Nonlinear stability of periodic traveling wave solutions to the Schrödinger and the modified Korteweg–de Vries equations

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Abstract

This work is concerned with stability properties of periodic traveling waves solutions of the focusing Schrödinger equation

\[ iu_t + u_{xx} + |u|^2 u = 0 \]

posed in \( \mathbb{R} \), and the modified Korteweg–de Vries equation

\[ u_t + 3u^2u_x + u_{xxx} = 0 \]

posed in \( \mathbb{R} \). Our principal goal in this paper is the study of positive periodic wave solutions of the equation \( \phi''_\omega + \phi^3_\omega - \omega \phi_\omega = 0 \), called dnoidal waves. A proof of the existence of a smooth curve of solutions with a fixed fundamental period \( L \), \( \omega \in (2\pi^2/L^2, +\infty) \to \phi_\omega \in H^1_{\text{per}}([0, L]) \), is given. It is also shown that these solutions are nonlinearly stable in the energy space \( H^1_{\text{per}}([0, L]) \) and unstable by perturbations with period \( 2L \) in the case of the Schrödinger equation.

Keywords: Schrödinger equation; Modified Korteweg–de Vries equation; Periodic traveling waves; Jacobian elliptic functions; Nonlinear stability

\[ \star \]

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1. Introduction

The classical models of the focusing Schrödinger equation (NLS equation henceforth)

\[ iu_t + u_{xx} + |u|^2 u = 0 \]  \hspace{1cm} (1.1)

posed in \( \mathbb{R} \) with \( u = u(x, t) \in \mathbb{C}, x, t \in \mathbb{R} \), and the modified Korteweg–de Vries equation (mKdV equation henceforth)

\[ u_t + 3u^2 u_x + u_{xxx} = 0, \]  \hspace{1cm} (1.2)

posed in \( \mathbb{R} \) with \( u = u(x, t) \in \mathbb{R} \), have been the focus of an extensive study in the last coupled decades in the particular issues about well-posedness of solutions (see [8,10,11,19], and reference therein) and about the existence and nonlinear stability of localized solutions. For instance, we have the bound state solutions in the case of the NLS equation, namely, solutions of the form

\[ u(x, t) = e^{i\omega t} \phi_\omega(x), \]  \hspace{1cm} (1.3a)

with \( |\phi_\omega(x)| \to 0 \) as \( |x| \to \infty \), and the solitary wave solutions in the case of the mKdV equation

\[ u(x, t) = \psi(x - \omega t), \]  \hspace{1cm} (1.3b)

where \( \psi(\xi) \to 0 \) as \( |\xi| \to \infty \). The physical importance and practical applications of this localized solutions (case of the NLS model) in the optical fibre technology or in the analysis of nonlinear water wavepackets is well known, as well as, in optical nonlinear for the mKdV model. A basic issue in these practical fields is the existence and linear or nonlinear stability of these localized solutions. A extensive development of mathematical stability theory for this type of solutions has been obtained in the past years by Benjamin, Bona, Weinstein, Grillakis, Shatah and Strauss [4,5,14–16,23,24]. In general form, this theory has been applied when the profiles \( \phi_\omega, \psi \), decay to zero in the infinity. Recently a well succeeded application of this theory has been obtained when we have the boundary periodic conditions on the profile \( \psi \) in (1.3b), such as in the case of the Korteweg–de Vries equation [3],

\[ u_t + u_{xxx} + uu_x = 0, \]

the Hirota–Satsuma systems [2]

\[
\begin{align*}
&u_t + u_{xxx} + 6uu_x = 2bv
&v_t + v_{xxx} + 3uv_x = 0,
\end{align*}
\]

for \( b > 0 \), and the Benjamin equation [1]

\[ u_t + u_{xxx} + l\mathcal{H}u_{xx} + uu_x = 0, \]

where \( l \in \mathbb{R} \) and \( \mathcal{H} \) is the Hilbert transform. We note that with regard to the Korteweg–de Vries equation a different stability approach for periodic traveling waves solutions has been obtained in [22].
Our main interest in this paper is to study the existence and the nonlinear stability/instability of localized solutions when $\phi_\omega$ and $\psi$ above satisfy periodic boundary conditions. More precisely, in the case of the NLS equation, we show the existence of at least three smooth branch of periodic traveling waves solutions of the form (1.3a) where $\phi_\omega : \mathbb{R} \to \mathbb{R}$ is a periodic smooth function with prescribe period $L > 0$ and where $\omega$ belongs to a determined interval in $\mathbb{R}$. Our approach is based in the equation

$$\left[ \phi_\omega' \right]^2 = \frac{1}{2} \left[ -\phi_\omega^4 + 2\omega \phi_\omega^2 + 4B \phi_\omega \right],$$

(1.4)

where $B \phi_\omega$ is a needed integration constant different of zero. Our curves arise of the specific form of the roots associated to the polynomial of fourth-order that appears in the right-hand side of (1.4), namely, $y \equiv F_{\phi_\omega}(t) = -t^4 + 2\omega t^2 + 4B \phi_\omega$. Initially, by considering the zeros of $F_{\phi_\omega}$ being reals and symmetric with regard to the axis $y$, we find, via the implicit function theorem, a smooth curve

$$\omega \in \left( \frac{2\pi^2}{L^2}, +\infty \right) \to \phi_\omega \in H^\infty_{\text{per}}([0, L])$$

of periodic solutions for

$$\phi''_\omega + \phi^3_\omega - \omega \phi_\omega = 0,$$

(1.5)

which are depending of the Jacobian elliptic $\text{dn}$ function (see Appendix A) in the form

$$\phi_\omega(\xi) = \eta_1 \text{dn} \left( \frac{\eta_1}{\sqrt{2}} \xi; k \right),$$

(1.6)

where $\eta_1$ and the modulus $k$ depend smoothly of $\omega$ (see Theorem 2.1 below).

Next, by using the Floquet theory associated to the linear operators

$$L_{\text{dn}} = -\frac{d^2}{dx^2} + \omega - 3\phi^2_\omega, \quad L^+_{\text{dn}} = -\frac{d^2}{dx^2} + \omega - \phi^2_\omega,$$

and the stability framework established in [4,5] and [24], we show that the orbit $O_{\phi_\omega} = \{ e^{i\theta} \phi_\omega(\cdot + y) : (y, \theta) \in \mathbb{R} \times [0, 2\pi] \}$ generated by the dnoidal waves $\phi_\omega$ will be nonlinearly stable in $H^4_{\text{per}}([0, L])$ with regard to the periodic flow of the NLS equation (see Theorem 4.2 below). Also, via the theory of instability given by Grillakis in [14], we show that the orbit $O_{\phi_\omega}$ will be nonlinearly unstable by perturbations of periodic functions of period $2L$ (see Theorem 4.3 below).

Now, by considering that $F_{\phi_\omega}$ has an imaginary pure root and two real roots, we find other two smooth curves of solutions $\phi_{\omega,i}$, $i = 1, 2$, of (1.5) of the form $\omega \in (0, +\infty) \to \phi_{\omega,1} \in H^\infty_{\text{per}}([0, L])$ and $\omega \in (-\frac{4\pi^2}{L^2}, 0) \to \phi_{\omega,2} \in H^\infty_{\text{per}}([0, L])$, which depend of the Jacobian elliptic $\text{cn}$ function in the form

$$\phi_{\omega,i}(\xi) = b_i \text{cn}(a_i \xi; k_i),$$

(1.7)

where $a_i, b_i, k_i$ are depending of $\omega$ (see Theorem 2.3 below). Unfortunately, we do not know yet if the generated orbit by these solutions of type cnoidal wave are stable by the flow of the
NLS equation. In Section 4, we give some arguments which show that the theories established by Grillakis in [14] and Grillakis, Shatah and Strauss in [16] do not give a conclusive answer of stability or instability for this case.

Other classes of periodic solutions of the NLS can be obtained as a special case of the solutions found by Bronski et al. in [6], to the following attractive cubic nonlinear Schrödinger equation with an external periodic potential depending on the Jacobian elliptic sn function,

\[ iu_t + \frac{1}{2}u_{xx} + |u|^2u + V_0 \text{sn}^2(x; k)u = 0. \tag{1.8} \]

It is known that this equation models a quasi-one-dimensional attractive dilute-gas Bose–Einstein condensate, where \( u \) represents the macroscopic wave function of the condensate. In [6] two explicit types of solutions of (1.8) of the form

\[ u(x, t) = r(x)e^{-iwt} \tag{1.9} \]

are considered. The initial ansatz \( r_1^2(x) = A \text{sn}^2(x; k) + B \) gives solutions in the form \( r_{1,1}(x) = \sqrt{-V_0 + k^2} \text{sn}(x; k) \) with \( \omega = (1 + k^2)/2 \), \( r_{1,2}(x) = \sqrt{V_0 + k^2} \text{cn}(x; k) \) with \( \omega = \frac{1}{2} - V_0 - k^2 \) and \( r_{1,3}(x) = \sqrt{V_0 + k^2} \text{dn}(x; k) \) with \( \omega = -1 - \frac{V_0}{k^2} + \frac{k^2}{2} \). The other type of ansatz considered in [6], \( r_2^2(x) = a_1 \text{sn}(x; k) + b_1 \) or \( r_3^2(x) = a_2 \text{dn}(x; k) + b_2 \) do not produce in the case \( V_0 = 0 \) nontrivial periodic solutions. We note that the instability of the solutions (1.9) with \( r = r_{1,3} \), is also obtained for \( V_0 \geq -k^2 \). So, from our stability theory in Section 4, the solutions \( r_{1,3} \) with \( V_0 = 0 \) are essentially different from our solutions (1.6). In the end of Section 2 we give other argument that shows the difference between the solutions \( r_{1,3} \) (\( V_0 = 0 \)) and (1.6). We note that in Bronski et al. [7] it is considered the case of the self-focusing Schrödinger equation with an elliptic function potential

\[ iu_t + \frac{1}{2}u_{xx} - |u|^2u + V_0 \text{sn}^2(x; k)u = 0, \]

and it is shown that this equation has a branch of solutions of the form (1.9) with

\[ r(x) = \sqrt{-\left(1 + \frac{V_0}{k^2}\right)} \text{dn}(x, k) \quad \text{and} \quad \omega = \frac{k^2}{2} - \frac{V_0}{k^2} - 1, \]

which are linearly stable.

With regarding to the mKdV equation (1.2), substitution of (1.3b) into (1.2) implies that the profile \( \psi_\omega \) satisfies the ordinary differential equation

\[ \psi_\omega'' + \psi_\omega^3 - \omega \psi_\omega = A_{\psi_\omega}, \tag{1.10} \]

where \( A_{\psi_\omega} \) is an integration constant which we assume to be zero in our theory. So, (1.10) takes the form of (1.5) and hence a parallel theory those for the NLS equation of existence and stability of dnoidal waves solutions can be established in the case of the mKdV equation (see Theorem 5.2 below). In this case, we do not know yet if the orbit generated by the dnoidal wave solutions (1.6), \( O_{\phi_\omega} = \{\phi_\omega(\cdot + y) : y \in \mathbb{R}\} \), is unstable by the flow of the mKdV equation by perturbations of periodic functions of period \( 2L \).
This paper is organized as follows. Section 2 is devoted to the existence of smooth curves of periodic traveling waves solutions associated to the NLS and mKdV equations. Section 3 is concerned with a short review of the Floquet theory and a detailed study of the spectrum of certain self-adjoint operators associated to these periodic solutions. Then, in Sections 4 and 5, the stability theory of the dnoidal waves solutions is established for the NLS and mKdV equations. In Appendix A, we briefly review the basic definitions and properties of the Jacobian elliptic functions.

**Notation**

The set of all real numbers is denoted by \( \mathbb{R} \). The norm of a function \( f \in L^p(\Omega) \) (equivalence class), for \( \Omega \) an open subset of \( \mathbb{R} \), is written \(|f|_{L^p(\Omega)}^p = \int_\Omega |f|^p \, dx\), \( p \geq 1 \). The inner product in \( L^2(\Omega) \) of two functions \( f, g \) is written as \((f, g) = \int_\Omega f \overline{g} \, dx\). To define the Sobolev spaces (of \( L^2 \) type) of periodic functions, we use the definitions and notions in Iorio and Iorio [18]. Let \( \mathcal{P} = C_0^\infty([−\ell, \ell]) = \{ f : \mathbb{R} \to \mathbb{C} \mid f \in C_0^\infty(\mathbb{R}) \text{ and periodic with period } 2\ell > 0 \} \). \( \mathcal{P}' \) (the topological dual of \( \mathcal{P} \)) is the collection of all continuous linear functionals from \( \mathcal{P} \) into \( \mathbb{C} \). \( \mathcal{P}' \) is called the set of all periodic distributions. If \( \Psi \in \mathcal{P}' \) we denote the value of \( \Psi \) in \( \phi \) by \( \langle \Psi, \phi \rangle = \langle \Psi, \phi \rangle \).

