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Journal of Computational and Applied Mathematics 236 (2012) 3016–3025

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A note on the application of the generalized finite difference method to seismic wave propagation in 2D

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ARTICLE INFO

MSC:
65M06
65M12
74S20
80M20

Keywords:
Meshless methods
Generalized finite difference method
Moving least squares
Seismic waves
Instructions

ABSTRACT

This paper shows the application of generalized finite difference method (GFDM) to the problem of seismic wave propagation. We investigated stability and star dispersion in 2D.

We obtained independent stability conditions and star dispersion of the phase velocity for the P and S waves. Also, P and S waves group velocity dispersion have been obtained.
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1. Introduction

The generalized finite difference method (GFDM) is evolved from classical finite difference method (FDM). GFDM can be applied over general or irregular clouds of points. The basic idea is to use moving least squares (MLS) approximation to obtain explicit difference formulae which can be included in partial differential equation to establish, together with an explicit method, a recursive relationship. The authors have made many contributions to the development of this method [1–7].

In this paper, this meshless method is applied to seismic wave propagation. Stability conditions and grid dispersion relations in 2D are derived.

2. Explicit generalized difference schemes for the seismic wave propagation problem for a perfectly elastic, homogeneous and isotropic medium

2.1. Equation of motion

The equation of motion and Hooke's law for a perfectly elastic, homogeneous, isotropic medium in 2D are

$$\begin{cases} \dfrac{\partial^2 U(x,y,t)}{\partial t^2} = \alpha^2 \dfrac{\partial^2 U(x,y,t)}{\partial x^2} + \beta^2 \dfrac{\partial^2 U(x,y,t)}{\partial y^2} + (\alpha^2-\beta^2)\dfrac{\partial^2 V(x,y,t)}{\partial x \partial y} \\ \dfrac{\partial^2 V(x,y,t)}{\partial t^2} = \beta^2 \dfrac{\partial^2 V(x,y,t)}{\partial x^2} + \alpha^2 \dfrac{\partial^2 V(x,y,t)}{\partial y^2} + (\alpha^2-\beta^2)\dfrac{\partial^2 U(x,y,t)}{\partial x \partial y} \end{cases} \tag{1}$$

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doi:10.1016/j.cam.2011.04.005

with the initial conditions

$$\begin{aligned} &U(x,y,0)=f_1(x,y); \qquad V(x,y,0)=f_2(x,y) \\ &\frac{\partial U(x,y,0)}{\partial t}=f_3(x,y); \qquad \frac{\partial V(x,y,0)}{\partial t}=f_4(x,y) \end{aligned} \tag{2}$$

and the boundary condition

$$\begin{cases} a_1U(x_0,y_0,t)+b_1\dfrac{\partial U(x_0,y_0,t)}{\partial n}=g_1(t) \\ a_2V(x_0,y_0,t)+b_2\dfrac{\partial V(x_0,y_0,t)}{\partial n}=g_2(t) \end{cases} \quad \text{en } \Gamma \tag{3}$$

where $f_1(x,y), f_2(x,y), f_3(x,y), f_4(x,y), g_1(t)$ y $g_2(t)$ are known functions,

$$\alpha=\sqrt{\frac{\lambda+2\mu}{\rho}}, \qquad \beta=\sqrt{\frac{\mu}{\rho}}$$

ρ is the density, λ and μ are Lamé elastic coefficients and Γ is the boundary of Ω.

2.2. Explicit generalized difference schemes

The aim is to obtain explicit linear expressions for the approximation of partial derivatives in the points of the domain. First of all, an irregular grid or cloud of points is generated in the domain $\Omega \cup \Gamma$. On defining the central node with a set of nodes surrounding that node, the star then refers to a group of established nodes in relation to a central node. Every node in the domain has an associated star assigned to it.

Following [1,3,5–7], the explicit difference formulae for the spatial derivatives are obtained,

$$\begin{cases} \dfrac{\partial^2 U(x_0,y_0,n\Delta t)}{\partial t^2}=\dfrac{u_0^{n+1}-2u_0^n+u_0^{n-1}}{(\Delta t)^2} \\ \dfrac{\partial^2 V(x_0,y_0,n\Delta t)}{\partial t^2}=\dfrac{v_0^{n+1}-2v_0^n+v_0^{n-1}}{(\Delta t)^2} \end{cases} \tag{4}$$

