The Tau method and a new preconditioner

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Abstract

In relations to the order of linear ordinary differential equations, using a modified form of the Chebyshev or Legendre and Gegenbauer polynomials, some particular integral operators are introduced. These are used to give a factorization of the operators arising from the application of the Chebyshev or Legendre Tau method. The New-Tau method presented in this article is then compared with the standard Tau method and preconditioned method of Cabos. The New-Tau method shows a superior performance. An analysis of error and a bound for condition number is given. Numerical examples applying iterative solvers show dramatic reduction in condition number and improved convergence for the Tau method with the new preconditioner.

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1. Introduction

The spectral methods are very much successful for the numerical solution of ordinary or partial differential equations [3]. Spectral methods have become increasingly popular in recent years, especially since the development of fast transform methods, with applications in numerical weather prediction, numerical simulations of turbulent flows, and other problems where high accuracy is desired for complicated solutions.

The Tau method can be described as a spectral method for the solution of differential equations. The approximate solution \( u_N(x) \), of an equation on the interval \([-1, 1]\) is represented as a
finite series

\[ u_N(x) = \sum_{j=0}^{N} a_j \phi_j(x), \]

where the \( \phi_j(x) \) are global (base) functions on \([-1,1]\), e.g. trigonometric functions (in Fourier spectral methods) or Chebyshev (Legendre) polynomials. The coefficients \( a_j \) are the unknowns one solves for. One characteristic of the Tau method is that the expansion functions \( \phi_j(x) \) do not satisfy boundary conditions in relation to the supplementary conditions imposed together with the differential equation. For an introduction to the Tau method see [3,8,13].

In linear problems the coefficients \( a = (a_0, a_1, \ldots, a_N)^T \) have to be determined from linear matrix equation \( Ma = F \). In this method the matrix \( M \) of order \( N+1 \) usually has \( O(N^2) \) nonzero elements and so it is rather expensive to solve this equation by a direct system solver. On the other hand iterative solution of the system \( Ma = F \) is obtained with difficulty as the condition number of \( M \) rises rapidly with \( N \) [3].

The coefficients of the Chebyshev (Legendre) polynomials grow rapidly with a rise in degree and number of differentiation. Therefore, in the standard Chebyshev (or Legendre) Tau method the elements of columns in the final coefficient matrix grow rapidly with columns indices and the resulted matrix is poorly conditioned. Hence in order to obtain a suitable preconditioner we try to modify the Chebyshev (Legendre) polynomials in such a way that their corresponding entries in \( M \) do not grow rapidly with \( N \). On the other hand, the test functions are chosen in such a way that they are orthogonal to the base functions arising from the higher order terms in ODE. Doing this, rapid growth of the resulting coefficients due to the difficult terms of ODE, i.e., terms of higher derivatives are controlled. A discussion on certain formal properties of orthogonal families and reformulation of differential equations based on integral operators that we shall need can be found in any of the recent references (e.g. [5,4]). It is also worth mentioning that most of our analyses are carried out for Chebyshev polynomials, because of their optimal approximation properties as well as the applicability of the fast Fourier transform. Further details and numerical results are given in next sections.

In Section 2 some properties of the classical orthogonal polynomials are discussed. In Section 3 the base and test functions and the integral operator \( I_n \) are introduced. These play an essential role in preconditioning Tau method which in turn is developed in this section. Also in this section a few paragraphs are devoted to the preconditioned method of Cabos, or Cabos-Tau for short, outlining her approach and the differences with the New-Tau method presented in this article.

Section 3.2 defines the Tau method in operator form. Section 3.3 contains an error analysis for the method. Section 4 addresses a detailed conditioning analysis of the New-Tau method for the advective and diffusion operators. It is also considered a general second order constant coefficient differential operator for which we present the condition number for various numbers of coefficients and approximation degrees. It is found that the condition numbers of the corresponding system matrices are independent of \( N \), once \( N \) is taken large enough to resolve the problems. Section 5 is devoted to numerical solution of several examples showing superior performance of the New-Tau method. Calculations were performed on a PC running Mathematica software. Based on their different ways of calculating the results under this software, two categories of calculations were carried out, namely 16 and 32 decimal digit accuracies. Comparison of 16 and 32 decimal digit computations,
or respectively 16Dec. and 32Dec. for short, reveals that the standard Tau and Cabos-Tau are very much sensitive to rounding, while the New-Tau performs well (and generally the same) in both categories.

2. Preliminary definitions and results

In this section some useful notations and also results concerning the Chebyshev, Legendre and Gegenbauer polynomials are introduced.

2.1. Ultraspherical polynomials

The ultraspherical polynomials, \( G_{n}^{(m-1/2)}(x) \), appear as eigensolutions to a singular Sturm–Liouville problem in the finite domain \( x \in [-1, 1] \), [1,10,16], with the first two being \( G_{0}^{(m-1/2)}(x) = 1 \), \( G_{1}^{(m-1/2)}(x) = 2mx \) and the remaining being defined via the recurrence relations,

\[
x G_{n}^{(m-1/2)}(x) = \frac{n + 2m - 1}{2n + 2m} G_{n-1}^{(m-1/2)}(x) + \frac{n + 1}{2n + 2m} G_{n+1}^{(m-1/2)}(x).
\]  

(1)

A relation between these polynomials and first and second derivatives appears as

\[
G_{n}^{(m-1/2)}(x) = \frac{-1}{2n + 2m} \frac{d}{dx} G_{n-1}^{(m-1/2)}(x) + \frac{1}{2n + 2m} \frac{d}{dx} G_{n+1}^{(m-1/2)}(x),
\]

\[
G_{n}^{(m-1/2)}(x) = C_{n-2} \frac{d^2}{dx^2} G_{n-2}^{(m-1/2)}(x) + C_{n} \frac{d^2}{dx^2} G_{n}^{(m-1/2)}(x) + C_{n+2} \frac{d^2}{dx^2} G_{n+2}^{(m-1/2)}(x),
\]

where

\[
C_{n-2} = \frac{1}{4(n + m)(n + m - 1)}, \quad C_{n} = \frac{-1}{2(n + m - 1)(n + m + 1)}, \quad C_{n+2} = \frac{1}{4(n + m)(n + m + 1)}.
\]

The following lemma shows relation between the \( k \)th derivative of \( G_{n}^{(m-1/2)}(x) \) and \( G_{n-k}^{(m-1/2)}(x) \).

Lemma 1. The following relation holds:

\[
\frac{d^k}{dx^k} (G_{n}^{(m-1/2)}(x)) = 2^k \frac{(m+k-1)!}{(m-1)!} G_{n-k}^{(m+k-1/2)}(x).
\]  

(2)

Proof. We use the following relation (see [6,7]):

\[
\frac{d^k}{dx^k} (G_{n}^{(m-1/2)}(x))|_{x=1} = 2^k \binom{n+k+2m-1}{n-k} \prod_{j=0}^{k-1} (m+j),
\]
and consider the first \((n + 1)\) terms of the Taylor series expansion for \(G_n^{(m-1/2)}(x)\) about \(x = 1\),

\[
G_n^{(m-1/2)}(x) = \binom{n + 2m - 1}{n} + \sum_{k=1}^{n} \frac{2^k}{k!} \binom{n + k + 2m - 1}{n - k} \prod_{j=0}^{k-1} (m + j) (x - 1)^k.
\]

Taking the \(k\)th derivative of \(G_n^{(m-1/2)}(x)\), we have

\[
\frac{d^k}{dx^k}(G_n^{(m-1/2)}(x)) = \sum_{i=0}^{n-k} \frac{2^{k+i}}{i!} \binom{n + k + 2m - 1 + i}{n - k - i} \prod_{j=0}^{k-1+i} (m + j) (x - 1)^i.
\] (3)

Similarly, for \(G_{n-k_1}^{(m+k_1-1/2)}(x)\) one has to consider

\[
\frac{d^k}{dx^k}(G_{n-k_1}^{(m+k_1-1/2)}(x))|_{x=1} = 2^k \binom{n - k_1 + k + 2(m + k_1) - 1}{n - k_1 - k} \prod_{j=0}^{k-1} (m + k_1 + j),
\]

and the first \((n - k_1 + 1)\) terms of its Taylor series expansion

\[
G_{n-k_1}^{(m+k_1-1/2)}(x) = \binom{n + 2m + k_1 - 1}{n - k_1} + \sum_{k=1}^{n-k_1} \frac{2^k}{k!} \binom{n + k_1 + 2m - 1}{n - k_1 - k} \prod_{j=0}^{k-1} (m + j) (x - 1)^k.
\]

On multiplying through by \(2^k(m + k - 1)!(m - 1)!\), adjusting the indices, this becomes

\[
2^k \frac{(m + k - 1)!}{(m - 1)!} G_{n-k}^{(m+k-1/2)}(x) = \sum_{i=0}^{n-k} \frac{2^{k+i}}{i!} \binom{n + k + i + 2m - 1}{n - i - k} \prod_{j=0}^{i+k-1} (m + j) (x - 1)^i.
\] (4)

Comparing (3) and (4) will complete the proof. □

Let \((\cdot, \cdot)\) denote the Gegenbauer scalar product on \([-1,1]\),

\[
(G_n^{(m-1/2)}(x), G_s^{(m-1/2)}(x)) = \int_{-1}^{1} G_n^{(m-1/2)}(x)G_s^{(m-1/2)}(x)(1 - x^2)^{(m-1/2)} dx.
\]

Utilizing the orthogonality of the ultraspherical polynomials, we have

\[
(G_n^{(m-1/2)}(x), G_s^{(m-1/2)}(x)) = \gamma_n^m \delta_{ns},
\]

where the continuous normalization factor, \(\gamma_n^m\), is

\[
\gamma_n^m = 2^{2m} \frac{\Gamma^2 \left(m + \frac{1}{2}\right) \Gamma(n + 2m)}{n!(2n + 2m)\Gamma^2(2m)},
\]