Define the functions \( \Theta_k(x) = \exp(ik\pi x/\ell) \), \( k \in \mathbb{Z} \), \( x \in \mathbb{R} \). The Fourier transform of \( \Psi \in \mathcal{P}' \) is the function \( \hat{\Psi} : \mathbb{Z} \to \mathbb{C} \) defined by the formula

\[
\hat{\Psi}(k) = \frac{1}{2\ell} \langle \Psi, \Theta_{−k} \rangle, \quad k \in \mathbb{Z}.
\]

So, if \( \Psi \) is a periodic function with period \( 2\ell \), for example \( \Psi \in L^2(−\ell, \ell) \), we have \( \hat{\Psi}(k) = \frac{1}{2\ell} \int_{−\ell}^\ell \Psi(x) \exp(−ik\pi x/\ell) \, dx \). Let \( s \in \mathbb{R} \). The Sobolev space \( H^s_{\text{per}}([−\ell, \ell]) \) is the set of all \( f \in \mathcal{P}' \) such that

\[
\| f \|_{s}^2 = 2\ell \sum_{k=−\infty}^{\infty} (1 + |k|^2)^s |\hat{f}(k)|^2 < \infty.
\]

\( H^s_{\text{per}}([−\ell, \ell]) \) is a Hilbert space with respect to the inner product

\[
(f|g)_s = 2\ell \sum_{k=−\infty}^{\infty} (1 + |k|^2)^s \hat{f}(k) \overline{\hat{g}(k)}.
\]

In the case \( s = 0 \), we obtain a Hilbert space that is isometrically isomorphic to a subspace of \( L^2(−\ell, \ell) \), so we have \( (f|g)_0 = (f, g) = \int_{−\ell}^{\ell} f \overline{g} \, dx \). \( H^0_{\text{per}}([−\ell, \ell]) \) will be denoted by \( L^2_{\text{per}}([−\ell, \ell]) \) with \( \| \cdot \|_0 = \| \cdot \| \). Since \( H^s_{\text{per}}([−\ell, \ell]) \subset L^2_{\text{per}}([−\ell, \ell]) \) for every \( s \geq 0 \), we have via the Parseval (or Plancherel’s) identity that for every \( n \in \mathbb{N} \)

\[
\| f \|_{n}^2 = \sum_{j=0}^{n} \| f^{(j)} \|_{2}^2 = \sum_{j=0}^{n} \int_{−\ell}^{\ell} \left| f^{(j)}(x) \right|^2 \, dx,
\]
where \( f^{(j)} \) represents the \( j \)th derivative of \( f \) taken in the sense of \( P' \). Moreover, \((H^s_{\text{per}}([-\ell, \ell]))'\), the topological dual of \( H^s_{\text{per}}([-\ell, \ell]) \), is isometrically isomorphic to \( H^{-s}_{\text{per}}([-\ell, \ell]) \) for all \( s \in \mathbb{R} \). The duality is implemented by the pairing

\[
(f, g)_s = 2l \sum_{k=-\infty}^{\infty} \hat{f}(k) \overline{\hat{g}(k)}, \quad f \in H^{-s}_{\text{per}}([-\ell, \ell]), \quad g \in H^s_{\text{per}}([-\ell, \ell]).
\]

So, if \( f, g \in L^2_{\text{per}}([-\ell, \ell]) \) it follows that \((f, g)_s = (f, g)\).

2. Existence of smooth curves of periodic traveling waves solutions for the NLS

This section is devoted essentially to establish the existence of some smooth curves of periodic traveling wave solutions to the Schrödinger equation (1.1) of the form

\[
u(x, t) = e^{i\omega t} \phi_\omega(x),
\]

where \( \omega \in \mathbb{R} \) and \( \phi_\omega : \mathbb{R} \rightarrow \mathbb{R} \) is a smooth periodic function. Initially, we shall show that for every fixed period \( L > 0 \) there is a smooth mapping, \( \omega \in (\frac{2\pi}{L^2}, \infty) \rightarrow \phi_\omega \in H^\infty_{\text{per}}([0, L]), \) such that \( \phi_\omega \) satisfies

\[
\phi''_\omega + \phi^3_\omega - \omega \phi_\omega = 0.
\]

First of all, we show that Eq. (2.1) has an explicit periodic solution which will depend of Jacobian elliptic functions. In fact from (2.1), \( \phi_\omega \) must satisfy the first-order equation (in quadrature form)

\[
\left[ \phi'_\omega \right]^2 = \frac{1}{2} \left[ -\phi^4_\omega + 2\omega \phi^2_\omega + 4B_{\phi_\omega} \right] = \frac{1}{2} (\eta_1^2 - \phi^2_\omega)(\phi^2_\omega - \eta_2^2),
\]

where \( B_{\phi_\omega} \) is an integration constant and \( -\eta_1, \eta_1, -\eta_2, \eta_2 \) are the zeros of the polynomial \( F_{\phi_\omega}(t) = -t^4 + 2\omega t^2 + 4B_{\phi_\omega} \). We suppose without losing generality that \( \eta_1 > \eta_2 > 0 \). Hence, we need to have that \( \eta_2 \leq \phi_\omega \leq \eta_1 \) and the \( \eta_i 's \) satisfy

\[
\begin{cases}
2\omega = \eta_1^2 + \eta_2^2, \\
4B_{\phi_\omega} = -\eta_1^2 \eta_2^2.
\end{cases}
\]

Define, \( \varphi = \phi_\omega / \eta_1 \) and \( k^2 = (\eta_1^2 - \eta_2^2) / \eta_1^2 \) then (2.2) becomes

\[
(\varphi')^2 = \frac{\eta_1^2}{2} (1 - \varphi^2)(\varphi^2 - 1 + k^2).
\]

Now, define a further variable \( \psi \) via the relation \( \varphi^2 = 1 - k^2 \sin^2 \psi \), so we get

\[
(\psi')^2 = \frac{\eta_1^2}{2} (1 - k^2 \sin^2 \psi).
\]
Then we obtain for \( l = \frac{\eta_1}{\sqrt{2}} \), that
\[
\int_{0}^{\psi(\xi)} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} = l \xi.
\]

It follows from the definition of the Jacobian elliptic function \( y = \text{sn}(u; k) \) (see Appendix A), that \( \sin \psi = \text{sn}(l \xi; k) \) and hence
\[
\varphi(\xi) = \sqrt{1 - k^2 \text{sn}^2(l \xi; k)} = \text{dn}(l \xi; k).
\]

Then by returning to the initial variable, we obtain the so-called dnoidal wave solutions
\[
\phi_\omega(\xi) \equiv \phi_\omega(\xi; \eta_1, \eta_2) = \eta_1 \text{dn} \left( \frac{\eta_1}{\sqrt{2}} \xi; k \right)
\]
(2.4)

with
\[
k^2 = \frac{\eta_1^2 - \eta_2^2}{\eta_1^2}, \quad \eta_1^2 + \eta_2^2 = 2 \omega, \quad 0 < \eta_2 < \eta_1.
\]
(2.5)

Now, since \( \text{dn} \) has fundamental period \( 2K \), \( \text{dn}(u + 2K; k) = \text{dn}(u; k) \), where \( K = K(k) \) represents the complete elliptic integral of first kind, it follows that the dnoidal wave solution \( \phi_\omega \) in (2.4) has fundamental period (wavelength), \( T_{\phi_\omega} \), given by
\[
T_{\phi_\omega} = \frac{2\sqrt{2}}{\eta_1} K(k).
\]
(2.6)

The expression for the dimensionless wavelength (2.6) can be manipulated into an easy form using (2.5), and so we can obtain basic information about the behavior of the fundamental period \( T_{\phi_\omega} \). In fact, from (2.5) we get that given a fixed \( \omega > 0 \), then \( 0 < \eta_2 < \sqrt{\omega} < \eta_1 < \sqrt{2\omega} \) and we can see (2.6) as a function of a unique variable \( \eta_2 \), namely,
\[
T_{\phi_\omega}(\eta_2) = \frac{2\sqrt{2}}{\sqrt{2\omega - \eta_2^2}} K(k(\eta_2)) \quad \text{with} \quad k^2(\eta_2) = \frac{2\omega - 2\eta_2^2}{2\omega - \eta_2^2}.
\]
(2.7)

We are now in position to examine in more details the wave form (2.4) and the wavelength (2.6). Initially, we will see that
\[
T_{\phi_\omega} > \frac{\sqrt{2}}{\sqrt{\omega}} \pi.
\]
(2.8)

If \( \eta_2 \to 0 \) then \( k(\eta_2) \to 1^- \), and so \( K(k(\eta_2)) \to +\infty \). Therefore, \( T_{\phi_\omega}(\eta_2) \to +\infty \) as \( \eta_2 \to 0 \). Now, if \( \eta_2 \to \sqrt{\omega} \) then \( k(\eta_2) \to 0 \), and so \( K(k(\eta_2)) \to \pi/2 \). Therefore, \( T_{\phi_\omega}(\eta_2) \to \pi \sqrt{2}/\sqrt{\omega} \) as \( \eta_2 \to \sqrt{\omega} \). Finally, since \( \eta_2 \to T_{\phi_\omega}(\eta_2) \) is a strictly decreasing function (see proof of Theorem 2.1) it follows (2.8).
Equation (2.4) represents, in a general form, solutions of (2.1) all of which are positive. Formula (2.4) contains, at least formally, two basic solutions of the NLS equation which are obtained as approximation of periodic solutions. In fact, by considering $\eta_2$ tends to zero, $\eta_1^2 \to 2\omega^-$ and $k \to 1^-$. The Jacobian elliptic function $dn$ and their periods also simplify in this limit, with $dn(u; 1^-) \sim \text{sech } u$ and $K(k) \to +\infty$ as $k \to 1^-$. The dnoidal wave loses its periodicity in this limit and we obtain a wave form with a single hump and with “infinity period” of the form

$$
\phi_\omega(\xi; \sqrt{2\omega}, 0) = \sqrt{2\omega} \text{sech}(\sqrt{\omega} \xi),
$$

which is exactly the classical ground state solutions for the NLS equation with speed $\omega$. Next, if we consider the limiting case $\eta_2 \to \eta_1$, we obtain that $\eta_2 \sim \omega^-$, $k^2$ is small and from (2.5), that $dn(u; 0^+) \sim 1$. At this limit we obtain the constant wave form

$$
\phi_\omega(\xi; \sqrt{\omega}, \sqrt{\omega}) = \sqrt{\omega}.
$$

We note that this constant wave form is the expected constant solution of Eq. (2.1).

Next we will construct, for a fixed period $L$, a smooth curve of dnoidal wave solutions for Eq. (2.1). We start by showing the existence of a family of dnoidal waves with a fixed period. In fact, it considers $L>0$ arbitrary but fixed, and $\omega>0$ such that $\sqrt{\omega} > \frac{\pi}{\sqrt{2}L}$. By the analysis made above we know that the mapping $\eta_2 \in (0, \sqrt{\omega}) \to T_{\phi_\omega}(\eta_2)$ is strictly decreasing, so by (2.8) there is a unique $\eta_2 \equiv \eta_2(\omega) \in (0, \sqrt{\omega})$ such that the fundamental period of the dnoidal wave, $\phi_\omega(\cdot; \eta_1(\omega), \eta_2(\omega))$, will be $T_{\phi_\omega}(\eta_2(\omega)) = L$. Note that in this case the modulus $k$ can be seen as a function of $\omega$, since by (2.3) we have

$$
k^2(\omega) = \frac{2\omega - 2\eta_2^2}{2\omega - \eta_2^2} \quad (2.9)
$$

with $\eta_2 = \eta_2(\omega)$. Now, we show that at least locally this choice of $\eta_2(\omega)$ depends of a smooth form of $\omega$.