$$\begin{aligned} &\frac{\partial^2 U(x_0,y_0,n\Delta t)}{\partial x^2}=-m_0u_0^n+\sum_{j=1}^{N}m_ju_j^n; \qquad \frac{\partial^2 V(x_0,y_0,n\Delta t)}{\partial x^2}=-m_0v_0^n+\sum_{j=1}^{N}m_jv_j^n \\ &\frac{\partial^2 U(x_0,y_0,n\Delta t)}{\partial y^2}=-\eta_0u_0^n+\sum_{j=1}^{N}\eta_ju_j^n; \qquad \frac{\partial^2 V(x_0,y_0,n\Delta t)}{\partial y^2}=-\eta_0v_0^n+\sum_{j=1}^{N}\eta_jv_j^n \\ &\frac{\partial^2 U(x_0,y_0,n\Delta t)}{\partial x\partial y}=-\zeta_0u_0^n+\sum_{j=1}^{N}\zeta_ju_j^n; \qquad \frac{\partial^2 V(x_0,y_0,n\Delta t)}{\partial x\partial y}=-\zeta_0v_0^n+\sum_{j=1}^{N}\zeta_jv_j^n \end{aligned} \tag{5}$$

where N is the number of nodes in the star whose central node has the coordinates (x_0, y_0) (in this work $N = 8$ and the are selected by using the four quadrants criteria [1,6]).

m_0, η_0, ζ_0 are the coefficients that multiply the approximate values of the functions U and V at the central node for the time $n\Delta t$ (u_0^n and v_0^n respectively) in the generalized finite difference explicit expressions for the space derivatives.

m_j, η_j, ζ_j are the coefficients that multiply the approximate values of the functions U and V at the rest of the star nodes for the time $n\Delta t$ (u_j^n and v_j^n respectively) in the generalized finite difference explicit expressions for the space derivatives.

The replacement in Eq. (1) of the explicit expressions obtained for the partial derivatives leads to

$$\begin{cases} u_0^{n+1}=2u_0^n-u_0^{n-1}+(\Delta t)^2\Bigg[\alpha^2\left(-m_0u_0^n+\displaystyle\sum_1^N m_ju_j^n\right)+\beta^2\left(-\eta_0u_0^n+\displaystyle\sum_1^N\eta_ju_j^n\right) \\ \qquad +(\alpha^2-\beta^2)\left(-\zeta_0v_0^n+\displaystyle\sum_1^N\zeta_jv_j^n\right)\Bigg] \\ v_0^{n+1}=2v_0^n-v_0^{n-1}+(\Delta t)^2\Bigg[\beta^2\left(-m_0v_0^n+\displaystyle\sum_1^N m_jv_j^n\right)+\alpha^2\left(-\eta_0v_0^n+\displaystyle\sum_1^N\eta_jv_j^n\right) \\ \qquad +(\alpha^2-\beta^2)\left(-\zeta_0u_0^n+\displaystyle\sum_1^N\zeta_ju_j^n\right)\Bigg]. \end{cases} \tag{6}$$

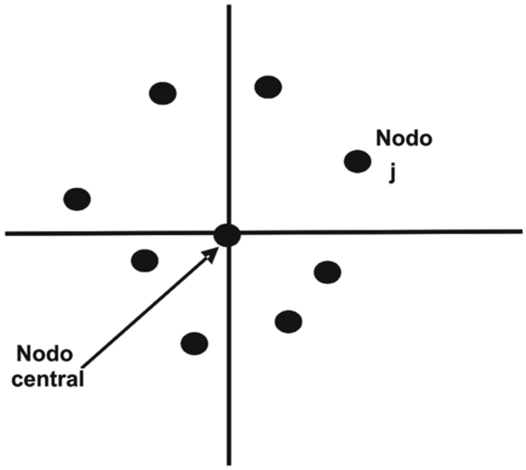

Fig. 1. Irregular star (9 nodes).

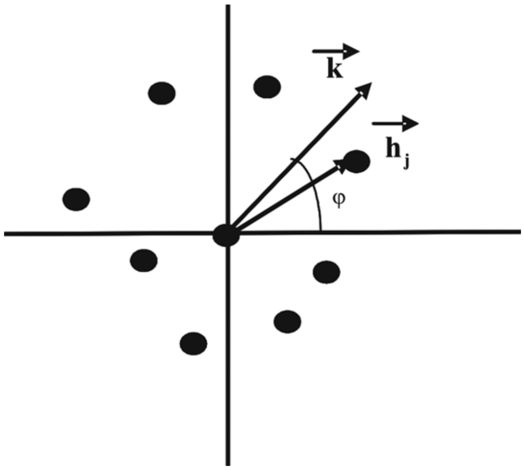

Fig. 2. The wavenumber $\vec{\boldsymbol{k}}$ and $\vec{\boldsymbol{h}}_{\boldsymbol{j}}$ position vector of the node j.

