Finite unions of weak $\bar{\theta}$-refinable spaces and products of ordinals

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In this note, we show that if $X$ is the union of a finite collection \( \{X_i : i = 1, \ldots, k\} \) of weak $\bar{\theta}$-refinable subspaces and \( e(X) = \omega \), then $X$ is a Lindelöf space. We also show that the product of two ordinals is dually discrete. The last conclusion gives a positive answer to a question of Alas, Junqueira and Wilson.

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0. Introduction

The notion of $D$-space was introduced by van Douwen [cf. [9]]. A neighborhood assignment for a space $X$ is a function $\phi$ from $X$ to the topology of the space $X$, such that $x \in \phi(x)$ for any $x \in X$. A space $X$ is called a $D$-space, if for any neighborhood assignment $\phi$ for $X$ there exists a closed discrete subspace $D$ of $X$ such that $X = \bigcup \{\phi(d) : d \in D\}$ [cf. [10]]. In [10], it was proved that a finite product of the Sorgenfrey line is a $D$-space. By results of [6], we know that all semi-stratifiable spaces are $D$-spaces. So we know that all metrizable spaces and all Moore spaces are $D$-spaces. In [7], Buzyakova proved that every strong $\Sigma$-space is a $D$-space. In 2004 (cf. [3]), Arhangel'skii proved that if $X$ is the union of a finite collection of spaces with a point-countable base, then $X$ is a $D$-space. In 2008, Peng proved that if $X$ is the union of a finite collection of Moore spaces, then $X$ is a $D$-space [cf. [16]]. The idea of $D$-spaces was then further developed in recent years, the properties of $\alpha D$-spaces and dually discrete spaces were discussed.

The concept of an $\alpha D$-space was introduced in [5]. A space $X$ is an $\alpha D$-space if for each closed subset $F$ of $X$ and each open covering $U$ of $X$ there exists a locally finite in $F$ subset $A$ of $F$ and a mapping $\phi$ of $A$ into $U$ such that $a \in \phi(a)$ for each $a \in A$, and the family $\phi(A) = \{\phi(a) : a \in A\}$ covers $F$. By results of [6], we know that every subparacompact space is an $\alpha D$-space. In [5], it was proved that if a regular space $X$ is the union of a finite collection of paracompact subspaces then $X$ is an $\alpha D$-space. In [3], it was proved that if $X$ is the union of a finite collection of subparacompact subspaces then $X$ is an $\alpha D$-space. Peng also proved that $X$ is an $\alpha D$-space if $X$ is the union of finite collection of $\theta$-refinable subspaces [cf. [17, Theorem 10]]. We also know that very $\delta\theta$-refinable space is an $\alpha D$-space [cf. [4, Theorem 1.15]].

The concept of weak $\bar{\theta}$-refinable spaces was introduced in [19]. It was proved that every $\bar{\theta}$-refinable space is a weak $\bar{\theta}$-refinable space and a weak $\bar{\theta}$-refinable space $X$ with countable extent (every countable closed discrete subspace of $X$ is
countable, usually be denoted by $e(X) = \omega$ is Lindelöf (cf. [19, Theorem 3.3]). By results of [14], we know that every weak $\tilde{\theta}$-refinable space is an $\alpha D$-space. The following question is open: Is the union of a finite collection $\{X_i: i \leq n\}$ of weak $\tilde{\theta}$-refinable subspaces an $\alpha D$-space? We know that every countably compact $\alpha D$-space is compact. Although the above question is not answered in this note, we show that if $X$ is the union of a finite collection $\{X_i: i \leq n\}$ of weak $\tilde{\theta}$-refinable subspaces and $e(X) = \omega$, then $X$ is Lindelöf.

The concept of dually discrete spaces was introduced in [13]. A space $X$ is called dually discrete, if for any neighborhood assignment $\phi$ for $X$ there exists a discrete subspace $D$ of $X$ such that $X = \bigcup \{ \phi(d): d \in D \}$. Every $D$-space is dually discrete. There has been much work on dual properties (cf. [1,2,8,13]).

In [8], it was proved that every ordinal is dually discrete. In [1], it was proved that a finite product of regular cardinals is dually discrete. The following problem appeared in [1]: Is the product of two ordinals (hereditarily) dually discrete? In this note, we show that a finite product of ordinals is dually discrete.

