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Prandtl's formulation for the Saint–Venant's torsion of homogeneous piezoelectric beams

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ABSTRACT

The Saint–Venant torsional problem for homogeneous, monoclinic piezoelectric beams is formulated in terms of Prandtl's stress function and electric displacement potential function. The analytical approach presented in this paper generalizes the known formulation of Prandtl's solution which refers to homogeneous elastic beams. The Prandtl's stress function and electric displacement potential function satisfy the so called coupled Dirichlet problem (CDP) in the cross-sectional domain. A direct and a variational formulation are developed. Exact analytical solutions for solid elliptical cross-section and hollow circular cross-section and an approximate solution based on a variational formulation for thin-walled closed cross-section are presented.

1. Introduction

Saint–Venant's torsion of a homogeneous, isotropic, elastic cylindrical body is a classical problem of elasticity (Lurje, 1970; Sadd, 2005; Sokolnikoff, 1956), which was solved using a semi-inverse method by assuming a state of pure shear in the cylindrical body such that it gives rise to a resultant torque over the end cross-sections. Extension of more complicated cases of anisotropic or non-homogeneous materials has been considered by Lekhnitskii (1971, 1981), Rooney and Ferrari (1995), Daví (1996), Bisegna (1998, 1999), Horgan and Chan (1999), Rovenski et al. (2006, 2007) and Horgan (2007). In all cases of Saint–Venant's torsion mention above the states of strains and stresses are independent of the axial coordinate.

In this paper, the Saint–Venant's torsional problem is formulated in the framework of the linear theory of piezoelectricity for homogeneous, monoclinic piezoelectric cylinders with arbitrary cross-sectional geometry. The specified loads considered in this study are torque resultants prescribed at the cylinder's end cross-sections. Following Saint–Venant it is assumed that the character of elastic and electric fields depends only in a secondary way of the exact distribution of the tractions on the ends of cylinder so that the end torques are introduced in an integral manner in the case of torsional problem. The formulation of the Saint–Venant's theory of uniform torsion for the piezoelectric beams has been analysed by Bisegna (1998, 1999), Daví (1996), Rovenski et al. (2006, 2007), Yang (2005) and Zehetner (2008). Bisegna (1998, 1999), Daví (1996) and Rovenski et al. (2006, 2007) studied the Saint–Venant's problem including axial force, bending and torsional moments, and shear forces in the framework of linear theory of piezoelectricity for homogeneous, monoclinic piezoelectric cylinders. A relaxed version of this problem including the torsion is also formulated and solved by Bisegna (1998, 1999). The papers by Bisegna (1998, 1999) use the Prandtl' s stress function and electric displacement potential function formulation for simply-connected cross-sections which is based on Clebsch-type hypotheses. Daví (1997) obtained the coupled boundary-value problem for the torsional function and for the cross-sectional electric potential function from a constrained three-dimensional static problem by the application of the usual assumptions of the Saint–Venant's theory. Rovenski et al. (2006, 2007) give the torsional and electric potential function formulation of the Saint–Venant's torsional problem for monoclinic piezoelectric beams. In papers by Rovenski et al. (2006, 2007) a coupled Neumann problem is derived for the torsional and electric potential functions, where exact and numerical solutions for elliptical and rectangular cross-sections are presented. The compensation of torsional deformations in rods with the help of thin integrated piezoelectric actuator layers based on the Saint–Venant's theory of uniform torsion has been analysed by Zehetner (2008). Torsion of circular cylinders made of ceramics with tangential poling is studied by Yang (2005). In the book by Yang (2005), the cylindrical surfaces are unelectroded while the end faces are electroded, the end electrodes can be either opened or shorted. For both two cases the expressions of torsional rigidity are derived by Yang (2005).

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In the present paper, the Prandtl's stress function and electric displacement potential function formulation is developed for multiply-connected cross-sections which leads to a coupled Dirichlet boundary-value problem (CDP). Relationships between the Prandtl's stress function-electric displacement potential function and the torsional function, electric potential function are derived. A stress-electric displacement based variational formulation is used to derive a Bredt-type solution for thin-walled close cross-sections. For simply-connected cross-sections a direct formulation of the Saint–Venant's torsional problem which uses the Prandtl's stress function and electric displacement potential function was presented by Bisegna (1998, 1999). The Bisegna's results about the Saint–Venant's torsion of solid cross-sections are recovered in Section 3 of the present paper. The structure of the present study is: Section 2 formulates the governing field equations and boundary conditions of the Saint–Venant's torsional problem for piezoelectric beams by the use of results of Rovenski et al. (2006, 2007). In Section 3, the Prandtl's stress function and electric displacement potential function are introduced. Here, the expressions for torsional and electric potential functions in terms of Prandtl's stress function and eletric displacement potential function are also presented with the equations of CDP. Formulas for torsional rigidity and electric torsional rigidity are derived in Section 4. A variational formulation of the Saint–Venant's torsional problem for beams made of homogeneous piezoelectric materials is presented in Section 5. Section 6 contains three examples: exact analytical solutions for solid elliptical cross-section and hollow circular cross-section and an approximate solution for thin-walled closed cross-section, which is based on the demonstrated variational formulation. Some conclusions are given in Section 7.

2. Saint–Venant torsion of piezoelectric beams

The analytical solution of the Saint–Venant's torsional problem originates form the next displacement and electric potential hypothesis

$$u = -\vartheta yz, \quad v = \vartheta xz, \quad w = \vartheta\omega(x,y), \quad \varphi = \vartheta\phi(x,y), \tag{1}$$

where u, v, w are the displacements in x, y, and z directions (Fig. 1), ϑ is the rate of twist with respect to axial coordinate z, $\omega = \omega(x,y)$ is the torsional function and $\varphi = \varphi(x,y)$ is the electric potential function (Rovenski et al., 2006, 2007). Fig. 1 shows the considered twisted piezoelectric beam whose cross-section A may be simply connected or multiply-connected bounded plane domain.

The boundary curve of A is indicated by $\partial A = \partial A_0 \cup \partial A_1 \cup \partial A_2 \ldots \partial A_p$, where ∂A_0 is the outer boundary curve and the inner boundary curves are $\partial A_i(i = 1 - p)$, furthermore the outward unit normal vector to ∂A is denoted by $\mathbf{n} = n_x\mathbf{e}_x + n_y\mathbf{e}_y$ (Fig. 2).

