \) have property R-N, \( \mathcal{H} \) is a ring and that \( \mathcal{H} \subseteq \text{cabv}(\mathcal{H}, \mathcal{X}) \) satisfies conditions (1)-(3). Using the Stone representation, we can assume that \( \mathcal{H} \subseteq \text{cabv}(\mathcal{H}, \mathcal{X}) \) and properties (1)-(3) still remain valid. Let \( \mathcal{I} = \sigma(\mathcal{H}) \) and for each \( m \in \text{cabv}(\mathcal{H}, \mathcal{X}) \) let \( m' \in \text{cabv}(\mathcal{I}, \mathcal{X}) \) be the extension of \( m \); set \( \mathcal{H}' = \{m': m \in \mathcal{H}\} \). Since the extension mapping \( m \to m' \) is an isometry, with respect to the variation norms, \( \mathcal{H}' \) is bounded in \( \text{cabv}(\mathcal{I}, \mathcal{X}) \). Since \( |m'| \) is the extension of \( |m| \), and \( |\mathcal{H}'| \) is uniformly strongly additive on \( \mathcal{H} \), by Theorem 2.2 we deduce that \( |\mathcal{H}'| \) is uniformly strongly additive on \( \mathcal{I} \). To complete the proof, we have to show that \( \mathcal{H}' \) satisfies condition (2) on \( \mathcal{I} \), that is, if \( A \in \mathcal{I} \), then \( \mathcal{H}'(A) \) is relatively weakly compact in \( \mathcal{I} \). To prove this, let \( \{m_n\} \) be a sequence from \( \mathcal{H}' \). By Theorem 2.3 there exists a positive, bounded, countably additive measure \( \mu \) on \( \mathcal{I} \) such that \( \mathcal{H}' \hookrightarrow \mu \) uniformly. Choose sets \( B_i \) from \( \mathcal{H} \) such that \( \mu(B_i \triangle A) \to 0 \).

Then \( m_n(B_i) \to m_n(A) \), uniformly in \( n \). Using hypothesis (2) and a diagonal process, we can assume that the sequence \( (m_n(B_i))_{n=1}^\infty \) is weakly convergent for each \( i \). Let \( I \) be the set of positive integers and define \( f, f_i : I \to \mathcal{I} \) by \( f_i(n) = m_n(B_i) \) and \( f(n) = m_n(A) \). By (2), \( f_i(I) \) is relatively weakly compact for each \( i \); since \( f_i \to f \) uniformly on \( I \), it follows from Theorem 2 in [26] that \( f(I) \) is relatively weakly compact. This in turn implies that \( \mathcal{H}'(A) \) is relatively weakly compact. From the remark at the beginning of the proof, it follows that \( \mathcal{H}' \) is relatively weakly compact in \( \text{cabv}(\mathcal{I}, \mathcal{X}) \); hence, \( \mathcal{H} \) is relatively weakly compact in \( \text{cabv}(\mathcal{H}, \mathcal{X}) \).

We wish to thank C. Swartz for pointing out Theorem 2 in [26] to us, which enabled the authors to dispense with a weak sequential condition that was earlier assumed on \( \mathcal{X} \).

We now turn to the space \( \text{fafa}(\mathcal{H}, \mathcal{X}) \) of finitely additive measures with local finite variation, endowed with the family of seminorms \( (q_A)_{A \in \mathcal{H}} \), where \( q_A(m) = |m|(A) \). If for \( A, B \in \mathcal{H} \), with \( A \subseteq B \), we consider the restriction mapping \( \rho_{AB} : \text{fafa}(\mathcal{H} \cap B, \mathcal{X}) \to \text{fafa}(\mathcal{H} \cap A, \mathcal{X}) \) defined by \( \rho_{AB}(m) = m|\mathcal{H} \cap A \), then \( \text{fafa}(\mathcal{H}, \mathcal{X}) \) is the projective limit of the spaces \( \text{fafa}(\mathcal{H} \cap A, \mathcal{X}), A \in \mathcal{H} \), with respect to the mappings \( \rho_{AB} \). The projection \( \rho_A : \text{fafa}(\mathcal{H}, \mathcal{X}) \to \text{fafa}(\mathcal{H} \cap A, \mathcal{X}) \) is given by \( \rho_A(m) = m_A \). Note that a set \( \mathcal{H} \) contained in \( \text{fafa}(\mathcal{H}, \mathcal{X}) \) is bounded if and only if sup \( \{m|(A): m \in \mathcal{H}\} \) is finite for every \( A \in \mathcal{H} \).

