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Joint Spectral Radius and Hölder Regularity of Wavelets

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Abstract—We give preliminary results on the Hölder exponent of wavelets of compact support. In particular, we give a nearly complete map of this exponent for the family of four-coefficient multiresolution analyses and determine the smoothest one. This will resolve two conjectures by Colella and Heil. Wavelets of compact support can be generated via infinite products of certain matrices. The rate of growth of these products determines the regularity of the wavelet. This rate can be determined via joint, generalized, or common spectral radius of the given set of matrices. We outline a method for calculating this radius for a given set of matrices. The method relies on guessing the particular finite optimal product which satisfies the finiteness conjecture and exhibits the fastest growth. Then, we generate an optimal unit ball by taking the convex hull of the action of the semigroup of matrices, scaled by their joint radius, on the invariant ball of the scaled optimal product. If this process terminates in a finite number of steps and the convex hull does not grow, then the guessed optimal product is confirmed and the joint radius is determined.

Keywords—Generalized spectral radius, Hölder exponent, Finiteness conjecture.

1. INTRODUCTION

Wavelets can be used to perform various data transforms. Some applications such as image analysis [1] can benefit from a knowledge of smoothness of the wavelet. Various classifications for the smoothness of wavelets are in use. Among them, the Hölder exponent furnishes a precise indicator of regularity. A real function f is said to have *a* Hölder exponent α if there is $C_\alpha > 0$ such that $|f(x) - f(y)| \leq C_\alpha |x - y|^\alpha$. We say f has *the* maximal Hölder exponent h, denoted $h = \mathrm{Hex}(f)$, if $h = \sup \alpha$.

For construction of wavelets of compact support, one can start from a dilation equation which relates a function $f(x)$ to a finite linear combination of integer translates of its two-scale $\{f(2x - k)\}$. The wavelet, in turn, is given by a linear combination of the translates of the scaling function, the solution of the dilation equation. The solution of this equation can be determined from infinite products of certain fixed matrices whose entries depend on the coefficient of the dilation equation. The rate of convergence of these products determines the differentiability order of the wavelet and the Hölder exponent of the last well-defined derivative.

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The rate of growth of consecutive powers of a single matrix is determined by its spectral radius. For the long products of a bounded collection of matrices, the maximal rate of growth is determined by the Spectral Radius of the Set of matrices (SRS). There are several equivalent definitions for this notion. Joint Spectral Radius (JSR) uses supremum of norms, and Generalized Spectral Radius (GSR) uses supremum of absolute value of eigenvalues of products of length n of the given matrices. In either case, the quantity is normalized by taking the nth root, and then the lim sup of the result is calculated as n tends to infinity. (Equivalently, JSR is the infimum of positive numbers r such that after dividing the matrices by r, the norm of any product of the resulting matrices is uniformly bounded by some $C_r > 0$. Similarly, GSR is the infimum of positive numbers r such that after dividing the matrices by r, the absolute value of any eigenvalue of any product of the resulting matrices is uniformly bounded by some $C_r > 0$.) Common Spectral Radius (CSR) does not use products; instead, given a matrix norm, it calculates the supremum of the norms of the given matrices and then, on the space of all matrix norms, finds the infimum of the results.

Calculating SRS by direct application of any of the above definitions is extremely inefficient. The branch-and-bound method of Daubechies and Lagarias [2] significantly reduces the cost of upper estimates. A refinement in this method, including considerable savings for estimating lower bounds, has been proposed by Gripenberg [3]. A reduction in the cost of the lower estimates using equivalence classes was investigated by the author [4]. The difficulties in calculation of SRS prompted some attempts [5] at proving that the quantity is not effectively computable. Here, we outline a very fast (but not completely proven) method for exact calculation of the joint radius. The method uses all three definitions of spectral radius and relies on the finiteness conjecture [6,7] of Daubechies and Lagarias. This conjecture asserts that in the definition of GSR, the lim sup is attained at a particular finite product called the *optimal product*.

Our method starts by guessing the optimal product and hence the spectral radius. (In fact, we have a conjecture which specifies the optimal product for all matrices associated with four-coefficient multiresolution analyses.) Then, we scale all matrices by the joint radius. Next, we identify the ball (typically of a subspace) which is invariant under the scaled optimal product. Then, we find the convex hull of action of semigroup of scaled matrices on the invariant ball. If the convex hull does not grow after products of a certain finite length are applied, our guess is confirmed and the joint radius is obtained.

This method has been successfully applied to the one-parameter family of the pair of matrices associated with the four-coefficient multiresolution analyses. We have experimental (but exact) determination of values of joint spectral radius and hence the Hölder exponent for nearly all parameter values for this case.

2. THE SPECTRAL RADIUS OF A SET OF MATRICES

The concept of iteration is a cornerstone of many mathematical disciplines as well as a favorite tool for approximation. Linear iterative systems are perhaps the most common type, and they occur in a cascade algorithm for wavelets of compact support [2,6,8], a refinement algorithm for computer aided design [9,10], image analysis techniques [11,12], Markov chains [13], asynchronous processes in control theory [5], fractal generation [11,12,14], and other areas.