The unit vectors of the Cartesian coordinate system used are $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$, the length of the beam is L. The strain–displacement and

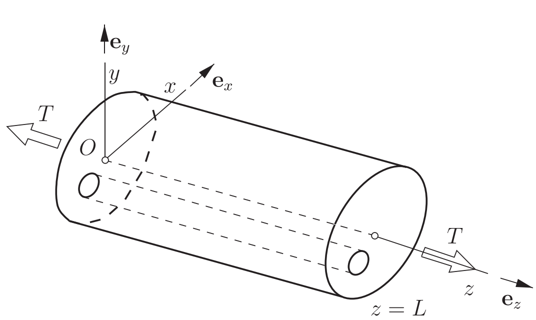

Fig. 1. Saint–Venant torsion of piezoelectric beam.

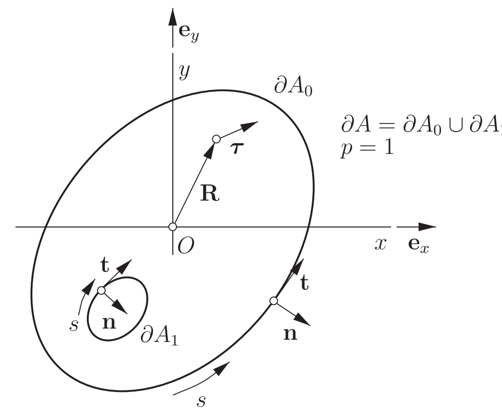

Fig. 2. Cross-section and its geometry.

electric field-electric potential relationships give (Cady, 1964; Yang, 2005, 2006; Rovenski et al., 2006, 2007)

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0, \quad \gamma_{xz} = \vartheta\left(\frac{\partial\omega}{\partial x} - y\right), \quad \gamma_{yz} = \vartheta\left(\frac{\partial\omega}{\partial y} + x\right), \tag{2}$$

$$E_x = -\vartheta\frac{\partial\phi}{\partial x}, \quad E_y = -\vartheta\frac{\partial\phi}{\partial y}, \quad E_z = 0. \tag{3}$$

In Eq. (2), ε_x, ε_y, ε_z, are the longitudinal strains, γ_{xy}, γ_{yz}, γ_{xz}, are the shearing strains, and in Eq. (3) E_x, E_y, E_z are the components of electric field vector $\mathbf{E}$. In the present problem the mechanical equilibrium and Gauss equation can be written in the form (Rovenski et al., 2006, 2007)

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} = 0, \quad \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0 \quad \text{in } A, \tag{4}$$

where τ_{xz}, τ_{yz} are the shearing stresses D_x, D_y are the components of electric displacement vector $\mathbf{D} = D_x\mathbf{e}_x + D_y\mathbf{e}_y + D_z\mathbf{e}_z$. Here, we note (Rovenski et al., 2006, 2007)

$$\sigma_x = \sigma_y = \sigma_z = \tau_{x_y} = 0, \quad \text{and} \quad D_z = 0, \tag{5}$$

in all points of the twisted piezoelectric beam, where σ_x, σ_y, σ_z are the normal stresses and according to Eq. (4) there are no present body forces and body charges. The mantle of the beam is stress and charge free, that is we have

$$\tau_{xz}n_x + \tau_{yz}n_y = 0, \quad D_xn_x + D_yn_y = 0 \quad \text{on} \quad \partial A. \tag{6}$$

The g-form of the constitutive equations (Ikeda, 1990) is used which gives the next result for the torsional problem assuming that the considered beam made of monoclinic piezoelectric material

$$\begin{aligned}\gamma_{xz} &= s_{55}\tau_{xz} + s_{45}\tau_{yz} + g_{15}D_x + g_{25}D_y,\\ \gamma_{yz} &= s_{45}\tau_{xz} + s_{44}\tau_{yz} + g_{14}D_x + g_{24}D_y,\end{aligned} \tag{7}$$

$$\begin{aligned}E_x &= -\frac{\partial\varphi}{\partial x} = -g_{15}\tau_{xz} - g_{14}\tau_{yz} + \eta_{11}D_x + \eta_{12}D_y,\\ E_y &= -\frac{\partial\varphi}{\partial y} = -g_{25}\tau_{xz} - g_{24}\tau_{yz} + \eta_{12}D_x + \eta_{22}D_y.\end{aligned} \tag{8}$$

In Eqs. (7) and (8) s_{55}, s_{45}, s_{44} are the flexibility (elastic complience) coefficients and g_{15}, g_{25}, g_{14}, g_{24} are the piezoelectric impermeability coefficients and η_{11}, η_{12}, η_{22} are the dielectric impermeability coefficients (Cady, 1964; Ikeda, 1990; Yang, 2005, 2006).

3. Prandtl's stress function and electric displacement potential function formulation

The equation of mechanical equilibrium and Gauss Eq. (4) and boundary conditions formulated in Eq. (6) can be written in the form

$$\nabla\cdot\boldsymbol{\tau}=0,\quad \nabla\cdot\mathbf{D}=0\quad \text{in } A,$$
$$\mathbf{n}\cdot\boldsymbol{\tau}=0,\quad \mathbf{n}\cdot\mathbf{D}=0\quad \text{on}\quad \partial A, \tag{9}$$

where $\nabla=\frac{\partial}{\partial x}\mathbf{e}_x+\frac{\partial}{\partial y}\mathbf{e}_y$ is del operator, $\boldsymbol{\tau}=\tau_{xz}\mathbf{e}_x+\tau_{yz}\mathbf{e}_y$, $\mathbf{D}=D_x\mathbf{e}_x+D_y\mathbf{e}_y$ and in Eq. (9), the scalar product of two vectors is denoted by dot. Let $U=U(x,y)$ and $F=F(x,y)$ be such functions whose second order mixed partial derivatives are the same according to Young' s theorem, but they are otherwise arbitrary functions. The general solution of Eq. $(9)_{1,2}$ by these functions can be represented as (Prandtl, 1903)

$$\boldsymbol{\tau}=\vartheta\nabla U\times\mathbf{e}_z,\quad \mathbf{D}=\vartheta\nabla F\times\mathbf{e}_z, \tag{10}$$

where the cross between two vectors is the sign of vectorial product. $U=U(x,y)$ is called the Prandtl's stress function and the name of $F=F(x,y)$ is the electric displacement potential function. From boundary condition $(9)_3$ it follows that (Fig. 2)