The following theorem extends Theorem 4.1, and, as was mentioned in the introduction, a result of Dieudonné.
Theorem 4.2. If $\mathcal{K} \subset \text{favo}(\mathcal{A}, \mathcal{X})$ is relatively weakly compact, then:

1. $\mathcal{K}$ is bounded in $\text{favo}(\mathcal{A}, \mathcal{X})$;
2. $\mathcal{K}(A)$ is relatively weakly compact in $\mathcal{X}$, for each $A \in \mathcal{A}$;
3. $\mathcal{X}$ is locally uniformly strongly additive.

Conversely, assume that $\mathcal{X}$ and $\mathcal{X}^*$ have property R-N and $\mathcal{K} \subset \text{favo}(\mathcal{A}, \mathcal{X})$ satisfies conditions (1)-(3) above. Then $\mathcal{K}$ is relatively weakly compact. If $\mathcal{X}$ is reflexive, then condition (2) is superfluous.

Proof. From the general theory of projective limits, it is known that the weak topology on $\text{favo}(\mathcal{A}, \mathcal{X})$ is the projective limit of the weak topologies on $\text{favo}(\mathcal{A} \cap A, \mathcal{X})$, and that $\mathcal{K} \subset \text{favo}(\mathcal{A}, \mathcal{X})$ is relatively weakly compact if and only if $\mathcal{K}_A$ is relatively weakly compact in $\text{favo}(\mathcal{A} \cap A, \mathcal{X})$ for each $A \in \mathcal{A}$. The result then follows by Theorem 4.1.

Theorem 4.3. Let $\mathcal{D}$ be a $\delta$-ring and let $\mathcal{K} \subset \text{favo}(\mathcal{D}, \mathcal{X})$ be a relatively weakly compact set. Then there exists a positive, locally $\sigma$-finite, countably additive measure $\mu$ on $\mathcal{D}$ such that $\mathcal{K} \ll \mu$ locally uniformly. In addition, $\mu(A) = 0$ if and only if $A$ is $\mathcal{K}$-null.

Proof. By Theorem 4.2, $\mathcal{X}$ is locally uniformly strongly additive. The result then follows by applying Theorem 3.2.

5. Strong Compactness with Respect to the Semi-Variation Topology

In this section we shall consider strongly compact sets in the Banach space $ba(\mathcal{A}, \mathcal{X})$ consisting of bounded finitely additive measures $m: \mathcal{A} \rightarrow \mathcal{X}$, endowed with the quasivariation norm, that is, the norm of $m$ is $\|m\| = \overline{m}(T)$. Note that an equivalent norm is given by the semivariation of $m$ on $T$ [16].

The subspace of strongly additive measures is denoted by $sa(\mathcal{A}, \mathcal{X})$.

Remark 5.1. The subspace $sa(\mathcal{A}, \mathcal{X})$ is closed in $ba(\mathcal{A}, \mathcal{X})$. This observation follows from the fact that $sa(\mathcal{A}, \mathcal{X})$ is isometrically isomorphic to the Banach space $ca(\mathcal{A}, \mathcal{X})$, where $\mathcal{A}$ is the $\sigma$-ring generated by the Stone ring determined by $\mathcal{A}$.

The connection between strong additivity and compactness is given by the following theorem.

Theorem 5.1. Let $\mathcal{K} \subset sa(\mathcal{A}, \mathcal{X})$ be a relatively compact set. Then $\mathcal{K}$ is uniformly strongly additive. Furthermore, there exists a positive bounded measure $\mu$ such that $\mathcal{K} \ll \mu$ uniformly and $\mu \ll \sup \mathcal{K}$. If $\mathcal{K}$ consists of countably
additive measures, then \( \mu \) is countably additive. Finally, if \( \mathcal{N} \ll \lambda \) for some positive measure \( \lambda \), then \( \mathcal{N} \ll \lambda \) uniformly and \( \mu \ll \lambda \).

Proof. If \( \mathcal{N} \) is not uniformly strongly additive, there exists an \( \epsilon > 0 \), \( \mathbf{m}_n \in \mathcal{N} \) and disjoint sets \( E_n \) such that \( |\mathbf{m}_n(E_n)| > \epsilon \). Since \( \mathcal{N} \) is relatively compact, we may assume that \( \mathbf{m}_n \to \mathbf{m} \) in \( sa(\mathcal{R}, \mathcal{X}) \) (see Remark 5.1). This leads to a contradiction since \( |\mathbf{m}(E_n)| - |\mathbf{m}_n(E_n) - \mathbf{m}(E_n)| < \epsilon \) for \( n \) sufficiently large. The remaining parts of the theorem follow from Theorems 2.1 and 2.3. \( \square \)

The next theorem gives necessary and sufficient conditions for compactness. First we establish a framework for these conditions.