All the spaces in this note are assumed to be $T_1$-spaces. The set of all natural numbers is denoted by $N$, $\omega$ is $N \cup \{0\}$. In notation and terminology we will follow [11].

1. On finite unions of weak $\tilde{\theta}$-refinable spaces

Let $X$ be a space and let $\mathcal{U}$ be a family of open subsets of $X$. We denote $\text{ord}(\mathcal{U}, X) = |\{V: x \in V \text{ and } V \in \mathcal{U}\}|$.

**Lemma 1.1.** (Cf. [18, Lemma 3]) Let $\mathcal{U} = \{U_\lambda: \lambda < \eta\}$ be an open cover of $X$ and $X_n = \{x \in X: \text{ord}(\mathcal{U}, x) \leq n\}$ for each $n \in N$. Then $X_n$ is a closed subset of $X$ and $\mathcal{F}_n = \{E(\lambda_1, \ldots, \lambda_n): \lambda_1 < \cdots < \lambda_n < \eta\}$ is a discrete cover of the subspace $X_n \setminus X_{n-1}$ for each $n \geq 1$, where $E(\lambda_1, \ldots, \lambda_n) = \bigcap \{U_\lambda \cap (X_n \setminus X_{n-1}): i \leq n\}$ and $X_0 = \emptyset$.

**Lemma 1.2.** (Cf. [16, Lemma 1]) Suppose $X = \bigcup \{X_i: 1 \leq i \leq n\}$ and $\mathcal{F}$ is a locally finite family of subsets of $X_i$ for some $i \leq n$ and $A = \{x: x \in X \text{ and } \mathcal{F} \text{ is not locally finite at } x\}$. Then $A$ is a closed subset of $X$ and $A \subset X \setminus X_i$.

**Lemma 1.3.** Let $X = X_1 \cup X_2$ and $e(X) = \omega$ and let $Y \subset X_1$ be a closed subspace of $X_1$. Let $\mathcal{U}$ be an open cover of $X$ and let $\mathcal{V} = \bigcup \{V_\eta: n \in N\}$ be an open refinement of $\mathcal{U}$, such that $Y \subset \bigcup V_\eta$ for each $n \in N$ and for each $y \in Y$ there exists some $n_y \in N$ such that $1 \leq \text{ord}(x, V_{n_y}) < \omega$. If each closed subspace $F$ of $X$ which is contained in $X_2 \setminus X_1$ is Lindelöf, then the family of $\{\bigcup \{U_\eta: F \in \mathcal{F}_n\}: F \in \mathcal{F}_1\}$ is a countably family $\mathcal{U}_F \subset \mathcal{U}$ such that $Y \subset \bigcup \mathcal{U}_F$.

**Proof.** For each $m \in N$ and $n \in N$, let $F_{mn} = \{x: x \in Y \text{ and } \text{ord}(X, m) \leq n\}$. By Lemma 1.1, we know that $F_{mn}$ is a closed subspace of $Y$ and $F_{m(n+1)} \setminus F_{mn} = \bigcup \mathcal{F}_{m(n+1)}$, where $\mathcal{F}_{m(n+1)}$ is a closed open (in $F_{m(n+1)} \setminus F_{mn}$) cover of $F_{m(n+1)} \setminus F_{mn}$ and for each $F \in \mathcal{F}_{m(n+1)}$ there is some $V_\eta \subset \mathcal{V}_\eta$ such that $F \subset V_\eta$.

For each $m \in N$, the set $\mathcal{F}_m$ is a closed discrete family of subsets of $Y$ and $Y$ is a closed in $X_1$, so $\mathcal{F}_m$ is a discrete family in $X_1$. Let $A_m = \{x: x \in X \text{ and } \mathcal{F}_m \text{ is not locally finite at } x\}$. Then $A_m$ is a closed subset of $X$ and $A_m \subset X_2 \setminus X_1$ by Lemma 1.2. Thus $A_m$ is a closed Lindelöf subspace of $X$. Thus there exists a countable family $\mathcal{U}_F \subset \mathcal{U}$ such that $A_m \subset \bigcup \mathcal{U}_F$. Thus $\mathcal{F}_{m+1} \setminus \bigcup \mathcal{U}_F = \bigcup \{F \setminus \bigcup \mathcal{U}_F: F \in \mathcal{F}_1\}$. The family $\{\bigcup \mathcal{U}_F: F \in \mathcal{F}_1\}$ is locally finite in $X$, so $\{\bigcup \mathcal{U}_F: F \in \mathcal{F}_1\}$ is countable since $e(X) = \omega$.