3. Stability criterion

For the stability analysis the first idea is to make a harmonic decomposition of the approximated solution at grid points and at a given time level (n). Then we can write the finite difference approximation in the nodes of the star at time n, as

$$u_0^n = A\xi^n e^{i\boldsymbol{k}^T\boldsymbol{x_0}}; \qquad u_j^n = A\xi^n e^{i\boldsymbol{k}^T\boldsymbol{x_j}}; \qquad v_0^n = B\xi^n e^{i\boldsymbol{k}^T\boldsymbol{x_0}}; \qquad v_j^n = B\xi^n e^{i\boldsymbol{k}^T\boldsymbol{x_j}} \tag{7}$$

where $\boldsymbol{x_0}$ is the position vector of the central node of the star, $\boldsymbol{x_j}$, $j = 1, \dots, N$ are the position vectors of the rest of the nodes in the star and $\boldsymbol{h_j}$ are the relative position vectors of the nodes in the star in respect to the central node whose coordinates are $h_{jx} = x_j - x_0$, $h_{jy} = y_j - y_0$ (Fig. 1).

ξ is the amplification factor whose value will determine the stability condition, w is the angular frequency in the grid.

$$\boldsymbol{x_j} = \boldsymbol{x_0} + \boldsymbol{h_j}; \qquad \xi = e^{-iw\Delta t}$$

$\boldsymbol{k}$ (see Fig. 2) is the column vector of the wave numbers

$$\boldsymbol{k} = \begin{Bmatrix} k_x \\ k_y \end{Bmatrix} = k \begin{Bmatrix} \cos\varphi \\ \sin\varphi \end{Bmatrix}.$$

Then we can write the stability condition as: $\|\xi\| \leq 1$.

Substituting Eq. (7) into Eq. (6), cancellation of $\xi^n e^{i\boldsymbol{\nu}^T \boldsymbol{x_0}}$, leads to

$$\begin{aligned} A\xi &= 2A - \frac{A}{\xi} + (\Delta t)^2 \Bigg[\alpha^2 \left(-Am_0 + A \sum_1^N m_j e^{i\boldsymbol{k}^T \boldsymbol{h_j}} \right) + \beta^2 \left(-A\eta_0 + A \sum_1^N \eta_j e^{i\boldsymbol{k}^T \boldsymbol{h_j}} \right) \\ &\quad + (\alpha^2 - \beta^2) \left(-B\zeta_0 + B \sum_1^N \zeta_j e^{i\boldsymbol{k}^T \boldsymbol{h_j}} \right) \Bigg] \\ B\xi &= 2B - \frac{B}{\xi} + (\Delta t)^2 \Bigg[\beta^2 \left(-Bm_0 + B \sum_1^N m_j e^{i\boldsymbol{k}^T \boldsymbol{h_j}} \right) + \alpha^2 \left(-B\eta_0 + B \sum_1^N \eta_j e^{i\boldsymbol{k}^T \boldsymbol{h_j}} \right) \\ &\quad + (\alpha^2 - \beta^2) \left(-A\zeta_0 + A \sum_1^N \zeta_j e^{i\boldsymbol{k}^T \boldsymbol{h_j}} \right) \Bigg] \end{aligned} \tag{8}$$

where

$$m_0 = \sum_1^N m_j; \qquad \eta_0 = \sum_1^N \eta_j; \qquad \zeta_0 = \sum_1^N \zeta_j. \tag{9}$$

Substituting Eq. (9) into Eq. (8), the system of equations is obtained

$$\begin{aligned} &A \left[\xi - 2 + \frac{1}{\xi} + (\Delta t)^2 \alpha^2 \sum_1^N m_j (1 - e^{i\boldsymbol{k}^T \boldsymbol{h_j}}) + (\Delta t)^2 \beta^2 \sum_1^N \eta_j (1 - e^{i\boldsymbol{k}^T \boldsymbol{h_j}}) \right] \\ &\quad + B(\Delta t)^2 (\alpha^2 - \beta^2) \sum_1^N \zeta_j (1 - e^{i\boldsymbol{k}^T \boldsymbol{h_j}}) = 0 \\ &A(\Delta t)^2 (\alpha^2 - \beta^2) \sum_1^N \zeta_j (1 - e^{i\boldsymbol{k}^T \boldsymbol{h_j}}) + B \Bigg[\xi - 2 + \frac{1}{\xi} + (\Delta t)^2 \beta^2 \sum_1^N m_j (1 - e^{i\boldsymbol{k}^T \boldsymbol{h_j}}) \\ &\quad + (\Delta t)^2 \alpha^2 \sum_1^N \eta_j (1 - e^{i\boldsymbol{k}^T \boldsymbol{h_j}}) \Bigg] = 0. \end{aligned} \tag{10}$$