The measurement of the rate of growth of the iterates is an important question in each case. This concept has been defined for linear iterative systems through natural generalizations of the spectral radius of a matrix. Suppose Σ is a bounded collection of square matrices with complex entries and of same size. Let $\mathcal{L}_n = \mathcal{L}_n(\Sigma)$ be the set of all products of length n of the elements of Σ, $\mathcal{L}_0$ the identity, and $\mathcal{L}_*$ the generated semigroup, i.e., the set of products of any finite length including identity. We will use various notions of norm of a matrix. By norm we imply any norm, while a matrix norm is assumed to be submultiplicative, i.e., $\|AB\| \leq \|A\|\,\|B\|$. Given a vector norm $\|\cdot\|$ on a *complex* space, we define its induced or operator norm as $\|A\| = \max_{\|x\|=1} \|Ax\|$.

An operator norm is submultiplicative, but matrix norms are not necessarily induced. Any two norms, $\|\cdot\|$ and $\|\cdot\|'$, on a finite dimensional space are equivalent in the sense that there are constants $a > b > 0$ such that for any element of the space A, we have $a\|A\| \geq \|A\|' \geq b\|A\|$. The norm of a set is the supremum of the norm of its elements.

Rota and Strang [15] defined *Joint Spectral Radius* (JSR), $\hat{\rho}(\Sigma)$, through the limit of normalized norms of products of Σ

$$\hat{\rho}(\Sigma) = \limsup_{n\to\infty} \hat{\rho}_n(\Sigma, \|\cdot\|), \tag{1}$$

$$\hat{\rho}_n(\Sigma, \|\cdot\|) = \sup_{A\in\mathcal{L}_n} \|A\|^{1/n}, \tag{2}$$

where lim sup can in fact be replaced with lim or inf, see (7), and $\|\cdot\|$ is an arbitrary norm. They also gave another definition, which we refer to as *Common Spectral Radius* (CSR), $\tilde{\rho}(\Sigma)$, by

$$\tilde{\rho}(\Sigma) = \inf_{\|\cdot\|} \|\Sigma\| = \inf_{\|\cdot\|} \sup_{A\in\Sigma} \|A\|, \tag{3}$$

where the infimum is over all matrix norms. They proved

$$\hat{\rho}(\Sigma) = \tilde{\rho}(\Sigma) \tag{4}$$

by first establishing that Σ is product bounded if and only if Σ is in the unit ball of a matrix norm (i.e., $\|\mathcal{L}_*\|' < K$ for some $K < \infty$ and some norm $\|\cdot\|'$ if and only if there is a matrix norm $\|\cdot\|$ such that $\|\Sigma\| \leq 1$). In fact, they gave an explicit construction of a vector norm, namely $\|x\| = \sup_{A\in\mathcal{L}_*} \|Ax\|'$, whose induced operator norm is the required matrix norm.

Daubechies and Lagarias [2,6] defined *Generalized Spectral Radius* (GSR), $\check{\rho}(\Sigma)$, in terms of the limit of normalized spectral radii of products of Σ

$$\check{\rho}(\Sigma) = \limsup_{n\to\infty} \check{\rho}_n(\Sigma), \tag{5}$$

$$\check{\rho}_n(\Sigma) = \sup_{A\in\mathcal{L}_n} \rho(A)^{1/n}, \tag{6}$$

where, by (7), lim sup can be replaced with sup. They established that for any matrix norm $\|\cdot\|$,

$$\check{\rho}_n(\Sigma) \leq \check{\rho}(\Sigma) \leq \hat{\rho}(\Sigma) \leq \hat{\rho}_n(\Sigma, \|\cdot\|), \tag{7}$$

conjectured that for any finite set of matrices

$$\check{\rho}(\Sigma) = \hat{\rho}(\Sigma), \tag{8}$$

and gave examples of unbounded sets for which $\check{\rho}(\Sigma) < \hat{\rho}(\Sigma)$. The conjecture was extended to any bounded set of matrices and proved by Berger and Wang [16]. A simplified proof based on analytical methods was also provided by Elsner [17]. Therefore JSR, CSR, and GSR are equal, and we may talk of *the* Spectral Radius of a bounded Set of matrices (SRS), or simply the spectral radius, and denote it by

$$\rho(\Sigma) = \tilde{\rho}(\Sigma) = \hat{\rho}(\Sigma) = \check{\rho}(\Sigma). \tag{9}$$

All of the various concepts of radius mentioned above, with the exception of $\hat{\rho}_n$, are invariant under similarity transformations. Moreover, Heil and Strang [18] showed that $\rho(\Sigma)$ is a continuous function of Σ.

The finiteness conjecture [6,7] states that for each finite Σ, there is a finite n such that $\rho(\Sigma) = \check{\rho}_n(\Sigma)$, i.e., there is an *optimal product* P of length n satisfying $\rho(\Sigma) = \rho(P)^{1/n}$. There is also the closely associated normed finiteness conjecture which states that for a finite Σ with $\|\Sigma\| \leq 1$ with respect to a matrix norm, either $\rho(\Sigma) < 1$ or $\rho(\Sigma) = \check{\rho}_n(\Sigma) = 1$ for a finite n. Lagarias

and Wang [7] proved that the finiteness conjecture is true if and only if the normed finiteness conjecture is true.