$$\mathbf{n}\cdot\boldsymbol{\tau}=\vartheta(\mathbf{e}_z\times\mathbf{n})\cdot\nabla U=\vartheta\mathbf{t}\cdot\nabla U=0\quad\text{on}\quad\partial A, \tag{11}$$

where, $\mathbf{t}=\mathbf{e}_z\times\mathbf{n}$ is the unit tangential vector to the boundary curve ∂A (Fig. 2). It means that

$$U=U_i=\text{constant}\quad\text{on}\quad\partial A(i=0-p). \tag{12}$$

Similar result can be derived for $F=F(x,y)$ from Eq. $(9)_4$:

$$F=F_i=\text{constant}\quad\text{on}\quad\partial A(i=0-p). \tag{13}$$

Since $U+C_U$, $F+C_F$ with arbitrary constants C_U, C_F and U, F give the same shearing stress and electric displacement vector fields it can be prescribed

$$U=U_0=0\quad\text{and}\quad F=F_0=0\quad\text{on}\quad\partial A_0. \tag{14}$$

The combination of Eqs. (2) and (3) with Eqs. (7) and (8) and Eq. (10) gives

$$\mathbf{e}_z\times\nabla\omega-\mathbf{R}=\mathbf{M},\quad \mathbf{M}=\mathbf{S}\cdot\nabla U+\mathbf{G}\cdot\nabla F,\quad \mathbf{R}=x\mathbf{e}_x+y\mathbf{e}_y, \tag{15}$$
$$\mathbf{e}_z\times\nabla\phi=\mathbf{N},\quad \mathbf{N}=\mathbf{G}^T\cdot\nabla U-\mathbf{H}\cdot\nabla F. \tag{16}$$

In Eqs. (15) and (16) matrixes of Cartesian tensors $\mathbf{S}$, $\mathbf{G}$ and $\mathbf{H}$ are as follows

$$\mathbf{S}=\begin{pmatrix} s_{44} & -s_{45}\\ -s_{45} & s_{55}\end{pmatrix},\quad \mathbf{G}=\begin{pmatrix} g_{24} & -g_{14}\\ -g_{25} & g_{15}\end{pmatrix},\quad \mathbf{H}=\begin{pmatrix} \eta_{22} & -\eta_{12}\\ -\eta_{12} & \eta_{11}\end{pmatrix}, \tag{17}$$

and $\mathbf{G}^T$ is the transpose of $\mathbf{G}$. Eqs. (15) and (16) formulate the connection in differential form between the torsional function $\omega=\omega(x,y)$, electric potential function $\phi=\phi(x,y)$ and Prandtl's stress function $U=U(x,y)$, electric displacement potential function $F=F(x,y)$. Next, starting from Eqs. (15) and (16) two integral type relations will be derived. Fig. 3 shows the curve $\widehat{P_0P}$ whose all points are in $A\cup\partial A$. From Eq. (15) it follows that (Fig. 3)

$$-\mathbf{t}\cdot\nabla\omega-\mathbf{R}\cdot\mathbf{n}=\mathbf{n}\cdot(\mathbf{S}\cdot\nabla U+\mathbf{G}\cdot\nabla F), \tag{18}$$

since $\mathbf{t}=\mathbf{e}_z\times\mathbf{n}$. Integration of Eq. (18) along the curve $\widehat{P_0P}$ yields (Fig. 3)

$$\omega(P)-\omega(P_0)=-\int_{\widehat{P_0P}}\mathbf{n}\cdot(\mathbf{S}\cdot\nabla U+\mathbf{G}\cdot\nabla F)\mathrm{d}s-a(P_0,P),$$
$$a(P_0,P)=\int_{\widehat{P_0P}}\mathbf{n}\cdot\mathbf{R}\mathrm{d}s. \tag{19}$$

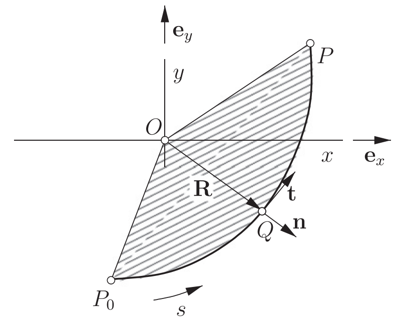

Fig. 3. $\widehat{P_0P}$ curve in the cross-sectional domain.

By the same method as which was used to obtain Eq. (19) starting from Eq. (16) the expression of electric potential in terms of U and F is derived as

$$\phi(P)-\phi(P_0)=-\int_{\widehat{P_0P}}\mathbf{n}\cdot(\mathbf{G}^T\cdot\nabla U-\mathbf{H}\cdot\nabla F)\mathrm{d}s. \tag{20}$$

In Eqs. (19) and (20) s is an arc-length defined on curve $\widehat{P_0P}$ and the equation of curve $\widehat{P_0P}$ is $\vec{OQ}=\mathbf{R}(s)$ (Fig. 3). The torsional function $\omega=\omega(x,y)$ and the electric potential function $\phi=\phi(x,y)$ are one valued functions so that we have

$$\oint_c\mathbf{t}\cdot\nabla\omega\mathrm{d}s=0,\quad \oint_c\mathbf{t}\cdot\nabla\phi\mathrm{d}s=0, \tag{21}$$

for any closed curve c which is in $A\cup\partial A$. A detailed form of Eq. (21) is as follows

$$\oint_c\mathbf{n}\cdot(\mathbf{S}\cdot\nabla U+\mathbf{G}\cdot\nabla F)\mathrm{d}s=-2A(c),$$
$$\oint_c\mathbf{n}\cdot(\mathbf{G}^T\cdot\nabla U-\mathbf{H}\cdot\nabla F)\mathrm{d}s=0, \tag{22}$$

where $A(c)$ is the area enclosed by curve c and $\mathbf{n}$ is the outer unit normal vector to the closed curve c. The local and global conditions of single valuedness for functions $\omega=\omega(x,y)$ and $\phi=\phi(x,y)$ with the Dirichlet boundary conditions (12)–(14) leads to the next (CDP) for $U=U(x,y)$ and $F=F(x,y)$:

$$\nabla\cdot(\mathbf{S}\cdot\nabla U+\mathbf{G}\cdot\nabla F)=-2,\quad \nabla\cdot(\mathbf{G}^T\cdot\nabla U-\mathbf{H}\cdot\nabla F)=0\quad\text{in } A, \tag{23}$$

$$U=0,\quad F=0\quad\text{on}\quad\partial A_0, \tag{24}$$

$$U=U_i=\text{constant}\quad\text{on}\quad\partial A_i\ F=F_i=\text{constant}\quad\text{on}\quad\partial A_i(i=1-p), \tag{25}$$

$$\oint_{\partial A_i}\mathbf{n}\cdot(\mathbf{S}\cdot\nabla U+\mathbf{G}\cdot\nabla F)\mathrm{d}s=2A_i,$$
$$\oint_{\partial A_i}\mathbf{n}\cdot(\mathbf{G}^T\cdot\nabla U-\mathbf{H}\cdot\nabla F)\mathrm{d}s=0(i=1-p). \tag{26}$$