If B is obtained from the second equation and is included into the first equation, then

$$\begin{aligned} &\left[2\cos w\Delta t - 2 + (\Delta t)^2 \left(\alpha^2 \sum_1^N m_j (1 - e^{i\boldsymbol{k}^T \boldsymbol{h_j}}) + \beta^2 \sum_1^N \eta_j (1 - e^{i\boldsymbol{k}^T \boldsymbol{h_j}}) \right) \right] \Bigg[2\cos w\Delta t - 2 + (\Delta t)^2 \\ &\quad \times \left(\beta^2 \sum_1^N m_j (1 - e^{i\boldsymbol{k}^T \boldsymbol{h_j}}) + \alpha^2 \sum_1^N \eta_j (1 - e^{i\boldsymbol{k}^T \boldsymbol{h_j}}) \right) \Bigg] = (\Delta t)^4 (\alpha^2 - \beta^2)^2 \left[\sum_1^N \zeta_j (1 - e^{i\boldsymbol{k}^T \boldsymbol{h_j}}) \right]^2. \end{aligned} \tag{11}$$

Operating, the following conditions are obtained:

Real part

$$\begin{aligned} &(1 - \cos w\Delta t)^2 - 2(1 - \cos w\Delta t) \frac{(\Delta t)^2}{4} (\alpha^2 + \beta^2) \sum_1^N (m_j + \eta_j)(1 - \cos \boldsymbol{k}^T \boldsymbol{h_j}) \\ &\quad + \frac{(\Delta t)^4}{4} \Bigg[\left(\alpha^2 \sum_1^N m_j (1 - \cos \boldsymbol{k}^T \boldsymbol{h_j}) + \beta^2 \sum_1^N \eta_j (1 - \cos \boldsymbol{k}^T \boldsymbol{h_j}) \right) \Bigg(\beta^2 \sum_1^N m_j (1 - \cos \boldsymbol{k}^T \boldsymbol{h_j}) \\ &\quad + \alpha^2 \sum_1^N \eta_j (1 - \cos \boldsymbol{k}^T \boldsymbol{h_j}) \Bigg) - \left(\alpha^2 \sum_1^N m_j \sin \boldsymbol{k}^T \boldsymbol{h_j} + \beta^2 \sum_1^N \eta_j \sin \boldsymbol{k}^T \boldsymbol{h_j} \right) \Bigg(\beta^2 \sum_1^N m_j \sin \boldsymbol{k}^T \boldsymbol{h_j} \\ &\quad + \alpha^2 \sum_1^N \eta_j \sin \boldsymbol{k}^T \boldsymbol{h_j} \Bigg) - (\alpha^2 - \beta^2)^2 \left[\left(\sum_1^N \zeta_j (1 - \cos \boldsymbol{k}^T \boldsymbol{h_j}) \right)^2 - \left(\sum_1^N \zeta_j \sin \boldsymbol{k}^T \boldsymbol{h_j} \right)^2 \right] \Bigg] = 0. \end{aligned} \tag{12}$$

Imaginary part

$$2(1-\cos w\Delta t)(\alpha^2+\beta^2)\sum_1^N(m_j+\eta_j)\sin \boldsymbol{k}^T\boldsymbol{h_j}-(\Delta t)^2\Bigg[\Bigg(\alpha^2\sum_1^N m_j(1-\cos \boldsymbol{k}^T\boldsymbol{h_j})$$
$$+\beta^2\sum_1^N\eta_j(1-\cos \boldsymbol{k}^T\boldsymbol{h_j})\Bigg)\Bigg(\beta^2\sum_1^N m_j\sin \boldsymbol{k}^T\boldsymbol{h_j}+\alpha^2\sum_1^N\eta_j\sin \boldsymbol{k}^T\boldsymbol{h_j}\Bigg)$$
$$+\Bigg(\alpha^2\sum_1^N m_j\sin \boldsymbol{k}^T\boldsymbol{h_j}+\beta^2\sum_1^N\eta_j\sin \boldsymbol{k}^T\boldsymbol{h_j}\Bigg)\Bigg(\beta^2\sum_1^N m_j(1-\cos \boldsymbol{k}^T\boldsymbol{h_j})$$
$$+\alpha^2\sum_1^N\eta_j(1-\cos \boldsymbol{k}^T\boldsymbol{h_j})\Bigg)+2(\alpha^2-\beta^2)^2\Bigg(\sum_1^N\zeta_j(1-\cos \boldsymbol{k}^T\boldsymbol{h_j})\sum_1^N\zeta_j\sin \boldsymbol{k}^T\boldsymbol{h_j}\Bigg)\Bigg]=0. \tag{13}$$

Operating with the Eqs. (12) and (13), cancelling with conservative criteria, the condition for stability of star is obtained as

$$\Delta t<\sqrt{\frac{4}{(\alpha^2+\beta^2)[(|m_0|+|\eta_0|)+\sqrt{(m_0+\eta_0)^2+\zeta_0^2}]}}. \tag{14}$$

4. Star dispersion

The GFDM needs the discretization of the domain using a cloud of nodes, and if the distance between nodes becomes too large in comparison with the source wavelength (and in our case even the position of the nodes), waves disperse in the grid and produce a variation of velocity with different frequencies. The existence of grid dispersion of the phase velocity implies the existence of the grid group velocity in the grid and its dispersion.