Consider the set \( (\pi) \) of all finite families \( \pi = (E_i)_{i=1}^n \) of disjoint sets in \( \mathcal{R} \). If \( \pi' = (E'_i) \), we write \( \pi \ll \pi' \) if every \( E_j' \) is either contained in some \( E_i \) or disjoint from all the \( E_i \) and \( \bigcup E_i \subseteq \bigcup E'_j \). This makes \( (\pi) \) a directed set. If \( \nu \) is a positive finitely additive measure on \( \mathcal{R} \) and \( \pi \) is given as above, then for every measure \( \mathbf{m} \in ba(\mathcal{R}, \mathcal{X}) \), with \( \mathbf{m} \ll \nu \), we set

\[
\mathbf{m}_{\pi(\nu)} = \sum_{i=1}^n \left( \mathbf{m}(E_i)/\nu(E_i) \right) \nu_{E_i},
\]

where \( \nu_{E_i}(A) = \nu(A \cap E_i) \), for \( A \in \mathcal{R} \), and \( \mathbf{m}(E_i)/\nu(E_i) = 0 \) if \( \nu(E_i) = 0 \). Note that \( \mathbf{m}_{\pi(\nu)} \ll \nu \).

**Theorem 5.2.** Let \( \mathcal{N} \subseteq ba(\mathcal{R}, \mathcal{X}) \). Suppose that

1. \( \mathcal{N}(A) \) is relatively compact in \( \mathcal{X} \), for each \( A \in \mathcal{R} \);
2. There exists a bounded, finitely additive, positive measure \( \nu \) on \( \mathcal{R} \) such that \( \mathcal{N} \ll \nu \) and \( \lim_{n} \mathbf{m}_{\pi(n)} = \mathbf{m} \) in \( ba(\mathcal{R}, \mathcal{X}) \) uniformly for \( \mathbf{m} \in \mathcal{N} \).

Then \( \mathcal{N} \) is relatively compact.

Conversely, if \( \mathcal{N} \) is relatively compact, then condition (1) is satisfied. If, in addition, \( \mathcal{N} \) consists of measures with relatively norm compact range, then condition (2) is also satisfied.

Proof. To prove the first part, consider the measure \( \nu \) in condition (2) and for every \( \pi \in (\pi) \) define the linear operator \( U_{\pi, \mathbf{m}} = \mathbf{m}_{\pi(\nu)} \), for \( \mathbf{m} \in ba(\mathcal{R}, \mathcal{X}, \nu) \), where \( ba(\mathcal{R}, \mathcal{X}, \nu) \) is the Banach space \( \{ \mathbf{m} \in ba(\mathcal{R}, \mathcal{X}) : \mathbf{m} \ll \nu \} \). Then \( \| U_{\pi} \| \leq 4 \), since for every \( A \in \mathcal{R} \),

\[
|\mathbf{m}_{\pi(n)}(A)| = \sup_{|z| \leq 1} \left| \sum E_i \left( \mathbf{m}_z(E_i)/\nu(E_i) \right) \nu(A \setminus E_i) \right| \leq \sup_{|z| \leq 1} \sum_i |\mathbf{m}_z(E_i)|
= \sup_{|z| \leq 1} |\mathbf{m}_z|(T) \leq 4\mathbf{m}(T),
\]

where \( \mathbf{m}_z \).
where
\[ m_+(A) = \langle m(A), x \rangle, \quad \text{for } x \in X^*. \]

For a fixed \( E \in \mathcal{A} \),
\( \mathcal{A} \)
the mapping \( x \rightarrow (\nu(E))^{-1} x \nu_E \) of \( X \) into \( ba(\mathcal{A}, X, \nu) \) is an isometry. To see this, note that
\[ ||(\nu(E))^{-1} x \nu_E|| = \sup_{A \in \mathcal{A}} |(\nu(E))^{-1} x \nu_E(A)| - |x|. \]

By condition (1), the sets \( \mathcal{H}(E_i) \) are relatively compact in \( X \). Consequently, in view of (\#), we see that the set
\[ \sum_{i=1}^{n} (\nu(E_i))^{-1} \mathcal{H}(E_i) \nu_{E_i} \]
is relatively compact in \( ba(\mathcal{A}, X, \nu) \); consequently, \( U_+(\mathcal{H}) \) is relatively compact. Using condition (2) and the Phillips lemma [24; 19, IV.5.4], it follows that \( \mathcal{H} \) is relatively compact. (We remark that the boundedness condition on \( \mathcal{H} \) in [19, IV.5.4] is superfluous.)