In Eq. $(26)_1$, A_i is the area enclosed by closed curve $\partial A_i(i=1-p)$. The derivation of Eq. (23) is based on Eqs. (15) and (16). In the present problem the local conditions of single valuedness for ω and ϕ in terms of U and F are formulated in Eq. (23) and the global conditions of single valuedness for ω and ϕ in terms of U and F are given by Eq. (26). A detailed analysis of the local and global conditions for single valuedness for a function in multiply-connected plane domain can be found in the book by Muskhelishvili (1953).

4. Torsional rigidity

The torsional rigidity of the piezoelectric beam is obtained from following equation (Fig. 1)

$$T = \mathbf{e}_z \cdot \int_A \mathbf{R} \times \boldsymbol{\tau} dA = \vartheta \mathbf{e}_z \cdot \int_A \mathbf{R} \times (\nabla U \times \mathbf{e}_z) dA$$
$$= -\vartheta \int_A \mathbf{R} \cdot \nabla U dA. \tag{27}$$

The definition of the torsional rigidity is $S = T/\vartheta$ which gives

$$S = -\int_A \mathbf{R} \cdot \nabla U dA. \tag{28}$$

Following Lurje (1970), starting from Eq. (27) it can be proven that

$$S = 2\left(\int_A U dA + \sum_{i=1}^{p} U_i A_i\right). \tag{29}$$

By the application of Leibniz's rule of the differentiation of product function and divergence theorem of Gauss–Stokes and boundary conditions (25), it can be written

$$-\int_A \mathbf{R} \cdot \nabla U dA = -\int_A \nabla \cdot (\mathbf{R} U) dA + \int_A U \nabla \cdot \mathbf{R} dA$$
$$= -\oint_{\partial A} \mathbf{n} \cdot \mathbf{R} U ds + 2\int_A U dA$$
$$= 2\left(\int_A U dA + \sum_{i=1}^{p} U_i A_i\right), \tag{30}$$

which proves the validity of formula (29). Next, a new formula will be proven for the torsional rigidity. From Eq. (23) it follows that

$$2\int_A U dA + \int_A U \nabla \cdot \mathbf{M} dA + \int_A F \nabla \cdot \mathbf{N} dA$$
$$= 2\int_A U dA + \oint_{\partial A} \mathbf{n} \cdot \mathbf{M} U ds - \int_A \nabla U \cdot \mathbf{M} dA + \oint_{\partial A} \mathbf{n} \cdot \mathbf{N} F ds$$
$$- \int_A \nabla F \cdot \mathbf{N} dA = 2\left(\int_A U dA + \sum_{i=1}^{p} U_i A_i\right) - \int_A (\nabla U \cdot \mathbf{S} \cdot \nabla U + 2\nabla U \cdot$$
$$\times \mathbf{G} \cdot \nabla F - \nabla F \cdot \mathbf{H} \cdot \nabla F) dA = 0. \tag{31}$$

The combination of Eq. (29) with Eq. (31) gives

$$S = \int_A (\nabla U \cdot \mathbf{S} \cdot \nabla U + 2\nabla U \cdot \mathbf{G} \cdot \nabla F - \nabla F \cdot \mathbf{H} \cdot \nabla F) dA. \tag{32}$$

By the same method as it has been used to show the validity of formula (32) it can be proven that

$$\int_A F \nabla \cdot (\mathbf{G}^T \cdot \nabla U - \mathbf{H} \cdot \nabla F) dA = \int_A \nabla U \cdot \mathbf{G} \cdot \nabla F dA$$
$$- \int_A \nabla F \cdot \mathbf{H} \cdot \nabla F dA = 0. \tag{33}$$

The substitution of Eq. (33) into Eq. (32) yields to an another new formula for the torsional rigidity which was derived for simply connected cross-sections by Bisegna (1999)

$$S = \int_A (\nabla U \cdot \mathbf{S} \cdot \nabla U + \nabla F \cdot \mathbf{H} \cdot \nabla F) dA. \tag{34}$$

Formula (34) shows that S is always positive since $\mathbf{S}$ and $\mathbf{H}$ are second order two-dimensional positive definite tensors (Ikeda, 1990). This statement is in accordance with the mechanical meaning of S. Next, the concept of electrical torsional rigidity is introduced. The torque of electric displacement field is defined as (Rovenski et al., 2006, 2007)

$$T_D = \mathbf{e}_z \cdot \int_A \mathbf{R} \times (D_x \mathbf{e}_x + D_y \mathbf{e}_y) dA. \tag{35}$$

In the case of Saint–Venant torsion the next formula can be written

$$T_D = \vartheta \mathbf{e}_z \cdot \int_A \mathbf{R} \times (\nabla F \times \mathbf{e}_z) dA = -\vartheta \int_A \mathbf{R} \cdot \nabla F dA. \tag{36}$$

The electrical torsional rigidity S_D of a piezoelectric beam is obtained as (Rovenski et al., 2006, 2007)

$$S_D = \frac{T_D}{\vartheta} = -\int_A \mathbf{R} \cdot \nabla F dA. \tag{37}$$

By the same method as which was used to derive formula (29) it can be proven that

$$S_D = 2\left(\int_A F dA + \sum_{i=1}^{p} F_i A_i\right). \tag{38}$$

An another expression can be available for S_D by the use of following identity which follows from Eqs. (23)–(26)

$$2\left(\int_A F dA + \sum_{i=1}^{p} F_i A_i\right) - \int_A (\nabla F \cdot \mathbf{S} \cdot \nabla U + \nabla F \cdot \mathbf{G} \cdot \nabla F) dA$$
$$= 0. \tag{39}$$

Comparison of Eq. (38) with Eq. (39) gives the next result

$$S_D = \int_A (\nabla F \cdot \mathbf{S} \cdot \nabla U + \nabla F \cdot \mathbf{G} \cdot \nabla F) dA. \tag{40}$$

5. Variational formulation

In this section, a variational formulation is presented for the torsional deformation of homogeneous, linearly piezoelectric, monoclinic beams. Variational formulation uses the Prandtl's stress function and electric displacement potential function as the independent quantities of the considered variational function. The mechanical meaning of the presented variational functional is analysed. Variational principles are useful in solving many complicated boundary-value problems through two major approaches: approximate solutions of boundary-value problems and FEM implementations. Another important applications of variational principles are the derivation of approximate equations for given boundary-value problems (Reddy, 1986, 2002; Washizu, 1968).