4.1. Star dispersion relations for the P and S waves

The Eq. (12) leads to

$$\omega=\frac{1}{\Delta t}\arccos\Phi \tag{15}$$

where

$$\Phi=1-\frac{(\Delta t)^2}{4}\Big((\alpha^2+\beta^2)(a_1+a_3)+((\alpha^2+\beta^2)^2(a_1+a_3)^2$$
$$+4[(\alpha^2-\beta^2)^2(a_5^2-a_6^2)+(\alpha^2a_2+\beta^2a_4)(\beta^2a_2+\alpha^2a_4)-(\alpha^2a_1+\beta^2a_3)(\beta^2a_1+\alpha^2a_3)])^{\frac{1}{2}}\Big) \tag{16}$$

with

$$\begin{aligned}
a_1&=\sum_1^N m_j(1-\cos \boldsymbol{k}^T\boldsymbol{h_j})\Rightarrow\frac{\partial a_1}{\partial k}=a_{1,k}=\sum_1^N m_jd\sin kd\\
a_2&=\sum_1^N m_j\sin \boldsymbol{k}^T\boldsymbol{h_j}\Rightarrow\frac{\partial a_2}{\partial k}=a_{2,k}=\sum_1^N m_jd\cos kd\\
a_3&=\sum_1^N \eta_j(1-\cos \boldsymbol{k}^T\boldsymbol{h_j})\Rightarrow\frac{\partial a_3}{\partial k}=a_{3,k}=\sum_1^N \eta_jd\sin kd\\
a_4&=\sum_1^N \eta_j\sin \boldsymbol{k}^T\boldsymbol{h_j}\Rightarrow\frac{\partial a_4}{\partial k}=a_{4,k}=\sum_1^N \eta_jd\cos kd\\
a_5&=\sum_1^N \zeta_j(1-\cos \boldsymbol{k}^T\boldsymbol{h_j})\Rightarrow\frac{\partial a_5}{\partial k}=a_{5,k}=\sum_1^N \zeta_jd\sin kd\\
a_6&=\sum_1^N \zeta_j\sin \boldsymbol{k}^T\boldsymbol{h_j}\Rightarrow\frac{\partial a_6}{\partial k}=a_{6,k}=\sum_1^N \zeta_jd\cos kd
\end{aligned} \tag{17}$$

and

$$\boldsymbol{k}^T \boldsymbol{h_j} = k(h_{jx} \cos\varphi + h_{jy} \sin\varphi) = kd.$$

Is known

$$\omega = 2\pi \frac{c^{\text{grid}}}{\lambda^{\text{grid}}} \tag{18}$$

where c^{grid} and λ^{grid} are the phase velocity (α^{grid} or β^{grid}) and the wavelength (λ_P^{grid} or λ_S^{grid}) in the star respectively.

Defining the relations:

$$s = \frac{2}{\lambda_S^{\text{grid}} \sqrt{(r^2+1)[(|m_0| + |\eta_0|) + \sqrt{(m_0+\eta_0)^2 + \zeta_0^2}]}} \tag{19}$$

$$s_P = \frac{2}{\lambda_P^{\text{grid}} \sqrt{(r^2+1)[(|m_0| + |\eta_0|) + \sqrt{(m_0+\eta_0)^2 + \zeta_0^2}]}} \tag{20}$$

$$p = \frac{\beta \Delta t \sqrt{(r^2+1)[(|m_0| + |\eta_0|) + \sqrt{(m_0+\eta_0)^2 + \zeta_0^2}]}}{2} \tag{21}$$

$$r = \frac{\alpha}{\beta} \tag{22}$$

$$s_P = \frac{s}{r}. \tag{23}$$

Substituting Eqs. (15), (20), (21) and (23) into Eq. (18), the star dispersion relations for P and S waves are obtained:

$$\frac{\alpha^{\text{grid}}}{\alpha} = \frac{\arccos \Phi}{2\pi sp} \tag{24}$$

$$\frac{\beta^{\text{grid}}}{\beta} = \frac{\arccos \Phi}{2\pi sp}. \tag{25}$$

4.2. Star dispersion for group velocity

By definition the group velocity is the derivative of w (see Eq. (15)) with respect to k, thus