For each $F \in \mathcal{F}_{m+1}$, there exists some $U_F \subset \mathcal{U}$ such that $F \setminus \bigcup \mathcal{U}_F \subset U_F$ if $F \setminus \bigcup \mathcal{U}_F \neq \emptyset$. Thus there exists a countable family $\mathcal{U}_F^{\omega} \subset \mathcal{U}$ such that $F \setminus \bigcup \mathcal{U}_F^{\omega} \subset U_\eta \subset U \subset \mathcal{U}$ and $\|U_\eta\| \leq \omega$ such that $F \subset \bigcup \mathcal{U}_F^{\omega}$.

For each $n \in N$, we assume that there is a countable subfamily $\mathcal{U}_n \subset \mathcal{U}$ such that $F_{nk} \subset \bigcup \mathcal{U}_n: j \leq k$. For each $k \leq n$, we have $F_{m(n+1)+1} \setminus F_{mn} = \bigcup \mathcal{F}_{m(n+1)+1}$ and $F_{mn} \subset \bigcup \mathcal{U}_n: j \leq n$. If $\mathcal{F}_{m(n+1)+1} = \{F \setminus \bigcup \mathcal{U}_n: j \leq n\}$, then $F_{m(n+1)+1}$ is a discrete family of subsets of $X$. So $F_{m(n+1)+1}$ is a discrete family in $X_1$. Let $A_{m(n+1)+1} = \{x: x \in X \text{ and } F_{m(n+1)+1} \text{ is not locally finite at } x\}$. Thus $A_{m(n+1)+1}$ is a closed subset of $X$ and $A_{m(n+1)+1} \subset X_2 \setminus X_1$ by Lemma 1.2. So $A_{m(n+1)+1}$ is a closed Lindelöf subspace of $X$. Thus $A_{m(n+1)+1}$ can be covered by a countable family $U_{n+1} \subset \mathcal{U}$. So $F_{m(n+1)+1} = \{B \setminus \bigcup \mathcal{U}_n: B \in F_{m(n+1)+1}\}$ is a locally finite family in $X$. Since $e(X) = \omega$, we have $|F_{m(n+1)+1}| \leq \omega$. For each $C \in F_{m(n+1)+1}$ there is some $U_C \subset \mathcal{U}$ such that $C \subset U_C$. So there is a countable subfamily $U_{n+1}^{\omega} \subset \mathcal{U}$ such that $\bigcup U_{n+1}^{\omega} \subset U_{n+1}$. If $U_{n+1} = U_{n+1}^{\omega} \subset U_{n+1} \subset \mathcal{U}$ and $|U_{n+1}| \leq \omega$ such that $F_{m(n+1)+1} \subset \bigcup \mathcal{U}_n: j \leq n+1$. Thus $\{\bigcup \mathcal{U}_m: m \in N\}$ can be covered by a countable subfamily $\mathcal{U}_n$ of $\mathcal{U}$ by induction.

The set $Y = \bigcup \{U_m: m \in N\}$, thus there exists some countable subfamily $\mathcal{U}_F \subset \mathcal{U}$ such that $Y \subset \bigcup \mathcal{U}_F$. □

**Definition 1.4.** (Cf. [19].) A space $X$ is called weak $\tilde{\theta}$-refinable if for any open cover $\mathcal{U}$ of $X$ there is an open refinement $\mathcal{V} = \bigcup \{V_i: i \in N\}$ such that:

1. $\bigcup \{V_i: i \in N\}$ is point-finite.
2. For any $x \in X$ there is some $i \in N$ such that $1 \leq \text{ord}(x, V_i) < \infty$.

The refinement $\mathcal{V} = \bigcup \{V_i: i \in N\}$ of $\mathcal{U}$ is said to be a weak $\tilde{\theta}$-refinement of $\mathcal{U}$. □
Lemma 1.5. (Cf. [19, Theorem 3.3.]) If X is a weak \( \bar{\theta} \)-refinable space and \( e(X) = \omega \), then X is Lindelöf.

Theorem 1.6. If X is the union of a finite collection of weak \( \bar{\theta} \)-refinable subspaces and \( e(X) = \omega \), then X is a Lindelöf space.

Proof. Let \( X = \bigcup \{X_i : i \leq k \} \) and \( k \in \mathbb{N} \), where \( X_i \) is a weak \( \bar{\theta} \)-refinable subspace of X for each \( i \leq k \).