\(\delta_{ns}\) is the Kronecker delta and \(\Gamma(x)\) is the Gamma function [1].
We define the scaled Gegenbauer polynomials $g_n^{(m-1/2)}$,

$$g_n^{(m-1/2)}(x) = \frac{1}{\sqrt{\gamma_n^m}} G_n^{(m-1/2)}(x),$$  \hspace{1cm} (5)

to form the orthogonal basis of $\mathcal{L}_2^{G^{(m-1/2)}}[-1,1]$, the space of all functions $f : [-1,1] \to \mathbb{R}$ with $\|f\|_{G^{(m-1/2)}}^2 < \infty$, and define

$$\|g\|_{G^{(m-1/2)}} = (g,g).$$  \hspace{1cm} (6)

$H_m^{G^{(m-1/2)}}$ denotes the Sobolev space of all functions $u(x)$ on $[-1,1]$ such that $u(x)$ and all its weak derivatives up to order $m$ are in $\mathcal{L}_2^{G^{(m-1/2)}}$. The norm of $H_m^{G^{(m-1/2)}}$ is defined by,

$$\|u(x)\|_{m,G^{(m-1/2)}}^2 = \sum_{k=0}^{m} \left( \frac{d^k}{dx^k} u(x) \right)_{G^{(m-1/2)}}^2.$$

### 2.2. Chebyshev polynomials

The Chebyshev polynomials appear as the special case of the ultraspherical polynomials,

$$T_n(x) = n \lim_{m \to 0} \Gamma(2m)G_n^{(m-1/2)}(x) = \cos(n \arccos(x)).$$  \hspace{1cm} (7)

The limit is taken since $\Gamma(2m)$ has a simple pole for $m = 0$, however, the limit exists and the Chebyshev polynomials are recovered (see [10,16]).

**Lemma 2.** The following relation holds:

$$\frac{d^k}{dx^k} T_n(x) = n 2^{k-1}(k-1)! G_n^{(k-1/2)}(x).$$  \hspace{1cm} (8)

**Proof.** Using (2) and (7), we obtain

$$\frac{d^k}{dx^k} T_n(x) = n \lim_{m \to 0} \Gamma(2m) \frac{d^k}{dx^k} G_n^{(m-1/2)}(x) = n \lim_{m \to 0} \Gamma(2m)m2^k \frac{(m + k - 1)!}{m!} G_n^{(m+k-1/2)}(x).$$

From this, one easily obtains

$$\frac{d^k}{dx^k} T_n(x) = n 2^{k-1}(k-1)! G_n^{(k-1/2)}(x),$$

since $\lim_{m \to 0} \Gamma(2m)m = 1/2$, completing the proof. □

Chebyshev polynomials $T_n(x)$ are, also, recovered from (1) with

$$T_0(x) = 1, T_1(x) = x,$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \hspace{1cm} n \geq 1.$$
The following relations establish a connection between $T_n(x)$ and its first and second order derivatives (see [5])

$$T_n(x) = \frac{-1}{2(n-1)} T'_{n-1}(x) + \frac{c_n}{2(n+1)} T'_{n+1}(x), \quad T_0' = 0, \quad T_1' = T_0,$$

$$T_n(x) = \frac{1}{4(n-1)(n-2)} T''_{n-2}(x) - \frac{1}{2(n-1)(n+1)} T''_{n}(x) + \frac{c_n}{4(n+1)(n+2)} T''_{n+2}(x),$$

$$T_0'' = 0, \quad T_1'' = 0, \quad T_2'' = 4T_0, \quad n = 1, \ldots,$$

where $T_n'$, $T_n''$ represent the first and second derivatives of the Chebyshev polynomials respectively.

Note that for $k = 2$, using (8), we obtain

$$T_n''(x) = 2nG_{n-2}^{(3/2)}(x),$$

$$T_n'(x) = -G_{n-3}^{(3/2)}(x) + c_n G_{n-1}^{(3/2)}(x),$$

$$T_n(x) = \frac{1}{2(n-1)} G_{n-4}^{(3/2)}(x) - \frac{n}{(n-1)(n+1)} G_{n-2}^{(3/2)}(x) + \frac{c_n}{2(n+1)} G_n^{(3/2)}(x), \quad (9)$$

where $c_0 = 2$, and $c_n = 1$, otherwise.

Similarly, higher order derivatives of the Chebyshev polynomials can be obtained.

The leading and maximum coefficients of the Chebyshev polynomial $T_n(x)$, which we shall indicate them respectively, by $l_{T_n}$ and $\mu_{T_n}$ are as follows:

$$\mu_{T_n} = \left( \frac{k!(n-k)!}{n(n-k-1)!} \right)^{-1} \left( \frac{2^{k-n+1}}{2^{k-n+1}} \right)^{-1} \quad \text{with } k = \left[ \frac{n+3}{7} \right],$$

$$l_{T_n} = 2^{n-1}, \quad n \geq 1. \quad (10)$$

2.3. Legendre polynomials

The Legendre polynomials $L_n(x)$ relate to the ultraspherical polynomials and defined as

$$L_0(x) = 1, \quad L_1(x) = x,$$

$$L_{n+1}(x) = \frac{2n+1}{n+1} xL_n(x) - \frac{n}{n+1} L_{n-1}(x), \quad n \geq 1.$$

**Lemma 3.** The following relation for Legendre polynomials $L_n(x)$ holds,

$$\frac{d^k}{dx^k} L_n(x) = (2k-1)!! G_{n-k}^{(k)}(x),$$

where $(2k-1)!! = 1 \times 3 \times 5 \times \cdots \times (2k-1)$. 


Proof. Using (2) for \( m = 1/2 \), we obtain
\[
\frac{d^k}{dx^k} L_n(x) = \frac{1}{(2n+1)(2n+1)} L_n''(x) - \frac{2}{(2n-1)(2n+3)} L_n''(x) + \frac{1}{(2n+1)(2n+3)} L_n''(x),
\]
which completes the proof.

As for the Chebyshev polynomials, the following relations for the Legendre polynomials hold:
\[
L_n(x) = \frac{1}{2n+1} L_{n-1}'(x) + \frac{1}{2n+1} L_{n+1}'(x), \quad L_0' = 0, \quad L_1' = L_0,
\]
\[
L_n(x) = \frac{1}{2n+1} \frac{1}{(2n+1)(2n+1)} L_n''(x) - \frac{2}{(2n-1)(2n+3)} L_n''(x) + \frac{1}{(2n+1)(2n+3)} L_n''(x),
\]
\[
L_0'' = 0, \quad L_1'' = 0, \quad L_2'' = 3L_0, \quad n = 1, \ldots,
\]
and
\[
L_2''(x) = 3G_{n-2}^{(2)}(x),
\]
\[
L_1''(x) = \frac{3}{2n+1} G_{n-2}(x) + \frac{3}{2n+1} G_{n-1}(x),
\]
\[
L_n(x) = \frac{3}{2n+1} \frac{1}{(2n+1)(2n+1)} G_{n-4}(x) - \frac{6}{(2n-1)(2n+3)} G_{n-2}(x) + \frac{3}{(2n+1)(2n+3)} G_{n}(x).
\]
Similar results for higher derivatives hold.

The leading and maximum coefficients of the Legendre polynomial \( L_n(x) \) which we shall indicate them respectively by \( l_{L_n} \) and \( \mu_{L_n} \), are as follows:
\[
\mu_{L_n} = \frac{2^{-n}(2n-2k)!}{k!(n-k)!(n-2k)!} \quad \text{with} \quad k = \left\lfloor \frac{n+2}{7} \right\rfloor, \quad (13)
\]
\[
l_{L_n} = \frac{(2n)!}{(n!)^2} 2^{-n}. \quad (14)
\]
Relations (10), (11), (13) and (14) are obtained from the corresponding Rodriguez formula (see [16]).

3. The New-Tau method

The development of the New-Tau method is centered around the results obtained in this section. Suitable base and test functions are introduced. Some theoretical results concerning the applicability of these functions and necessary operators are given.

Throughout this section \( m \) will denote the order of the differential operator.
As the Cabos-Tau is used for comparison purpose we first give some detail of that approach and try to mark explicitly the differences with the new method discussed in the following parts of this
paper. Cabos has first chosen the Chebyshev polynomials $T_j(x)$ as test functions and then by using the following linear integral operator:

$$w(x) = I_m(v) = \int_x^x \int_x^x \cdots \int_x^x v(t) \, dt = \sum_{i=0}^{\infty} \hat{w}_i T_i(x), \quad \forall v \in L^2_T,$$

with $\hat{w}_0 = \hat{w}_1 = \cdots = \hat{w}_{m-1} = 0$, has introduced the base functions as some particular expansions of the Chebyshev polynomials. In applying $I_m$ on $T_j(x)$, the $m$ constants of integrations were chosen such that $I_m(T_j)$ are expressed only in terms of Chebyshev polynomials. For example,

$$I_1(T_0) = \int_x^x T_0(t) \, dt = T_1(x),$$

$$I_1(T_1) = \int_x^x T_1(t) \, dt = \frac{1}{4} T_2(x),$$

$$I_1(T_j) = \int_x^x T_1(t) \, dt = \frac{1}{2(j+1)} T_{j+1}(x) - \frac{1}{2(j-1)} T_{j-1}(x), \quad j \geq 2,$$

$$I_2(T_0) = \int_x^x \int_x^x T_0(t) \, dt = \frac{1}{4} T_2(x),$$

$$\cdots$$

$$I_m(T_j) = \sum_{i=m}^{m+j} \hat{t}_i T_i(x).$$

Using the above-mentioned base functions the exact solution, $u(x)$, of the problem is then approximated by a truncated series such as the following:

$$u(x) \simeq u_n(x) = \sum_{j=0}^{m-1} e_j T_j + I_m(v) = \sum_{j=0}^{m-1} e_j T_j + \sum_{i=0}^{n-m} a_i \left( \sum_{j=m}^{m+i} \hat{t}_j T_j \right),$$

where $v(x) = \sum_{i=0}^{n-m} a_i T_i(x)$, and $n$ is the degree of approximant. Further detail can be obtained from [2]. But in the New-Tau, the topic of this article, we modify the standard Tau method in greater generality through the use of a particular type of the Gegenbauer polynomials as our test functions which are chosen according to the order of given ODE. Then we introduce a suitable linear integral operator that its application, as outlined in the subsequent subsections, provides us with all the means we need to introduce our desirable base functions in such a way that they are some appropriately scaled Chebyshev (or Legendre) polynomials. So, all the differences made by New-Tau over the standard Tau and Cabos-Tau have their roots in introduction of its test and base functions.
3.1. Base and test functions