**Theorem 2.1.** Let $L>0$ arbitrary but fixed. Consider $\omega_0 > \frac{2\pi^2}{L^2}$ and the unique $\eta_{2,0} = \eta_2(\omega_0) \in (0, \sqrt{\omega_0})$ such that $T_{\phi_{\omega_0}} = L$. Then,

1. there exist an interval $I(\omega_0)$ around $\omega_0$, an interval $B(\eta_{2,0})$ around $\eta_2(\omega_0)$, and a unique smooth function $\Lambda : I(\omega_0) \to B(\eta_{2,0})$ such that $\Lambda(\omega_0) = \eta_{2,0}$ and

$$
\frac{2\sqrt{2}}{2\omega - \eta_2^2} K(k) = L, \quad (2.10)
$$

where $\omega \in I(\omega_0)$, $\eta_2 = \Lambda(\omega)$, and $k^2 \equiv k^2(\omega) \in (0, 1)$ is defined by (2.9);

2. the dnoidal wave solution in (2.4), $\phi_\omega(\cdot; \eta_1, \eta_2)$, determined by $\eta_1 \equiv \eta_1(\omega)$, $\eta_2 \equiv \eta_2(\omega)$, has fundamental period $L$ and satisfies (2.1). Moreover, the mapping

$$
\omega \in I(\omega_0) \to \phi_\omega \in H^1_{\text{per}}([0, L]) \quad (2.11)
$$

is a smooth function;

3. $I(\omega_0)$ can be chosen as $(\frac{2\pi^2}{L^2}, +\infty)$.
Proof. The idea of the proof is to apply the implicit function theorem. In fact, it considers the open set \( \Omega = \{(\eta, \omega) : \omega > \frac{2\pi^2}{L^2}, \eta \in (0, \sqrt{\omega}) \} \subseteq \mathbb{R}^2 \) and define \( \Psi : \Omega \to \mathbb{R} \) by

\[
\Psi(\eta, \omega) = \frac{2\sqrt{2}}{\sqrt{2\omega - \eta^2}} K(k(\eta, \omega)),
\]

(2.12)

where \( k^2(\eta, \omega) = \frac{2\omega - 2\eta^2}{2\omega - \eta^2} \). By hypotheses \( \Psi(\eta_2, 0, \omega_0) = L \). Next, we affirm that \( \partial_\eta \Psi(\eta, \omega) < 0 \). In fact, it is immediate that

\[
\partial_\eta \Psi(\eta, \omega) = \frac{2\sqrt{2}{\eta}}{(2\omega - \eta^2)^{3/2}} K(k) + \frac{2\sqrt{2}}{\sqrt{2\omega - \eta^2}} \frac{dK}{dk} \frac{\partial k}{\partial \eta}.
\]

Now, from (2.7) it follows that \( k(\eta, \omega) \) is a strictly decreasing function of \( \eta \)

\[
\frac{\partial k}{\partial \eta} = -\frac{2\eta \omega}{k(2\omega - \eta^2)^2}.
\]

Now, by using the relation

\[
\frac{dK(x)}{dx} = \frac{E(x) - (1 - x^2)K(x)}{x(1 - x^2)}, \quad \text{for any } x \in (0, 1),
\]

where \( E \) is the complete elliptic integral of second kind we have the following equivalences

\[
\partial_\eta \Psi(\eta, \omega) < 0 \quad \Leftrightarrow \quad k(2\omega - \eta^2)K(k) < 2\omega \frac{dK}{dk} \quad \Leftrightarrow \quad (1 - k^2)(4\omega - 2\eta)K(k) < 2\omega E(k)
\]

\[
\Leftrightarrow \quad \eta^2 K(k) < \omega E(k) \quad \Leftrightarrow \quad 2(1 - k^2)K(k) < (2 - k^2)E(k).
\]

(2.12a)

Next, by defining \( \beta^2 = 1 - k^2 \) we have that \( \beta \) is an increasing function of \( \eta \in (0, \sqrt{\omega}) \) with \( \beta \in (0, 1) \), \( \beta(0) = 0 \) and \( \beta(1) = 1 \). Therefore, from (2.12a) it follows

\[
\partial_\eta \Psi(\eta, \omega) < 0 \quad \Leftrightarrow \quad f(\beta) \equiv (1 + \beta^2)E(\sqrt{1 - \beta^2}) - 2\beta^2 K(\sqrt{1 - \beta^2}) > 0.
\]

Since \( f(1) = 0 \) is sufficient to show that \( f \) is a decreasing (strictly) function. In fact, using that

\[
x \frac{dE(x)}{dx} = E(x) - K(x)
\]

we have

\[
f'(\beta) < 0 \quad \Leftrightarrow \quad (1 - \beta^2)E(\sqrt{1 - \beta^2}) < (1 + \beta^2)K(\sqrt{1 - \beta^2}),
\]

but since \( E(x) < K(x) \) it follows that \( f'(\beta) < 0 \) and so we obtain our affirmation.

Therefore, by the implicit function theorem there is a unique smooth function, \( \Lambda \), defined in a neighborhood \( I(\omega_0) \) of \( \omega_0 \), such that \( \Psi(\Lambda(\omega), \omega) = L \) for every \( \omega \in I(\omega_0) \). So, we obtain (2.10).

Since \( \omega_0 \) was chosen arbitrarily in the interval \( \mathcal{I} = \left(\frac{2\pi^2}{L^2}, +\infty\right) \), it follows from the uniqueness of the function \( \Lambda \), that we can extend its domain of definition to all the interval \( \mathcal{I} \). This completes the proof. \( \square \)
**Corollary 2.2.** Consider the mapping \( \Lambda : I(\omega_0) \to B(\eta_2,0) \) determined by Theorem 2.1. Then, \( \Lambda \) is a strictly decreasing function in \( I(\omega_0) \).

**Proof.** By the proof of Theorem 2.1, we know that \( \Psi(\Lambda(\omega), \omega) = L \) for every \( \omega \in I(\omega_0) \). So, it follows that

\[
\frac{d}{d\omega} \Lambda(\omega) = - \frac{\partial \Psi}{\partial \omega} \frac{\partial \Psi}{\partial \eta}.
\]

(2.13)

Hence, we only need to show that \( d\Psi/d\omega < 0 \). In fact, by using the relation \( \eta^2 = (2\omega - \eta^2)(1 - k^2) \equiv (2\omega - \eta_2^2)k'^2 \), we obtain the following equivalences

\[
\frac{d\Psi}{d\omega} < 0 \iff (2\omega - \eta^2)K > \frac{\eta^2 dK}{k} \iff K > \frac{k'^2 dK}{k}. \]

(2.14)

Then, from relation \( dK/dk = (E - k'^2 K)/(kk') \) and (2.14) it follows that

\[
\frac{d\Psi}{d\omega} < 0 \iff k^2 K > E - k'^2 K \iff (k^2 + k'^2)K > E \iff K > E.
\]

This completes the corollary. \( \square \)

Next we shall show the existence of other periodic solutions associated to Eq. (2.1). It is immediate from (2.1) that wherever \( \phi_\omega \) is a solution then \(-\phi_\omega\) is so. We do not know when the dnoidal wave solution (2.4) is the unique positive solution, modulo translation, of (2.1). We note that by obtaining the dnoidal wave solutions we have assumed in (2.2) that the polynomial \( F_{\phi_\omega}(t) = -t^4 + 2\omega t^2 + 4B_{\phi_\omega} \) has the real zeros \(-\eta_1, \eta_1, -\eta_2, \eta_2\) (note that \( F_{\phi_\omega} \) is even). Now, we will consider the case when we have a pure imaginary root and the other two roots are real. So, we must have from (2.1) and (2.2) that

\[
[\phi'_\omega]^2 = \frac{1}{2}(a^2 + \phi_\omega^2)(b^2 - \phi_\omega^2).
\]

(2.15)

By supposing \( b > 0 \), we need to have that \(-b \leq \phi_\omega \leq b\) and

\[
\begin{cases}
2\omega = b^2 - a^2, \\
4B_{\phi_\omega} = a^2b^2 > 0.
\end{cases}
\]

(2.16)

Define \( \chi = \phi_\omega / b \) (suppose \( \chi(0) = 0 \)) and \( k^2 = b^2/(a^2 + b^2) \), then (2.15) becomes

\[
(\chi')^2 = \frac{b^2}{2} \left(1 - \chi^2\right) \left(\frac{a^2}{b^2} + \chi^2\right).
\]

Now, defining \( \psi \) as \( \chi^2 = 1 - \sin^2 \psi \), we get that

\[
(\psi')^2 = \frac{a^2 + b^2}{2} \left(1 - k^2 \sin^2 \psi\right).
\]
Then for $\beta = \sqrt{\frac{a^2 + b^2}{2}}$, we obtain from via integration and the definition of the Jacobian elliptic function $\text{sn}$, that $\sin \psi(\xi) = \text{sn}(\beta \xi; k)$. Therefore, $\chi^2 = 1 - \text{sn}^2(\beta \xi; k) = \text{cn}^2(\beta \xi; k)$. So, we obtain that

$$\phi_{\omega}(\xi) = b \, \text{cn}(\beta \xi; k)$$  \hspace{2cm} (2.17)

is a solution of (2.1) with

$$k^2 = \frac{b^2}{a^2 + b^2} \quad \text{and} \quad \beta = \sqrt{\frac{a^2 + b^2}{2}}.$$  \hspace{2cm} (2.18)

Since $\text{cn}(\cdot; k)$ has fundamental period $4K(k)$, it follows that the cnoidal wave solution (2.17) has fundamental period (wavelength), $T_{\phi_{\omega}}$, given by

$$T_{\phi_{\omega}} = \frac{4\sqrt{2}K(k)}{\sqrt{b^2 - 2\omega}}.$$  \hspace{2cm} (2.19)

We note that (2.17) represents a nonpositive periodic solution of (2.1). Moreover from (2.16), $\omega$ can be positive or negative.

Now, our interest will be show that there is a smooth curve of cnoidal waves for (1.1) depending of the parameter $\omega$. Initially, if $\omega > 0$ then we have from (2.16), (2.18) and $0 < k^2 < 1$, that $b^2 > 2\omega$. So, it follows that for $k^2 = b^2/(2b^2 - 2\omega)$ we have $k^2 \in \left(0, \frac{1}{2}\right)$ and

$$T_{\phi_{\omega}}(b) = \frac{4\sqrt{2}K(k)}{\sqrt{2b^2 - 2\omega}} \rightarrow \begin{cases} +\infty & \text{if } b \rightarrow \sqrt{2\omega}, \\ 0 & \text{if } b \rightarrow +\infty. \end{cases}$$

So, since $b \in (\sqrt{2\omega}, +\infty) \rightarrow T_{\phi_{\omega}}(b)$ is a strictly decreasing function, we have that for $\omega > 0$ fixed and given $L > 0$ there is a unique $b = b(\omega)$ such that $T_{\phi_{\omega}}(b) = L$. Similarly, we have that if $\omega < 0$ then $a^2 > -2\omega$ and so for $k^2 = (a^2 + 2\omega)/(2a^2 + 2\omega)$ we have $k^2 \in (0, \frac{1}{2})$ and

$$T_{\phi_{\omega}}(a) = \frac{4}{\sqrt{a^2 + \omega}}K(k) \rightarrow \begin{cases} \frac{2\pi}{\sqrt{2\omega}} & \text{if } a \rightarrow \sqrt{2\omega}, \\ 0 & \text{if } a \rightarrow +\infty. \end{cases}$$

Therefore, for $\omega < 0$ fixed and $0 < L < \frac{2\pi}{\sqrt{2\omega}}$ there is a unique $a = a(\omega)$ such that $T_{\phi_{\omega}}(a) = L$. Hence, we have the following theorem of existence.

**Theorem 2.3.** Let $L > 0$ arbitrary but fixed. Then,

1. for every $\omega > 0$ there is a unique $b = b(\omega)$ such that $b \in (\sqrt{2\omega}, +\infty)$, $\omega \in (0, +\infty) \rightarrow b(\omega)$ is a strictly increasing smooth function and $\frac{4K(k)}{\sqrt{b^2 - \omega}} = L$. Here the modulo $k = k(\omega)$ satisfies that $k^2 = b^2/(2b^2 - 2\omega)$ and $k'(\omega) > 0$. Moreover, the cnoidal wave

$$\phi_{\omega,1}(\xi) = b \, \text{cn}(\sqrt{b^2 - \omega \xi}; k)$$  \hspace{2cm} (2.20)
is a solution of (2.1) with fundamental period \( L \), and the mapping

\[
\omega \in (0, +\infty) \rightarrow \phi_{\omega, 1} \in H^1_{\text{per}}([0, L])
\]

is a smooth function.

(2) For every \( \omega \in (-\frac{4\pi^2}{L^2}, 0) \) there is a unique \( a = a(\omega) \) such that \( a \in (\sqrt{-2\omega}, +\infty) \), \( \omega \in (-\frac{4\pi^2}{L^2}, 0) \rightarrow a(\omega) \) is a strictly decreasing smooth function and \( \frac{4K(k)}{\sqrt{a^2 + \omega}} = L \). Here the modulo \( k = k(\omega) \) satisfies that \( k^2 = \frac{(a^2 + 2\omega)}{2a^2 + 2\omega} \) and \( k'(\omega) > 0 \). Moreover, the cnoidal wave

\[
\phi_{\omega, 2}(\xi) = \sqrt{a^2 + 2\omega} \cn(\sqrt{a^2 + \omega} \xi; k)
\]

is a solution of (2.1) with fundamental period \( L \), and the mapping

\[
\omega \in \left(-\frac{4\pi^2}{L^2}, 0\right) \rightarrow \phi_{\omega, 2} \in H^1_{\text{per}}([0, L])
\]

is a smooth function.