The proof is by induction.

1. If \( k = 1 \) then X is Lindelöf by Lemma 1.5.

2. Assume now that the statement holds for \( k = n \), for some \( n \in \mathbb{N} \), and let us show that it is also true for \( k = n + 1 \).

Assume \( X \subseteq \bigcup \{X_i : i \leq n + 1 \} \). If \( X_1 = X_1 \) and \( X_2 = \bigcup \{X_i : 2 \leq i \leq n + 1 \} \) then \( X = X_1 \cup X_2 \). If C is a closed subset of X and \( C \subseteq X_1 \cup X_2 \) then \( C = \bigcup \{X_i \cap C : 2 \leq i \leq n + 1 \} \). The set \( X_i \cap C \) is a weak \( \bar{\theta} \)-refinable subspace for each \( 2 \leq i \leq n + 1 \) and \( e(C) = \omega \), so we know that C is Lindelöf by induction.

Let \( \mathcal{U} \) be an open cover of X, so \( \mathcal{U} (X_1) = \{U \cap X_1 : U \in \mathcal{U} \} \) is an open cover of \( X_1 \). The subspace \( X_1 \) is weak \( \bar{\theta} \)-refinable, thus \( \mathcal{U} (X_1) \) has a weak \( \bar{\theta} \)-refinement \( \mathcal{V} = \bigcup \{V_n : n \in \mathbb{N} \} \). Let \( V_n = \bigcup V_n \) for each \( n \in \mathbb{N} \). Let \( E_n = \{x : x \in X_1 \text{ and } \theta_n(x) \leq m \} \) and let \( E_n \setminus E_n - 1 = \bigcup \mathcal{F}_m \) for each \( m \in \mathbb{N} \), where \( E_0 = \emptyset \) and \( \mathcal{F}_m \) is a discrete cover of \( E_n \setminus E_n - 1 \) and for each \( F \in \mathcal{F}_m \) there exists some \( V_n \in \mathcal{V} \) for each \( i \leq m \) such that \( F = \bigcap \{V_n : i \leq m \} \). Let \( E_1 \) be the first set \( \mathcal{U} (X_1) \) that is a Lindelöf space.

Thus \( \mathcal{U} \) is a Lindelöf space.

Assume we use \( E_1 \subseteq \mathcal{U} \) and \( |E_1| \leq \omega \) for each \( i \leq n \) such that \( E_1 \subseteq \bigcup \{U_i : i \leq n \} \).

Let \( \bigcup \mathcal{F}_{n+1} = \bigcup \{F \subseteq \bigcup \mathcal{U}_i : i \leq n \} : F \in \mathcal{F}_{n+1} \). Then \( \mathcal{F}_{n+1} \) is a discrete family of closed subsets of \( X_1 \). If \( A_{n+1} = \{x : x \in X_1 \text{ and } \theta_n(x) \neq \omega \} \) is not locally finite at x), then \( A_{n+1} \subseteq X_1 \setminus \mathcal{U}_n \) and \( A_{n+1} \) is a closed subset of \( X_1 \). Then \( A_{n+1} \) is a Lindelöf subspace of \( X_1 \). So there exists a countable subfamily \( \mathcal{U}_{n+1} \) of \( \mathcal{U} \) covers \( A_{n+1} \). For each \( B \subseteq \mathcal{U}_{n+1} \), the set \( B \setminus \bigcup \mathcal{U}_{n+1} \) is a closed subset of \( X_1 \) and \( B \setminus \bigcup \mathcal{U}_{n+1} \subseteq \bigcup \mathcal{U}_{n+1} \) is a locally finite family in \( X_1 \). Thus \( |B \setminus \bigcup \mathcal{U}_{n+1} : B \subseteq \mathcal{U}_{n+1} \} | \leq \omega \).

For each \( i \leq n \) and for each \( m \leq n + 1 \), there is some \( i_m \in \mathbb{N} \) such that \( C \subseteq \bigcap \{V_n : m \leq m + 1 \} \) and for each \( x \) \( B \subseteq \mathbb{N} \) there is some \( m \leq n + 1 \) such that \( 1 \leq \theta_n(x, V_{i_m}) \leq 2 \).