We now define suitable sequences of orthogonal polynomials to be used as test and base functions. Corresponding to the Chebyshev and Legendre New-Tau methods, test functions, \( t_{Tn}(x) \), \( t_{Ln}(x) \) and base functions \( b_{Tn}(x) \), \( b_{Ln}(x) \), are defined as follows:

\[
t_{Tn-m}(x) = 2^{m-1} l_{Tn}^{-1}(m-1)! n G_{n-m}^{(m-1/2)}(x), \quad n \geq 0,
\]

\[
t_{Ln-m}(x) = (2m-1)!! l_{Ln}^{-1} G_{n-m}^{(m)}(x), \quad n \geq 0,
\]

\[b_{Tn}(x) = l_{Tn}^{-1} T_n(x), \quad n \geq 0,\]

\[b_{Ln}(x) = l_{Ln}^{-1} L_n(x),\]  \(15\)

where \( l_{Tn} \) and \( l_{Ln} \) are given by (11) and (14), respectively. Note that using (10) instead of (11) or (13) instead of (14) in defining the base and test functions, from their structure, one can generally expect to obtain better numerical results in the case of ODEs with high degree coefficients. This is demonstrated in the results of Problem 6 in Section 5.

As the Gegenbauer polynomials are orthogonal, so are \( t_{Tn}(x) \), \( t_{Ln}(x) \), \( b_{Tn}(x) \) and \( b_{Ln}(x) \). Thus, the connection between these two functions, after differentiating \(16\) \( m \) times and comparing with (8) (or (12)) and (15), is as follows:

\[
\frac{d^m}{dx^m}(b_{Tn}(x)) = t_{Tn-m}(x),
\]

\[
\frac{d^m}{dx^m}(b_{Ln}(x)) = t_{Ln-m}(x),
\]

Denoting the \( m \)th order linear integral operator by \( I_m \), we define

\[I_m : \mathcal{L}^2_{G^{(\bar{z})}} \rightarrow \mathcal{L}^2,\]

\[I_m(t_{n-m}(x)) = b_n(x), \quad n \geq m \geq 0,\]  \(17\)

where \( \bar{z} \) is \( m - 1/2 \) or \( m \), respectively, for the Chebyshev or Legendre method.

\( I_m(v(x)) \) means \( m \) repeated integrations of function \( v(x) \). According to this definition each \( v(x) \in \mathcal{L}^2_{G^{(\bar{z})}} \) is projected onto a finite scaled Gegenbauer expansion, in which the \( m \) constants of integrations are chosen appropriately. Due to their extensive use, we consider the results of the general analysis to the case of Chebyshev and Legendre Tau approximations. Note that in Eq. (17) and some parts of the following sections \( t_{n-m}(x) \) is used as a general name for \( t_{Tn-m}(x) \) or \( t_{Ln-m}(x) \). A similar role is played by \( b_n(x) \) in place of \( b_{Tn}(x) \) or \( b_{Ln}(x) \).

3.2. The New-Tau method in operator form

Assume that the \( m \)th order ordinary differential equation

\[
Lu = f, \quad u \in H^m_{G^{(\bar{z})}}, \quad g(x) \in \mathbb{R}^m, \quad f \in \mathcal{L}^2_{G^{(\bar{z})}},
\]

\[Bu = g,\]  \(18\)

Where \( L \) is a linear differential operator, \( B \) is a linear boundary operator, and \( u \) is the unknown function.
on $[-1, 1]$ has to be solved, where $Lu = L((d^m/dx^m)u, (d^{m-1}/dx^{m-1})u, \ldots, u, x)$ and $B$ is some bounded boundary operator which accounts for $m$ necessary supplementary conditions.

From now on, $n$ will be a fixed natural number which represents the degree of the Tau approximant. Let $Q_n$ be an orthogonal projection operator from $y = \mathbb{R}^m \times L^2$, onto $y_n = \mathbb{R}^m \times P_{n-m}$, such that $P_n$ is the space of polynomials of degree less or equal to $n$. The solution $u_n(x)$ of the Tau equation

$$Q_n \left( \begin{array}{c} B \\ L \end{array} \right) u_n(x) = Q_n \left( \begin{array}{c} g \\ f \end{array} \right), \quad u_n \in P_n,$$  

(19)

will be called "$n$th degree Tau approximant" in the sequel (see [2]).

We now continue to present the New-Tau method in the following way: Let $Iem$ be the integral operator $I_m$ extended to operate on $y$,

$$Iem: y \rightarrow L^2, \quad m \geq 1,$$

$$Iem \left( \begin{array}{c} x \\ v \end{array} \right) = (E_{m-1}, I_m) \left( \begin{array}{c} x \\ v \end{array} \right) = \sum_{j=0}^{m-1} x_j b_j(x) + I_m(v(x)),$$  

(20)

for any $v \in L^2_{G(x)}$, where $E_m$ and $x$ are defined as the row vectors $E_m = (b_0(x), b_1(x), \ldots, b_m(x))$ and $x = (x_0, x_1, \ldots, x_{m-1})$. Setting $u(x) = Iem \left( \begin{array}{c} x \\ v \end{array} \right)$, one obtains, from (20),

$$\frac{d^m}{dx^m} u(x) = v(x).$$  

(21)

According to the role of base functions in the Tau method and (17), (20) we have

$$u_n(x) = \sum_{i=0}^{n} a_i b_i(x) = \sum_{i=0}^{m-1} a_i b_i(x) + \sum_{i=m}^{n} a_i I_m(t_{i-m}),$$

$$u_n(x) = \sum_{i=0}^{m-1} a_i b_i(x) + I_m \left( \sum_{i=m}^{n} a_i t_{i-m} \right) = Iem \left( \begin{array}{c} a^{(n)} \\ v^{(n)} \end{array} \right),$$  

(22)

where

$$a^{(n)} = (a_0, a_1, \ldots, a_{m-1}), \quad v^{(n)} = \sum_{i=m}^{n} a_i t_{i-m}.$$

Comparing (19) and (22) we obtain the following result which will be called the New-Tau equation:

$$Q_n \left( \begin{array}{c} B \\ L \end{array} \right) Iem \left( \begin{array}{c} a^{(n)} \\ v^{(n)} \end{array} \right) = Q_n \left( \begin{array}{c} g \\ f \end{array} \right), \quad \left( \begin{array}{c} a^{(n)} \\ v^{(n)} \end{array} \right) \in y_n.$$  

(23)

This has several advantages over solving (19) directly. At first $\left( \begin{array}{c} a^{(n)} \\ v^{(n)} \end{array} \right)$ is determined from (23) and in the second step the $n$th degree New-Tau approximant, $u_n(x)$, satisfying (19) can be recovered from $\left( \begin{array}{c} a^{(n)} \\ v^{(n)} \end{array} \right)$ by setting $u_n = Iem \left( \begin{array}{c} a^{(n)} \\ v^{(n)} \end{array} \right)$. 
3.3. Error analysis

Let \( L \) be the linear differential operator
\[
Lu = \frac{d^m}{dx^m} u(x) + \sum_{j=0}^{m-1} a_j(x) \frac{d^j}{dx^j} u(x), \quad a_j(x) \in \mathcal{L}^2[-1, 1].
\] (24)

From (21) and (24) we obtain
\[
Lu = v(x) + L_1(u(x)),
\]
where \( L_1 := \sum_{j=0}^{m-1} a_j(x) d/dx^j \).

With \( u(x) = Ie_m(x) \) the operator \( (B/L)Ie_m \) can be written as the sum of the operator \( k \) and the identity operator \( id \) as follows: we clearly have
\[
Lu = id(v(x)) + L_1E_{m-1}z + L_1I_m v,
\]
\[
Bu = BE_{m-1}z + BI_m v.
\]

Thus
\[
\begin{pmatrix} B \\ L \end{pmatrix} Ie_m \begin{pmatrix} z \\ v \end{pmatrix} = (id + k) \begin{pmatrix} z \\ v \end{pmatrix}, \quad k = \begin{pmatrix} BE_{m-1} - id & BI_m \\ L_1E_{m-1} & L_1I_m \end{pmatrix}.
\] (25)

We show that \( k \) is compact. To do this it is sufficient to prove that the operator \( I_m, m \geq 1 \), is compact (see [2]).

**Lemma 4.** Let \( \{x_n\} \) be an orthonormal basis of Hilbert space \( H \). Bounded linear operator \( \Phi : H \rightarrow H \) is compact if \( \sum_n \|\Phi x_n\|^2 < \infty \).

**Proof.** See [11]. \( \square \)

The following lemma and theorem are considered for Chebyshev polynomials \( T_n(x) \); they can likewise be expressed for Legendre polynomials.