Conversely, assume that \( \mathcal{H} \) is relatively compact. For \( A \in \mathcal{A} \), the mapping \( m \rightarrow m(A) \) of \( ba(\mathcal{A}, X) \) into \( X \) is continuous; hence, \( \mathcal{H}(A) \) is relatively compact, and condition (1) is satisfied.

Now let \( \nu \) be a bounded, positive, finitely additive measure on \( \mathcal{A} \). For every scalar measure \( \mu \in ba(\mathcal{A}, C, \nu) \), we have \( \lim_{n} U_+(\mu) = \mu \) in \( ba(\mathcal{A}, C) \) (see Brooks [4]). This equality remains true for every \( m \) in the closure \( C(\nu) \) in \( ba(\mathcal{A}, X) \) of the set of “step measures” of the form \( \sum x_i \mu_i \), where \( x_i \in X \) and \( \mu_i \in ba(\mathcal{A}, C, \nu) \). By the Phillips lemma, this limit is uniform on every relatively compact subset of \( C(\nu) \).

Next we remark that if \( m \ll \nu \), and \( m \) has relatively norm compact range, then \( m \in C(\nu) \). In fact, using the Stone representation we can assume that \( m \) and \( \nu \) are countably additive on \( \mathcal{A} \). Let \( m' \) and \( \nu' \) be their extensions to \( \sigma(\mathcal{A}) \); we still have \( m' \ll \nu' \). Also \( m' \) has relatively norm compact range on \( \sigma(\mathcal{A}) \). To prove this it suffices to show that \( m'(\sigma(\mathcal{A})) \) is contained in the closure of \( m'(\mathcal{A}) \). To this end, let \( A \in \sigma(\mathcal{A}) \); there exists a sequence \( (B_n) \) of sets from \( \mathcal{A} \) such that \( \nu'(A \Delta B_n) \rightarrow 0 \). This implies that \( m'(B_n) \rightarrow m'(A) \), which establishes the assertion.

The construction in the proof of Theorem 3.1 in [21] shows that since \( m' \) has the above properties, \( m' \) can be approximated by step measures in \( ca(\sigma(\mathcal{A}), X, \nu') \). It then follows that \( m \) can be approximated in \( ba(\mathcal{A}, X) \) by step measures in \( ba(\mathcal{A}, X, \nu) \). Hence, \( m \in C(\nu) \).

Now assume that \( \mathcal{H} \subset sa(\mathcal{A}, X) \) is relatively compact and that \( \mathcal{H} \) consists of measures with relatively norm compact range. Since \( \mathcal{H} \) is relatively...
compact, by Theorem 5.1, there exists a bounded, positive, finitely additive
measure \( v \) on \( \mathcal{R} \) such that \( \mathcal{H} \ll v \) uniformly. By the above, we have \( \mathcal{H} \subset C(v) \);
consequently, \( \lim U_n(m) = m \) uniformly for \( m \in \mathcal{H} \).

Remark 5.2. From the above proof, we single out the following result,
which extends Theorem 3.1 in [21]:
The set of step measures of the form \( \sum x_i \mu_i \) with \( x_i \in \mathcal{X} \) and \( \mu_i \in ba(\mathcal{R}, C) \)
is dense in the Banach space \( CM(\mathcal{R}, \mathcal{X}) \) of strongly additive measures with
relatively norm compact range.
Furthermore, if \( m \in CM(\mathcal{R}, \mathcal{X}) \) and \( m \ll v \), then \( m_n \to m \).

Theorem 5.3. Suppose that \( \mathcal{X} \) has property \( R-N \) and that \( \mathcal{H} \subset sa(\mathcal{R}, \mathcal{X}) \)
consists of countably additive measures with local \( \sigma \)-finite variation. If \( \mathcal{H} \) is
relatively compact, then there is a bounded positive measure \( v \) such that \( \mathcal{H} \ll v \)
uniformly and \( \lim m_n(\cdot) = m \) in \( sa(\mathcal{R}, \mathcal{X}) \) uniformly for \( m \in \mathcal{H} \).