**Proof.** [Sketch of the proof for item (1).] It considers \( \Omega_0 = \{(b, \omega): \omega > 0, b \in (\sqrt{2\omega}, +\infty)\} \) and define \( \Phi : \Omega_0 \rightarrow \mathbb{R} \) by

\[
\Phi(b, \omega) = \frac{4\sqrt{2}}{\sqrt{2b^2 - 2\omega}} K(k(b, \omega)),
\]

where \( k^2(b, \omega) = \frac{b^2}{2b^2 - 2\omega} \). Since \( \partial_b k = -2b\omega/[k(2b^2 - 2\omega)^2] < 0 \) we obtain immediate that

\[
\partial_b \Phi(b, \omega) = -\frac{8\sqrt{2} b}{(2b^2 - 2\omega)^{3/2}} K(k) + \frac{4\sqrt{2}}{\sqrt{2b^2 - 2\omega}} \frac{dK}{dk} \frac{\partial k}{\partial b} < 0.
\]

So, from the implicit function theorem we can obtain a smooth function \( b = b(\omega) \) such that \( \Phi(b(\omega), \omega) = L \) for \( \omega > 0 \). Next, since \( \partial_\omega k = b^2/[k(2b^2 - 2\omega)^2] > 0 \) it follows

\[
\partial_\omega \Phi(b, \omega) = \frac{4\sqrt{2}}{2b^2 - 2\omega} K(k) + \frac{4\sqrt{2}}{\sqrt{2b^2 - 2\omega}} \frac{dK}{dk} \frac{\partial k}{\partial \omega} > 0,
\]

and so \( \frac{d}{d\omega} b(\omega) = -\partial_\omega \Phi/\partial_b \Phi > 0 \). Now we see that \( k'(\omega) > 0 \). Indeed, from \( 2kk' = 2b(2b - 2\omega b'(\omega))/(2b^2 - 2\omega)^2 \) it is sufficient to show that \( b - 2\omega b'(\omega) > 0 \), so

\[
b - 2\omega b'(\omega) > 0 \iff b\partial_b \Phi + 2\omega \partial_\omega \Phi < 0 \iff \frac{8\sqrt{2}(\omega - b^2)}{(2b^2 - 2\omega)^{3/2}} K < 0.
\]

This completes the proof. A similar argument shows item (2). \( \square \)
We finish this section by giving an argument that shows that our dnoidal wave solutions found in Theorem 2.1 are essentially different from that for (1.8) established in [6] with \( V_0 = 0 \). Indeed, considering our basic model (1.1) we can find solutions of it in the form \( u(x,t) = r_\omega(x) e^{i\omega t} \) with

\[
r_\omega(x) = \text{dn}\left(\frac{x}{\sqrt{2}}; k_\omega\right) \quad \text{and} \quad \omega = 1 - \frac{k_\omega^2}{2}.
\]

So, \( k_\omega(\omega) = 2(1 - \omega) \) with \( \omega \in J = (\frac{1}{2}, 1) \) and \( T_\omega = 2\sqrt{2}K(k_\omega) \in (\pi\sqrt{2}, +\infty) \). Therefore, for every fixed period \( L > \pi\sqrt{2} \) there is a unique \( \omega \in J \) such that \( T_\omega = L \). Now, from Corollary 2.2 it follows that the modulus function \( k \) in (2.7) associated to our dnoidal waves solutions (2.4) is a strictly increasing function of \( \omega \) and so \( k_\omega \) and \( k \) will intersect at most in one point. Since the Jacobian elliptic functions depend of the modulus we obtain that \( \phi_\omega \neq r_\omega \) for every \( \omega \in J \) except possibly in a point.

3. Spectral analysis

In this section we study the spectral properties associated to the linear operators \( L_{dn} = -\frac{d^2}{dx^2} + \omega - 3\phi_\omega^2 \) and \( L_{dn}^+ = -\frac{d^2}{dx^2} + \omega - \phi_\omega^2 \), where \( \phi_\omega \) is the dnoidal wave solution (2.4) given by Theorem 2.1 with fundamental period \( L \) and \( \omega \in \left(\frac{2\pi^2}{L^2}, +\infty\right) \). This information will be basic in our nonlinear stability theory for the NLS equation (1.1) and the mKdV equation (1.2). Also, an analysis of the self-adjoint operators \( L_i = -\frac{d^2}{dx^2} + \omega - 3\phi_{\omega,i}^2 \), \( i = 1, 2 \), with \( \phi_{\omega,i} \) given by Theorem 2.3 is done. The principal interest of this last analysis is to show that a result of stability or instability of these periodic traveling waves cannot be obtained into the framework of the Grillakis and Grillakis, Shatah and Strauss’s theories [14–16].

Initially, our analysis will be on the periodic eigenvalue problem considered on \([0, L]\)

\[
\begin{align*}
L_{dn}\chi &= \lambda\chi, \\
\chi(0) &= \chi(L), \\
\chi'(0) &= \chi'(L).
\end{align*}
\]

(3.1)

We shall show that problem (3.1) determines exactly the existence of a single negative eigenvalue \( \lambda = \lambda_0 \) which is simple, that \( \lambda = 0 \) is also an eigenvalue simple with eigenfunction \( \phi_\omega \), and the remainder of the spectrum is a discrete set of eigenvalues which are double and converging to infinity. In this point we will use the Floquet theory (see Ince [17], Eastham [12], Erdélyi [13], Magnus and Winkler [20]). In fact, initially since

\[
L_{dn} = \left(-\frac{d^2}{dx^2} + \omega\right) + (-3\phi_\omega^2) \equiv L + M
\]

with \( L \) having a discrete spectrum and \( M \) being relatively compact with regard to \( L \), it follows from the Weyl’s essential spectral theorem [21] that \( \sigma_{\text{ess}}(L_{dn}) = \sigma_{\text{ess}}(L) = \emptyset \). Moreover, from the theory of compact self-adjoint operators we have that (3.1) determines that the spectrum of \( L_{dn} \) is a countable infinity set of eigenvalues \( \{\lambda_n | n = 0, 1, 2, \ldots\} \) with

\[
\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots
\]

(3.2)

where double eigenvalue is counted twice and \( \lambda_n \to \infty \) as \( n \to \infty \). We shall denote by \( \chi_n \) the eigenfunction associated to the eigenvalue \( \lambda_n \). By the conditions \( \chi(0) = \chi(L), \chi'(0) = \chi'(L), \ldots \)
\( \chi_n \) can be extended to the whole of \((-\infty, \infty)\) as a continuously differentiable function with period \( L \). Further, the double eigenvalues (if any) are the values of \( \lambda \) which all solutions of (3.1) have period \( L \), that means the existence of two linearly independent periodic solutions of period \( L \). In this case we say that we have coexistence of solutions to problem (3.1) with the value \( \lambda \).

Now from Floquet theory, we know that a study of the periodic eigenvalue problem (3.1) is related to the study of the following semi-periodic eigenvalue problem considered on \([0, L] \)

\[
\begin{aligned}
\mathcal{L}_{dn} \xi &= \mu \xi, \\
\xi(0) &= -\xi(L), \\
\xi'(0) &= -\xi'(L),
\end{aligned}
\] (3.3)

which is also a self-adjoint problem and therefore determines a sequence of eigenvalues \( \{\mu_n | n = 0, 1, 2, 3, \ldots\} \), with

\[
\mu_0 \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \mu_4 \leq \cdots,
\] (3.4)

where double eigenvalue is counted twice and \( \mu_n \to \infty \) as \( n \to \infty \). We shall denote by \( \xi_n \) the eigenfunction associated to the eigenvalue \( \mu_n \). A function \( f \) with the property that \( f(x + L) = -f(x) \) for all \( x \) is said to be semi-periodic with semi-period \( L \). Clearly such a function has period \( 2L \). So, we have that the equation

\[
\mathcal{L}_{dn} f = \gamma f
\] (3.5)

has a solution of period \( L \) if and only if \( \gamma = \lambda_n, n = 0, 1, 2, \ldots \), as well as, it has a solution of period \( 2L \) if and only if \( \gamma = \mu_n, n = 0, 1, 2, \ldots \). If all solutions of (3.5) are bounded we say that they are stable; otherwise we say that they are unstable.

Next, from the oscillation theorem associated to the differential equation (3.5) (see [20]) the sequences in (3.2) and (3.4) are so interlaced that

\[
\lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \mu_2 \leq \mu_3 < \lambda_3 \leq \cdots
\] (3.6)

It can be seen that the solutions of (3.5) are stable in the intervals \( (\lambda_0, \mu_0), (\mu_1, \lambda_1), \ldots \). These intervals are called intervals of stability. At the endpoints of these intervals the solutions of (3.5) are, in general, unstable. This is always true for \( \gamma = \lambda_0 \) (\( \lambda_0 \) is always simple). The solutions of (3.5) are stable for \( \gamma = \lambda_{2n+1} \) or \( \gamma = \lambda_{2n+2} \) if and only if \( \lambda_{2n+1} = \lambda_{2n+2} \) (double eigenvalue), and they are stable for \( \gamma = \mu_{2n} \) or \( \gamma = \mu_{2n+1} \) if and only if \( \mu_{2n} = \mu_{2n+1} \). The intervals, \( (-\infty, \lambda_0), (\mu_0, \mu_1), (\lambda_1, \lambda_2), (\mu_2, \mu_3), \ldots \), are called intervals of instability, omitting however any interval which is absent as a result of having a double eigenvalue. The interval of instability \( (-\infty, \lambda_0) \) will always be present. We note that the absence of an instability interval means that there is a value of \( \gamma \) for which all solutions of (3.5) have either period \( L \) or semi-period \( L \), in other words, coexistence of solutions of (3.5) with period \( L \) or period \( 2L \) occurs for that value of \( \gamma \).

Before establishing our theorems, we note that the number of zeros of \( \chi_n \) and \( \xi_n \) is determined in the following form:

(i) \( \chi_0 \) has no zeros in \([0, L]\).

(ii) \( \chi_{2n+1} \) and \( \chi_{2n+2} \) have exactly \( 2n + 2 \) zeros in \([0, L]\).

(iii) \( \xi_{2n} \) and \( \xi_{2n+1} \) have exactly \( 2n + 1 \) zeros in \([0, L]\).
So, we have the following theorem.

**Theorem 3.1.** Let $\phi_\omega$ be the dnoidal wave solution given by Theorem 2.1 and $\omega \in \left(\frac{2\pi}{L^*}, +\infty\right)$. Then, the linear operator

$$L_{\text{dn}} = -\frac{d^2}{dx^2} + \omega - 3\phi_\omega^2$$

defined on $H^2_{\text{per}}([0, L])$ has exactly its three first eigenvalues simple, being the eigenvalue zero, the second one with eigenfunction $\phi_\omega'$. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues which are double.

**Proof.** From (3.6) we will show initially that $0 = \lambda_1 < \lambda_2$. In fact, since $L_{\text{dn}} \phi_\omega' = 0$ and $\phi_\omega'$ has 2 zeros in $[0, L)$ then (3.7) implies that the eigenvalue zero is either $\lambda_1$ or $\lambda_2$. Next, by using the transformation $T_\gamma x(x) \equiv x(\gamma x)$ with $\gamma = \frac{2}{\eta}$, the explicit form (2.4) to $\phi_\omega$ and the relation

$$k^2 \text{sn}^2 x + \text{dn}^2 x = 1,$$

we have for $\Phi \equiv T_\gamma x(x)$ that problem (3.1) is equivalent to the eigenvalue problem

$$\begin{aligned}
\frac{d^2}{dx^2} \Phi + \left[\rho - 6k^2 \text{sn}^2(x)\right] \Phi &= 0, \\
\Phi(0) &= \Phi(2K), \quad \Phi'(0) = \Phi'(2K),
\end{aligned}$$

(3.8)

where the relation between $\rho$ and $\lambda$ is given by

$$\rho = \frac{2}{\eta_1^2} [\lambda + 3\eta_1^2 - \omega].$$

(3.9)

We recall that the second order differential equation in (3.8) is called the Jacobian form of Lamé’s equation. Now, from Floquet theory [20, Theorem 7.8] it follows that the Lamé’s equation has exactly 3 intervals of instability: $(−\infty, \rho_0), (\mu_0', \mu_1'), (\rho_1, \rho_2)$ (where for $i \geq 0$, $\rho_i$ are the eigenvalues associated to the periodic problem and $\mu_i'$ are the eigenvalues associated to the semi-periodic problem determined by the Lamé’s equation). Therefore, it follows that the three first eigenvalues $\rho_0, \rho_1, \rho_2$ will be simple and the rest of the eigenvalues $\rho_3 \leq \rho_4 \leq \rho_5 \leq \rho_6 \leq \cdots$, satisfy that $\rho_3 = \rho_4, \rho_5 = \rho_6, \ldots$, in other words, they are double eigenvalues.