**Lemma 5.** Operator \( I_m : \mathcal{L}^2_{G^{(m-1)/2}} \rightarrow \mathcal{L}^2_T \) is compact.

**Proof.** Operator \( I_m \) is linear and continuous therefore it is bounded [12]. Scaled Gegenbauer polynomials \( \{g_n^{(m-1)/2}\} \) are orthonormal base for \( \mathcal{L}^2_{G^{(m-1)/2}} \). Using (6) and (17),
\[
\|I_m(g_n^{(m-1)/2})\|^2_T = \frac{2^{-2m}}{\Gamma^m((m-1)!)^2} \int_{-1}^{1} G_n^{(-1/2)}(1-x^2)^{-1/2} \text{d}x.
\]

Thus,
\[
\|I_m(g_n^{(m-1)/2})\|^2_T = C(m) \frac{n!}{(n+m)(n+2m-1)!}.
\]
such that
\[ C(m) = \frac{2^{-4m} I^2(2m) I^2(1/2)}{I^2(m + 1/2)((m - 1)!)^2}, \]
and \[ \sum_n \|I_m(g_n^{(m-1/2)})\|_T < \infty, \] because it is a \( p \) series with \( p \geq 2. \)

From (22) and (24),
\[ Q_n(id + k) \left(\begin{array}{c} a^{(n)} \\ v^{(n)} \end{array}\right) = Q_n \left(\begin{array}{c} g \\ f \end{array}\right). \]

Clearly, from (23) and simple manipulation that leads to (25), the above-mentioned operator equation means that the operator \( id + Q_n k \) has to be inverted. The following theorem shows some results concerning this operator.

**Theorem 3.** Let \( B : \mathcal{L}^2_T \rightarrow \mathbb{R}^n \) be a linear boundary operator and \( L : D(L) \subset \mathcal{L}^2_T \rightarrow \mathcal{L}^2_G^{(m-1/2)} \) be the linear differential operator given by (24). If \( u^{(0)} \) uniquely solves (18) then the Tau equations (19), (23) are uniquely solvable for \( n \) large enough. Rate of convergence can be estimated by
\[ \|u_n - u^{(0)}\|_T \leq (1 + \|kQ^{(n)}\| \cdot \|(id + Q_n k)^{-1}\|)\|P^{(n-m)} \frac{d^n}{dx^n} u^{(0)}\|_T, \] (26)
and the term in brackets converges to one, as it can be easily seen that \( \|kQ^{(n)}\| \cdot \|(id + Q_n k)^{-1}\| \rightarrow 0 \) as \( n \rightarrow \infty. \) Here we have used the abbreviations \( Q^{(n)} = id - Q_n, \) \( P^{(n)} = id - P_n. \)

There is an upper bound for the condition number of the Tau operator \( id + Q_n k \) which converges to \( \|id + k\| \cdot \|(id + k)^{-1}\| \) as \( n \rightarrow \infty,
\[ \|id + Q_n k\| \cdot \|(id + Q_n k)^{-1}\| \leq \frac{(\|id + k\| + \|Q^{(n)} k\|) \cdot \|(id + k)^{-1}\|}{1 - \|Q^{(n)} k\| \cdot \|(id + k)^{-1}\|}. \]

**Proof.** See [2].

From (24) and truncation error bound of Chebyshev polynomials (see [1]) we obtain
\[ \|u_n - u^{(0)}\|_T \leq \text{const} \cdot n^{m-k} \|u\|_{k,T}. \]
This inequality shows that the Tau approximant converges faster than any inverse power of \( n, \) if \( u^{(0)} \) is infinitely differentiable (see [3,8]).

4. Properties of the preconditioning Tau method (New-Tau)

We shall discuss here the qualitative behavior of the eigenvalue spectrum for Chebyshev and Legendre New-Tau approximations of the first order hyperbolic operator \( Lu = du/dx \) and the second order diffusion operator \( Lu = d^2u/dx^2 \) and a general second order ODE with constant coefficients. Also, a description of what is exactly done follows for each case.
4.1. Condition number of the advective operator

Let us consider the first order problem

\[
\frac{d}{dx} u(x) = f(x),
\]

\[u(1) = 0.\]  

(27)

Our objective here is to solve the above equation, using the Chebyshev and Legendre New-Tau methods, which were described in Section 3, and then we want to obtain the condition number of corresponding coefficients matrix. In the case of standard Chebyshev Tau method (see [14]), the coefficients matrix is obtained by \([V B V^{-1}]^T\), in which \(V\) is the matrix of coefficients of Chebyshev polynomials \(T_0(x), T_1(x), \ldots, T_n(x)\) (Chebyshev base functions), \(V^{-1}\) is the inverse matrix of coefficients of \(T_0(x), T_1(x), \ldots, T_n(x)\) (Chebyshev test functions) and \(B\) is as follows:

\[
B = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 3 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & n & 0
\end{bmatrix}.
\]

Similarly, in Chebyshev New-Tau method presented in Section 3, \(V\) is defined to be the matrix of coefficients of the base functions \(b_{T_n}\), such that

\[
[b_{T_0}, b_{T_1}, \ldots, b_{T_n}] = VX,
\]

where \(X^T = [1, x, x^2, \ldots, x^n]\), and \(g\) is the matrix of coefficients of test functions \(t_{T_n}\), such that

\[
[t_{T_0}, t_{T_1}, \ldots, t_{T_n}] = gX. \tag{28}
\]

Then the matrix of coefficients of the system of Chebyshev New-Tau method will be as

\[
M_T = [V B g^{-1}]^T.
\]

Clearly,

\[
\begin{bmatrix}
\frac{d}{dx} b_{T_0}, & \frac{d}{dx} b_{T_1}, & \cdots, & \frac{d}{dx} b_{T_n}
\end{bmatrix} = V B X. \tag{29}
\]

Relations (8), (15), (16) show \((d/dx)b_{T_n} = t_{T_{n-1}}, n \geq 1\).

Thus (28) yields

\[
gX = \begin{bmatrix}
\frac{d}{dx} b_{T_1}, & \frac{d}{dx} b_{T_2}, & \cdots, & \frac{d}{dx} b_{T_{n+1}}
\end{bmatrix}.
\]
Comparing this with (29), one can easily see that matrices $g$ and $V\beta$ have $n$ equal rows. It means, except the first row of matrix $V\beta$ which is zero, the other rows of this matrix are equal to the rows of matrix $g$. So, $M_T$ is a matrix with upper first diagonal elements equal to unity and its other elements are zero. Then on applying the boundary condition $u(1) = \sum_{i=0}^{n} a_i b_{T_i}$, the following final matrix is obtained:

$$M_T = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 1 \\
1 & 2^0 & 2^{-1} & 2^{-2} & \cdots & 2^{1-n}
\end{bmatrix}.$$ 

Therefore, $M_T M_T^T$ is as follows:

$$M_T M_T^T = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 2^0 \\
0 & 1 & 0 & \cdots & 0 & 2^{-1} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 1 \\
2^0 & 2^{-1} & 2^{-2} & \cdots & 2^{1-n} & \alpha
\end{bmatrix},$$

such that

$$\alpha = 1 + \sum_{i=1}^{n} 2^{2-2i}.$$ 

Since, $n$ diagonal elements of the above matrix are 1, through a simple calculation, one can see that $n - 1$ elements of the spectrum of this matrix are 1, and the remaining are as follows:

$$\lambda_{\pm} = 2^{-2n}(\beta + 2^{2n} \pm \sqrt{\beta(\beta + 2^{2n+1})}),$$

where $\beta = \sum_{i=1}^{n} 2^{2i-1} = \frac{2}{3}(2^{2n} - 1)$.

Equivalently,

$$\lambda_{\pm} = \gamma + 1 \pm \sqrt{\gamma^2 + 2\gamma},$$

$$\gamma = \frac{2}{3}(1 - 2^{-2n}).$$

Clearly, $\forall n$, $n \in \mathbb{N}$, $\lambda_+ > 1$, $\lambda_- < 1$, we thus have

$$\lambda_{\max}(M_T M_T^T) = \lambda_+,$$

$$\lambda_{\min}(M_T M_T^T) = \lambda_-.$$
Therefore, we can estimate the customary condition number of matrix $M_T$ with $L^2$ norm,

$$\text{cond}(M_T) = \frac{\lambda_{\text{max}}(M_T M_T^T)}{\lambda_{\text{min}}(M_T M_T^T)} = \sqrt{\frac{\gamma^2 + 2\gamma}{\gamma + 1}} = \sqrt{\gamma + 1 + \sqrt{\gamma^2 + 2\gamma}} = \gamma + 1 + \sqrt{\gamma^2 + 2\gamma}.$$ 

As $n \to \infty$, $\gamma \to 2/3$, we thus have $\text{cond}(M_T) \to 3$. Then for large values of $n$, the condition number of the resulting coefficient matrix will not exceed 3.

Similarly, for the Legendre New-Tau method, presented in this article, we can calculate the condition number of the final matrix of coefficients. The matrix of coefficients related to this method for solving the above first order problem will be as follows:

$$M_L = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 1 \\
1 & 1 & 2/3 & 2/5 & \ldots & \frac{2^n(n!)^2}{(2n)!}
\end{bmatrix}.$$ 

In the Legendre New-Tau method, also, $n - 1$ eigenvalues of matrix $M_L M_L^T$ are unity and the other two eigenvalues will be obtained as follows:

$$\lambda_{\pm} = \frac{q + 2^{2n-1} p \pm \sqrt{q^2 + 2^{2n}qp}}{2^{2n-1} p},$$

$$p = \prod_{i=0}^{n-1} \left( \frac{2n - 2i}{n - i} \right)^2,$$

$$q = \sum_{j=0}^{n-1} \frac{4^n j^2}{\left( \frac{2j+2}{j+1} \right)^2}.$$ 

This can equivalently be written as

$$\lambda_{\pm} = q' + 1 \pm \sqrt{q'^2 + 2q'},$$

$$q' = \frac{q}{2^{2n-1} p} = \sum_{j=0}^{n-1} \frac{2^{2j+1}}{\left( \frac{2j+2}{j+1} \right)^2}.$$ 

It is then clear that, $\forall n$, $n \in \mathbb{N}$, $\lambda_+ > 1$, $\lambda_- < 1$. Thus, $\lambda_{\text{max}}(M_L M_L^T) = \lambda_+$, $\lambda_{\text{min}}(M_L M_L^T) = \lambda_-$. 