Proof. It is enough, in view of Theorem 5.2, to prove that \( \mathcal{H} \) consists
of measures with relatively norm compact range. Let \( m \in \mathcal{H} \) and let \( m' \)
be the extension of \( m \) to the \( \sigma \)-ring \( \mathcal{I} = \sigma(\mathcal{R}) \). There exists a set \( T' \in \mathcal{I} \)
(Lemma 1.1) such that \( m' \) vanishes outside \( T' \). Since \( m' \) has local \( \sigma \)-finite
variation (see Remark 2.2(ii)), there is a sequence \( (T_n) \) of disjoint sets from \( \mathcal{I} \)
such that \( T' = \bigcup T_n \) and \( m' \) has finite variation on \( \mathcal{I} \cap T_n \) for each \( n \). Let
\( \mu' \) be a control measure for \( m' \). Since \( \mathcal{H} \) has property \( R-N \), for each \( n \) there
is a Bochner integrable function \( g_n: T \to \mathcal{X} \) vanishing outside \( T_n \) such that
\( m'(A) = \int_A g_n \, d\mu' \), for \( A \in \mathcal{I} \cap T_n \). The function \( g = \sum g_n \) is strongly
measurable and by a result of Dinculeanu and Uhl [17], \( g \) is a Pettis integrable
function such that \( m'(A) = (\text{Pettis}) \int_A g \, d\mu' \), for \( A \in L \). Define the operator
\( U_g: \mathcal{L}^{\omega} (\mu') \to \mathcal{X} \) induced by \( g \) as follows: \( U_g(f) = \int f g \, d\mu' \) for \( f \in \mathcal{L}^{\omega} (\mu') \).
By a theorem of Pettis [19, Theorem 6.2], \( U_g \) is a compact operator. This
implies that \( m' \) (hence \( m \)) has relatively compact range. By the second part
of Theorem 5.2, the result follows.

From the above proof, we have the following result.

Theorem 5.4. If \( \mathcal{X} \) has property \( R-N \), then every countably additive
measure \( m \in sa(\mathcal{R}, \mathcal{X}) \) with local \( \sigma \)-finite variation has relatively norm compact
range.

The following example shows that the above theorem is false if \( \mathcal{X} \) does not
have property \( R-N \).

Example 5.4. Let \( \mu \) be Lebesgue measure on \([0, 1]\), and let \( \mathcal{B} \) be the
class of Borel sets. Define \( m: \mathcal{B} \to \mathcal{L}^1 (\mu) \) by \( m(A) = \xi_A \), for \( A \in \mathcal{B} \). Then \( m \)
is countably additive with finite variation. Let \( \mathcal{H} = \{ m(A): A \in \mathcal{B} \} \subset \mathcal{L}^1 (\mu) \).
Since $\mathcal{A}$ can be identified with the noncompact pseudo metric space $(\mathcal{A}, \rho)$, where $\rho(A, B) = \mu(A \Delta B)$, it follows that $\mathbf{m}$ does not have relatively norm compact range.

We now turn to the space $\text{loc} \, \text{ba}(\mathcal{R}, \mathcal{X})$ consisting of the locally bounded measures $\mathbf{m}: \mathcal{R} \rightarrow \mathcal{X}$, endowed with the family of seminorms $(\tilde{q}_A)_{A \in \mathcal{R}}$, where $\tilde{q}_A(\mathbf{m}) = \tilde{m}(A)$. Note that $\text{loc} \, \text{ba}(\mathcal{R}, \mathcal{X})$ is the projective limit of the spaces $\text{ba}(\mathcal{R} \cap A, \mathcal{X}), A \in \mathcal{R}$, with respect to the usual restriction mappings. In view of this, we have the following theorem.

**Theorem 5.5.** If $\mathcal{H} \subseteq \text{loc} \, \text{sa}(\mathcal{R}, \mathcal{X})$ is relatively compact, then $\mathcal{H}$ is locally uniformly strongly additive.

If, in addition, $\mathcal{H} \subseteq \text{loc} \, \text{sa}(\mathcal{D}, \mathcal{X})$, where $\mathcal{D}$ is a $\delta$-ring, and $\mathcal{H}$ consists of countably additive measures, then there exists a positive, locally $\sigma$-finite, countably additive measure $\nu$ on $\mathcal{D}$ such that $\mathcal{H} \subseteq \nu$ locally uniformly.

Theorems 5.1 and 3.2 are used to prove the above theorem.