The class of matrices for which SRS has been exactly calculated is small. There are four known classes.

(1) Demonstration examples [18] based on sets of nilpotent matrices which generate a finite semigroup, e.g., $\Sigma = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$. Notice, the spectral radius of each matrix is 0, but $\rho(\Sigma) = 1$.
(2) Sets which, through a similarity transformation, can be simultaneously reduced to one of normal, Hermitian, symmetric, or triangular forms. In this case, SRS is the same as the largest spectral radius of the matrices in the set.
(3) Examples of rotation-like matrices, developed in [7], which are used to show that the critical exponent n in the finiteness conjecture can be arbitrarily large.
(4) Certain matrices associated with four-coefficient multiresolution analyses and wavelets of compact support. (Here, SRS determines the Hölder regularity of the wavelet.) The conjectures of Colella and Heil [1, p. 242; 19, p. 879; 20, p. 181,194] were related to a particular set of matrices. We settle the conjectures and provide a nearly complete description of this case below.

3. THE OPTIMUM UNIT BALL

Our method for the calculation of spectral radius emphasizes CSR, unlike the Daubechies-Lagarias-Gripenberg's branch-and-bound method which focuses on the multiplicative nature of JSR and GSR. CSR minimizes the matrix norm of the set of matrices. In order to make this approach more efficient, we need to limit the space of the matrix norms which is considered. The following two lemmas show that if Σ is real, we may limit our search to operator norms induced from real vector norms rather than all matrix norms.

LEMMA 1. *For every matrix norm $\|\cdot\|_m$, there is an operator norm $\|\cdot\|_o$ such that $\|A\|_o \le \|A\|_m$ for all A.*

PROOF. Given a vector x, define a matrix X whose columns are identical with x. Define a vector norm as $\|x\|_v = \|X\|_m$; then the induced norm satisfies $\|A\|_o = \max_{x\neq 0}(\|Ax\|_v/\|x\|_v) = \max_{x\neq 0}(\|AX\|_m/\|X\|_m) \le \|A\|_m$ as required. ∎

COROLLARY 1. *Common spectral radius can be defined using operator norms*

$$\tilde{\rho}(\Sigma) = \inf_{\|\cdot\|_o} \sup_{A\in\Sigma} \|A\|_o.$$

PROOF. For every matrix norm there is an operator norm whose values do not exceed that of the matrix norm. Hence, taking the infimum over the operator norms will do (and is more efficient). ∎

Occasionally it happens that all elements of Σ have the same lower block triangular structure, perhaps after a similarity transformation. In that case, the calculation of CSR may be segmented accordingly as explained below.

LEMMA 2. *If each $A \in \Sigma$ has the same lower triangular block structure with diagonal blocks A_i, $i = 1, \ldots, k$, and $\Sigma_i = \{A_i, A \in \Sigma\}$, then $\tilde{\rho}(\Sigma) = \max_i \tilde{\rho}(\Sigma_i)$ where $\tilde{\rho}(\Sigma_i) = \inf_{\|\cdot\|_o} \max_{A_i\in\Sigma_i} \|A_i\|_o$.*

PROOF. This is essentially the same as the block triangularization lemmas of the same nature that are used for JSR or GSR [16,20]. The result follows from equivalence of CSR, JSR, and GSR mentioned above. ∎

COROLLARY 2. *If Σ is real, then $\tilde{\rho}(\Sigma) = \inf_{\|\cdot\|_o} \max_{A\in\Sigma} \|A\|_o$, where the operator norm is induced from a real vector norm $\|A\|_o = \max_{\|x\|=1,\, x\in\mathbb{R}^q} \|Ax\|$.*

PROOF. Notice that if A is a $q \times q$ complex matrix, then it can be mapped into a $2q \times 2q$ real matrix. In particular, if A is real then the result of mapping can be permuted into a block diagonal matrix with two identical blocks. Therefore, we can map each $A \in \Sigma$ into a $2q \times 2q$ block diagonal matrix with identical blocks and apply Lemma 2. ∎

In order to calculate the common spectral radius efficiently, we need a norm with respect to which the radius is attained. This a not always possible, not even when Σ consists of only one matrix.

DEFINITION 1. *A bounded set Σ is called ρ-diagonal if, for a given norm,*

$$\sup_{A\in\mathcal{L}_n} \|A\| = \mathcal{O}(\rho(\Sigma)^n),$$

as n tends to infinity.

The following conditions are known to be equivalent [15].

(a) Σ is ρ-diagonal.
(b) There is a matrix norm $\|\cdot\|$ for which $\rho(\Sigma) = \|\Sigma\|$.

In addition, when Σ consists of a single matrix A, then it is well known that the above conditions are equivalent to the following.

(c) Each eigenvalue λ of A with $|\lambda| = \rho$ (here referred to as a *leading eigenvalue*) has the same algebraic and geometric multiplicity; i.e., the corresponding Jordan block is a diagonal matrix.

