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Global dynamics of a discretized SIRS epidemic model with time delay

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ABSTRACT

We derive a discretized SIRS epidemic model with time delay by applying a nonstandard finite difference scheme. Sufficient conditions for the global dynamics of the solution are obtained by improvements in discretization and applying proofs for continuous epidemic models. These conditions for our discretized model are the same as for the original continuous model.

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1. Introduction

For understanding the spread of infectious diseases in populations, mathematical epidemic models with ordinary differential equations are frequently considered. Hethcote [6] proposed a continuous SIR epidemic model. SIR epidemic models assume that the recovery class has permanent immunity to the disease. It is well known that the continuous model exhibits threshold behavior such that the disease dies out if a key parameter $\sigma \leqslant 1$ and the disease limits to endemic equilibrium if $\sigma > 1$. McCluskey [10] considered continuous SIR epidemic models with a discrete and distributed time delay. The discrete delay is used to express the fact that an individual may not be infectious until some time after becoming infected. The distributed delay allows infectivity to be a function of the duration since infection, up to some maximum duration. It may be more realistic to consider the distributed delay rather than the discrete delay. In [10], the global dynamics of continuous SIR epidemic models with the discrete delay and distributed delay were completely analyzed using the same threshold behavior as in [6]. There also exist various types of continuous epidemic models (see for example, [4,11,20,7]).

In continuous epidemic models, the models assuming that the recovery class has only temporary immunity are so-called "SIRS epidemic models". Zhang and Teng [19] obtained the following continuous SIRS epidemic model with a distributed time delay:

$$\begin{cases} S'(t) = \lambda - \mu_1 S(t) - \beta(I)S(t)\int_0^{\omega} I(t-s)\,d\eta(s) + \delta R(t), \\ I'(t) = \beta(I)S(t)\int_0^{\omega} I(t-s)\,d\eta(s) - (\mu_2+\gamma)I(t), \\ R'(t) = \gamma I(t) - (\mu_3+\delta)R(t), \end{cases} \tag{1}$$

where $\beta(I)$ is the probability per unit time that is an incidence rate. The system (1) is a kind of continuous SIRS epidemic model about incidence rates (for various incidence rates, see for example, [9,18]). A little complicated epidemic model as the system (1) may be helpful for understanding more realistic phenomenon of diseases. In [19], sufficient conditions for

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global asymptotic stability of the disease free equilibrium and permanence of the system (1) were obtained. It is difficult to analyze global dynamics in continuous SIRS epidemic models with a time delay, compared with SIR models.

On the other hand, numerical approaches have been frequently used in continuous models. Traditional numerical schemes such as the Euler and Runge–Kutta sometimes fail, by generating oscillations, bifurcations, chaos and false steady states (see [2,5]). However, as one of numerical schemes, the nonstandard finite-difference scheme is well known and has been applied to various problems in science (for example [1,3,15,16,13]). Mickens [12] concludes that the use of this scheme leads to asymptotic dynamics and numerical results that are always qualitatively the same as the corresponding solutions of several ordinary differential equations for any positive step size. Throughout this paper, we call the discretization in the nonstandard finite-difference scheme the "Mickens' nonstandard discretization". Discretized epidemic models using Mickens' nonstandard discretization have also been studied. For example, Jódar et al. [8] considered a discretized mathematical model for influenza with temporary immunity. However, global stability of the solution in the discretized models was not mentioned. From the viewpoint of stability analysis, it is unknown whether the solutions of the discretized models have the same properties as the original continuous models.