Define for Saint–Venant torsion of piezoelectric beam the functional $\Pi = \Pi(U, F)$ as

$$\Pi(U,F) = \int_A (\nabla U \cdot \mathbf{S} \cdot \nabla U + 2\nabla U \cdot \mathbf{G} \cdot \nabla F - \nabla F \cdot \mathbf{H} \cdot \nabla F) dA$$
$$- 4\left(\int_A U dA + \sum_{i=1}^{p} U_i A_i\right). \tag{41}$$

The independent quantities subject to variation in functional (41) are U and F with the following subsidary conditions

$$U = 0 \quad \text{on} \quad \partial A_0, \quad U = U_i = \text{constant} \quad \text{on} \quad \partial A_i (i = 1 - p), \tag{42}$$
$$F = 0 \quad \text{on} \quad \partial A_0, \quad F = F_i = \text{constant} \quad \text{on} \quad \partial A_i (i = 1 - p). \tag{43}$$

Theorem 1. *The stationary condition of functional (41) with respect to U and F under the conditions (42) and (43) yields Eqs. (23) and (26).*

Proof. The stationary condition of $\Pi = \Pi(U,F)$ with respect to U and F under the conditions (42) and (43) can be written in the form

$$\begin{aligned}\delta\Pi &= 2\left\{\int_A [\nabla\delta U\cdot\mathbf{M}+\nabla\delta F\cdot\mathbf{N}]\mathrm{d}A - 2\left(\int_A \delta U\mathrm{d}A+\sum_{i=1}^{p}\delta U_i A_i\right)\right\}\\
&= 2\left\{\oint_{\partial A}\delta U\mathbf{n}\cdot\mathbf{M}\mathrm{d}s - \int_A \delta U\nabla\cdot\mathbf{M}\mathrm{d}A + \oint_{\partial A}\delta F\mathbf{n}\cdot\mathbf{N}\mathrm{d}s\right.\\
&\quad\left. - \int_A \delta F\nabla\cdot\mathbf{N}\mathrm{d}A - 2\left(\int_A \delta U\mathrm{d}A+\sum_{i=1}^{p}\delta U_i A_i\right)\right\}\\
&= 2\left\{\sum_{i=1}^{p}\left[\delta U_i\oint_{\partial A_i}\mathbf{n}\cdot\mathbf{M}\mathrm{d}s + \delta F_i\oint_{\partial A_i}\mathbf{n}\cdot\mathbf{N}\mathrm{d}s\right] - \int_A \delta U\nabla\cdot\mathbf{M}\mathrm{d}A\right.\\
&\quad\left. - \int_A \delta F\nabla\cdot\mathbf{N}\mathrm{d}A - 2\left(\int_A \delta U\mathrm{d}A+\sum_{i=1}^{p}\delta U_i A_i\right)\right\}\\
&= 2\left\{\sum_{i=1}^{p}\delta U_i\left[\oint_{\partial A_i}\mathbf{n}\cdot\mathbf{M}\mathrm{d}s - 2A_i\right] - \int_A \delta U[\nabla\cdot\mathbf{M}+2]\mathrm{d}A\right.\\
&\quad\left. + \sum_{i=1}^{p}\delta F_i\oint_{\partial A_i}\mathbf{n}\cdot\mathbf{N}\mathrm{d}s - \int_A \delta F\nabla\cdot\mathbf{N}\mathrm{d}A\right\} = 0. \qquad (44)\end{aligned}$$

Here, the followings are applied $\mathbf{S} = \mathbf{S}^T$, $\mathbf{H} = \mathbf{H}^T$. The validity of statement formulated in Theorem 1 follows from Eq. (44) and the fundamental lemma of the calculus of variation (Elsgolts, 1977) since δU and δF are arbitrary functions in A and δU_i, δF_i are arbitrary constants defined on $\partial A_i (i = 1 - p)$ and $\delta U = 0$, $\delta F = 0$ on ∂A_0. □

Theorem 2 gives the mechanical meaning of functional defined by Eq. (41).

Theorem 2. *Let $U = U(x,y)$ and $F = F(x,y)$ be the solution of the CDP formulated by* Eqs. (23)–(26), *then we have*

$$S = -\Pi(U,F). \qquad (45)$$

Proof. From Eqs. (29), (32) and (41) it follows that

$$\Pi(U,F) = S - 2S = -S, \qquad (46)$$

according to Eq. (45). □

6. Examples

6.1. Torsion of a solid elliptical cross-section

The equation of the boundary contour of the solid elliptical cross-section shown in Fig. 4 is

$$\alpha x^2 + 2\gamma xy + \beta y^2 = 1 \quad (x,y)\in\partial A, \qquad (47)$$

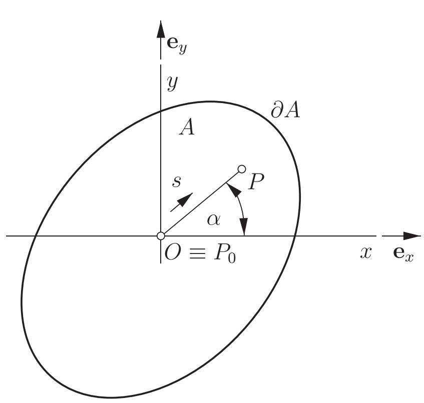

Fig. 4. Solid elliptical cross-section.

where $\alpha > 0$, $\alpha\beta - \gamma^2 > 0$. The assumed forms of the Prandtl's stress function and electric displacement potential function are as follows

$$\begin{aligned}U(x,y) &= C_U(\alpha x^2 + 2\gamma xy + \beta y^2 - 1), \quad F(x,y)\\ &= C_F(\alpha x^2 + 2\gamma xy + \beta y^2 - 1) \quad (x,y)\in A\cup\partial A, \qquad (48)\end{aligned}$$

where C_U and C_F are unknown constants. With arbitrary values of C_U and C_F the prescribed boundary conditions formulated in Eq. (24) are satisfied.