Next, we establish an explicit formula for the three first eigenvalues $\rho_0, \rho_1, \rho_2$ and its corresponding eigenfunctions $\Phi_0, \Phi_1, \Phi_2$. We start with that $\rho_1 = 4 + k^2$ satisfies $L_{\text{dn}} \Phi_1 = \rho_1 \Phi_1$ with $\Phi_1(x) = \text{sn}(x) \text{cn}(x) = \text{const} \cdot T_\gamma \phi_\omega'(x)$. Therefore, we have from (3.9) that $\lambda = 0$ is a simple eigenvalue to (3.1) with eigenfunction $\phi_\omega'$. Now, from Ince [17] we have that the functions (called Lamé polynomials)

$$\Phi_0(x) = 1 - \left(1 + k^2 - \sqrt{1 - k^2 + k^4}\right) \text{sn}^2(x),$$

$$\Phi_2(x) = 1 - \left(1 + k^2 + \sqrt{1 - k^2 + k^4}\right) \text{sn}^2(x),$$

with period $2K$, satisfy that $L_{\text{dn}} \Phi_0 = \rho_0 \Phi_0$ and $L_{\text{dn}} \Phi_1 = \rho_2 \Phi_1$, with

$$\rho_0 = 2\left[1 + k^2 - \sqrt{1 - k^2 + k^4}\right] \quad \text{and} \quad \rho_2 = 2\left[1 + k^2 + \sqrt{1 - k^2 + k^4}\right].$$
Note that $\Phi_0$ has no zeros in $[0, 2K]$ and $\Phi_2$ has exactly 2 zeros in $[0, 2K)$, then $\rho_0$ is the first eigenvalue to (3.8). Since $\rho_0 < \rho_1$ for every $k^2 \in (0, 1)$, we obtain from (3.9) that

$$2 \lambda_0 = -2 \omega - 2 \eta_1^2 \sqrt{1 - k^2} + k^4 < 0 \iff \rho_0 < \rho_1.$$ 

Therefore, $\lambda_0$ given by

$$\lambda_0 = -\omega - \frac{2 \omega}{2 - k^2} \sqrt{1 - k^2} + k^4$$

will be the first negative eigenvalue to $L_{\text{dn}}$ with eigenfunction $\chi_0(x) = \Phi_0(\frac{1}{\gamma} x)$. Now, since $\rho_1 < \rho_2$ for every $k^2 \in (0, 1)$, we obtain from (3.9) that

$$2 \lambda_2 = -2 \omega + 2 \eta_1^2 \sqrt{1 - k^2} + k^4 > 0 \iff \rho_1 < \rho_2,$$

and so $\lambda_2$ is the third eigenvalue to $L_{\text{dn}}$ with eigenfunction $\chi_2(x) = \Phi_2(\frac{1}{\gamma} x)$.

To finish the proof, we will establish the two first eigenvalues associated to the semi-periodic problem (3.3), $\mu_0, \mu_1$, which will be used later in our theory of stability. So, by using the Lamé’s equation in (3.8) with the conditions $\Phi(0) = -\Phi(2K)$ and $\Phi'(0) = -\Phi'(2K)$, we find that the eigenvalues $\mu_i'$ associated to semi-periodic problem are related to the $\mu_i$ via the relation

$$\mu_i' = \frac{2}{\eta_1^2} [\mu_i + 3 \eta_1^2 - \omega]. \quad (3.10)$$

Now, we can see that $\mu_0' = 1 + k^2$ and $\mu_1' = 1 + 4k^2$ are the two first eigenvalues to the Lamé’s equation in (3.8) in the semi-periodic case. Moreover, the eigenfunctions associated are

$$\Phi_{0,\text{sm}}(x) = cn \ x \ dn \ x \quad \text{and} \quad \Phi_{1,\text{sm}}(x) = sn \ x \ dn \ x, \quad (3.11)$$

respectively. Therefore, $\xi_0(x) = \Phi_{0,\text{sm}}(\frac{1}{\gamma} x)$ and $\xi_1(x) = \Phi_{1,\text{sm}}(\frac{1}{\gamma} x)$ are the two first eigenfunctions of (3.3) associated to the eigenvalues $\mu_0 = -3\omega/(2 - k^2)$ and $\mu_1 = 3\omega(k^2 - 1)/(2 - k^2)$. □

Now, our interest is about the periodic problem

$$\begin{cases}
L_i \chi \equiv \left[-\frac{d^2}{dx^2} + \omega - 3\Phi_{\omega,i}^2\right] \chi = \lambda \chi, \quad i = 1, 2, \\
\chi(0) = \chi(L), \quad \chi'(0) = \chi'(L),
\end{cases} \quad (3.12)$$

where $\Phi_{\omega,i}$ are given by Theorem 2.3. So, via the transformation $T_{\gamma_1} \chi(x) \equiv \chi(\gamma_1 x)$ for $\gamma_1^2 = 1/(b^2 - \omega)$, we have from (2.19)–(2.20) for $\Psi_1 \equiv T_{\gamma_1} \chi$ that (3.12) with $i = 1$, is equivalent to the eigenvalue problem

$$\begin{cases}
\frac{d^2}{dx^2} \Psi_1 + \left[\theta - 6k^2 sn^2(x)\right] \Psi_1 = 0, \\
\Psi_1(0) = \Psi_1(4K), \quad \Psi_1'(0) = \Psi_1'(4K),
\end{cases} \quad (3.13)$$
where the relation between $\theta$ and $\lambda$ is given by

$$\theta = \frac{\lambda - \omega + 3b^2}{b^2 - \omega}.$$ 

Since $\mathcal{L}_1 \phi'_{\omega,1} = 0$ and $\phi'_{\omega,1}$ has two zeros in $[0, L)$, then the eigenvalue zero is the second or third eigenvalue for (3.12). Next, we shall show that zero is the third eigenvalue and that it is simple. From (3.11) we have that $\Phi_{0,sm}, \Phi_{1,sm}$ are eigenfunctions of (3.13) associated to the simple eigenvalues $1 + k^2$ and $1 + 4k^2$ respectively, and each one of these functions has two zeros in $[0, 4K)$. So, since $\lambda = 0$ implies $\theta = 1 + 4k^2$ then we have that zero is simple. Moreover, $\theta = 1 + k^2$ implies $\lambda = -3\omega k^2/(2k^2 - 1) < 0$ with eigenfunction $\chi = T_{\gamma_1} \Phi_{0,sm}$. So we have the following theorem.

**Theorem 3.2.** Let $L > 0$ and $\phi_{\omega,i}$, $i = 1, 2$, the cnoidal wave solutions given by Theorem 2.3. Then, the linear operators

$$\mathcal{L}_i = -\frac{d^2}{dx^2} + \omega - 3 \phi_{\omega,i}^2$$

defined on $H^2_{\text{per}}([0, L])$ have exactly two negative eigenvalues which are simple. The eigenvalue zero is the third one, which is simple with eigenfunction $\phi'_{\omega,i}$. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

Next, we have the following result about $\mathcal{L}^+_{\text{dn}}$.

**Theorem 3.3.** Let $\phi_\omega$ be the dnoidal wave solution given by Theorem 2.1 and $\omega \in (\frac{2\pi^2}{L^2}, +\infty)$. Then, the linear operator $\mathcal{L}^+_{\text{dn}} = -\frac{d^2}{dx^2} + \omega - \phi_\omega^2$ defined on $H^2_{\text{per}}([0, L])$ is a nonnegative operator which has a spectrum constituted by a discrete set of eigenvalues. The eigenvalue zero is simple and the remainder of eigenvalues are double. Moreover, there is a positive value $\beta$ such that

$$\inf \{(\mathcal{L}^+_{\text{dn}} \varphi, \varphi): \varphi \in H^1_{\text{per}}([0, L]), \|\varphi\| = 1, (\varphi, \phi_\omega^3) = 0\} = \beta. \quad (3.14)$$

**Proof.** Initially from the Weyl’s theorem we get that the spectrum of $\mathcal{L}^+_{\text{dn}}$ is discrete and the eigenvalues satisfy a relation as in (3.2). From (2.1) we have that $\mathcal{L}^+_{\text{dn}} \phi_\omega = 0$. Since $\phi_\omega > 0$ in $[0, L]$ it follows from (3.7) that zero is the first eigenvalue of $\mathcal{L}^+_{\text{dn}}$, and so the operator is nonnegative with zero being a simple eigenvalue. Finally, by using Floquet theory we have that there are exactly two intervals of instability associated to the periodic eigenvalue problem generated by $\mathcal{L}^+_{\text{dn}}$, and then the remainder of the spectrum of $\mathcal{L}^+_{\text{dn}}$ is constituted by eigenvalues which are double.

Next, let $\beta = \inf \{(\mathcal{L}^+_{\text{dn}} \varphi, \varphi): \|\varphi\| = 1, (\varphi, \phi_\omega^3) = 0\}$. So, because of the above analysis we have that $\beta \geq 0$. Suppose that $\beta = 0$ and let $\{\varphi_j\}$ be a sequence of $H^1_{\text{per}}([0, L])$-functions with $\|\varphi_j\| = 1$, $(\varphi_j, \phi_\omega^3) = 0$, and $(\mathcal{L}^+_{\text{dn}} \varphi_j, \varphi_j) \to 0$ as $j \to \infty$. It follows then that $\|\varphi_j\|_1$ is uniformly bounded as $j$ varies. So, there is a subsequence of $\varphi_j$, which we denote again by $\varphi_j$, and a function $\varphi \in H^1_{\text{per}}([0, L])$ such that $\varphi_j \rightharpoonup \varphi$ weakly in $H^1_{\text{per}}([0, L])$. Now, since the embed-
Since is constituted by a discrete set of eigenvalues converging to infinity. Moreover, since weak convergence is lower semi-continuous it follows that

\[ 0 \leq (L^+_{\text{dn}} \varphi, \varphi) \leq \liminf_{j \to \infty} (L^+_{\text{dn}} \varphi_j, \varphi_j) = 0. \]

Therefore, the minimum \( \beta \) is attained at an admissible function \( \varphi \neq 0 \) and there is \( (\lambda, \theta) \in \mathbb{R}^2 \) such that

\[ L^+_{\text{dn}} \varphi = \lambda \varphi + \theta \phi^3_{\omega}. \tag{3.15} \]

Thus, from \( (L^+_{\text{dn}} \varphi, \varphi) = 0 \) we have \( \lambda = 0 \). Now, taking the inner product of (3.15) with \( \phi_{\omega} \), it is deduced from the self-adjoint property of \( L^+_{\text{dn}} \) that \( \theta = 0 \). So, since zero is simple it follows that \( \varphi = \eta \phi_{\omega} \) for \( \eta \neq 0 \). Therefore, we deduce that \( \varphi \) is orthogonal to \( \phi^3_{\omega} \) which is a contradiction. This completes the theorem.\( \square \)

Now, we have the following spectral structure associated to the solutions \( \phi_{\omega,i} \).

**Theorem 3.4.** Let \( \phi_{\omega,i}, \ i = 1, 2, \) be the cnoidal wave solutions given by Theorem 2.3. Then, the linear operators \( T_i = -\frac{d^2}{dx^2} + \omega - \phi^2_{\omega,i} \) defined on \( H^2_{\text{per}}([0, L]) \) have exactly one negative eigenvalue which is simple, the eigenvalue zero is also simple and the remainder of the spectrum is constituted by a discrete set of eigenvalues converging to infinity.

**Proof.** Since \( T_i \phi_{\omega,i} = 0 \) and \( \phi_{\omega,i} \) has exactly two zeros in \([0, L]\), then zero will be the second or the third eigenvalue. Next, we shall show that zero is simple and is the second one eigenvalue for each operator \( T_i \). In fact, for \( i = 1 \) and \( k^2_1 = b^2/(2b^2 - 2\omega) \), it is easy to see that \( \lambda_{0,1} = \omega(k^2_1 - 1)/(2k^2_1 - 1) < 0 \) and \( \lambda_{2,1} = \omega k^2_1/(2k^2_1 - 1) > 0 \) are eigenvalues of \( T_1 \) with eigenfunctions \( \chi_{0,1}(x) = \text{dn}(\sqrt{b^2 - \omega x}; k_1) \) and \( \chi_{2,1}(x) = \text{sn}(\sqrt{b^2 - \omega x}; k_1) \), respectively. So, from the Floquet theory we have that zero is simple and \( \lambda_{0,1} \) is the first negative eigenvalue of \( T_1 \). Similarly, we obtain for \( i = 2 \) and \( k^2_2 = (a^2 + 2\omega)/(2a^2 + 2\omega) \), that \( \lambda_{0,2} = -\omega(1 - k^2_2)/(2k^2_2 - 1) < 0 \) and \( \lambda_{2,2} = \omega k^2_2/(2k^2_2 - 1) > 0 \) are eigenvalues of \( T_2 \) with eigenfunctions \( \chi_{0,2}(x) = \text{dn}(\sqrt{a^2 + \omega x}; k_2) \) and \( \chi_{2,2}(x) = \text{sn}(\sqrt{a^2 + \omega x}; k_2) \) respectively. This completes the theorem.\( \square \)

**Theorem 3.5.** Let \( \phi_{\omega} \) be the dnoidal wave solution given by Theorem 2.1 and \( \omega \in (\frac{2\pi}{L^2}, +\infty) \). Then, the linear operator \( L_{\text{dn}} = -\frac{d^2}{dx^2} + \omega - 3\phi^2_{\omega} \) satisfies that

\[
\begin{align*}
(\text{a}) & \quad \inf \{ (L_{\text{dn}} \varphi, \varphi) : \varphi \in H^1_{\text{per}}([0, L]) , \| \varphi \| = 1, (\varphi, \phi_{\omega}) = 0 \} \equiv \gamma = 0, \\
(\text{b}) & \quad \inf \{ (L_{\text{dn}} \varphi, \varphi) : \varphi \in H^1_{\text{per}}([0, L]) , \| \varphi \| = 1, (\varphi, \phi_{\omega}) = 0, (\varphi, \phi^2_{\omega} \phi'_{\omega}) = 0 \} \equiv \zeta > 0.
\end{align*}
\tag{3.16}
\]

**Proof.** For part (a). Because \( \phi_{\omega} \) is bounded, it is inferred that \( \gamma \) is finite. Since \( (\phi^3_{\omega}, \phi_{\omega}) = 0 \) and \( L_{\text{dn}} \phi^3_{\omega} = 0 \) it follows that \( \gamma \leq 0 \). Now we show that the inf in (3.16)(a) is attained. In fact, using the same analysis as that in the proof of (3.14) we can guarantee a sequence \( \{ \psi_j \} \)
such that \((L_{dn}\psi_j, \psi_j) \to \gamma\) as \(j \to \infty\), and that \(\psi_j \to \psi\) weakly in \(H^1_{\text{per}}([0, L])\). So, \(\|\psi\| = 1\), \((\psi, \phi_\omega) = 0\) and
\[
\gamma \leq (L_{dn}\psi, \psi) \leq \liminf_{j \to \infty} (L_{dn}\psi_j, \psi_j) = \gamma.
\]
Therefore, \(\psi\) is a minimum.