Our aim in this paper is to show the global behavior of the solution in a discretized SIRS epidemic model with time delay, applying Mickens' nonstandard discretization. The main idea is an application of the method given in Wang [17] to the discretized epidemic model. In [17], the eventual lower bound of the infected class in a continuous epidemic model with time delay was shown. By Wang's technique, we obtain the same sufficient condition for permanence of the model as for the continuous model. Sekiguchi [14] also applied Wang's technique to a discretized SIR epidemic model with time delay. But, the sufficient condition for positivity of the solution is not sharp, compared with that in the original continuous model. Moreover, it is difficult to apply the discretization used in [14] to SIRS epidemic models. In this paper, we can derive a discretized SIRS epidemic model with a time delay from Mickens' nonstandard discretization, where the condition for positivity of the solution is the same as that for the original continuous model. When positivity holds, the stability of the solution can be discussed and the sufficient conditions can be obtained, corresponding to those in the original continuous model.

This paper is organized as follows. In Section 2, we derive a discretized SIRS epidemic model with time delay from the continuous model (1). In the original continuous model, positivity and boundedness of the solution were clearly obtained. They are also obtained in our discretized model. We obtain a sufficient condition for the global asymptotic stability of the disease free equilibrium in Section 3. Applying Wang's technique, we consider the permanence of the discrete epidemic model in Section 4. In the discretized epidemic model, sufficient conditions for global asymptotic stability and permanence are the same as for the original continuous epidemic model. Numerical examples for different epidemic parameters are shown in Section 5.

2. A discretized SIRS epidemic model with time delay

In this section, applying Mickens' nonstandard discretization to the continuous model (1), we first derive the following discretized SIRS epidemic model with a distributed time delay:

$$\begin{cases} S_{n+1} - S_n = \lambda - \mu_1 S_{n+1} - \beta(I_n) S_{n+1} \sum_{k=0}^{\omega} I_{n-k}\eta_k + \delta R_{n+1}, \\ I_{n+1} - I_n = \beta(I_n) S_{n+1} \sum_{k=0}^{\omega} I_{n-k}\eta_k - (\mu_2 + \gamma) I_{n+1}, \\ R_{n+1} - R_n = \gamma I_{n+1} - (\mu_3 + \delta) R_{n+1}, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{2}$$

where S_n is the susceptible class, I_n is the infective class and R_n is the recovered class at nth step. Since the sufficient conditions can be obtained, independently of the choice of a time step-size, we let the time step-size be one for the sake of simplicity. The notions of all parameters in the system (2) are similar to those in [19]. The nonnegative constants μ_1, μ_2 and μ_3 denote the death rates of the susceptible, infected and recovered classes, respectively. The constant $\lambda > 0$ denotes the immigration rate, assuming all newborns to be susceptible. The constant $\gamma > 0$ is the recovery rate. The recovered class becomes susceptible again at a constant rate $\delta \geqslant 0$. $\beta(I)$ is the probability per unit time and the incidence is used with the form $\beta(I_n) S_{n+1} \sum_{k=0}^{\omega} I_{n-k}\eta_k$, which includes various delays. By a natural biological meaning, we assume that $\beta(I)$ is a positive function and that there exists a constant $I_\beta > 0$ such that $\beta(I)$ is nondecreasing on the interval $[0, I_\beta]$. The integer $\omega \geqslant 0$ is the time delay. The sequence $\{\eta_k\}$: $-\infty < \eta_k < \infty$ $(k = 0, 1, \ldots, \omega)$ is nondecreasing and has bounded variation.

The initial conditions of the system (2) are given by

$$S_n = \psi_n^{(1)}, \qquad I_n = \psi_n^{(2)}, \qquad R_n = \psi_n^{(3)} \quad \text{for } n = -\omega, -\omega + 1, \ldots, 0, \tag{3}$$

where $\psi_n^{(i)} \geqslant 0$ $(n = -\omega, -\omega + 1, \ldots, 0,\ i = 1, 2, 3)$. Again by a biological meaning, we further assume that $\psi_0^{(i)} > 0$ for $i = 1, 2, 3$.