The construction of a unit ball for the single matrix case shows that there are infinitely many matrix norms which produce the spectral radius. However, the construction along the eigenvectors corresponding to the spectral radius is usually rigid. The leading eigenvalues of A are of the form $\rho\exp(i\alpha\pi)$. If α is rational with $\alpha = p/q$, $\gcd(p,q) = 1$, and p even (odd), then a q-gon ($2q$-gon) will be present on the unit ball; but if α is irrational, a circle will be present. (In the process, the regular polygons and circles are typically mapped on an ellipse by a linear transformation.) This section of the unit ball can be described through the invariant ball of $A/\rho(A)$.

DEFINITION 2. *Given a matrix P and a vector norm $\|\cdot\|$, the invariant unit ball of P is the maximal set $\mathcal{G}$ such that $P\mathcal{G} = \mathcal{G}$ and $\sup_{g\in\mathcal{G}} \|g\| = 1$.*

It is easy to see that if P has an invariant unit ball then $\rho(P) = 1$, P is ρ-diagonal, and $\mathcal{G}$ is in fact a unit ball of a subspace (termed *the leading subspace*). The ball can have only polyhedral and ellipsoidal components. The solutions of $P^n v = v$, $\|v\| = 1$, generate the vertices of the polyhedral components through $\{v, \ldots, P^{n-1}v\}$. The solutions of $P^*SP = S$ generate the ellipsoidal components through $\{x \mid x^*Sx = 1\}$. (These sets need to be scaled to ensure the correct radius and maximality. P^* and x^* indicate Hermitian adjoints.) To obtain a complete unit ball, we take the convex hull of the various components alongside a small ball of the complementary space of the leading subspace.

Convexity plays a central role in our construction. Here, we detail some notation which will be used. For a set of points D, let $\mathcal{C}(D)$ indicate the convex hull of D. The closure of $\mathcal{C}(D)$ is shown by $\bar{\mathcal{C}}(D)$. Suppose x, y, z belong to a closed convex set and $0 < \alpha < 1$. If $z = \alpha x + (1-\alpha)y$ implies $x = y = z$, then z is called an *extreme* point of the set [21]. A compact set is the convex hull of its extreme points (Krein-Milman theorem). The vertices of a nondegenerate polyhedron are its extreme points. (When a vertex becomes degenerate, e.g., when the point becomes an interior point of an edge or a face, then it is no longer an extreme point. The concept of degenerate vertices arises in parametric problems where for a special value of the parameter, a vertex ceases to be an extreme point. These points are important for us since they indicate that the construction of the ball has reached a critical stage.)

Suppose Σ is product bounded and $\mathcal{U}$ is a ball; then $\mathcal{U}' = \bar{\mathcal{C}}(\mathcal{L}_*\mathcal{U})$ is also a ball. Let $\|\cdot\|'$ be the norm in which $\mathcal{U}'$ is a unit ball; then $\max\{\|A\|', A \in \Sigma\} \leq 1$. This is the *alternative construction for the norms* given in [15]. Here, however, we do not start with an arbitrary ball; instead we identify the invariant unit ball $\mathcal{G}$ of P and create the optimal ball $\bar{\mathcal{C}}(\mathcal{L}_*\mathcal{G})$ from it. The experimentally observed advantage of using $\mathcal{G}$ is that its extreme points generate the vertices (usually extreme points) of the optimum ball. In contrast, an arbitrary starting point will mark a set of nondescript points on the boundary of the optimal ball. Moreover, $\mathcal{G}$ has a minimality property in the sense that if $\mathcal{U}$ is a ball with respect to whose norm $\|\cdot\|$ we have $\rho(\Sigma) = \|\Sigma\|$, then there is a $\mathcal{G}$ whose extreme points are on the boundary of $\mathcal{U}$.

The method used by the author is called Continuous Adaptive Generation of Optimal Unit Balls (CAGOUB). When CAGOUB is used to calculate the spectral radius of a given set of matrices, it requires several steps. The steps for the generation of the optimum ball are outlined below.

ALGORITHM 1. This algorithm verifies that a given product is the optimal product satisfying the finiteness conjecture by generating an optimum ball.

(1) Use a numerical method (e.g., Gripenberg's or a version of CAGOUB) to arrive at a guessed optimal product P, say of length n, and hence $\rho(\Sigma) = \rho(P)^{1/n}$.
(2) Scale all matrices so that the guessed SRS is one; i.e., define $\Sigma^* = \Sigma/\rho(\Sigma)$ and $P^* = P/\rho(P)$.
(3) Find $\mathcal{G}_0$, the invariant unit ball of P^*.
(4) For $q \geq 1$, define $\mathcal{G}_q$ as the convex hull of $\mathcal{G}_{q-1} \cup \Sigma^*\mathcal{G}_{q-1}$.
(5) If at a certain stage $\mathcal{G}_q = \mathcal{G}_{q-1}$, then P is indeed the optimal product and $\mathcal{G} = \mathcal{G}_q$ is the optimal ball.

When $\mathcal{G}_0$ is ellipsoidal (with no polyhedral subset, i.e., P is ergodic on the ellipsoid), we need only to check $\mathcal{G}_1 = \mathcal{G}_0$. For the general case we have the following conjecture.