Now, we want to show that \(\gamma \geq 0\). In this case, we will apply Lemma E1 in Weinstein \([23]\) in the case that \(A = L_{dn}\) and \(R = \phi_\omega\). In fact, from Theorem 3.1 we have that \(L_{dn}\) has the necessary spectral properties required by Lemma E1. Next, we find a \(\chi\) such that \(L_{dn}\chi = \phi_\omega\) and \((\chi, \phi_\omega) \leq 0\). In fact, from Theorem 2.1 we have that the mapping \(\omega \in (\frac{2\pi^2}{L^2}, +\infty) \to \phi_\omega\) is of class \(C^1\), so by taking differentiation with regard to \(\omega\) in (2.1) we obtain that \(\chi = -\frac{d}{d\omega}\phi_\omega\) satisfies that
\[
L_{dn}\left(-\frac{d}{d\omega}\phi_\omega\right) = \phi_\omega.
\]
Therefore we get that
\[
(L_{dn}^{-1}\phi_\omega, \phi_\omega) = -\frac{1}{2} \frac{d}{d\omega} \int_0^L \phi_\omega^2(\xi) \, d\xi \leq 0 \quad \Leftrightarrow \quad \frac{d}{d\omega} \int_0^L \phi_\omega^2(\xi) \, d\xi \geq 0.
\]
Next we will show that
\[
\frac{d}{d\omega} \int_0^L \phi_\omega^2(\xi) \, d\xi = 4 \frac{d}{L \, dk} \left[K(k) E(k)\right] \frac{dk}{d\omega} > 0 \quad \text{for} \quad \text{K'(k) > 0.} \tag{3.17}
\]
Indeed, by (2.4), (2.5), (2.9) and (2.10) we have that
\[
\|\phi_\omega\|^2 = \sqrt{2} \eta_1 \int_0^\frac{\eta_1 L}{\sqrt{2}} \, dn^2(x; k) \, dx = \sqrt{2} \eta_1 \int_0^{2K} \, dn^2(x; k) \, dx = \frac{8K(k)}{L} \int_0^K \, dn^2(x; k) \, dx, \tag{3.18}
\]
where we have used that the Jacobian elliptic \(dn\) function has fundamental period \(2K\) and it is an even function. Now, by using that (see [9])
\[
\int_0^K \, cn^2(x; k) \, dx = \frac{1}{k^2} [E(k) - (1 - k^2)K(k)]
\]
and \(dn^2(x; k) = 1 - k^2 + k^2 \, cn^2(x; k)\), it follows from (3.18) that
\[
\frac{1}{2} \int_0^L \phi_\omega^2(\xi) \, d\xi = \frac{4}{L} K(k) E(k). \tag{3.19}
\]
Now, from Theorem 2.1 and Corollary 2.2 we have that the map \( \omega \rightarrow \Lambda(\omega) \equiv n_2(\omega) \) is a strictly decreasing function, so we obtain from (2.9) for \( \eta_2 = \eta_2(\omega) \), that
\[
\frac{dk}{d\omega} = \frac{1}{2k} \left[ \frac{2\eta_2^2 - 4\omega\eta_2n_2'}{(2\omega - \eta_2^2)^2} \right] > 0.
\]
(3.20)
Therefore, since the map \( k \in (0,1) \rightarrow K(k)E(k) \) is a strictly increasing function, affirmation (3.17) follows from (3.19) and (3.20). Then, from Lemma E1 we obtain that \( \gamma \geq 0 \). This finishes the proof of (3.16)(a).

For part (b). From part (a) it is inferred that \( \zeta \geq 0 \). Suppose that \( \zeta = 0 \). Then we can find a function \( \varphi \) such that \( \| \varphi \| = 1 \), \( (\varphi, \phi_\omega) = 0 \), \( (\varphi, \phi_\omega^2 \phi_\omega') = 0 \) and \( (L_{dn} \varphi, \varphi) = 0 \). Therefore, there are \( \lambda, \theta, \mu \) such that
\[
L_{dn} \varphi = \lambda \varphi + \theta \phi_\omega + \mu \phi_\omega^2 \phi_\omega'.
\]
(3.21)
So, \( \lambda = \mu = 0 \). Therefore \( L_{dn} \varphi = \theta \phi_\omega \). Now, since \( L_{dn} \chi = \phi_\omega \) with \( \chi = -\frac{d}{d\omega} \phi_\omega \), it follows that \( L_{dn} (\varphi - \theta \chi) = 0 \) and so there is a \( \alpha \in \mathbb{R} \) such that \( \varphi - \theta \chi = \alpha \phi_\omega \). From (3.17) we have that \( (\chi, \phi_\omega) \neq 0 \), so \( \theta = 0 \). Then \( \varphi = \alpha \phi_\omega \) and hence \( \phi_\omega' \) is orthogonal to \( \phi_\omega^2 \phi_\omega' \) which is a contradiction. Therefore the minimum in (3.16)(b) is positive and the proof of the theorem is completed.

We note that from (3.14) and from the specific form of \( L_{dn}^+ \) we have that if \( \varphi \perp \phi_\omega^3 \), then
\[
(L_{dn}^+ \varphi, \varphi) \geq \beta_1 \| \varphi \|_1.
\]
(3.22)

4. Nonlinear stability of the dnoidal wave solutions for the NLS

In this section we shall use the Lyapunov method for studying the nonlinear stability of the solutions \( u(x,t) = e^{i\omega t} \phi_\omega(x) \) with \( \phi_\omega \) given by Theorem 2.1. So, a use of the conservation laws for (1.1)
\[
E_{Sch}(u) = \frac{1}{2} \int_0^L \left[ |u'|^2 - \frac{1}{2} |u|^4 \right] dx,
\]
\[
F(u) = \frac{1}{2} \int_0^L |u|^2 dx
\]
will be required. Moreover, since the NLS equation has phase and translations symmetries, i.e., if \( u(x,t) \) is solution of (1.1) then \( e^{i\theta} u(x+y,t) \) solves (1.1) for any \( (y,\theta) \in \mathbb{R} \times [0,2\pi) \), then the stability that we will study here it will be modulo these symmetries, namely, the orbital stability. More precisely, we shall show that the orbit generated by \( \phi_\omega \),
\[
O_{\phi_\omega} = \left\{ e^{i\theta} \phi_\omega (\cdot + y) : (y, \theta) \in \mathbb{R} \times [0,2\pi) \right\}
\]
(4.1)
is stable by the periodic flow generated by the NLS equation. Namely, for every \( \epsilon > 0 \) there exists \( \delta(\epsilon) > 0 \) such that if
\[
\inf_{(y,\theta)\in\mathbb{R}\times[0,2\pi)} \| u_0 - e^{i\theta} \phi_\omega (\cdot + y) \|_1 < \delta
\]
(4.2a)
then the solution \( u(x,t) \) of the NLS equation (1.1) with initial data \( u_0 \) satisfies that
\[ \left\| u(t) - e^{i\theta} \phi_\omega(y + y) \right\|_1 < \epsilon \] (4.2b)

for all \( t \in \mathbb{R}, \theta = \theta(t) \) and \( y = y(t) \).

Before establishing our main theorem, we show the following result about the periodic initial value problem associated to the NLS equation.

**Theorem 4.1.** The Cauchy problem (1.1) is globally well-posed for every data \( u_0 \in H^s_{\text{per}}([0, L]), s \geq 0 \). Namely, there is a unique solution \( u \in C(\mathbb{R}; H^s_{\text{per}}([0, L])) \) of Eq. (1.1) with \( u(x, 0) = u_0(x) \), and such that for every \( T > 0 \) the mapping \( u_0 \to u(t) \) is continuous from \( H^s_{\text{per}}([0, L]) \to C([-T, T]; H^s_{\text{per}}([0, L])) \).

**Proof.** See Bourgain [8]. □

**Theorem 4.2 (Stability).** Let \( \phi_\omega \) be the dnoidal wave solution given by Theorem 2.1 and \( \omega \in \left( \frac{2\pi^2}{L^2}, +\infty \right) \). Then the orbit \( O_{\phi_\omega} \) is nonlinearly stable in \( H^1_{\text{per}}([0, L]) \) with regard to the periodic flow of the NLS equation.

**Proof.** Since it will be based in the ideas developed by Benjamin, Bona and Weinstein [4,5,24] we will give an outline of the proof. Initially, if we define for \( y \in [0, L], \theta \in [0, 2\pi) \) and \( t \in \mathbb{R}, \)

\[ \Omega_t(y, \theta) = \| u'(\cdot + y, t)e^{i\theta} - \phi'_\omega \|^2 + \omega \| u(\cdot + y, t)e^{i\theta} - \phi_\omega \|^2 \]

then the deviation of the solution \( u(t) \) from \( O_{\phi_\omega} \) is measure by

\[ \left[ \rho_\omega(u(t), O_{\phi_\omega}) \right]^2 = \inf_{(y, \theta) \in [0, L] \times [0, 2\pi)} \Omega_t(y, \theta). \] (4.3)

So, by using standard arguments (see [4,5]) there exists an interval of time \( I = [0, T] \) such that the inf \( \Omega_t(y, \theta) \) is attained in \( (y, \theta) \equiv (y(t), \theta(t)) \) for every \( t \in I \). Hence, we have that

\[ \left[ \rho_\omega(u(t), O_{\phi_\omega}) \right]^2 = \Omega_t(y(t), \theta(t)). \] (4.3)

We consider the perturbation of the periodic traveling wave \( \phi_\omega \)

\[ u(x + y, t)e^{i\theta} \equiv \phi_\omega(x) + w(x, t) \quad \text{and} \quad w = p + iq \] (4.4)

for \( t \in [0, T] \) and \( y = y(t), \theta = \theta(t) \) determined by (4.3). So, by the property of minimum of \( (y, \theta) \) we obtain from (4.4) that \( (p, q) \) satisfies the compatibility relations

\[
\begin{cases}
\int_0^L \phi_\omega^2(x) \phi'_\omega(x) p(x, t) \, dx = 0, \\
\int_0^L \phi_\omega^3(x) q(x, t) \, dx = 0
\end{cases}
\] (4.5)

for all \( t \in [0, T] \).
Next using that, $E_{\text{Sch}}$, $F$ are invariant by translations and rotations, the representation (4.4), the classical embedding $H^1_{\text{per}}([0, L]) \hookrightarrow L^r_{\text{per}}([0, L])$ for every $r \geq 2$, and the fact that $\phi_{\omega}$ satisfies (2.1), we have the following variation for $E(u) = E_{\text{Sch}}(u) + \omega F(u)$

$$
\Delta E = E(u_0) - E(\phi_{\omega}) = E(u(t)) - E(\phi_{\omega}) = E(\phi_{\omega} + w(t)) - E(\phi_{\omega})
\geq \frac{1}{2} (\mathcal{L}_{dn} p, p) + \frac{1}{2} (\mathcal{L}_{dn}^+ q, q) - \beta_2 \|w\|^3_1 - \beta_3 \|w\|^4_1,
$$

(4.6)

where $\mathcal{L}_{dn} = -\frac{d^2}{dx^2} + \omega - 3\phi_{\omega}^2$, $\mathcal{L}_{dn}^+ = -\frac{d^2}{dx^2} + \omega - \phi_{\omega}^2$ (see Section 3) and $\beta_i$ are positive constants. Now we obtain a suitable lower bound on the quadratic forms in (4.6). Initially, we consider the normalization $\|u(t)\|^2 = \|\phi_{\omega}\|^2$ for every $t \in [0, T]$. By (4.4) it follows $(p, \phi_{\omega}) = -\frac{1}{2} (\|p\|^2 + \|q\|^2)$. Without loss of generality, we suppose that $\|\phi_{\omega}\| = 1$. Define $p_\parallel$ and $p_\perp$ to be $p_\parallel = (u, \phi_{\omega}) \phi_{\omega} = -\frac{1}{2} (\|p\|^2 + \|q\|^2) \phi_{\omega}$ and $p_\perp = p - p_\parallel$. So, $p_\perp \perp \phi_{\omega}$ and $p_\perp \perp \phi_{\omega}'$. Therefore, it follows from (3.16)(b) and $(\mathcal{L}_{dn}\phi_{\omega}, \phi_{\omega}) < 0$ that

$$(\mathcal{L}_{dn} p_\perp, p_\perp) \geq \xi \|p_\perp\|^2 - D_1 \|w\|^3_1 - D_2 \|w\|^4_1 \geq \xi \|p\|^2 - D_1 \|w\|^3_1 - D_3 \|w\|^4_1,
$$