So the condition number of matrix $M_L$ will be as follows:

$$\text{cond}(M_L) = \sqrt{\frac{q' + 1 + \sqrt{q'^2 + 2q'}}{q' + 1 - \sqrt{q'^2 + 2q'}}} = q' + 1 + \sqrt{q'^2 + 2q'}.$$  

As $n \to \infty$, $q'$ is a convergent series and tends to 1. (It is clear that $q' < \sum_{j=0}^{n-1} 1/(j - 1)^2$, and because it is convergent to $\pi^2/6$, then its upper bound will be $\pi^2/6$). Therefore the condition number of matrix $M_L$ is approximately $2 + \sqrt{3}$. Then the maximum of the condition number will not exceed $2 + \sqrt{3}$.

We computed the condition numbers of matrix $M$, with 16Dec. and 32Dec., and observed that the performance for the standard Tau, Cabos-Tau, and New-Tau follows the same results discussed in the next sections. So, to keep the paper to a reasonable length we do not give here any numerical table.

### 4.2. Condition number of the diffusive operator

Consider the second order problem

$$\frac{d^2}{dx^2} u(x) = f(x),$$

$$u(\pm 1) = 0.$$  \hspace{1cm} (30)

In this section the objective is also the calculation of the condition number of matrix of coefficients in the Chebyshev and Legendre New-Tau methods presented in Section 3, for the above second order problem. First, we recall that in standard Chebyshev Tau method the coefficient matrix of the final system is obtained by $[V^2 V^{-1}]^T$, in which $V$ and $B$, are as defined in the previous section.

Since the problem is of the second order, test and the base functions introduced in Section 3 will follow $(d^2/dx^2)b_{T_n} = T_{n-2}$. Like the previous section, the matrix of coefficients in this method will be as $[V^2 g^{-1}]^T$, so that $V$ is the matrix of coefficients of base functions $b_{T_0}, b_{T_1}, \ldots, b_{T_n}$, and $g$, is the matrix of coefficients of test functions, $T_0, T_1, \ldots, T_n$.

We see that matrices $g$ and $V^2$ except in the first two rows of $V^2$, which are zero, are equal. This means they are equal in $n - 1$ rows. Therefore, $M_T = [V^2 g^{-1}]^T$, is a matrix with upper second diagonal elements equal to unity, and other elements are zero. Applying the boundary conditions $u(\pm 1) = \sum_{i=0}^{n} a_i b_{T_i}$, we finally get the following matrix:

$$M_T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 2^0 & 2^{-1} & 2^{-2} & 2^{-3} & \ldots & 2^{1-n} \\ 1 & -2^0 & 2^{-1} & -2^{-2} & 2^{-3} & \ldots & (-1)^{n-1} 2^{1-n} \end{bmatrix}.$$
In order to estimate the condition number of the matrix $M_T$ we consider

$$M_T M_T^T = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 1/2 & -1/2 \\
0 & 1 & 0 & \cdots & 0 & 1/4 & 1/4 \\
0 & 0 & 1 & \cdots & 0 & 1/8 & -1/8 \\
\vdots & \ddots & \ddots & & \vdots & \vdots & \\
0 & \cdots & \cdots & 0 & 1 & 1 \frac{1}{2^{n-2}} \frac{(-1)^{n-1}}{2^{n-1}} \\
1/2 & 1/4 & 1/8 & \cdots & 1 \frac{1}{2^{n-1}} & \delta & \theta \\
-1/2 & 1/4 & -1/8 & \cdots & 1 \frac{(-1)^{n-1}}{2^{n-1}} & \theta & \delta 
\end{bmatrix},$$

such that $\delta = 1 + \sum_{i=1}^{n} 2^{2-2i}$, $\theta = 1 + \sum_{i=1}^{n} (-1)^i 2^{2-2i}$.

Calculating the eigenvalues of this matrix one can see that $n - 3$ elements out of $n + 1$ eigenvalues of this matrix are 1; among 4 remaining eigenvalues, those with maximum and minimum absolute values are as follows:

$$\lambda_{\pm} = 2^{-2n-1}(\gamma + 3 \times 2^{2n} \pm \sqrt{\gamma^2 + 2^{4n} + 3 \times \gamma 2^{2n+1}}),$$

where

$$\gamma = \sum_{i=1}^{(n-1)/2} 2^{4i+1} \text{ for } n \text{ odd},$$

$$\gamma = \sum_{i=1}^{n/2} 2^{4i-1} \text{ for } n \text{ even}.$$ Equivalently,

$$\lambda_{\pm} = q + 1 \pm \sqrt{q^2 + 3q + 1/4},$$

$$q = \frac{8}{15} (1/2 - 2^{-2n-1}) \text{ for } n \text{ even},$$

$$q = \frac{32}{15} (1/8 - 2^{-2n-1}) \text{ for } n \text{ odd}.$$ So, the condition number of $M_T$ is

$$\text{cond}(M_T) = \frac{\sqrt{q + 3/2 + \sqrt{q^2 + 3q + 1/4}}}{q + 3/2 - \sqrt{q^2 + 3q + 1/4}} = \frac{\sqrt{2}}{2} (q + 3/2 + \sqrt{q^2 + 3q + 1/4}).$$

As $n \to \infty$, $q \to 4/15$. Thus we approximately have $\text{cond}(M_T) \to 2$. 

For the Legendre New-Tau method, one can likewise see that the coefficients matrix is

\[
M_L = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 1 & 2/3 & 2/5 & 8/35 & \cdots & \frac{2^n(n!)^2}{(2n)!} \\
1 & -1 & 2/3 & -2/5 & 8/35 & \cdots & \frac{2^n(n!)^2}{(-1)^{n-1}(2n)!}
\end{bmatrix},
\]

and its spectrum consists of \( n - 3 \) occurrences of unity and

\[
\lambda_{\pm} = q + 3 \times 2^{2n-5} p \pm \sqrt{-2^{4n-7} p^2 + (q + 3 \times 2^{2n-5} p)^2},
\]

where, for \( n \) odd,

\[
q = \sum_{i=1}^{(n-1)/2} \frac{4^{n+2i-2} p}{\left(\frac{4i}{2i}\right)^2},
\]

\[
p = \prod_{i=1}^{(n-1)/2} \left( \frac{2n - 4i + 2}{n - 2i + 1} \right)^2,
\]

and, for \( n \) even,

\[
q = \sum_{i=1}^{n/2} \frac{4^{n+2i-2} p}{\left(\frac{4i}{2i}\right)^2},
\]

\[
p = \prod_{i=1}^{n/2} \left( \frac{2n - 4i + 4}{n - 2i + 2} \right)^2.
\]

Equivalently,

\[
\hat{\lambda}_{\pm} = q' + 3/2 \pm \sqrt{q'^2 + 3q' + 1/4},
\]

\[
q' = \sum_{i=1}^{n/2} \frac{2^{4i}}{\left(\frac{4i}{2i}\right)^2} \text{ for } n \text{ even},
\]
Finally, the condition number of the matrix $M_L$ is as follows:

$$\text{cond}(M_L) = \sqrt{q' + 3/2 + \sqrt{q'^2 + 3q' + 1/4}} = \frac{\sqrt{2}}{2} (q' + 3/2 + \sqrt{q'^2 + 3q' + 1/4}).$$

As $n \to \infty$, series $q'$, roughly speaking, tends to 1, then $\text{cond}(M_L) \to 3$. So, for $n$ sufficiently large, the maximum of the condition number will be 3.

The numerical results presented in the final section of this article clarify and confirm the efficiency of the New-Tau method.

We computed the condition numbers of matrix $M$, with 16Dec. and 32Dec., and observed that the performance for the standard Tau, Cabos-Tau, and New-Tau follows the same results discussed in the next sections. So, to keep the paper to a reasonable length we do not give here any numerical table.

### 4.3. Condition number of a second order differential operator

For the sake of simplicity, a general second order linear ordinary differential operator with constant coefficients is considered. The analysis, here proceeds only for the Chebyshev case of the New-Tau method and it can likewise be applied for the Legendre case. Also a comparison is made with the result of the preconditioning method discussed by Cabos (see [2]).

Let us consider

$$Lu = \frac{d^2}{dx^2} u(x) + k_1 \frac{d}{dx} u(x) + k_0 u(x), \quad k_0, k_1 \in \mathbb{R}, \quad u(x) \in H^2_T.$$

The approximate solution $u_n(x)$ is considered to be as

$$u_n(x) = \sum_{j=0}^{n} a_j b_j(x) = \sum_{j=0}^{n} a_j l^{-1}_{T_j} T_j(x),$$

where $a_j, j \geq 0$, are the unknowns to be determined. Therefore,

$$Lu_n(x) = \sum_{j=2}^{n} a_j l^{-1}_{T_j} T_j''(x) + k_1 \sum_{j=0}^{n} a_j l^{-1}_{T_j} T_j'(x) + k_0 \sum_{j=0}^{n} a_j l^{-1}_{T_j} T_j(x).$$

As the order of the differential equation is 2 then for the test functions we choose $m = 2$. Using (9) we obtain

$$Lu_n(x) = \sum_{j=2}^{n} a_j l^{-1}_{T_j} (2j)G_{j-2}^{(3/2)} + k_1 \sum_{j=1}^{n} a_j l^{-1}_{T_j} (G_{j-1}^{(3/2)} - G_{j-3}^{(3/2)})$$

$$+ k_0 \sum_{j=0}^{n} a_j l^{-1}_{T_j} \left( \frac{1}{2j - 2} G_{j-4}^{(3/2)} - \frac{j}{j^2 - 1} G_{j-2}^{(3/2)} + \frac{1}{2j + 2} G_{j+2}^{(3/2)} \right).$$
Adjusting the indices we have the following result:

\[
L_{u_n}(x) = \sum_{j=0}^{n} a_{j+2} l_{T_{j+2}}^{-1} 2(j + 2) G_j^{(3/2)} + k_1 \sum_{j=0}^{n} a_{j+1} l_{T_{j+1}}^{-1} G_j^{(3/2)}
- k_1 \sum_{j=0}^{n} a_{j+3} l_{T_{j+3}}^{-1} G_j^{(3/2)} + k_0 \sum_{j=0}^{n} a_{j+4} l_{T_{j+4}}^{-1} \frac{1}{2j + 6} G_j^{(3/2)}
- k_0 \sum_{j=0}^{n} a_{j+2} l_{T_{j+2}}^{-1} \frac{j + 2}{(j + 2)^2 - 1} G_j^{(3/2)} + k_0 \sum_{j=0}^{n} a_{j} l_{T_{j}}^{-1} \frac{1}{2j + 2} G_j^{(3/2)}.
\]