CONJECTURE 1. *If P is optimal, then $\mathcal{G}_n = \mathcal{G}_{n-1}$ for some finite n.*

Some steps toward the proof of the above conjecture have been taken. Suppose $v \neq 0$, $\mathcal{F}_0 = \mathcal{C}(\{\pm v\})$, and $\mathcal{F}_n = \mathcal{C}(\Sigma^+\mathcal{F}_{n-1})$ for $n \geq 1$, where Σ^+ is the set Σ augmented with identity. Let $V_0 = \{\pm v\}$ and for $n \geq 1$ define V_n, the *vertices* of $\mathcal{F}_n$, to be the points of Σ^+V_{n-1} which are on the boundary of $\mathcal{F}_n$.

CONJECTURE 2. *If there are $v' \in V_n$ and $m > n$ such that v' is in the interior of $\mathcal{F}_m$, then $\rho(\Sigma) > 1$.*

This conjecture implies that none of the vertices at a given stage of iteration will be overtaken by the next set of vertices. Consequently, if this occurs we do not have an optimal product. One case of this conjecture is simple, namely when v' is v itself. This is the subject of the next lemma.

DEFINITION 3. *Given a vector $v \neq 0$, a compact set of vectors V, and a finite set of matrices Σ, we say v is dominated by Σ acting on V if there are $\alpha > 1$ and $w \in \mathcal{C}(\Sigma V)$ such that $v = w/\alpha$.*

LEMMA 3. *If v is dominated by Π, a finite subset of $\mathcal{L}_*(\Sigma)$, acting on $\{v\}$, then $\rho(\Sigma) > 1$.*

PROOF. First assume $\Pi = \Sigma$ and consider any norm in which $\|v\| = 1$. Then, for $w' \in \mathcal{C}(\Sigma v)$ we have $\max|w'\| \geq \alpha$. On a compact convex set, the maximum of a convex function is attained at an extremum point. Norm function is convex, and hence $\max\|w'\|$ occurs at $\|Av\|$ for an $A \in \Sigma$. Hence, $\max\|A\| \geq \alpha > 1$. Since this occurs for any norm, by the definition of CSR, we have $\rho(\Sigma) > 1$. For general Π we have $\mathcal{L}_*(\Pi) \subset \mathcal{L}_*(\Sigma)$, and hence $\rho(\Sigma) \geq \rho(\Pi) > 1$. ∎

When CAGOUB is used to calculate the radius for a continuously parameterized family of matrices, its performance improves. This is done by using a continuity method to form a new optimal ball as the parameters change. Generally, for a range of values of the parameters, the optimal product stays the same while the generator and the optimal ball change continuously. At

certain values, the ball might undergo major changes when a new face is born during convex hull calculations. CAGOUB easily adapts to these changes. At certain other values, the ball exhibits a "lack of closure". That happens when the optimal product itself changes. It is not entirely clear how one predicts a new optimal product. This section of the algorithm is still under construction.

4. HÖLDER REGULARITY OF WAVELETS

Wavelets of compact support can be constructed from a linear combination of integer translates of scaling functions. These functions are solutions of the dilation or two-scale equation. The most commonly studied example of dilation in one dimension is the following:

$$\phi(x) = c_0\phi(2x) + c_1\phi(2x-1) + \cdots + c_m\phi(2x-m), \tag{10}$$

where $\phi : \mathcal{R} \to \mathcal{R}$ and c_i, $i = 0, \ldots, m$, are given real coefficients. We always require $\Sigma_{k=0}^{m} c_k = 2$. This is necessary for an $\mathcal{L}^1$ solution to exist.

If ϕ determines a multiresolution, then the associated wavelet is given by $\psi(x) = \Sigma(-1)^k \times c_{1-k}\phi(2x-k)$. The regularity properties of the solutions of dilation equations have been extensively studied. In particular, nontrivial $\mathcal{L}^1$ solutions having compact support are characterized in [8] and shown to have their support in $[0, m]$. Moreover, it is shown that if ϕ is r times continuously differentiable, then $r < m - 1$. Hölder exponent and fractal structure of ϕ is determined in [2,19,20]. Continuous solutions are characterized in terms of the general and joint spectral radii of a family of matrices in [6].

The compact support and linearity of the dilation equation allows us to rewrite that in matrix notation. Define the vector Φ and matrices T_0 and T_1 by

$$\Phi(x) = [\phi(x), \phi(x+1), \ldots, \phi(x+m-1)]^t, \qquad \text{for } 0 \le x \le 1, \tag{11a}$$

$$c_k = 0, \qquad \text{for } k < 0 \text{ or } k > m, \tag{11b}$$

$$(T_d)_{ij} = c_{2i-j+d-1}, \qquad \text{for } 1 \le i, j \le m \text{ and } d = 0 \text{ or } 1, \tag{11c}$$

$$T_0 = \begin{pmatrix} c_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_2 & c_1 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_4 & c_3 & c_2 & c_1 & c_0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & c_m & c_{m-1} & c_{m-2} & c_{m-3} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_m & c_{m-1} \end{pmatrix},$$

$$T_1 = \begin{pmatrix} c_1 & c_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_3 & c_2 & c_1 & c_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_5 & c_4 & c_3 & c_2 & c_1 & c_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_m & c_{m-1} & c_{m-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_m \end{pmatrix}.$$

Notice that the definition of Φ is based on dividing the interval $[0, m]$ into m *cells*, $[i-1, i]$, $1 \le i \le m$. We define a pair of vectors or a function $f : \{0, 1\} \to \mathcal{R}^m$ to be *shift continuous* if $f(0)_i = f(1)_{i-1}$ for $1 < i \le m$. Obviously $\phi(x)$ is continuous on $[0, m]$ iff Φ is continuous on $[0, 1]$ and is shift continuous.