When $\beta(I) = \beta > 0$ is a constant, the disease free equilibrium of the system (2) is

$$E^0 = \big(S^0, 0, 0\big), \quad S^0 = \frac{\lambda}{\mu_1}.$$

For the system (2), we define a positive constant $A \equiv \sum_{k=0}^{\omega} \eta_k$ and a threshold value σ,

$$\sigma \equiv \beta(0)A\frac{\lambda}{\mu_1(\mu_2+\gamma)}.$$

If $\sigma > 1$, an endemic equilibrium $E^* = (S^*, I^*, R^*)$ of the system (2) exists, where

$$S^* = \frac{\mu_2+\gamma}{\beta A}, \qquad I^* = \frac{(\mu_3+\delta)(\lambda-\mu_1 S^*)}{\mu_2(\mu_3+\delta)+\mu_3\gamma}, \qquad R^* = \frac{\gamma}{\mu_3+\delta}I^*.$$

Theorem 1. *Assume $\beta(I) = \beta > 0$ is a constant. If $\sigma \leqslant 1$, there exists only the disease free equilibrium E^0 and if $\sigma > 1$, the endemic equilibrium E^* appears and is unique except for E^0.*

2.1. Basic properties

We show that the sequences $\{S_n\}$, $\{I_n\}$ and $\{R_n\}$ of the system (2) are positive and are bounded above. The following results are independent of the initial values. For most continuous epidemic models, positivity of the solution is clear. Positivity of the solution for the discretized system (2) is easily obtained.

Lemma 2. *Any solution (S_n, I_n, R_n) of the system (2) is positive for all $n \in \mathbb{N}$.*

Proof. From the first equation of (2), we have

$$S_{n+1} = \frac{\lambda + S_n + \delta R_{n+1}}{1+\mu_1+\beta(I_n)\sum_{k=0}^{\omega} I_{n-k}\eta_k} \quad \text{for } n > 0.$$

Using the second and third equations of (2), we can rewrite

$$S_{n+1} = \frac{C_1(\lambda+S_n)+\gamma\delta I_n+\delta(1+\mu_2+\gamma)R_n}{C_2+C_3\beta(I_n)\sum_{k=0}^{\omega} I_{n-k}\eta_k},$$

where

$$C_1 = (1+\mu_2+\gamma)(1+\mu_3+\delta),$$
$$C_2 = (1+\mu_1)(1+\mu_2+\gamma)(1+\mu_3+\delta),$$
$$C_3 = (1+\mu_2)(1+\mu_3)+\delta(1+\mu_2)+\gamma(1+\mu_3).$$

From the initial condition (3) and $S_0 > 0$, it is easy to see $S_1 > 0$. From the second and third equations of (2), we obtain

$$I_{n+1} = \frac{\beta(I_n)S_{n+1}\sum_{k=0}^{\omega} I_{n-k}\eta_k + I_n}{1+\mu_2+\gamma}, \qquad R_{n+1} = \frac{R_n+\gamma I_{n+1}}{1+\mu_3+\delta} \quad \text{for } n > 0.$$

From the initial condition (3) and $S_1 > 0$, we have $I_1 > 0$ and $R_1 > 0$. The rest of this lemma can be proved in the same way. □

Now, we define the total population $N_n \equiv S_n + I_n + R_n$. Then, from the system (2), we know that

$$N_{n+1} - N_n = \lambda - \mu_1 S_{n+1} - \mu_2 I_{n+1} - \mu_3 R_{n+1}. \tag{4}$$

Throughout the paper, it is biologically natural to assume that $\mu_1 \leqslant \min(\mu_2, \mu_3)$, that is, the death rates of the infective and recovered classes may increase because of disease. From the hypothesis, we obtain

$$\begin{aligned} N_n &\leqslant \frac{\lambda + N_{n-1}}{1+\mu_1} \\ &\leqslant \frac{\lambda}{1+\mu_1} \times \left\{1 + \frac{1}{1+\mu_1} + \cdots + \left(\frac{1}{1+\mu_1}\right)^{n-1}\right\} + \left(\frac{1}{1+\mu_1}\right)^n N_0 \\ &= \frac{\lambda}{\mu_1}\left\{1 - \left(\frac{1}{1+\mu_1}\right)^n\right\} + \left(\frac{1}{1+\mu_1}\right)^n N_0 \\ &\leqslant \max\left\{\frac{\lambda}{\mu_1}, N_0\right\}. \end{aligned}$$