However, in order to be $U = U(x,y)$ and $F = F(x,y)$ given by Eq. (48) the valid Prandtl's stress function and electric displacement potential function for elliptical cross-section shown in Fig. 4, C_U and C_F must satisfy the following system of equations which is obtained from Eq. (23)

$$QC_U + GC_F = -1, \quad GC_U - HC_F = 0, \qquad (49)$$

$$\begin{aligned}Q &= s_{44}\alpha - 2s_{45}\gamma + s_{55}\beta, \quad G = g_{24}\alpha - (g_{14}+g_{25})\gamma + g_{15}\beta,\\ H &= \eta_{22}\alpha - 2\eta_{12}\gamma + \eta_{11}\beta. \qquad (50)\end{aligned}$$

Solution of system of Eqs. (49) for C_U and C_F is

$$C_U = -\frac{H}{QH+G^2}, \quad C_F = -\frac{G}{QH+G^2}. \qquad (51)$$

The stress and electric displacement fields can be computed by the applications of Eq. (10)

$$\tau_{xz} = -\frac{2\vartheta H}{QH+G^2}(\gamma x + \beta y), \quad \tau_{yz} = \frac{2\vartheta H}{QH+G^2}(\alpha x + \gamma y), \qquad (52)$$

$$D_x = -\frac{2\vartheta G}{QH+G^2}(\gamma x + \beta y), \quad D_y = \frac{2\vartheta G}{QH+G^2}(\alpha x + \gamma y). \qquad (53)$$

To obtain the torsional rigidity S formula (28) will be used. According to Eqs. (28) and $(48)_1$ it can be written

$$S = \frac{2H}{QH+G^2}\int_A(\alpha x^2 + 2\gamma xy + \beta y^2)\mathrm{d}A = \frac{H\pi}{(QH+G^2)\sqrt{\alpha\beta-\gamma^2}}. \qquad (54)$$

By the same method as is applied to get S it can be derived the following formula for the electrical torsional rigidity

$$S_D = \frac{G\pi}{(QH+G^2)\sqrt{\alpha\beta-\gamma^2}}. \qquad (55)$$

The determination of the torsional function $\omega = \omega(x,y)$ is based on Eq. (19), where $\overset{\frown}{P_0P}$ is an arbitrary curve in $\bar{A}$ connecting point P_0 to point P (Fig. 4). Let the point P_0 be the point O and let the curve $\overset{\frown}{P_0P}$ be the line segment $\overline{OP}$ as shown in Fig. 4. In this case

$$\begin{aligned}&s = \sqrt{x^2+y^2}, \quad x = s\cos\alpha, \quad y = s\sin\alpha,\\ &\mathbf{n} = -\sin\alpha\mathbf{e}_x + \cos\alpha\mathbf{e}_y. \qquad (56)\end{aligned}$$

In the present problem (Fig. 4) $\mathbf{R}\cdot\mathbf{n} = 0$ and $\alpha =$ constant on $\overline{P_0P}$. By a lengthy but elementary computations the following formula can be deduced from Eq. (19) for the torsional function of the solid elliptical cross-section shown in Fig. 4

$$\begin{aligned}\omega(x,y) = \frac{1}{QH+G^2}\{&H[x^2(-s_{45}\alpha + s_{55}\gamma) + xy(-s_{44}\alpha + s_{55}\beta)\\ &+ y^2(-s_{44}\gamma + s_{45}\beta)] + G[x^2(-g_{25}\alpha + g_{15}\gamma) + xy(-g_{24}\alpha + g_{14}\gamma\\ &- g_{25}\gamma + g_{15}\beta) + y^2(-g_{24}\gamma + g_{14}\beta)]\}, \qquad (57)\end{aligned}$$

assuming that $\omega(0,0) = 0$. By the same method as is used to obtain the torsional function $\omega = \omega(x,y)$, starting from Eq. (20) one gets for the electric potential function $\phi = \phi(x,y)$ the next result under the condition $\phi(0,0) = 0$

$$\begin{aligned}\phi(x,y) = \frac{1}{QH+G^2}\{&H[x^2(-g_{14}\alpha + g_{15}\gamma) + xy(-g_{24}\alpha + g_{25}\gamma - g_{14}\gamma + g_{15}\beta)\\ &+ y^2(-g_{24}\gamma + g_{25}\beta)] + G[x^2(-\eta_{11}\gamma + \eta_{12}\alpha) + xy(\eta_{22}\alpha - \eta_{11}\beta)\\ &+ y^2(\eta_{22}\gamma - \eta_{12}\beta)]\}. \qquad (58)\end{aligned}$$

6.2. Torsion of hollow circular cross-section

Fig. 5 shows the considered hollow circular cross-section which is bounded by two concentric circles with radii R_0 and R_1. Following the same technique as is applied in above it is assumed that

$$U(x,y) = C_U(R_0^2 - x^2 - y^2), \quad F(x,y) = C_F(R_0^2 - x^2 - y^2). \tag{59}$$

It is evident in the present problem

$$U = 0, \quad F = 0 \quad \text{on} \quad \partial A_0 \quad \text{and}$$
$$U = \text{constant} \quad \text{on} \quad \partial A_1, \quad F = \text{constant} \quad \text{on} \quad \partial A_1.$$

From Eq. (59) it is obtained that

$$C_U = \frac{1}{\bar{s} + \frac{\bar{g}^2}{\bar{\eta}}}, \quad C_F = \frac{1}{\bar{g} + \frac{\bar{s}\bar{\eta}}{\bar{g}}}, \quad \bar{s} = s_{44} + s_{55},$$
$$\bar{g} = g_{24} + g_{15}, \quad \bar{\eta} = \eta_{11} + \eta_{22}. \tag{61}$$