By induction, the set \( X_i \subseteq \mathcal{U} \) can be covered by a countable subfamily \( \mathcal{U}^* \cup \mathcal{U}^{**} \) of \( \mathcal{U} \). Since \( X \subseteq \bigcup \mathcal{U} \) and \( X \subseteq \mathcal{U} \) is a closed subspace of \( X \), the set \( X \setminus \mathcal{U}^* \subseteq \bigcup \mathcal{U}^* \) is a Lindelöf subspace of \( X \). So \( X \setminus \mathcal{U}^* \) can be covered by a countable subfamily \( \mathcal{U}^{**} \) of \( \mathcal{U} \).

So we have proved that \( X \) can be covered by a countable subfamily \( \mathcal{U}^* \cup \mathcal{U}^{**} \) of \( \mathcal{U} \). Thus X is a Lindelöf space.

Corollary 1.7. If \( X \) is a countably compact space and \( X = \bigcup \{X_i : i \leq k \} \), \( k \in \mathbb{N} \), where \( X_i \) is a weak \( \bar{\theta} \)-refinable subspace for each \( i \leq k \), then \( X \) is compact.

In what follows, we will discuss the dually discrete property of the product of two ordinals.

2. On products of ordinals

Proposition 2.1. If \( X \) is a dually discrete space, then every closed subspace of \( X \) is dually discrete.

Proposition 2.2. If \( X = \bigcup \{X_i : i \in \mathbb{N} \} \), where \( X_i \) is a closed dually discrete subspace of \( X \) for each \( i \in \mathbb{N} \), then \( X \) is dually discrete.

Proof. Let \( \phi \) be any neighborhood assignment on \( X \). Then there is a discrete subspace \( D_1 \subseteq X_1 \) such that \( X_1 \subseteq \bigcup \{\phi(x) : x \in D_1 \} \). For each \( i > 1 \), we have a discrete subspace \( D_i \subseteq X_i \setminus \bigcup \{\phi(x) : x \in \bigcup D_i : j \leq i - 1 \} \) such that \( X_i \setminus \bigcup \{\phi(x) : x \in D_1 \} \). We can see that \( D = \bigcup \{D_i : i \in \mathbb{N} \} \) is a discrete subspace of \( X \) such that \( X = \bigcup \{\phi(x) : x \in D \} \) and hence \( X \) is dually discrete.

Proposition 2.3. If \( X = X_1 \times X_2 \), where \( X_i \) is a dually discrete subspace of \( X \) for each \( i \in \{1, 2 \} \) and \( X_1 \) is closed in \( X \), then \( X \) is dually discrete.

Lemma 2.4. (Cf. [1, Corollary 3.10.]) If \( \lambda \) and \( \mu \) are regular cardinals, then \( \lambda \times \mu \) is dually discrete.
Lemma 2.5. (Cf. [8, Proposition 2.4].) If \( f : X \to Y \) is a perfect map, and \( Y \) is a dually discrete space, then \( X \) is dually discrete.

Lemma 2.6. (Cf. [8, Corollary 3.4].) Every ordinal is dually discrete.

Let \( \mu \) be an ordinal with \( cf \mu \geq \omega \), where \( cf \mu \) denotes the cofinality of \( \mu \). A strictly increasing function \( M : cf \mu \to \mu \) is said to be normal if \( M(\lambda) = \sup\{M(\lambda') : \lambda' < \lambda \} \) for each limit ordinal \( \lambda < cf \mu \), and \( \mu = \sup\{M(\lambda) : \lambda < cf \mu \} \) (cf. [12]).

Theorem 2.7. The product of two ordinals is dually discrete.

Proof. Let \( \alpha \times \beta \) be the product of two ordinals, \( \alpha \) and \( \beta \). If \( \alpha = 0 \), then the product is trivially dually discrete. If \( \alpha \neq 0 \), then we can write \( \alpha = \omega \cdot \tau + 1 \), where \( \tau \) is an ordinal less than \( \omega \).

The product \( \alpha \times \beta \) is then dually discrete because it is the product of two ordinals, which is dually discrete by Lemma 2.6.

Thus, the product of two ordinals is dually discrete.
In what follows, we will show that if $D$ is a discrete subspace of $X$, it is enough to prove that for each $x \in D_2$ there is some open set $T_x$ of $\mu \times v$ such that $x \in T_x$ and $T_x \cap D_1 = \emptyset$.