6. **Weak Compactness with Respect to the Quasivariation Topology**

In this section, we shall consider conditionally weakly compact sets in the space $\text{ba}(\mathcal{R}, \mathcal{X})$ endowed with the quasivariation norm. Since this space is not necessarily weakly sequentially complete, we shall have to study conditional weak compactness instead of relative weak compactness. A subset $\mathcal{K}$ of a topological vector space is **conditionally weakly compact** if every sequence of elements from $\mathcal{K}$ contains a Cauchy subsequence. Note that relative weak compactness implies conditional weak compactness. The two concepts coincide in $\text{ba}(\mathcal{R}, C)$, since this space, endowed with the quasivariation norm (which in this case is equivalent to the variation norm) is weakly sequentially complete. By the Eberlein–Smulian theorem, this implies the above assertion.

The following theorem extends criteria of conditional weak compactness established by Lewis [21].

**Theorem 6.1.** If $\mathcal{K} \subseteq \text{ba}(\mathcal{R}, \mathcal{X})$ is conditionally (respectively, relatively) weakly compact, then:

1. The set $x^* \mathcal{K} = \{x^* \mathbf{m}: \mathbf{m} \in \mathcal{K}\}$ is relatively weakly compact in $\text{ba}(\mathcal{R}, C)$ for every $x^* \in \mathcal{X}^*$;

2. The set $\mathcal{K}(A) = \{\mathbf{m}(A): \mathbf{m} \in \mathcal{K}\}$ is conditionally (respectively, relatively) weakly compact in $\mathcal{X}$ for every $A \in \mathcal{R}$. 

Conversely, assume that $\mathcal{H} \subset \text{sa}(\mathcal{R}, \mathcal{X})$ and that $\mathcal{H}$ consists of measures with relatively norm compact range. If $\mathcal{H}$ satisfies conditions (1) and (2) above, then $\mathcal{H}$ is conditionally weakly compact. If $\mathcal{X}$ is reflexive, condition (2) is superfluous.

Proof. The first part follows from the continuity of the maps $m \rightarrow x^*m$ and $m \rightarrow m(A)$ from $ba(\mathcal{R}, \mathcal{X})$ into $ba(\mathcal{R}, C)$ and $\mathcal{X}$ respectively. To prove the second part, we may assume, by passing to the Stone space, that $\mathcal{H}$ consists of countably additive measures. Every measure $m \in \mathcal{H}$ can be extended to a countably additive measure $m'$ on $\mathcal{I} = \sigma(\mathcal{R})$, by Theorem 2.2. Moreover, as the proof of Theorem 5.2 shows, $m'$ also has relative norm compact range. Let $\mathcal{H}' = \{m': m \in \mathcal{H}\}$. It follows that for every $x^* \in \mathcal{X}$, the set $x^*\mathcal{H}'$ is relatively weakly compact in $ca(\mathcal{I}, C)$. In fact, for every $m \in \mathcal{H}$, $x^*m'$ is the extension of $x^*m$ from $\mathcal{R}$ to $\mathcal{I}$, and this extension preserves the norms. We shall now prove that for every $A \in \mathcal{I}$, the set $\mathcal{H}'(A)$ is conditionally weakly compact in $\mathcal{X}$. Let $A \in \mathcal{I}$ and let $(m_n')$ be a sequence from $\mathcal{H}'$. Choose a sequence of sets $B_n$ from $\mathcal{R}$ such that $\mathcal{R}$ belongs to the $\sigma$-ring generated by $(B_n)$. Let $\mathcal{R}_0$ be the countable ring generated by $(B_n)$ and set $\mathcal{I}_0 = \sigma(\mathcal{R}_0)$. For each $B \in \mathcal{R}_0$, by hypothesis, the set $\{m_n'(B); n = 1, 2, \ldots\}$ is conditionally weakly compact, hence it contains a weakly Cauchy sequence. By a diagonal process, we can extract a subsequence (which we shall still denote by $(m_n')$) such that $m_n'(B)$ is weakly Cauchy for each $B \in \mathcal{R}_0$. Let $x^* \in \mathcal{X}$. Since $(x^*m_n')$ is uniformly countably additive, by the weak compactness of $x^*\mathcal{H}'$, and since $(x^*m_n')$ converges on $\mathcal{R}_0$, it follows by Lemma IV.8.8 in Dunford and Schwartz [19] that $(x^*m_n')$ converges on $\mathcal{I}_0$. In particular, $(x^*m_n'(A))$ converges. Hence $(m_n')$ is weakly Cauchy. This shows that $\mathcal{H}'(A)$ is conditionally weakly compact. To prove that $\mathcal{H}'$ is conditionally weakly compact, let $(m_n')$ be a sequence from $\mathcal{H}'$. By Lemma 1.1, there exists a set $T' \in \mathcal{I}$ such that all the measures $m_n'$ vanish outside $T'$. Hence, we can consider the $m_n'$ as being defined on the $\sigma$-algebra $\mathcal{I}' = \mathcal{I} \cap T'$ of subsets of $T'$. By the above arguments, $(m_n')$ satisfies the conditions of Corollary 3.2 in Lewis [21], which shows that $(m_n')$ is conditionally weakly compact. This implies that $\mathcal{H}'$ is conditionally weakly compact. Since the extension mapping $m \rightarrow m'$ is an isometry, it follows that $\mathcal{H}$ is conditionally weakly compact in $ba(\mathcal{R}, \mathcal{X})$. To prove the last assertion in the theorem, note that (1) implies that for every $A \in \mathcal{R}$, the set $\mathcal{H}'(A)$ is bounded (since $x^*\mathcal{H}$ is bounded; hence, $\{x^*m(A); m \in \mathcal{H}\}$ is bounded for every $x^* \in \mathcal{X}$). Since $\mathcal{X}$ is reflexive, it follows that $\mathcal{H}(A)$ is relatively weakly compact. 