(4.7)

with $D_i > 0$. Therefore from (3.22), (4.5), (4.7) and from the specific form of $\mathcal{L}_{dn}$, we have that (4.6) implies

$$\Delta E(t) \geq \gamma_0 \|w\|^2_1 - \gamma_1 \|w\|^3_1 - \gamma_2 \|w\|^4_1.
$$

(4.8)

Hence, from (4.3) it follows that for $t \in [0, T]$

$$\Delta E(t) \geq h\left(\rho_{\omega}(u(t), \Omega_{\phi_{\omega}})\right),
$$

(4.9)

where $h(x) = ax^2(1 - bx - cx^2)$ with $a, b, c > 0$. The essential properties of $h$ are $h(0) = 0$ and $h(x) > 0$ for $x$ small. The stability result is an immediate consequence from (4.9). In fact, let $\epsilon > 0$. Then, by using the properties that $E$ is continuous on $S = \{u_0 \in H^1_{\text{per}}([0, L]) : \|u_0\| = \|\phi_{\omega}\|\}$, $\Delta E(t)$ is constant in time and $t \rightarrow \rho_{\omega}(u(t), \Omega_{\phi_{\omega}})$ is a continuous function, we have that there is $\delta(\epsilon) > 0$ such that if $u_0 \in S$ and $\rho_{\omega}(u_0, \Omega_{\phi_{\omega}}) < \delta(\epsilon)$ then for $t \in [0, T]$, $h\left(\rho_{\omega}(u(t), \Omega_{\phi_{\omega}})\right) \leq \Delta E(0) < h(\epsilon) \Rightarrow \rho_{\omega}(u(t), \Omega_{\phi_{\omega}}) < \epsilon,
$$

(4.10)

which shows that $\Omega_{\phi_{\omega}}$ is orbitally stable in $H^1_{\text{per}}([0, L])$ relative to small perturbations which preserve the $L^2_{\text{per}}$-norm.

Next, that inequality (4.10) is still true for all $t > 0$ it is an immediate consequence of the continuity of the mapping $t \rightarrow \inf_{(y, \theta) \in [0, L] \times [0, 2\pi]} \Omega_t(y, \theta)$ (see [5]). To prove stability to general perturbations, we use that the mapping $\omega \in (\frac{2\pi}{L^2}, +\infty) \rightarrow \phi_{\omega} \in H^1_{\text{per}}([0, L])$ is continuous, the preceding theory and the triangle inequality (see [5,24]). Theorem 4.2 is now established. □

Theorem 4.2 shows us the nonlinear stability of periodic traveling wave solutions of the form $u(x, t) = e^{i\omega t} \phi_{\omega}(x)$, with $\phi_{\omega}$ being the cnoidal waves (2.4). The following natural question arising in this point it is to decide if the traveling waves $u_i(x, t) = e^{i\omega t} \phi_{\omega,i}(x)$ determined by the cnoidal waves $\phi_{\omega,i}$, defined by Theorem 2.3, are stable in $H^1_{\text{per}}([0, L])$. From Theorem 3.2 it follows immediately that we cannot apply the method used in the proof of Theorem 4.2. We
do not know if these periodic traveling waves are stable by perturbations of period $L$. Next, we would like to give some arguments which show that the abstract theories established by Grillakis in [14] and Grillakis, Shatah and Strauss in [16] do not give an information about the stability or instability problem:

(1) It follows from [16] that one of the principal informations for obtaining a result of instability of the solutions $u_i(x,t) = e^{i\omega t}\phi_{\omega,i}(x)$ is based in the calculate of the number of the negative eigenvalue associated to the linear operator (the linearized Hamiltonian)

$$H_{\omega,i} = \begin{pmatrix} L_i & 0 \\ 0 & T_i \end{pmatrix},$$

and in the sign of the quantity $d_2(\omega) = \frac{d}{d\omega} \|\phi_{\omega,i}\|^2$. In fact, if $n(H_{\omega,i})$ denotes the number of negative eigenvalues of $H_{\omega,i}$ and $\text{sgn}(d_2)$ denotes the sign of $d_2$,

$$\text{sgn}(r) = \begin{cases} -1, & \text{if } r < 0, \\ 1, & \text{if } r > 0, \\ 0, & \text{if } r = 0, \end{cases}$$

it follows from the instability theorem in [16], that the orbit generated by $u_i$ will be unstable if $n(H_{\omega,i}) - \text{sgn}(d_2)$ is odd. In our case this condition is not satisfied. Indeed, from Theorems 3.2 and 3.4 we have that for every $i = 1, 2$, $n(H_{\omega,i}) = 3$. Now, from the explicit form of $\phi_{\omega,i}$ in (2.20)--(2.21) we obtain that

$$\frac{L}{\int_0^L \phi_{\omega,i}^2(\xi) d\xi} = \frac{32}{L} K(k_i) \left[ E(k_i) - (1 - k_i^2) K(k_i) \right] = \frac{32}{L} k_i (1 - k_i^2) K(k_i) \frac{dK(k_i)}{dk_i}.$$

Hence, since $k_i'(\omega) > 0$ for each $i = 1, 2$ (see Theorem 2.3), it follows after some calculations that

$$\frac{d}{d\omega} \frac{L}{\int_0^L \phi_{\omega,i}^2(\xi) d\xi} = \frac{32}{L} k_i'(\omega) \left[ k_i (1 - k_i^2) \left( \frac{dK(k_i)}{dk_i} \right)^2 + k_i K^2 \right] > 0. \quad (4.11)$$

So, $n(H_{\omega,i}) - \text{sgn}(d_2) = 2$.

(2) It follows from [14] that for obtaining a result of nonlinear instability of the orbit $\Omega_{\phi_i} = \{e^{i\gamma}\phi_i: \gamma \in \mathbb{R}\}$ it is sufficient to show that the linearization of (1.1) around the orbit $\Omega_{\phi_i}$, has the zero solution unstable. So, it is necessary a study of the number of real eigenvalues of the linear operator

$$A_i = \begin{pmatrix} 0 & T_i \\ -L_i & 0 \end{pmatrix}.$$

The main theorems in [14] (Theorems 1.2 and 1.3) are based in the linear operators $R_i \equiv P_i L_i P_i$, where $P_i$ is the orthogonal projection on $[\text{ker} T_i]^\perp$. More precisely, if $I(A_i) = \# \text{ of pairs of nonzero real eigenvalues of } A_i$, then we have:
(i) if $|n(R_i) - n(T_i)| = m > 0$, then $I(A_i) \geq m$.

(ii) if $n(R_i) = n(T_i)$ and

$$\{ f \in L^2_{\text{per}}([0, L]): (R_i f, f) < 0 \} \cap \{ f \in L^2_{\text{per}}([0, L]): (T_i^{-1} f, f) < 0 \} = \emptyset,$$

then $I(A_i) \geq 1$. \hfill (4.12)

We will see that the conditions in (i) and (ii) are not satisfy in the case of the $\phi_{\omega,i}$. In fact, since $d_2(\omega) > 0$ we obtain that $n(R_i) = n(L_i) - 1$ and so by Theorems 3.2 and 3.4, we have that $n(R_i) = 1 = n(T_i)$. Next, since $\chi_{0,1}(x) = \text{dn}(\gamma x) \in \ker L_i^1 \setminus \ker L_i$ ($\gamma = \sqrt{b^2 - \omega}$) and $b^2 > 2\omega$ it follows that

$$(R_1 \chi_{0,1}, \chi_{0,1}) = (L_1 \chi_{0,1}, \chi_{0,1})$$

$$= -2b^2 \int_0^L \text{cn}^2(\gamma x) \text{dn}^2(\gamma x) \, dx + \left( \omega - \frac{b^2}{2} \right) \int_0^L \text{dn}^2(\gamma x) \, dx < 0.$$

Moreover, since $T_1 \chi_{0,1} = \lambda_{0,1} \chi_{0,1}$ with $\lambda_{0,1} < 0$, it follows that $(T_1^{-1} \chi_{0,1}, \chi_{0,1}) < 0$. So the criterion in (4.12) cannot be applied in the case of the solution $u_1$. Similarly, we can obtain the same conclusion in the case of $u_2$.

Next, we prove that the periodic dnodial wave solutions for the NLS equation determined by $\phi_{\omega}$ in (2.4) are unstable by perturbation of periodic functions of period $2L$.

**Theorem 4.3 (Instability).** It considers the solution $\phi_{\omega}$ determined by Theorem 2.1. Then the orbit $\{ e^{i\lambda \phi_{\omega}}: y \in \mathbb{R} \}$ is $H^1_{\text{per}}([0, 2L])$-unstable by the flow of the periodic NLS.

**Proof.** The idea of the proof is to apply the criterion given by (4.12) and so to obtain the linearized instability which immediately implies the nonlinear instability. It considers the orbit generated by the dnodial wave $\phi_{\omega}$, $\Omega_{\phi_{\omega}} = \{ e^{i\lambda \phi_{\omega}} : s \in \mathbb{R} \}$. Then, since $\phi_{\omega}$ has four zeros in $[0, 2L]$, it follows from the proof of Theorem 3.1 that $L_{\text{dn}}$ has exactly three negative eigenvalues. Moreover, by Theorem 3.3 it follows that $n(L_{\text{dn}}^+) = 0$. Now, by defining $R_0 \equiv P_0 L_{\text{dn}} P_0$ with $P_0$ being the orthogonal projection on $\ker L_{\text{dn}}^+$, we obtain from (3.17) that $n(R_0) = n(L_{\text{dn}}) - 1 = 2$. Therefore, it follows from (4.12)(i) that $I(A_0) \geq 2$ where

$$A_0 = \begin{pmatrix} 0 & L_{\text{dn}}^+ \\ -L_{\text{dn}} & 0 \end{pmatrix}$$

(by Corollary 1.1 in [14], since $n(L_{\text{dn}}^+) = 0$ then $I(A_0) = n(R_0) = 2$). So, we have that $\Omega_{\phi_{\omega}}$ is $H^1_{\text{per}}([0, 2L])$-unstable. \vspace{0.5cm}

Next, we will analyze the case of periodic traveling waves solutions of (1.1) of the general form

$$u(x, t) = e^{i\alpha t} \psi_{\omega,c}(x - \alpha t), \quad (4.13)$$

where $\omega, \alpha \in \mathbb{R}$ and $\psi_{\omega,c} : \mathbb{R} \rightarrow \mathbb{C}$. The function $\psi = \psi_{\omega,c}$ must satisfy the equation

$$\psi''(\xi) - i c \psi'(\xi) - \omega \psi(\xi) + \left| \psi(\xi) \right|^2 \psi(\xi) = 0. \quad (4.14)$$
Now, by considering
\[ \psi(\xi) = e^{i\frac{\xi}{2}} \phi_{\omega,c}(\xi), \] (4.15)
where \( \phi_{\omega,c} \) is real-valued, we obtain that \( \phi_{\omega,c} \) must satisfy the equation
\[ \phi_{\omega,c}'' + \phi_{\omega,c}^3 - \mu \phi_{\omega,c} = 0, \] (4.16)
where \( \mu \equiv \omega - \frac{c^2}{4} \). Since (4.16) is exactly Eq. (2.1), it follows from Theorem 2.1 that for every fixed period \( L > 0 \) there is a smooth curve \( \mu \in \left( \frac{2\pi^2}{L^2}, \infty \right) \to \phi_{\mu} \in H^1_{\text{per}}([0, L]) \) of solutions of (4.16). Now, (4.15) in general does not represent a periodic function. In fact, since \( e^{i\frac{\xi}{2}} \) has fundamental period \( P_1 = \frac{4\pi}{|c|} \) and \( \phi_{\mu} = \phi_{\omega,c} \) has period \( L \), we have that \( T \) is a period of \( \psi \) if and only if there are \( n, k \in \mathbb{N} \) such that \( T = nP_1 \) and \( T = kL \). Therefore
\[ \frac{n}{k} = \frac{L}{P_1} = \frac{cL}{4\pi} \in \mathbb{Q}. \] (4.17)
So, if we choose \( \omega, c \) such that \( \omega - \frac{c^2}{4} > \frac{2\pi^2}{L^2}, 4\pi/|c| < L \) and there is \( k \in \mathbb{N} \) such that \( 4\pi k = L|c| \), then (4.13) (also (4.15)) represents a periodic traveling wave solutions of the NLS equation with fundamental period \( L \). Next, we will see that in this case Theorem 4.2 implies that the solution (4.13) is orbitally stable in \( H^1_{\text{per}}([0, L]) \) by the flow of the NLS equation. In fact, it defines the linear operator \( Q_c : H^1_{\text{per}}([0, L]) \to H^1_{\text{per}}([0, L]) \) by
\[ (Q_c\varphi)(x) = e^{i\frac{\xi}{2}x} \varphi(x), \]
where \( \varphi \in H^1_{\text{per}}([0, L]) \). So, we obtain immediately that
\[ (1 + |c|)^{-1} \| \varphi \|_1 \leq \| Q_c \varphi \|_1 \leq (1 + |c|) \| \varphi \|_1. \] (4.18)
Next, a short calculation shows that if \( v \) is solution of Eq. (1.1) with initial data \( v_0 \in H^1_{\text{per}}([0, L]) \) then
\[ u(x, t) = e^{i(\frac{\xi}{2}x - \frac{1}{4}c^2t)} v(x - ct, t) \] (4.19)
is a solution of the initial value problem
\[
\begin{cases}
  iu_t + u_{xx} + |u|^2 u = 0, \\
  u(x, 0) = Q_c v_0.
\end{cases}
\]

**Theorem 4.4.** Consider \( \omega, c \) such that \( \omega - \frac{c^2}{4} > \frac{2\pi^2}{L^2} \).

(i) Suppose that \( 4\pi/|c| < L \) and there is \( k \in \mathbb{N} \) such that \( 4\pi k = L|c| \). Then (4.13)–(4.15) represent a traveling wave solution of period \( L \) which is orbitally stable in \( H^1_{\text{per}}([0, L]) \) with regard to the periodic flow of the NLS equation.