For each $x \in F$, we let $x = (a, b)$, where $a \in \mu$ and $b \in v$. For each $(\alpha, \beta) \in cf \mu \times cf v$, we have $(M(\alpha), N(\beta)) \notin F$. Thus $(M(\alpha), N(\beta)) \neq (a, b)$.

(1) Assume $a \neq M(\alpha)$ for each $\alpha \in cf \mu$.

If $[0, a] \cap \{M(\alpha): \alpha \in cf \mu\} = \emptyset$ then let $T_x = [0, a] \times v$. Then $x \in T_x$ and $T_x \cap ((M(\alpha), N(\beta)): \alpha \in cf \mu$ and $\beta \in cf v]) = \emptyset$.

So $T_x \cap D_1 = \emptyset$.

If there is some $\beta \in cf \mu$ such that $M(\beta) < a$. Let $A_\beta = [\beta, M(\beta) < a]$ and let $\gamma = sup A_\beta$. If $\gamma$ is a successor ordinal then $\gamma \in A_\beta$. Thus $M(\gamma) < a$. We let $T_x = (M(\gamma), a] \times v$. Then $x \in T_x$ and $T_x \cap D_1 = \emptyset$.

If $\gamma$ is a limit ordinal then $a \neq sup(M(\beta)): \beta \in A_\beta)$. The reason is that sup$(M(\beta): \beta \in A_\beta) = M(\gamma)$ by normal property of $M$. Thus sup$(M(\beta): \beta \in A_\beta) = M(\gamma) < a$. So we let $T_x = (M(\gamma), a] \times v$. Then $T_x \cap D_1 = \emptyset$ and $x \in T_x$.

(2) If $a = M(\alpha)$ for some $\alpha \in cf \mu$ then $b \neq N(\beta)$ for each $\beta \in cf v$. Similarly, we have an open set $T_x$ such that $T_x \cap D_1 = \emptyset$.

Thus for each $x \in F$, there is an open set $T_x$ of $\mu \times v$ such that $T_x \cap D_1 = \emptyset$. Thus $D = D_1 \cup D_2$ is a discrete subspace of $\mu \times v$ such that $\mu \times v = \bigcup (\phi(x): x \in D)$ and hence $\mu \times v$ is dually discrete.

Lemma 2.8. (cf. [1, Corollary 3.11]) A finite product of regular cardinals is dually discrete.

Theorem 2.9. A finite product of ordinals is dually discrete.

Proof. This is by induction and since the proof is analogous to that of Theorem 2.7, we omit it.

In [15], it was proved that if $X$ is countably compact and $X = \bigcup (X_n): n \in N$, where $X_n$ is a $D$-space for each $n \in N$, then $X$ is compact. In [13], it was pointed that $\omega_1$ is dually discrete. So $\omega_1$ is a dually discrete countably compact space, but it is not compact. A space is discretely complete if every discrete subspace of $X$ has a complete accumulation point in $X$ (cf. [1, p. 1421]). We know that every discretely complete space is countably compact. In [1], it was proved that a space is compact if and only if it is dually discrete and discretely complete. In fact, using a method similar to that of [1], we have:

Theorem 2.10. If $X$ is discretely complete and $X = \bigcup (X_n): n \in N$, where $X_n$ is a dually discrete subspace of $X$ for each $n \in N$, then $X$ is compact.

Proof. Suppose $X$ is not compact. Then for some cardinal $\kappa$ there is an open cover $U = \{U_\alpha: \alpha \in \kappa\}$ of $X$ which has no finite subcover. We may assume that $\kappa$ is minimal with respect to this property. So $U$ has no subcover of cardinality less than $\kappa$. $X$ is countably compact, so $cf(\kappa) > \omega$. Let $\phi(x) = U_\alpha$, where $\alpha_\kappa = min\{\alpha: x \in U_\alpha$ and $\alpha \in \kappa\}$. For each $n \in N$, there is a discrete subspace $D_n \subset X_n$ such that $X_n \subset \bigcup \{\phi(x): x \in D_n\}$ since $X_n$ is dually discrete. So there is some $n \in N$ such that $|D_n| = \kappa$. We may assume $\phi(x) \neq \phi(y)$ for any distinct points $x$ and $y$ of $D_n$. The set $D_n$ has a complete accumulation point $z \in X$. There is some $\beta \in \kappa$ such that $z \in U_\beta$. Thus $|U_\beta \cap D_n| = \kappa$. So for each $y \in U_\beta \cap D_n$, we have $\alpha_y \leq \beta$ which is a contradiction.

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