Corollary 6.2. Assume that $\mathcal{H} \subset \text{sa}(\mathcal{R}, \mathcal{X})$ consists of measures with relatively norm compact range. If
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(1) \( \mathcal{X} \) is bounded and uniformly strongly additive;
(2) \( \mathcal{X}(A) \) is conditionally weakly compact in \( \mathcal{X} \) for every \( A \in \mathcal{A} \);
then \( \mathcal{X} \) is conditionally weakly compact.

If \( \mathcal{X} \) is reflexive, then condition (2) is superfluous.

Proof. Let \( x^* \in \mathcal{X}^* \). Then \( x^* \mathcal{X} \) is bounded and uniformly strongly additive. By Theorem 1 in Brooks [3], \( x^* \mathcal{X} \) is relatively weakly compact. Note that strong additivity of \( x^* \mathcal{X} \) is equivalent to strong additivity of \( |x^* \mathcal{X}| \). In view of this, we can apply the above theorem.

Using Theorem 5.4 and the above results, we have the following.

Corollary 6.3. Assume that \( \mathcal{X} \) has property R-N and that \( \mathcal{X} \subset sa(\mathcal{B}, \mathcal{X}) \) is a set of countably additive measures with local \( \alpha \)-finite variation. If \( \mathcal{X} \) satisfies conditions (1) and (2) in Theorem 6.1 or Corollary 6.2, then \( \mathcal{X} \) is conditionally weakly compact.

It is natural to ask whether the converse of Corollary 6.2 is true, that is whether a conditionally weakly compact set in \( sa(\mathcal{B}, \mathcal{X}) \) is uniformly strongly additive. The answer is in the negative, as the following example shows.

Example 6.3. Let \( \mu \) denote Lebesgue measure on \([0, 1]\) and let \( \mathcal{B} \) be the Borel subsets of \([0, 1]\). Let \((e_n)\) be the unit vectors in \( \mathcal{X} = l_2 \). Choose \( E_n \in \mathcal{B} \) such that \( \mu(E_n) > 0 \) and \( E_n \not\subset \phi \), and define \( m_n = e_n(\mu(E_n))^{-1} \mu \).

Since \( e_n \to 0 \) weakly in \( \mathcal{X} \), one can show (see [21, Corollary 3.3]) that \( m_n \to 0 \) in \( ca(\mathcal{B}, \mathcal{X}) \). However, the countable additivity of \((m_n)\) is not uniform, since \( |m_n(E_n)| = 1 \).

In spite of the above example, we are still able to exhibit a control measure for conditionally weakly compact subsets of \( sa(\mathcal{B}, \mathcal{X}) \), when \( \mathcal{X} \) is separable.

Theorem 6.4. Assume that \( \mathcal{X} \) is separable. If \( \mathcal{X} \subset sa(\mathcal{B}, \mathcal{X}) \) is conditionally weakly compact, then there exists a bounded, positive, finitely additive measure \( \mu \) on \( \mathcal{B} \) such that \( \mathcal{X} \ll \mu \) and \( \mu \leq \sup \mathcal{X} \). If, in addition, \( \mathcal{X} \subset ca(\mathcal{B}, \mathcal{X}) \), then \( \mu \) is countably additive.