We search for the unique normalized continuous solution ϕ with support in $[0, m]$. This solution satisfies $\phi(x) = 0$ for $x \le 0$ or $x \ge m$, $\Phi(0)_1 = \Phi(1)_m = 0$ and

$$\Phi(x) = T_{x_1}\Phi(2x - x_1), \tag{12}$$

where x_1 is the first digit in the binary expansion of x. In particular, if we apply (12) to $x = 0 = 0.00\ldots$, $x = 1 = 0.11\ldots$, and $x = 1/2 = 0.100\cdots = 0.011\ldots$, respectively, then we get

$$T_0\Phi(0) = \Phi(0), \qquad T_1\Phi(1) = \Phi(1), \tag{13a}$$

$$T_0\Phi(1) = T_1\Phi(0) = \Phi\left(\frac{1}{2}\right). \tag{13b}$$

Once $\Phi(0)$ or $\Phi(1)$ is known, one can calculate Φ at dyadics by repeated applications of (12).

Now, suppose $0.x_1x_2\cdots x_qx_{q+1}\cdots$ indicates an infinite binary expansion of $x \in [0,1]$. Denote by $\bar{x}_q$ the residual after the qth digit, $\bar{x}_q = 0.x_{q+1}x_{q+2}\cdots$. Then, by repeated application of (12), we get

$$\Phi(x) = \prod_{\ell=1}^{q} T_{x_\ell}\Phi(\bar{x}_q). \tag{14}$$

We define $P_q(T_0, T_1, x) = \prod_{\ell=1}^{q} T_{x_\ell}$ and $P(T_0, T_1, x) = \lim_{q\to\infty} P_q(T_0, T_1, x)$ whenever the limit exists.

Dyadic numbers have two binary expansions, e.g., $x = 1/2 = 0.100\cdots = 0.011\cdots$. Therefore, in the definition of $P_q(T_0, T_1, x)$ a particular expansion of x should be prespecified. The consistency of (12) at dyadics, i.e., (13b), remedies this nonuniqueness for the infinite products, and the value of $P(T_0, T_1, x)$ is then determined independent of the choice of expansion for x.

DEFINITION 4. *The coefficients c_k are said to satisfy the moment or sum rules of order L if*

$$\sum_k c_k k^q (-1)^k = 0, \qquad \text{for } q = 0, \ldots, L. \tag{15}$$

Dyn and Levin [22] showed that the sum rules of order L are necessary for the nontrivial solutions to be L times continuously differentiable. Cavaretta and Micchelli [9] use the multivariate version of the moment conditions in the context of subdivision schemes and polynomial reproduction. Daubechies and Lagarias [2] use an explicit similarity transformation S which partially diagonalizes T_0 and T_1 simultaneously

$$ST_iS^{-1} = \begin{pmatrix} D & 0 \\ C_i & H_i \end{pmatrix}, \qquad i = 0, 1, \tag{16}$$

where S is given by

$$S_{ij} = \begin{cases} (i-1)!\dbinom{j-1}{i-1}, & i \le L+1, \\ L!\dbinom{j-1+L}{L}, & i > L+1, \end{cases} \tag{17}$$

its inverse is

$$S_{ij}^{-1} = \begin{cases} (-1)^{i+j}\dbinom{j-1}{i-1}[(j-1)!]^{-1}, & j \le L+1, \\ (-1)^{i+j}\dbinom{L+1}{i-j+L+1}(L!)^{-1}, & j > L+1, \end{cases} \tag{18}$$

and the diagonal part is $D = \operatorname{diag}(1, \ldots, 2^{-L})$. They prove the following theorem.

THEOREM 1. *If there are $1/2 \le \lambda < 1$, $0 \le \ell \le L$, and $C > 0$ such that for all k, $\|P_k(H_0, H_1, x)\| \le C\lambda^k 2^{-k\ell}$, then there is a nontrivial $\mathcal{L}^1$ solution ϕ to the dilation equation which is ℓ times continuously differentiable. If $\lambda > 1/2$, then the ℓth derivative is Hölder continuous with exponent at least $-\log_2\lambda$. If $\lambda = 1/2$, then ϕ^ℓ satisfies a Lipschitz type condition $|\phi^\ell(x+t) - \phi^\ell(x)| \le C|t|\,|\ln|t||$.*

The bound on the norm of long products implies that $\rho = \rho(\Sigma) \le 2^{-\ell}\lambda$ where $\Sigma = \{H_0, H_1\}$. By choosing $\lambda = 2^\ell\rho + \epsilon'$ where $\epsilon' > 0$, we see that the Hölder exponent is at least $-\ell - \log_2\rho - \epsilon$ for any $\epsilon > 0$.