$$\alpha_{\text{group}}^{\text{grid}} = \frac{\partial w}{\partial k} = \frac{\Delta t}{4} \frac{\beta^2 \Upsilon}{\sqrt{1-\Phi^2}} \tag{26}$$

where

$$\begin{aligned} \Upsilon = {} & (r^2+1)(a_{1,k} + a_{3,k}) + \frac{1}{2}[2(r^2+1)^2(a_1+a_3)(a_{1,k}+a_{3,k}) + 4[2(r^2-1)^2(a_5 a_{5,k} - a_6 a_{6,k}) \\ & + (r^2 a_{2,k} + a_{4,k})(a_2 + r^2 a_4) + (r^2 a_2 + a_4)(a_{2,k} + r^2 a_{4,k}) - (r^2 a_{1,k} + a_{3,k})(a_1 + r^2 a_3) \\ & - (r^2 a_1 + a_3)(a_{1,k} + r^2 a_{3,k})]] \times [(r^2+1)^2(a_1+a_3)^2 + 4[(r^2-1)^2(a_5^2 - a_6^2) \\ & + (r^2 a_2 + a_4)(a_2 + r^2 a_4) - (r^2 a_1 + a_3)(a_1 + r^2 a_3)]]^{-\frac{1}{2}}. \end{aligned} \tag{27}$$

Defining

$$\begin{aligned} F = {} & (r^2+1)(a_1+a_3) + [(r^2+1)^2(a_1+a_3)^2 \\ & + 4[(r^2-1)^2(a_5^2 - a_6^2) + (r^2 a_2 + a_4)(a_2 + r^2 a_4) - (r^2 a_1 + a_3)(a_1 + r^2 a_3)]]^{\frac{1}{2}} \end{aligned} \tag{28}$$

and substituting Eqs. (21) and (28) into Eq. (26), the star dispersion for P and S waves are

$$\frac{\alpha_{\text{group}}^{\text{grid}}}{\alpha} = \frac{1}{2\sqrt{2}r} \frac{\Upsilon}{\sqrt{F - \left(\frac{pF}{\sqrt{(r^2+1)[(|m_0|+|\eta_0|) + \sqrt{(m_0+\eta_0)^2 + \zeta_0^2}]}\sqrt{2}} \right)^2}} \tag{29}$$

$$\frac{\beta_{\text{group}}^{\text{grid}}}{\beta}=\frac{1}{2\sqrt{2}}\frac{\Upsilon}{\sqrt{F-\left(\frac{pF}{\sqrt{(r^2+1)[(|m_0|+|\eta_0|)+\sqrt{(m_0+\eta_0)^2+\zeta_0^2}]}\sqrt{2}}\right)^2}}. \tag{30}$$

5. Irregularity of the star (IIS) and dispersion

In this section we are going to define the index of irregularity of a star (IIS) and also the index of irregularity of a cloud of nodes (IIC).

The coefficients m_0, η_0, ζ_0 are functions of:

- The number of nodes in the star.
- The coordinates of each star node referred to the central node of the star.
- The weighting function (see [1,6]).

If the number of nodes by star is fixed, in this case 9 ($N = 8$), and the weighting function

$$w(h_{jx}, h_{jy}) = \frac{1}{(\sqrt{h_{jx}^2+h_{jy}^2})^3} \tag{31}$$

the expression

$$\frac{1}{\sqrt{(r^2+1)[(|m_0|+|\eta_0|)+\sqrt{(m_0+\eta_0)^2+\zeta_0^2}]}} \tag{32}$$

is function of the coordinates of each node of star referred to its central node.

The coefficients m_0, η_0, ζ_0, are functions of $\frac{1}{h_{jx}^2+h_{jy}^2}$.

Denoting τ_l as the average of the distances between of the nodes of the star l and its central node and denoting τ the average of the τ_l values in the stars of the mesh, then

$$\boldsymbol{h_j} = \tau \begin{Bmatrix} \overline{h_{jx}} \\ \overline{h_{jy}} \end{Bmatrix} \tag{33}$$

$$\overline{m_0} = m_0\tau^2; \qquad \overline{\eta_0} = \eta_0\tau^2; \qquad \overline{\zeta_0} = \zeta_0\tau^2. \tag{34}$$

The stability criterion can be rewritten

$$\Delta t < \frac{2\tau}{\beta\sqrt{(r^2+1)}\sqrt{(|\overline{m_0}|+|\overline{\eta_0}|)+\sqrt{(\overline{m_0}+\overline{\eta_0})^2+\overline{\zeta_0}^2}}} \tag{35}$$

where the bar over the letters means average.