With the help of variational form of the Tau method and the following inner product, the elements of the final matrix \( M \) with last two rows replaced by the two initial and boundary conditions is obtained:

\[
(L_{u_n}, t_k) = \frac{l_{T_{k+2}}}{(2k + 4)} \left\{ \frac{k_0 l_{T_k}^{-1}}{2k + 4} a_k + k_1 l_{T_{k+1}}^{-1} a_{k+1} + l_{T_{k+2}}^{-1} (k + 2) \left( 2 - \frac{k_0}{(k + 2)^2 - 1} \right) a_{k+2}
- k_1 l_{T_{k+3}}^{-1} a_{k+3} + k_0 \frac{l_{T_{k+4}}^{-1}}{2k + 6} a_{k+4} \right\}.
\]

Thus we have

\[
m_{i,i} = \frac{k_0}{(i + 1)(i + 2)}, \quad m_{i,i+1} = \frac{k_1}{i + 2}, \quad m_{i,i+2} = 1 - \frac{k_0}{2(i + 2)^2 - 2} \quad \text{for} \quad i = 0, 1, \ldots, n - 2,
\]

\[
m_{i,i+3} = -\frac{k_1}{4(i + 2)} \quad \text{for} \quad i = 0, 1, \ldots, n - 3,
\]

\[
m_{i,i+4} = \frac{k_0}{16(i + 3)(i + 2)} \quad \text{for} \quad i = 0, 1, \ldots, n - 4,
\]

\[
m_{n-1,0} = 1 \quad \text{and} \quad m_{n-1,i} = 2^{-i+1}, \quad i = 1, \ldots, n,
\]

\[
m_{n,0} = 1 \quad \text{and} \quad m_{n,i} = (-1)^i 2^{-i+1}, \quad i = 1, \ldots, n.
\]

To confirm our theoretic discussion (that the condition number in New-Tau is not growing very much with \( n \) particularly when \( n \) is large and this method performs strongly against rounding) we have compared numerically the condition number of \( M \), obtained in this section and that obtained by the method of Cabos, for some values of \( n \) and \( k_0, k_1 \) (see Tables 1 and 2 for calculations with 16Dec.).

From Table 2, one can see that the Cabos-Tau was quickly influenced by rounding as \( n \) increases, while the New-Tau performed well up to \( n = 128 \) and even for \( n = 256 \), although at this stage started to be influenced by rounding errors. However, with 32Dec. computation, the performance of computation for both methods was stable and very good.

Similarly, entries of the final matrix coefficient of the New-Tau Legendre method can be computed.
Table 1
Comparing the condition numbers of $M$ for $n = 32$ and 64 and different values of $k_0$, $k_1$ (with 16Dec.)

<table>
<thead>
<tr>
<th>$k_0$, $k_1$</th>
<th>$n = 32$</th>
<th>$n = 64$</th>
<th>$n = 32$</th>
<th>$n = 64$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cabos-Tau</td>
<td>New-Tau</td>
<td>Cabos-Tau</td>
<td>New-Tau</td>
</tr>
<tr>
<td>$-0.5$, $-50$</td>
<td>$5.4 \times 10^3$</td>
<td>$8.7 \times 10^5$</td>
<td>$6.9 \times 10^{13}$</td>
<td>$1.8 \times 10^6$</td>
</tr>
<tr>
<td>$0$, $-10$</td>
<td>$6.6 \times 10^2$</td>
<td>$3.0 \times 10^4$</td>
<td>$2.0 \times 10^{12}$</td>
<td>$3.0 \times 10^4$</td>
</tr>
<tr>
<td>$0.5$, $3$</td>
<td>$3.4 \times 10^1$</td>
<td>$4.0$</td>
<td>$1.4 \times 10^{12}$</td>
<td>$4.0$</td>
</tr>
<tr>
<td>$1$, $1$</td>
<td>$8.0$</td>
<td>$4.0$</td>
<td>$3.1 \times 10^{13}$</td>
<td>$4.0$</td>
</tr>
<tr>
<td>$3$, $0.5$</td>
<td>$5.4 \times 10^1$</td>
<td>$2.1 \times 10^1$</td>
<td>$1.6 \times 10^{13}$</td>
<td>$2.1 \times 10^1$</td>
</tr>
<tr>
<td>$10$, $0$</td>
<td>$3.1 \times 10^1$</td>
<td>$7.6 \times 10^2$</td>
<td>$6.8 \times 10^{13}$</td>
<td>$7.6 \times 10^2$</td>
</tr>
<tr>
<td>$50$, $-0.5$</td>
<td>$3.1 \times 10^1$</td>
<td>$1.7 \times 10^1$</td>
<td>$2.3 \times 10^{13}$</td>
<td>$1.7 \times 10^1$</td>
</tr>
<tr>
<td>$200$, $-1$</td>
<td>$9.5 \times 10^6$</td>
<td>$2.0 \times 10^8$</td>
<td>$2.8 \times 10^{15}$</td>
<td>$2.0 \times 10^8$</td>
</tr>
</tbody>
</table>

Table 2
Comparing the condition numbers of $M$ for $k_0 = 200$, $k_1 = -1/2$ and different values of $n$ (with 16Dec.)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n = 32$</th>
<th>$n = 64$</th>
<th>$n = 128$</th>
<th>$n = 256$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cabos-Tau</td>
<td>$1.0 \times 10^7$</td>
<td>$3.4 \times 10^{15}$</td>
<td>$1.7 \times 10^{63}$</td>
<td>$9.7 \times 10^{175}$</td>
</tr>
<tr>
<td>New-Tau</td>
<td>$2.4 \times 10^7$</td>
<td>$2.4 \times 10^8$</td>
<td>$7.4 \times 10^8$</td>
<td>$3.0 \times 10^{49}$</td>
</tr>
</tbody>
</table>

From Tables 1 and 2, one can obviously notice that the new method generally performs well, except possibly if $n$ is small and $k_0$ and $k_1$ very different. However, the case of small $n$ hardly deserves a special analysis as no preconditioning is needed in this case. Different behavior of the eigenvalue spectrum of the matrix $M$ may be expected as can be seen through elements of $M$ given above.

5. Numerical results

Six test problems were solved using the Chebyshev and Legendre Tau method with different modifications (i.e., with preconditioners) and also its standard version, or Tau for short. We use the Cabos-Tau and the New-Tau as the two preconditioned Tau methods. The test problems are not numerically complicated but for our purpose in showing the effect of the new Tau preconditioner seem to be adequate. In case of using preconditioning we have applied the new suggested method of this article, (denoted by New-Tau), and the one introduced by Cabos [2] (denoted by Cabos-Tau). Since involved linear systems are nonsymmetric, a modified conjugate gradients method [15] was used. For all problems, iteration was stopped as soon as $\|\text{residual}\| \leq \text{tol}$, where $\text{tol} \leq 10^{-10}$.

Zero vector was used as starting iteration vector.

All calculations were performed on a PC running Mathematica software with 16 and 32 decimal digit accuracies. In tables “maximal error” always refers to the maximal difference between approximation and exact solution at the Gauss Lobatto points. In all cases any non-polynomial coefficient
was replaced by a truncated Taylor series expansion of order $n$, where $n$ represents the degree of the Tau approximant. The condition number was calculated with the $L^2$ norm.

Table 3
Iterative solution of problem 1 for $n = 32$ and 64 (Chebyshev basis) (with 16Dec.)

<table>
<thead>
<tr>
<th>Method</th>
<th>$n = 32$</th>
<th>$n = 64$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iteration steps</td>
<td>Condition number</td>
</tr>
<tr>
<td>Tau</td>
<td>45</td>
<td>$4.9 \times 10^4$</td>
</tr>
<tr>
<td>Cabos-Tau</td>
<td>33</td>
<td>$5.9 \times 10^3$</td>
</tr>
<tr>
<td>New-Tau</td>
<td>22</td>
<td>$6.1 \times 10^2$</td>
</tr>
</tbody>
</table>

Table 4
Iterative solution of problem 1 for $n = 32$ and 64 (Legendre basis) (with 16Dec.)

<table>
<thead>
<tr>
<th>Method</th>
<th>$n = 32$</th>
<th>$n = 64$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iteration steps</td>
<td>Condition number</td>
</tr>
<tr>
<td>Tau</td>
<td>39</td>
<td>$1.7 \times 10^4$</td>
</tr>
<tr>
<td>New-Tau</td>
<td>23</td>
<td>$5.3 \times 10^2$</td>
</tr>
</tbody>
</table>

Problem 1. Let us consider

$$u'' - 20xu' - 20u = 0, \quad u(-1) = u(1) = 1,$$

which is taken from Cabos [2], with exact solution $u(x) = e^{10x^2-10}$.

Computations with 16Dec. and 32Dec. had some differences in convergence and accuracy for the cases of standard Tau and Cabos-Tau (where the New-Tau had already obtained stable results with 16Dec. computation). So, we first report the numerical results for the case of 16Dec. in Tables 3, 4 and then discuss the effect of 32Dec. computation.