Proof. Using the Stone representation, we can assume that \( \mathcal{X} \subset ca(\mathcal{B}, \mathcal{X}) \). Let \( \mathcal{J} = \sigma(\mathcal{B}) \) and for each \( m \in sa(\mathcal{B}, \mathcal{X}) \) which is countably additive, let \( m' \) be its extension to \( \mathcal{J} \), which exists by Theorem 2.2. The set \( \mathcal{X}' = \{m' : m \in \mathcal{X}\} \) is conditionally weakly compact in \( ca(\mathcal{J}, \mathcal{X}) \). Since \( \mathcal{X} \) is separable, we can choose a sequence \((x_n^*)\) which is \( w^* \)-dense in \( \mathcal{X}_1^* \), the unit ball of \( \mathcal{X}^* \). By Theorem 6.1, \( x_n^* \mathcal{X}'' \) is relatively weakly compact in \( ca(\mathcal{J}, \mathcal{C}) \) for each \( n \). Therefore, there exists a positive, bounded, countably additive measure \( \mu_n' \) on \( \mathcal{B} \) such that \( x_n^* \mathcal{X}'' \ll \mu_n' \) uniformly, and \( \mu_n' \leq \sup \{ |x_n^* m' | : m' \in \mathcal{X}' \} \leq \sup \mathcal{X}' \).
Let \( \mu' = \sum (2^n B_n)^{-1} \mu_n' \), where \( B_n \) is a positive bound for \( \mu_n' \). Then \( \mu' \) is positive, countably additive, \( \mu' \ll \sup \mathcal{N} \) and \( \mu_n' \ll \mu' \); hence, \( x_n \ast \mathcal{N} \ll \mu' \) uniformly for each \( n \). Now let \( y* \in \mathcal{X}^* \). Choose a subsequence \((y_n*)\) of \((x_n*)\) such that \( y_n* \to y* \) in the \( w^*\)-topology of \( \mathcal{X}^* \). If \( m' \in \mathcal{M} \), then for each \( A \in \mathcal{F} \), it follows that \( y_n* m'(A) \to y* m'(A) \). Since \( y_n* m' \ll \mu' \) for each \( n \), by the Vitali–Hahn–Saks theorem, we have \( y* m' \ll \mu' \). Hence, it follows that if \( \mu'(A) = 0 \), then \( m'(A) = 0 \). By a theorem of Pettis [19, Theorem IV.10.1], we have \( m' \ll \mu' \). The restriction \( \mu = \mu' \mid \mathcal{F} \) satisfies the requirements of the theorem.

Remark. Diestel [15] proved that if \( \mathcal{F} \) is separable, then every bounded, finitely additive measure on a \( \sigma \)-ring \( \mathcal{F} \) is strongly additive. In view of this, the preceding theorem remains true if \( \mathcal{N} \subseteq ba(\mathcal{F}, \mathcal{X}) \).

We now turn to the space \( ca(\mathcal{D}, \mathcal{X}) \), where \( \mathcal{D} \) is a \( \sigma \)-ring, endowed with the topology induced by the quasi-variation semi-norms. As we have seen, \( ca(\mathcal{D}, \mathcal{X}) \) is the projective limit of the Banach spaces \( ca(\mathcal{D} \cap A, \mathcal{X}), A \in \mathcal{D} \). Hence, if a set \( \mathcal{N} \) is conditionally weakly compact in \( ca(\mathcal{D}, \mathcal{X}) \), then for each \( A \in \mathcal{D} \), the set \( \mathcal{N}_A \) is conditionally weakly compact in \( ca(\mathcal{D} \cap A, \mathcal{X}) \). Using this fact, we deduce the following theorem.

**Theorem 6.5.** Assume that \( \mathcal{X} \) is separable and that \( \mathcal{N} \subseteq ca(\mathcal{D}, \mathcal{X}) \) is conditionally weakly compact. Then there exists a positive, locally \( \sigma \)-finite, countably additive measure \( \mu \) on \( \mathcal{D} \) such that \( \mathcal{N} \ll \mu \) locally.

**Proof.** For each \( A \in \mathcal{D} \), \( \mathcal{D} \cap A \) is a \( \sigma \)-ring and \( \mathcal{N}_A \subseteq ca(\mathcal{D} \cap A, \mathcal{X}) \) is conditionally weakly compact. Consequently, by Theorem 6.4 there exists a bounded, positive, countably additive measure \( \mu_A \) defined on \( \mathcal{D} \cap A \) such that \( \mathcal{N}_A \ll \mu_A \) and \( \mu_A \leq \sup \mathcal{N}_A \). We can then apply the synthesis theorem and obtain the required measure.

Remark. Example 6.3 shows that, in general, it is false that \( \mathcal{N} \ll \mu \) locally uniformly.

**References**