It can be shown by a direct computation the functions

$$U(x,y) = \frac{R_0^2 - x^2 - y^2}{\bar{s} + \frac{\bar{g}^2}{\bar{\eta}}}, \quad F(x,y) = \frac{R_0^2 - x^2 - y^2}{\bar{g} + \frac{\bar{s}\bar{\eta}}{\bar{g}}}, \tag{62}$$

satisfy the global conditions of compatibility formulated in Eq. (26). The stress and electric displacement fields are as follows

$$\tau_{xz} = -\frac{2\vartheta y}{\bar{s} + \frac{\bar{g}^2}{\bar{\eta}}}, \quad \tau_{yz} = \frac{2\vartheta x}{\bar{s} + \frac{\bar{g}^2}{\bar{\eta}}}, \quad D_x = -\frac{2\vartheta y}{\bar{g} + \frac{\bar{s}\bar{\eta}}{\bar{g}}}, \quad D_y = \frac{2\vartheta x}{\bar{g} + \frac{\bar{s}\bar{\eta}}{\bar{g}}}. \tag{63}$$

To get the values of torsional rigidity and electrical torsional rigidity we use Eqs. (28) and (37)

$$S = \frac{R_0^4 - R_1^4}{\bar{s} + \frac{\bar{g}^2}{\bar{\eta}}}\pi, \quad S_D = \frac{R_0^4 - R_1^4}{\bar{g} + \frac{\bar{s}\bar{\eta}}{\bar{g}}}\pi. \tag{64}$$

6.3. Torsion of thin-walled closed cross-section

Fig. 6 shows the cross-section of a thin-walled piezoelectric beam of closed profile. The middle curve of closed profile is denoted by c_m and the area enclosed by c_m is indicated by A_m. The arc-length defined on c_m is σ and the tangential and normal unit vectors to curve c_m are $\mathbf{m}$ and $\boldsymbol{\nu}$, respectively (Fig. 6). Equation of c_m is

$$\mathbf{R}_m = \vec{OP} = \mathbf{R}_m(\sigma), \tag{65}$$

so that we have

$$\mathbf{m} = \frac{\mathrm{d}\mathbf{R}_m}{\mathrm{d}\sigma}, \quad \boldsymbol{\nu} = \mathbf{m} \times \mathbf{e}_z. \tag{66}$$

An approximate solution of the torsional problem for thin-walled closed profile is formulated by the use of usual assumptions of

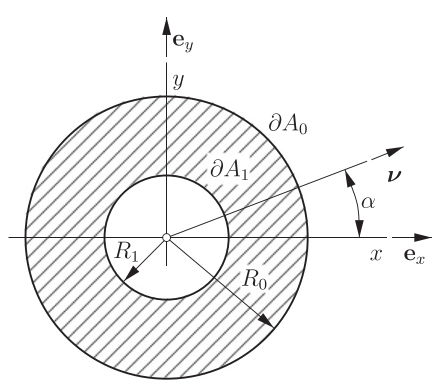

Fig. 5. Hollow circular cross-section.

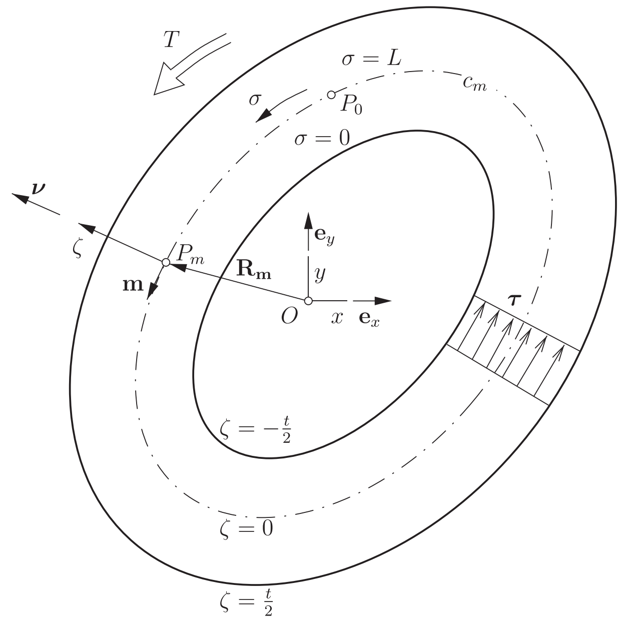

Fig. 6. Thin-walled closed cross-section.

Bredt's theory (Murray, 1985; Vlasov, 1961) and Theorem 1 of the present paper.

We assume that

- The shear stress and electric displacement vector do not depend on the thickness coordinate ς (Fig. 6) and they have the forms
$$\boldsymbol{\tau} = \tau(\sigma)\mathbf{m}, \quad \mathbf{D} = D(\sigma)\mathbf{m}. \tag{67}$$
- The Prandtl's stress function and electric displacement potential function according to Bredt's theory can be represented as
$$U = \frac{U_1}{2}\left(1 - \frac{2\varsigma}{t}\right), \quad F = \frac{F_1}{2}\left(1 - \frac{2\varsigma}{t}\right), \tag{68}$$
where $t = t(\sigma)$ is wall-thickness.
- The next approximations will be used for ∇U and ∇F

$$\nabla U = -\frac{U_1}{t}\boldsymbol{\nu}, \quad \nabla F = -\frac{F_1}{t}\boldsymbol{\nu}. \tag{69}$$

Following Murray (1985) it can be written

$$\int_A U\mathrm{d}A + A_1U_1 = U_1A_m, \quad \int_A F\mathrm{d}A + A_1F_1 = F_1A_m. \tag{70}$$

Introduction of assumptions above formulated into Eq. (41) we obtain

$$\Pi(U_1, F_1) = U_1^2 I_a + 2U_1F_1I_c - F_1^2 I_b - 4A_mU_1, \tag{71}$$

where

$$I_a = \oint_{c_m} \frac{a(\sigma)}{t}\mathrm{d}\sigma, \quad I_b = \oint_{c_m} \frac{b(\sigma)}{t}\mathrm{d}\sigma, \quad I_c = \oint_{c_m} \frac{c(\sigma)}{t}\mathrm{d}\sigma, \tag{72}$$
$$a(\sigma) = \boldsymbol{\nu}\cdot\mathbf{S}\cdot\boldsymbol{\nu}, \quad b(\sigma) = \boldsymbol{\nu}\cdot\mathbf{H}\cdot\boldsymbol{\nu}, \quad c(\sigma) = \boldsymbol{\nu}\cdot\mathbf{G}\cdot\boldsymbol{\nu}. \tag{73}$$