(ii) Suppose that \( 4\pi/|c| > L \) and (4.17) is true. Then the orbit \( \{ e^{iy} \psi : y \in \mathbb{R} \} \) is \( H^1_{\text{per}}([0, pL]) \)-unstable for some \( p \in \mathbb{N}, p \geq 2 \).
Proof. For part (i). Let \( u_0 \in H^1_{\text{per}}([0, L]) \) which lies close to \( \psi_{\omega,c} \) (defined in (4.15)) in the \( H^1_{\text{per}} \) norm. Define, \( v_0 = Q_{-c}u_0 \in H^1_{\text{per}}([0, L]) \). Let \( u, v \) be the solution of (1.1) with initial data \( u_0, v_0 \) respectively. It follows from (4.18) that
\[
\|v_0 - \phi_\mu\|_1 = \|Q_{-c}u_0 - Q_{-c}\psi_{\omega,c}\|_1 \leq (1 + |c|)\|u_0 - \psi_{\omega,c}\|_1.
\]
Let \( \epsilon > 0 \) be fixed. Then from the stability of \( \phi_\mu \) given by Theorem 4.2, there is a \( \delta > 0 \) such that the condition \( \|v_0 - \phi_\mu\|_1 < \delta \) implies that
\[
\|v(t) - e^{i\theta} \phi_\mu(\cdot + y)\|_1 < \frac{\epsilon}{1 + |c|}
\]
for all \( t \in \mathbb{R} \), \( \theta = \theta(t) \) and \( y = y(t) \). Therefore provided \( \|u_0 - \psi_{\omega,c}\|_1 < \delta/(1 + |c|) \), (4.19) and (4.20) imply that
\[
\|Q_{-c}(e^{-\frac{c^2}{4}t}u(\cdot + ct, t) - e^{i(\theta - \frac{c}{2}y)}\psi_{\omega,c}(\cdot + y))\|_1 < \frac{\epsilon}{1 + |c|}
\]
for all \( t \in \mathbb{R} \), and so it is concluded from (4.18) that
\[
\|u(\cdot, t) - e^{ip(t)}\psi_{\omega,c}(\cdot - q(t))\|_1 < \epsilon
\]
for all \( t \in \mathbb{R} \), where
\[
p(t) = \theta(t) + \frac{c^2}{4}t - \frac{c}{2}y(t), \quad q(t) = y(t) - ct.
\]
Hence \( \Omega_\psi \) is orbitally stable.

Proof of (ii). It follows from (4.17) that the fundamental period of (4.15) has the form \( pL \) for \( p \in \mathbb{N}, p \geq 2 \). So, the instability result is obtained similarly as in Theorem 4.3.

5. Nonlinear stability of periodic traveling wave solutions for the modified KdV

In this section we shall give an idea of the orbital stability of the periodic traveling wave solutions for the mKdV equation (1.2) of the form
\[
u(x, t) = \phi_\omega(x - \omega t).
\]
(5.1)
Since the details of the proof are parallel those for the NLS equation we only give an outline. Substitution of (5.1) into (1.2), implies the ordinary differential equation
\[
\phi_\omega'' + \phi_\omega^3 - \omega \phi_\omega = A_{\phi_\omega},
\]
(5.2)
where \( A_{\phi_\omega} \) is an integration constant, which we will consider equal to zero. So, (5.2) takes the form of (2.1) and Theorem 2.1 gives us the existence of a smooth curve of dnoideal waves solutions of the form (2.4)
\[
\omega \in \left(\frac{2\pi^2}{L^2}; +\infty\right) \to \phi_\omega \in H^1_{\text{per}}([0, L])
\]
Since the mKdV has translations symmetries, orbital stability is now modulo translations and therefore the orbit generated by $\phi_\omega$ is

$$O_{\phi_\omega} = \{ \phi_\omega(\cdot + y) : y \in \mathbb{R} \}. \quad (5.3)$$

So, we have the following theorems.

**Theorem 5.1.** Let $L > 0$ fixed. Then the periodic initial value problem associated to the mKdV equation (1.2) is globally well-posed in $H^s_{\text{per}}([0, L])$, $s \geq 1/2$.

**Proof.** See Colliander, Kell, Staffilani, Takaoka and Tao [11]. \(\square\)

**Theorem 5.2.** Let $\phi_\omega$ be the dnoidal wave solution given by Theorem 2.1 and $\omega \in \left(\frac{2\pi^2}{L^2}, +\infty\right)$. Then the orbit $O_{\phi_\omega}$ defined in (5.3) is orbitally stable in $H^1_{\text{per}}([0, L])$ with regard to the periodic flow of the mKdV equation, i.e., for every $\epsilon > 0$, there is $\delta(\epsilon) > 0$ such that if

$$\inf_{y \in \mathbb{R}} \| u_0 - \phi_\omega(\cdot + y) \|_1 < \delta$$

then the solution $u(x, t)$ of the mKdV equation (1.2) with initial data $u_0$ satisfies that

$$\| u(t) - \phi_\omega(\cdot + y) \|_1 < \epsilon$$

for all $t \in \mathbb{R}$ and $y = y(t)$.

**Proof.** We consider the perturbation $u(x + y, t) = \phi_\omega(x) + v(x, t)$, where $y = y(t)$ is chosen as being a minimum for $\Omega_t(y) = \| u'(\cdot + y, t) - \phi_\omega' \|^2 + \omega \| u(\cdot + y, t) - \phi_\omega \|^2$. Therefore, $v(t)$ satisfies the compatibility condition

$$\int_0^L \phi_\omega^2(x) \phi_\omega'(x) v(x, t) \, dx = 0. \quad (5.4)$$

The analogue of (4.6) is then given by

$$E_{\text{mKdV}}(u_0) - E_{\text{mKdV}}(\phi_\omega) \geq \frac{1}{2} (\mathcal{L}_{\text{dn}} v, v) - \beta_1 \| v \|_1^3 - \beta_2 \| v \|_1^4, \quad (5.5)$$

where

$$E_{\text{mKdV}}(u) = \frac{1}{2} \int_0^L \left[ (u')^2 - \frac{1}{2} u^4 \right] \, dx + \omega \int_0^L u^2 \, dx,$$

$$\mathcal{L}_{\text{dn}} = -\frac{d^2}{dx^2} + \omega - 3\phi_\omega^2$$

and $\beta_i$ are positive constants. So, by using Theorems 3.1, 3.5 and constraint (5.4), we obtain from (5.5) an analogue inequality to (4.8). This finishes the theorem. \(\square\)
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Appendix A

In this appendix we establish some basic properties about Jacobian elliptic functions (see Byrd and Friedman [9]). Initially, we define the normal elliptic integral of the first kind

\[
\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \equiv F(\varphi, k),
\]

where \( y = \sin \varphi \), and the normal elliptic integral of the second kind,

\[
\int_0^y \sqrt{\frac{1-k^2t^2}{1-t^2}} \, dt = \int_0^\varphi \sqrt{1-k^2\sin^2\theta} \, d\theta \equiv E(\varphi, k).
\]

In their algebraic forms, these two integrals have the following properties: the first is finite for all real (or complex) values of \( y \), including infinity; the second has a simple pole of order 1 for \( y = \infty \). The number \( k \) is called the modulus and belongs to the interval \((0, 1)\). The number \( k' = \sqrt{1-k^2} \) is called the complementary modulus. \( \varphi \) is called the argument of the normal elliptic integrals. It is usually understood that \( 0 < y \leq 1 \) or \( 0 < \varphi \leq \pi/2 \). When \( y = 1 \), the integrals above are said to be complete. In this case, one writes:

\[
\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\pi/2, k) \equiv K(k) \equiv K,
\]

and

\[
\int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}} \, dt = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta} \, d\theta = E(\pi/2, k) \equiv E(k) \equiv E.
\]

So, \( K(0) = E(0) = \pi/2 \), \( E(1) = 1 \) and \( K(1) = +\infty \). For \( k \in (0, 1) \), \( K'(k) > 0 \), \( K''(k) > 0 \), \( E'(k) < 0 \) and \( E''(k) < 0 \). Moreover, for \( k \in (0, 1) \) we have that \( E(k) < K(k) \), and \( E(k) + K(k) \), \( E(k)K(k) \) are strictly increasing functions.

The complete elliptic integrals \( K \) and \( E \) satisfy the following hypergeometric differential equations

\[
\begin{cases}
kk'k^2 \frac{d^2K}{dk^2} + (1-3k^2) \frac{dK}{dk} - kK = 0, \\
nk'k^2 \frac{d^2E}{dk^2} + k'k^2 \frac{dE}{dk} + kE = 0.
\end{cases}
\]
Now, we have some derivatives of the complete elliptical integrals $K$ and $E$, which we used in this work:

\[
\begin{align*}
\frac{dK}{dk} &= E - k'K, \\
\frac{dE}{dk} &= E - K, \\
\frac{d^2E}{dk^2} &= -\frac{E - k'K}{k k'}.
\end{align*}
\]

Next, we will define the Jacobian elliptic functions. Initially, we consider the elliptic integral

\[
u(y_1; k) \equiv u = \int_0^{y_1} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = F(\varphi, k)
\]

which is a strictly increasing function of variable $y_1$ (real), hence we can define its inverse function by $y_1 = \sin \varphi \equiv \text{sn}(u; k)$, or briefly $y_1 = \text{sn} u$, when it is not necessary to emphasize the modulus. The function $\text{sn} u$ is a odd elliptic function. Other two basic functions can also be defined by

\[
\begin{align*}
\text{cn}(u; k) &= \sqrt{1 - y_1^2} = \sqrt{1 - \text{sn}^2(u; k)}, \\
\text{dn}(u; k) &= \sqrt{1 - k^2 y_1^2} = \sqrt{1 - k^2 \text{sn}^2(u; k)},
\end{align*}
\]

requiring that $\text{sn}(0, k) = 0$, $\text{cn}(0, k) = 1$ and $\text{dn}(0, k) = 1$. The functions $\text{cn} u$ and $\text{dn} u$ are therefore even functions. The functions $\text{sn}(u; k)$, $\text{cn}(u; k)$, and $\text{dn}(u; k)$ are called Jacobian elliptic functions and are one-valued functions of the argument $u$. These functions have a real period, namely, $4K(k)$, $4K(k)$ and $2K(k)$ respectively. The more important properties of the Jacobian elliptic functions which have been used in this work are summarized by the following formulas:

\[
\begin{align*}
\text{sn}^2 u + \text{cn}^2 u &= 1, & k^2 \text{sn}^2 u + \text{dn}^2 u &= 1, & k^2 \text{sn}^2 u + \text{cn}^2 u &= \text{dn}^2 u, \\
1 \leq \text{sn} u &\leq 1, & -1 \leq \text{cn} u &\leq 1, & k' &\leq \text{dn} u \leq 1.
\end{align*}
\]

Also, we have some special values:

\[
\begin{align*}
\text{sn} \ 1 &= 1, & \text{cn} \ 0 &= 0, & \text{sn}(u + 4K) &= \text{sn} u, & \text{cn}(u + 4K) &= \text{cn} u, \\
\text{dn}(u + 2K) &= \text{dn} u, & \text{sn}(u + 2K) &= -\text{sn} u, & \text{cn}(u + 2K) &= -\text{cn} u, \\
\text{sn}(u, 0) &= \sin u, & \text{cn}(u, 0) &= \cos u, & \text{dn}(u, 0) &= 1, \\
\text{sn}(u, 1) &= \tanh u, & \text{cn}(u, 1) &= \text{sech} u, & \text{dn}(u, 1) &= \text{sech} u.
\end{align*}
\]

Finally, we have the following differentiation formulas,

\[
\begin{align*}
\frac{\partial}{\partial u} \text{sn} u &= \text{cn} u \text{ dn} u, & \frac{\partial}{\partial u} \text{cn} u &= -\text{sn} u \text{ dn} u, \\
\frac{\partial}{\partial u} \text{dn} u &= -k^2 \text{sn} u \text{ cn} u.
\end{align*}
\]
References