For the regular mesh case, the inequality (35) is

$$\Delta t < \frac{\tau}{\beta\sqrt{r^2+1}}\frac{2(\sqrt{2}-1)\sqrt{3}}{\sqrt{5}}. \tag{36}$$

Multiplying the right-hand side of inequality (36) by the factor

$$\frac{\sqrt{5}(\sqrt{2}+1)}{\sqrt{3(|\overline{m_0}|+|\overline{\eta_0}|+\sqrt{(\overline{m_0}+\overline{\eta_0})^2+\overline{\zeta_0}^2})}} \tag{37}$$

the inequality (35) is obtained.

For each one of the stars of the cloud of nodes, we define the IIS for a star with central node in (x_0, y_0) as Eq. (37)

$$\text{IIS}_{(x_0,y_0)} = \frac{\sqrt{5}(\sqrt{2}+1)}{\sqrt{3(|\overline{m_0}|+|\overline{\eta_0}|+\sqrt{(\overline{m_0}+\overline{\eta_0})^2+\overline{\zeta_0}^2})}} \tag{38}$$

that takes the value of one in the case of a regular mesh and $0 < \text{IIS} \leq 1$.

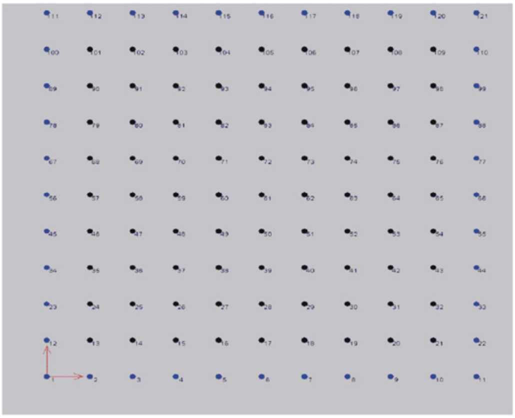

Fig. 3. Regular mesh (121 nodes).

If the index IIS decreases, then absolute values of $\overline{m_0}$, $\overline{\eta_0}$, $\overline{\zeta_0}$ increase and then according to Eq. (35), Δt decreases and star dispersion increases (see Eqs. (24), (25), (29) and (30)).

The irregularity index of a cloud of nodes (IIC) is defined as the minimum of all the IIS of the stars of a cloud of nodes

$$\mathrm{IIC} = \min\{\mathrm{IIS}_{(x_z,y_z)}/z = 1, \ldots, NT\} \tag{39}$$

where NT is the total number of nodes of the domain.

6. Numerical results

Let us solve the Eq. (1), in $\Omega = [0, 1] \times [0, 1] \subset \mathbf{R}^2$, with Dirichlet boundary conditions

$$\begin{cases} U(0, y, t) = 0 & \forall y \in [0, 1] \\ U(1, y, t) = \sin 1 \sin y \cos(\sqrt{2}\beta t) & \forall y \in [0, 1] \\ U(x, 0, t) = 0 & \forall x \in [0, 1] \\ U(x, 1, t) = \sin x \sin 1 \cos(\sqrt{2}\beta t) & \forall x \in [0, 1] \end{cases}$$
$$\begin{cases} V(0, y, t) = 0 & \forall y \in [0, 1] \\ V(1, y, t) = \cos 1 \cos y \cos(\sqrt{2}\beta t) & \forall y \in [0, 1] \\ V(x, 0, t) = 0 & \forall x \in [0, 1] \\ V(x, 1, t) = \cos x \cos 1 \cos(\sqrt{2}\beta t) & \forall x \in [0, 1] \end{cases} \tag{40}$$

and initial conditions

$$U(x, y, 0) = \sin x \sin y; \qquad V(x, y, 0) = \cos x \cos y; \qquad \frac{\partial U(x, y, 0)}{\partial t} = 0; \frac{\partial V(x, y, 0)}{\partial t} = 0 \tag{41}$$

using the regular meshes (see Fig. 3 with 121 nodes) and irregular meshes (see Figs. 4 and 5) with 121 nodes. The analytical solution is

$$U(x, y, t) = \cos(\sqrt{2}\beta t) \sin x \sin y; \qquad V(x, y, t) = \cos(\sqrt{2}\beta t) \cos x \cos y. \tag{42}$$

The weighting function is given by Eq. (31) and the criterion for the selection of star nodes is the quadrant criterion (see [1,4,5]). The global error is evaluated for each time increment, in the last time step considered, using the following formula

$$\text{Global error} = \frac{\sqrt{\frac{\sum_{j=1}^{NT} (\mathrm{sol}(j) - \mathrm{exac}(j))^2}{NT}}}{|\mathrm{exac}_{\max}|} \times 100 \tag{43}$$

where $\mathrm{sol}(j)$ is the GFDM solution at the node $j$exac(j) is the exact value of the solution at the node j, $\mathrm{exac}_{\max}$ is the maximum value of the exact solution in the cloud of nodes considered and NT is the total number of nodes of the domain.