For $n = 32$, the solution of the problem was found, in 45 steps with condition number $c(M) \simeq 4.9 \times 10^4$ by the Chebyshev Tau, in 33 steps with $c(M) \simeq 5.9 \times 10^3$ by the Cabos-Tau and in 22 steps with $c(M) \simeq 6.1 \times 10^2$ by the New-Tau.

For $n = 64$ neither the standard Tau method nor Cabos-Tau converged. Only the New-Tau method converged (in about 45 steps), see Table 3. For the Legendre basis see Table 4.

The following three matrices are the $M$ matrices corresponding to the standard Chebyshev Tau, Cabos-Tau and the New-Tau methods, respectively. As we expected theoretically, the entries of $M$ in the case of the New-Tau method are generally better organized than those in the other two methods.
Similar behavior happens for the case of the Legendre New-Tau method.

\[
\begin{bmatrix}
-20 & 0 & -36 & 0 & -48 & 0 & -12 & 0 & 96 \\
0 & -40 & 0 & -96 & 0 & -80 & 0 & 56 & 0 \\
0 & 0 & -60 & 0 & -112 & 0 & -48 & 0 & 160 \\
0 & 0 & 0 & -80 & 0 & -120 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -100 & 0 & -120 & 0 & 64 \\
0 & 0 & 0 & 0 & 0 & -120 & 0 & -112 & 0 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-20 & 0 & -9 & 0 & 5 & 0 & 0 & 0 & 0 \\
0 & -40 & 0 & -4 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & -15 & 0 & 7.67 & 0 & 0.83 & 0 & 0 \\
0 & 0 & 0 & -3.33 & 0 & 3.5 & 0 & 0.83 & 0 \\
0 & 0 & 0 & 0 & -2.08 & 0 & 2.33 & 0 & 0.75 \\
0 & 0 & 0 & 0 & 0 & -1.50 & 0 & 1.83 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1.17 & 0 & 1.57 \\
1 & -1 & 0.25 & -0.041 & -0.14 & 0.05 & 0.016 & -0.0059 & 0.0026 \\
1 & 1 & 0.25 & 0.041 & -0.14 & -0.05 & 0.016 & 0.0059 & 0.0026 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
-10 & 0 & 1 & 0 & 0.625 & 0 & 0 & 0 & 0 \\
0 & -6.67 & 0 & 1 & 0 & 0.416 & 0 & 0 & 0 \\
0 & 0 & -5 & 0 & 1 & 0 & 0.312 & 0 & 0 \\
0 & 0 & 0 & -4 & 0 & 1 & 0 & 0.25 & 0 \\
0 & 0 & 0 & 0 & -3.33 & 0 & 1 & 0 & 0.208 \\
0 & 0 & 0 & 0 & 0 & -2.85 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2.50 & 0 & 1 \\
1 & -1 & 0.5 & -0.25 & 0.125 & -0.0625 & 0.031 & -0.015 & 0.007 \\
1 & 1 & 0.5 & 0.25 & 0.125 & 0.0625 & 0.031 & 0.015 & 0.007 \\
\end{bmatrix}
\]

As it is seen, New-Tau needed less steps than the other two methods. In this method the condition number is also much less than those in the other two methods.
Table 5
Iterative solution of problem 2 for \( n = 16 \) and 32 (Chebyshev basis) (with 16Dec. and 32Dec.)

<table>
<thead>
<tr>
<th>Method</th>
<th>( n = 16 )</th>
<th>( n = 32 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iteration</td>
<td>Condition</td>
</tr>
<tr>
<td></td>
<td>steps</td>
<td>number</td>
</tr>
<tr>
<td>Tau</td>
<td>17</td>
<td>( 4.4 \times 10^4 )</td>
</tr>
<tr>
<td>Cabos-Tau</td>
<td>11</td>
<td>( 3.0 \times 10^3 )</td>
</tr>
<tr>
<td>New-Tau</td>
<td>9</td>
<td>( 2.5 \times 10^0 )</td>
</tr>
</tbody>
</table>

Table 6
Iterative solution of problem 2 for \( n = 16 \) and 32 (Legendre basis) (with 16Dec. and 32Dec.)

<table>
<thead>
<tr>
<th>Method</th>
<th>( n = 16 )</th>
<th>( n = 32 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iteration</td>
<td>Condition</td>
</tr>
<tr>
<td></td>
<td>steps</td>
<td>number</td>
</tr>
<tr>
<td>Tau</td>
<td>14</td>
<td>( 1.2 \times 10^4 )</td>
</tr>
<tr>
<td>New-Tau</td>
<td>8</td>
<td>( 2.8 \times 10^0 )</td>
</tr>
</tbody>
</table>

Note that as it is also clear from Table 4 the dramatic and comparative effect of our preconditioning in the New-Tau will become evident as it is applied for a large \( n \). Thus, solving with a large \( n \), the New-Tau would of course be more favorable.

The numbers of nonzero elements of \( M \) in New-Tau and Cabos-Tau are roughly the same, but much less than that in the standard Chebyshev Tau method. Similar conclusion follows for the case of Legendre basis.

In this example, computation with 32Dec. was able to recover the unacceptable performance of the Tau and Cabos-Tau for \( n = 64 \) and obtained numerical results the same as \( n = 32 \), but with higher accuracy. This confirms that the standard Tau and Cabos-Tau are very much sensitive to rounding.

**Problem 2.** Consider

\[
  u''(x) + 3xu'(x) + x^4u = 6x + 9x^3 + x^7, \quad u(-1) = 1, \quad u(1) = 1,
\]

with exact solution \( u(x) = x^3 \).

In this problem, computation with 16Dec. and 32Dec. produced the same numerical results, shown in Tables 5, 6. The condition numbers of \( M \) in Tau (standard) and Cabos-Tau are much larger than the condition number of \( M \) in the New-Tau method, particularly for \( n = 32 \).

It is seen that even in the case of Cabos-Tau the condition number grows rapidly with \( n \), while for the New-Tau it is unchanged. The reason for this is that according to our strategy the elements of \( M \) grow slowly with \( n \). For numerical results in the Legendre basis see Table 7. Similar conclusion follows for this case.
Table 7
Iterative solution of problem 3 for \( n = 16 \) and 32 (Chebyshev basis) (with 16Dec. and 32Dec.)

<table>
<thead>
<tr>
<th>Method</th>
<th>( n = 16 )</th>
<th></th>
<th></th>
<th>( n = 32 )</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iteration</td>
<td>Condition</td>
<td>Maximal</td>
<td>Iteration</td>
<td>Condition</td>
<td>Maximal</td>
</tr>
<tr>
<td></td>
<td>steps</td>
<td>number</td>
<td>error</td>
<td>steps</td>
<td>number</td>
<td>error</td>
</tr>
<tr>
<td>Tau</td>
<td>55</td>
<td>( 1.2 \times 10^4 )</td>
<td>( 4.3 \times 10^{-6} )</td>
<td>1016</td>
<td>( 1.1 \times 10^9 )</td>
<td>( 8.7 \times 10^{-11} )</td>
</tr>
<tr>
<td>Cabos-Tau</td>
<td>16</td>
<td>( 1.9 \times 10^3 )</td>
<td>( 4.3 \times 10^{-6} )</td>
<td>108</td>
<td>( 9.9 \times 10^7 )</td>
<td>( 6.1 \times 10^{-11} )</td>
</tr>
<tr>
<td>New-Tau</td>
<td>15</td>
<td>( 5.4 \times 10^0 )</td>
<td>( 4.3 \times 10^{-6} )</td>
<td>15</td>
<td>( 5.4 \times 10^0 )</td>
<td>( 6.5 \times 10^{-11} )</td>
</tr>
</tbody>
</table>

Table 8
Iterative solution of problem 3 for \( n = 16 \) and 32 (Legendre basis) (with 16Dec. and 32Dec.)

<table>
<thead>
<tr>
<th>Method</th>
<th>( n = 16 )</th>
<th></th>
<th></th>
<th>( n = 32 )</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iteration</td>
<td>Condition</td>
<td>Maximal</td>
<td>Iteration</td>
<td>Condition</td>
<td>Maximal</td>
</tr>
<tr>
<td></td>
<td>steps</td>
<td>number</td>
<td>error</td>
<td>steps</td>
<td>number</td>
<td>error</td>
</tr>
<tr>
<td>Tau</td>
<td>48</td>
<td>( 7.1 \times 10^3 )</td>
<td>( 4.3 \times 10^{-6} )</td>
<td>648</td>
<td>( 5.1 \times 10^7 )</td>
<td>( 6.1 \times 10^{-11} )</td>
</tr>
<tr>
<td>New-Tau</td>
<td>15</td>
<td>( 6.3 \times 10^0 )</td>
<td>( 4.6 \times 10^{-6} )</td>
<td>15</td>
<td>( 6.3 \times 10^0 )</td>
<td>( 6.5 \times 10^{-11} )</td>
</tr>
</tbody>
</table>

Problem 3. Consider

\[
\frac{d^3 u}{dx^3} + \frac{4}{\pi} \frac{d^2 u}{dx^2} - \frac{\pi^2}{8} \sec^2 \left( \frac{\pi}{4} x \right) u(x) = \sec \left( \frac{\pi}{4} x \right), \quad u(-1) = -1, \quad u(1) = 1,
\]

with exact solution \( u(x) = \tan \left( \frac{1}{4} \pi x \right) \).

Here the coefficients in the differential equation are not polynomials and therefore they must be replaced with appropriate polynomial approximations.

Solving this problem using three methods standard Tau, Cabos-Tau and New-Tau, its numerical results (with 16Dec. and 32Dec. computations, both of which are the same) are shown in Tables 7, 8 for \( n = 16 \) and 32, respectively. As it is seen, with \( n = 16 \) and 32 the New-Tau needs 15 iteration steps for convergence and \( M \) has a fixed condition number of 5.4. In contrast, the standard Tau and the Cabos-Tau need much more iteration steps (1016 and 108 steps, respectively) for \( n = 32 \). The condition numbers of \( M \) in these cases are \( 1.1 \times 10^9 \) and \( 9.9 \times 10^7 \), respectively. Thus, as we expected, the New-Tau was clearly superior. For numerical results in the Legendre basis see Table 8. Similar conclusion follows for this case.