5. HÖLDER REGULARITY OF FOUR-COEFFICIENT MRA

In this section, we give a nearly complete description of Hölder regularity for four-coefficient multiresolution analyses. Consider the two-scale real dilation equation in four coefficients

$$\phi(x) = c_0\phi(2x) + c_1\phi(2x-1) + c_2\phi(2x-2) + c_3\phi(2x-3). \tag{19}$$

One can construct $\phi(x)$ through infinite products of a pair of wavelet matrices, namely

$$T_0 = \begin{pmatrix} c_0 & 0 & 0 \\ c_2 & c_1 & c_0 \\ 0 & c_3 & c_2 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} c_1 & c_0 & 0 \\ c_3 & c_2 & c_1 \\ 0 & 0 & c_3 \end{pmatrix}. \tag{20}$$

Here, we only assume the regularity sum rule $c_0 + c_2 = c_1 + c_3 = 1$, i.e., $L = 0$. Then, the Hölder exponent of ϕ can be determined from two matrices obtained by restricting T_0 and T_1 to the space normal to the common left eigenvector $(1,1,1)$. These matrices are

$$H_0 = \begin{pmatrix} c_0 & 0 \\ -c_3 & 1-c_0-c_3 \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 1-c_0-c_3 & -c_0 \\ 0 & c_3 \end{pmatrix}. \tag{21}$$

Let $\Sigma = \{H_0, H_1\}$ and $\rho(\Sigma) < 1$; then ϕ is Hölder continuous with exponent $h \geq -\log_2 \rho(\Sigma) - \epsilon$ for any $\epsilon > 0$. To characterize the solutions that give a multiresolution [23] we further restrict the coefficients by the orthogonality rule which restricts (c_0, c_3) to the circle of orthogonality $(c_0 - 1/2)^2 + (c_3 - 1/2)^2 = 1/2$. (Each point of the circle except $(c_0, c_3) = (1,1)$ represents a multiresolution.)

The behavior of ϕ at the particular point $c^* = (c_0^*, c_3^*) = (0.6, -0.2)$ has been scrutinized with the expectation that it leads to the smallest $\rho(\Sigma)$ and the smoothest orthogonal scaling function [19] (note that $\check{\rho}_1(\Sigma)$ achieves its minimum at c^*). Let $\Sigma^* = \{H_0^*, H_1^*\}$ denote this particular value of Σ where

$$H_0^* = \begin{pmatrix} 0.6 & 0 \\ 0.2 & 0.6 \end{pmatrix}, \qquad H_1^* = \begin{pmatrix} 0.6 & -0.6 \\ 0 & -0.2 \end{pmatrix}. \tag{22}$$

Colella and Heil carried extensive computations and, based on products of length up to 30, conjectured

$$\rho(\Sigma^*) = \rho\left(H_1^* H_0^{*12}\right)^{1/13} \approx 0.659679. \tag{23}$$

Gripenberg's method produced the same result for products of lengths up to 243. We confirmed this conjecture and, by calculating $\rho(\Sigma)$ in the vicinity of Σ^*, disproved the statement that the smoothest MRA occurs at Σ^*. In fact, we have a detailed map of the behavior of $\rho(\Sigma)$ for nearly all points on the circle of orthogonality with the specification of the point at which $\rho(\Sigma)$ attains its minimum.

We have applied CAGOUB to over 150 subintervals which make up more than 99.9% of the circle of orthogonality. Here, we summarize our results. (We emphasize that all matrix calculations here can be done in exact arithmetic, and even when they are performed in finite precision there is no significant round-off error since the number of calculations is small.)

Consider the circle of orthogonality $(c_0 - 1/2)^2 + (c_3 - 1/2)^2 = 1/2$ in the (c_0, c_3) plane. We will travel on the half-circle below $c_0 = c_3$, from $(0,0)$ toward $(1,1)$ in the counterclockwise direction. (The properties on the upper half can be described similarly.) First, the optimal product is simply H_0 and the optimal ball is a quadrilateral. Then, starting at $(1/2, (1-\sqrt{2})/2)$, there is a critical strip on which the optimal product, we conjecture, is of the form $H_1 H_0^n$ where n starts at infinity, descends to 11, and goes back to infinity. On an interval where n is constant, there are typically three subintervals where the facial structure of the ball remains the same. However, two anomalous intervals have been detected, one at $n = 11$, where there are five subintervals,

and one at $n = 16$, where there are four subintervals. On the second stretch of the critical strip (when n goes from 11 to infinity) we pass through Heil-Colella point $(c_0, c_3) = (0.6, -0.2)$ which is on a subinterval where $n = 12$. The joint radius decreases throughout that interval and no minimum occurs. Next, there is a point on the border between $n = 22$ and $n = 23$ at which the smallest joint spectral radius and the smoothest multiresolution is realized. At this point, the ball has 54 sides, $c_0 = 0.64319821225683$, $c_3 = -0.19245524910022$, $\rho(\Sigma) = 0.64705462513820$, and the Hölder exponent of the resulting MRA is $h = 0.62804058345878$. As we leave the critical strip (at $c_3 = 1 - a^{1/3} - (1/3)a^{-1/3}$, where $a = 1/4 + 33^{1/2}/36$, i.e., $c_0 = 0.64779887126104$, and $c_3 = -0.19148788395312$), we enter an interval where once again the optimal product is of length one and the optimal ball is first a quadrilateral (Daubechies' D_4 is here), and then a hexagon. Finally, we arrive at $(1, 1)$.