Tables 1 and 2 show the global errors, with $\Delta t = 0.01$, for several values of α and β, in regular meshes (see Fig. 3).

Table 3 shows the values of the global error for several values of Δt, using the irregular mesh with 121 nodes (see Fig. 4), with IIC $= 0.6524$.

Table 4 shows the values of the global error for several values of Δt, using the irregular mesh with 121 nodes (see Fig. 5), with IIC $= 0.8944$.

Table 5 shows the results of the dispersion of the star with the greatest index of irregularity for different angles of propagation (see Fig. 2) and the values $p = 0.8$, $s = 0.2$ and $r = 2$.

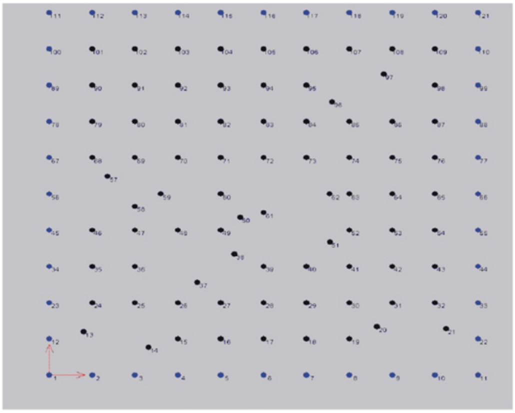

Fig. 4. Irregular mesh (IIC = 0.65).

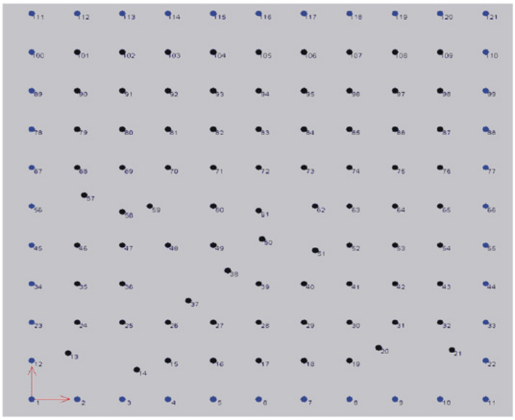

Fig. 5. Irregular mesh (IIC = 0.89).

Table 1
Influence of the number of nodes in the global error with $\alpha = 1$; $\beta = 0.6$.

N of nodes	Global error *U*	Global error *V*
121	0.002816	0.003851
289	0.001166	0.001618
441	0.000652	0.000896
676	0.000328	0.000443

Table 2
Influence of the number of nodes in the global error with $\alpha = 1$; $\beta = 0.5$.

N of nodes	Global error *U*	Global error *V*
121	0.001569	0.001754
289	0.000604	0.000679
441	0.000386	0.000431
676	0.000245	0.000275

Table 3
Influence of Δt in the global error with $\alpha = 1$; $\beta = 0.5$; IIC = 0.6524.

Δt	Global error *U*	Global error *V*
0.0316	0.015060	0.007900
0.0223	0.010170	0.006447
0.01	0.003566	0.003099
0.007	0.002245	0.001945

Table 4
Influence of Δt in the global error with $\alpha = 1$; $\beta = 0.5$; IIC = 0.8944.

Δt	Global error U	Global error V
0.0316	0.004900	0.010400
0.0223	0.003900	0.007300
0.01	0.002020	0.002460
0.007	0.001520	0.001530

Table 5
Relation between the IIC and the star dispersion.

φ/IIC	1.0000	0.8944	0.6524
30	0.9999	0.8254	0.3251
45	0.9998	0.8314	0.3078
60	0.9999	0.8146	0.3102

7. Conclusions

This paper shows a scheme in generalized finite differences, for seismic wave propagation in 2D. The von Neumann stability criterion has been expressed as a function of the coefficients of the star equation and the velocity ratio.

The investigated star dispersion has been related with the irregularity of the star using the irregularity indicator of the cloud of nodes. The use of irregular meshes, adjusted to the geometry of the problem, may create high dispersion in certain stars which is related to high values of the irregularity index of cloud of nodes (IIC). In this case the cloud of nodes can be redefined by an adaptive process [2] until a new cloud with suitable dispersion and irregularity index values is obtained.

Acknowledgement

The authors acknowledge the support from Ministerio de Ciencia e Innovación of Spain, project CGL2008-01757/CLI.

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