Stationary condition of functional (71) with respect to U_1 and F_1 yields the following system of equations

$$U_1I_a + F_1I_c = 2A_m, \quad U_1I_c - F_1I_b = 0. \tag{74}$$

Solution of system of Eq. (74) for U_1 and F_1 is as follows

$$U_1 = \frac{2A_m}{I_a + \frac{I_c^2}{I_b}}, \quad F_1 = \frac{2A_m}{\frac{I_aI_b}{I_c} + I_c}. \tag{75}$$

A simple computation, which is based on Eqs. (10) and (69), gives for the shear stress vector and electric displacement vector the following results:

$$\boldsymbol{\tau} = \frac{2A_m\vartheta}{\left(I_a + \frac{I_c^2}{I_b}\right)t}\mathbf{m}, \quad \mathbf{D} = \frac{2A_m\vartheta}{\left(\frac{I_aI_b}{I_c} + I_c\right)t}\mathbf{m}. \tag{76}$$

By the application of Theorem 2 gives for the torsional rigidity

$$S = -\Pi(U_1, F_1) = 2A_mU_1 = \frac{4A_m^2}{I_a + \frac{I_c^2}{I_b}}. \tag{77}$$

The electrical torsional rigidity S_D is computed from Eqs. (38) and $(70)_2$

$$S_D = \frac{4A_m^2}{\frac{I_aI_b}{I_c} + I_c}. \tag{78}$$

The concepts of the shear and electric displacement flows are introduced such as

$$q = \tau(\sigma)t(\sigma), \quad p = D(\sigma)t(\sigma). \tag{79}$$

A simple computation based on Eqs. (15) and (76) yields

$$q = \vartheta U_1 = \text{constant}, \quad p = \vartheta F_1 = \text{constant}, \tag{80}$$

according to the condition of mechanical equilibrium (Murray, 1985; Vlasov, 1961) and the charge balance condition which can be formulated in the present problem as (Fig. 7)

$$\int_{A^*} \nabla\cdot\mathbf{D}\,\mathrm{d}A = \int_{\partial A_0^*} \mathbf{n}\cdot\mathbf{D}\,\mathrm{d}s + \int_{\partial A_1^*} \mathbf{n}\cdot\mathbf{D}\,\mathrm{d}s + \int_{c'} \mathbf{n}\cdot\mathbf{D}\,\mathrm{d}s' + \int_{c''} \mathbf{n}\cdot\mathbf{D}\,\mathrm{d}s'' = p(\sigma') - p(\sigma'') = 0. \tag{81}$$

The plane domain A^* is in A bounded by $\partial A^* = \partial A_0^* \cup \partial A_1^* \cup c' \cup c''$, where on the curves c' and c'' $\sigma = \sigma'$ and $\sigma = \sigma''$, respectively as shown in Fig. 7. From Eqs. (27), (36) and (80) the next Bredt's-type formulae can be derived for p and q (Murray, 1985; Vlasov, 1961)

$$q = \frac{T}{2A_m}, \quad p = \frac{T_D}{2A_m}. \tag{82}$$

The application of the presented formulae is illustrated in the example of thin-walled circular tube with uniform wall-thickness. The cross-section shown in Fig. 5 can be considered as a thin-walled circular tube whose center line is a circle of radius $R = 0.5(R_0 + R_1)$ if $t = R_0 - R_1$ is small in comparison with R. A simple computation gives (Fig. 5)

$$a(\alpha) = \boldsymbol{\nu}\cdot\mathbf{S}\cdot\boldsymbol{\nu} = s_{44}\cos^2\alpha - 2s_{45}\cos\alpha\sin\alpha + s_{55}\sin^2\alpha, \tag{83}$$

$$b(\alpha) = \boldsymbol{\nu}\cdot\mathbf{H}\cdot\boldsymbol{\nu} = \eta_{22}\cos^2\alpha - 2\eta_{12}\cos\alpha\sin\alpha + \eta_{11}\sin^2\alpha, \tag{84}$$

$$c(\alpha) = \boldsymbol{\nu}\cdot\mathbf{G}\cdot\boldsymbol{\nu} = g_{24}\cos^2\alpha - (g_{14} + g_{25})\cos\alpha\sin\alpha + g_{15}\sin^2\alpha, \tag{85}$$

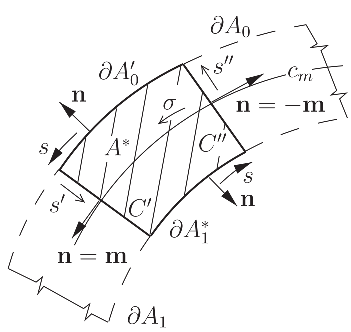

Fig. 7. Illustration of A^* and its boundary curve.

since $\boldsymbol{\nu} = \mathbf{e}_x\cos\alpha + \mathbf{e}_y\sin\alpha$, $\sigma = R\alpha$. From Eqs. (83)–(85) it follows that

$$I_a = \frac{R}{t}\bar{s}\pi, \quad I_b = \frac{R}{t}\bar{\eta}\pi, \quad I_c = \frac{R}{t}\bar{g}\pi. \tag{86}$$

Here, only formulae of S and S_D are derived

$$S = \frac{4R^3t\pi}{\bar{s} + \frac{\bar{g}^2}{\bar{\eta}}}, \quad S_D = \frac{4R^3t\pi}{\bar{g} + \frac{\bar{s}\bar{\eta}}{\bar{g}}}. \tag{87}$$

The above formulae for S and S_D can be obtained from Eq. (64) by putting in them

$$R_0 = R + \frac{t}{2}, \quad R_1 = R - \frac{t}{2}, \tag{88}$$

and by the use of the next approximation for $t \ll R(t/R \to 0)$

$$R_0^4 - R_1^4 = 4R^3t\left[1 + \frac{1}{4}\left(\frac{t}{R}\right)^4\right] \approx 4R^3t. \tag{89}$$

7. Conclusions

In this paper, the Saint–Venant's torsion is formulated in the framework of the linear theory of piezoelectricity for homogeneous, monoclinic piezoelectric cylinders with arbitrary cross-sectional geometry. The paper generalizes the known elastic solution of Saint–Venant's torsional problem developed by Prandtl to piezoelectric beams. A coupled Dirichlet problem (CDP) is derived for the Prandtl's stress function-electric displacement potential function. A direct and a variational formulation are developed. Exact analytical solutions for solid elliptical cross-section and for hollow circular cross-section and an approximate solution, based on the treated variational formulation, for thin-walled closed cross-section are presented. Some new formulae for the torsional rigidity and electric torsional rigidity are also derived.

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