At the two end points of the critical strip, the length of the optimal product and the number of sides on the optimal ball go to infinity. One might suspect that this gives a counterexample to the extremality conjecture of [7], which prescribes a piecewise-analytic ball with finite number of sides. However, there is no contradiction, since in the limit the ball with increasing number of sides approaches a quadrilateral. As a particular example of calculation of joint spectral radius, here we prove a conjecture of Heil and Colella.

PROPOSITION 1. *Let $\rho^* = \rho(\Sigma^*)$ and let $A = T^*/\rho^*$ and $B = T_1^*/\rho^*$. Then $\rho(\{A, B\}) = 1$. There is a neighborhood of Σ^* where $\rho(\Sigma) = \rho(H_1, H_0^{12})$, $\rho(\Sigma)$ is a strictly decreasing function of c_3, and the Hölder exponent of ϕ is a strictly increasing function of c_3.*

PROOF. Let $P = BA^{12}$. Note, P has an eigenvalue -1, and denote the corresponding eigenvector by v. Define a polygonal unit ball $\mathcal{U}$ with 30 sides whose vertices, in a counterclockwise direction, are labeled as $v_1, \ldots, v_{15}$ and $v_{-1}, \ldots, v_{-15}$, where $v_{-i} = -v_i$, $v_i = A^{i-1}v$ for $i = 1, \ldots, 14$, and $v_{15} = BA^{13}v$. One verifies that $\mathcal{U}$ is convex, and hence we can define a norm $\|\cdot\|_u$ based on it. Obviously, Av_i is a vertex of $\mathcal{U}$ for $i = 1, \ldots, 13$ and Bv_i is a vertex of $\mathcal{U}$ for $i = 13, 14$. One also verifies Av_i is in the interior of $\mathcal{U}$ for $i = 14, 15$ and Bv_i is in the interior of $\mathcal{U}$ for $i = 1, \ldots, 12, 15$. Therefore $\hat{\rho}_1(\{A, B\}, \|\cdot\|_u) = 1$. On the other hand, $\check{\rho}_{13}(\{A, B\}) \geq \rho(BA^{12})^{1/13} = 1$. We have $\check{\rho}_{13} \leq \rho(\{A, B\}) \leq \hat{\rho}_1$, and therefore $\rho(\{A, B\}) = 1$. The convexity of the ball can be indicated by a system of inequalities of the form $F(V) \leq 0$, where V is the vector of vertices of the ball, F is a vector of continuous functions, and the inequality is component-wise. We say the ball has *slack* if the vertices satisfy $F(V) < 0$. If any component of $F(V)$ is zero, then we say the ball is *critical*. A ball becomes critical, for example, if two adjacent sides are parallel or if two vertices coincide. Here, $\mathcal{U}$ has slack and the entire construction of the ball remains stable under small changes in, say, c_3 and we may obtain the spectral radius of $\rho(H_1 H_0^{12})^{1/13}$. One verifies that $\rho(\Sigma)$ is a decreasing function of c_3 in the vicinity of c^*, and hence the Hölder exponent is an increasing function. ∎

REMARK 1. If we use any of the 13 cyclical permutations of BA^{12}, we will obtain the same unit ball (up to a scale). Given $0 \leq n \leq 12$, let $P' = A^{12-n}BA^n$, and $P'v' = -v'$. Then, the vertices are given by $v_i' = A^{i-1}v'$ for $i = 1, \ldots, n+2$, $v_{n+3}' = BA^{n+1}v'$, and $v_i' = A^{i-n-4}BA^n v'$ for $i = n+4, \ldots, 15$. Here, v_{n+4}' corresponds to v_1 in Proposition 1.

REMARK 2. For the particular set of matrices Σ^*, the above steps result in a certain thirty-sided polygonal unit ball which acts as the optimal unit ball with respect to which the joint spectral radius is attained. In this case, the invariant unit ball is $\{tv,\ t \in [-1, 1]\}$, where v is the eigenvector of the optimal product P^* associated with eigenvalue -1. The critical index, the first value of m for which $\mathcal{G}_m = \mathcal{G}_{m+1}$, is 13.

The calculation of joint spectral radius is performed in a similar fashion for the remaining points on the circle of orthogonality. The main difficulty is on the critical strip. For any optimal product, which is valid for a specific interval on the circle, CAGOUB generates the optimal ball even as the facial structure of the ball changes. However, the optimal product itself follows a predictable pattern given by the following conjecture.

CONJECTURE 3. *The optimal product on the lower critical strip is $H_1H_0^n$, and on the upper critical strip it is $H_0H_1^n$. The value of n starts at infinity, descends to 11, and ascends to infinity again.*

The endpoints of the critical strips are easy to determine by using the asymptotic expansion of the average spectral radius of the optimal product in terms of n, i.e., $f(n) = \rho(H_0, H_1^n)^{1/n+1}$, and finding the limit of the root of $f(n) = f(n+1)$ as n tends to infinity.

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