Problem 4. To illustrate the increased effect of New-Tau in case of higher derivative terms in ODE,

\[
\frac{d^4 u}{dx^4} + \frac{d^3 u}{dx^3} + x^2 \frac{d^2 u}{dx^2} + u = 6 + 6x + x^3 + 3x^4, \quad u(-1) = -1, \quad u(1) = 1, \quad u(0) = 0,
\]

was considered with exact solution \( u(x) = x^3 \).

In this problem, computation with 16Dec. and 32Dec. produced the same numerical results, shown in Tables 9, 10. The numerical results show that the New-Tau was again superior.
Table 9
Iterative solution of problem 4 for \( n = 16 \) and 32 (Chebyshev basis) (with 16Dec. and 32Dec.)

<table>
<thead>
<tr>
<th>Method</th>
<th>Iteration steps</th>
<th>Condition number</th>
<th>Maximal error</th>
<th>Iteration steps</th>
<th>Condition number</th>
<th>Maximal error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tau</td>
<td>101</td>
<td>( 2.5 \times 10^5 )</td>
<td>( 1.5 \times 10^{-14} )</td>
<td>3685</td>
<td>( 1.1 \times 10^{11} )</td>
<td>( 2.7 \times 10^{-10} )</td>
</tr>
<tr>
<td>Cabos-Tau</td>
<td>18</td>
<td>( 1.1 \times 10^2 )</td>
<td>( 1.1 \times 10^{-11} )</td>
<td>48</td>
<td>( 1.6 \times 10^{10} )</td>
<td>( 1.2 \times 10^{-11} )</td>
</tr>
<tr>
<td>New-Tau</td>
<td>12</td>
<td>( 2.4 \times 10^0 )</td>
<td>( 7.1 \times 10^{-12} )</td>
<td>15</td>
<td>( 3.1 \times 10^0 )</td>
<td>( 1.8 \times 10^{-12} )</td>
</tr>
</tbody>
</table>

Table 10
Iterative solution of problem 4 for \( n = 16 \) and 32 (Legendre basis) (with 16Dec. and 32Dec.)

<table>
<thead>
<tr>
<th>Method</th>
<th>Iteration steps</th>
<th>Condition number</th>
<th>Maximal error</th>
<th>Iteration steps</th>
<th>Condition number</th>
<th>Maximal error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tau</td>
<td>88</td>
<td>( 13 \times 10^5 )</td>
<td>( 4.6 \times 10^{-15} )</td>
<td>2020</td>
<td>( 6.6 \times 10^9 )</td>
<td>( 2.2 \times 10^{-11} )</td>
</tr>
<tr>
<td>New-Tau</td>
<td>12</td>
<td>( 2.5 \times 10^0 )</td>
<td>( 4.9 \times 10^{-12} )</td>
<td>15</td>
<td>( 3.0 \times 10^0 )</td>
<td>( 9.4 \times 10^{-13} )</td>
</tr>
</tbody>
</table>

Problem 5. Consider the following stiff problem:

\[
n''(x) - ku(x) = \cos x, \quad u(-1) = 1, \quad u(1) = 1,
\]

where \( k \) is the stiffness parameter. This problem was used as a challenging test problem by different authors, for examples see [9,12]. The exact solution is

\[
u(x) = \frac{1}{k+1}(C_1e^{-\sqrt{k}x} + C_2e^{\sqrt{k}x} + \cos x),
\]

where \( C_1 = C_2 = (k + 1 + \cos(1))/(e^{-\sqrt{k}} + e^{\sqrt{k}}) \).

For large \( k \) the exact solution has very large gradients at the boundaries. Therefore, for \( k = 500 \) and 1000, \( n \) was raised to 64 and 128 in order to obtain a good approximation of the solution.

In this problem, computations with 16Dec. and 32Dec. had some differences in convergence and accuracy for the cases of Tau and Cabos-Tau (where the New-Tau had already obtained stable results with 16Dec. computation). So, we first report the numerical results for the case of 16Dec. in Tables 11, 12 and then discuss the effect of 32Dec. computation. For \( k = 500 \) and \( n = 64 \) neither the original (standard) Tau method nor Cabos-Tau reached convergence, (see Table 11). The same happened for \( k = 1000 \) and \( n = 128 \). Only the New-Tau method converged in about 55 steps, in the first case and 290 steps in the second case, see Table 11. For numerical results in the Legendre basis see Table 12. Similar conclusion follows for this case.

With 32Dec. computation, the standard Tau and Cabos-Tau could only converge for \( n = 64 \). In Chebyshev basis, they converged after 733 and 92 iterations and their condition numbers were
Table 11
Iterative solution of problem 5 for \( n = 64 \) and 128 (Chebyshev basis) (with 16Dec.)

<table>
<thead>
<tr>
<th>Method</th>
<th>( n = 64 )</th>
<th>( k = 500 )</th>
<th>( n = 128 )</th>
<th>( k = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iteration steps</td>
<td>Condition number</td>
<td>Maximal error</td>
<td>Iteration steps</td>
</tr>
<tr>
<td>Tau</td>
<td>No conv.</td>
<td>( 3.45 \times 10^{13} )</td>
<td>—</td>
<td>No conv.</td>
</tr>
<tr>
<td>Cabos-Tau</td>
<td>No conv.</td>
<td>( 4.23 \times 10^{12} )</td>
<td>—</td>
<td>No conv.</td>
</tr>
<tr>
<td>New-Tau</td>
<td>55</td>
<td>( 1.64 \times 10^{4} )</td>
<td>( 8.7 \times 10^{-15} )</td>
<td>290</td>
</tr>
</tbody>
</table>

Table 12
Iterative solution of problem 5 for \( n = 64 \) and 128 (Legendre basis) (with 16Dec.)

<table>
<thead>
<tr>
<th>Method</th>
<th>( n = 64 )</th>
<th>( k = 500 )</th>
<th>( n = 128 )</th>
<th>( k = 1000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iteration steps</td>
<td>Condition number</td>
<td>Maximal error</td>
<td>Iteration steps</td>
</tr>
<tr>
<td>Tau</td>
<td>No conv.</td>
<td>( 3.3 \times 10^{12} )</td>
<td>—</td>
<td>No conv.</td>
</tr>
<tr>
<td>New-Tau</td>
<td>54</td>
<td>( 1.3 \times 10^{4} )</td>
<td>( 4.6 \times 10^{-14} )</td>
<td>250</td>
</tr>
</tbody>
</table>

7.8 \times 10^{5} and 8.6 \times 10^{4}, respectively. For \( n = 128 \), those two methods could not converge but their condition numbers reduced to \( 1.1 \times 10^{7} \) and \( 3.2 \times 10^{5} \), respectively.

**Problem 6.** Consider the following problem:

\[
 u''(x) + (x^5 - 1)u'(x) + 9x^6u(x) = x^5(9x + 1)e^x, \quad u(-1) = e^{-1}, \quad u(1) = e,
\]

with exact solution \( u(x) = e^x \).

In this problem, ‘L-New-Tau’ refers to the New-Tau method based on the leading coefficients of Chebyshev or Legendre polynomials (i.e., Eq. (11) or (14), respectively) and ‘M-New-Tau’ plays the same role but based on the maximum coefficients of those polynomials (i.e., Eq. (10) or (13), respectively). Numerical results for the Tau, Cabos-Tau, L-New-Tau, and M-New-Tau are shown in Table 13 (with 16Dec. and 32Dec. computations, both of which are the same). They confirm our expectation stated in the paragraph following Eq. (16).

6. **Conclusions**

The standard Chebyshev (Legendre) Tau method applied to linear ordinary differential equations is generally badly conditioned.

In this article it has been shown that solving an \( m \)th order ordinary differential equation with \( m \) supplementary condition, using the new preconditioning Tau (New-Tau) method is instead well-conditioned with condition numbers bounded independent of \( n \) in many cases.
Table 13
Iterative solution of problem 6 for \( n = 32 \) and 64 (Chebyshev basis) (with 16Dec. and 32Dec.)

<table>
<thead>
<tr>
<th>Method</th>
<th>( n = 32 )</th>
<th>( n = 64 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iteration steps</td>
<td>Condition number</td>
</tr>
<tr>
<td>Tau</td>
<td>No conv.</td>
<td>( 1.2 \times 10^{14} )</td>
</tr>
<tr>
<td>Cabos-Tau</td>
<td>958</td>
<td>( 1.3 \times 10^{13} )</td>
</tr>
<tr>
<td>L-New-Tau</td>
<td>42</td>
<td>( 4.2 \times 10^{2} )</td>
</tr>
<tr>
<td>M-New-Tau</td>
<td>39</td>
<td>( 4.5 \times 10^{1} )</td>
</tr>
</tbody>
</table>

Numerical results show that the number of iterations required for finding the Tau approximation and also the condition number of the final matrix \( M \) can significantly be reduced by this new method (New-Tau).

Even comparing with Cabos-Tau (the preconditioning method of Cabos [2]) the New-Tau is clearly superior, particularly when \( n \) becomes large. It was demonstrated that the standard Tau and Cabos-Tau are very much sensitive to rounding errors. It was also shown that for ODEs with nonconstant coefficients even using calculations with 32Dec. cannot assist the standard Tau and Cabos-Tau in producing acceptable numerical results when \( n \) is large.

It is worth noting that the standard Tau method, preconditioned with a standard finite-difference method, was not capable of obtaining a comparative result over the New-Tau method. For diffusive operator, it could only reduce the condition number by approximately a factor of 10 to \( 10^{3} \) for \( n = 16, 32, 64 \). Further details of such comparison and an extension of the New-Tau method for 2D and 3D problems will be investigated in a new paper.

References