If $\lambda/\mu_1 \geqslant N_0$, it is easy to see that $N_n \leqslant \lambda/\mu_1 = S^0$ for all large n. If $\lambda/\mu_1 < N_0$, from Eq. (4), we obtain

$$N_1 \leqslant \frac{\lambda+N_0}{1+\mu_1} < \frac{\lambda}{\mu_1} = N_0.$$

Hence, we have $N_1 < N_0$ and there exists $i \in \mathbb{N}$ such that $N_i \leqslant \lambda/\mu_1 = S^0$. Therefore, we may use N_i as a starting value instead of N_0.

Theorem 3. *For any solution (S_n, I_n, R_n) of the system (2), the total population $N_n = S_n + I_n + R_n$ satisfies*

$$\limsup_{n\to+\infty} N_n \leqslant S^0 = \frac{\lambda}{\mu_1}.$$

3. Global asymptotic stability of the disease free equilibrium

In this section, we obtain a sufficient condition for global asymptotic stability of the disease free equilibrium E^0 in the system (2). We assume that $\beta(I)$ is independent of I, which implies that $\beta(I) = \beta > 0$ is a constant.

Using a Lyapunov function, we show a sufficient condition for global asymptotic stability of the disease free equilibrium E^0. The Lyapunov function is similar to that for the original continuous model in [19].

Theorem 4. *If $\sigma < 1$, the disease free equilibrium E^0 is globally asymptotically stable.*

Proof. We construct the following Lyapunov function:

$$V_n = I_n + c_1 R_n + c_2 \sum_{k=0}^{\omega} \left(\sum_{l=n-k}^{n} I_l \right) \eta_k + \frac{c_3}{2} \left(S_n - S^0 \right)^2,$$

where $c_i > 0$ $(i = 1, 2, 3)$ will be offered later and $S^0 = \lambda/\mu_1$. Using (2), the difference of V_n satisfies

$$\Delta V = V_{n+1} - V_n = I_{n+1} - I_n + c_1 (R_{n+1} - R_n) + c_2 \sum_{k=0}^{\omega} (I_{n+1}\eta_k - I_{n-k}\eta_k) + \frac{c_3}{2} \left\{ \left(S_{n+1} - S^0 \right)^2 - \left(S_n - S^0 \right)^2 \right\}.$$

From $S_n \leqslant S^0$ for all $n \geqslant 0$, we have

$$\begin{aligned}
\Delta V &\leqslant -\mu_1 c_3 \left(S_{n+1} - S^0 \right)^2 - \left\{ c_1(\mu_3 + \delta) - c_3 \delta \left(S_{n+1} - S^0 \right) \right\} R_{n+1} \\
&\quad + \left\{ \beta S_{n+1} - c_2 - c_3 \beta S_{n+1} \left(S_{n+1} - S^0 \right) \right\} \sum_{k=0}^{\omega} I_{n-k}\eta_k + \left\{ c_1\gamma + c_2 A - (\mu_2 + \gamma) \right\} I_{n+1} \\
&\leqslant -\mu_1 c_3 \left(S_{n+1} - S^0 \right)^2 - c_1(\mu_3 + \delta) R_{n+1} + \left\{ c_1\gamma + c_2 A - (\mu_2 + \gamma) \right\} I_{n+1} \\
&\quad + \left\{ \beta S_{n+1} - c_2 - c_3 \beta S_{n+1} \left(S_{n+1} - S^0 \right) \right\} \sum_{k=0}^{\omega} I_{n-k}\eta_k.
\end{aligned}$$

Let us choose $c_i > 0$ $(i = 1, 2, 3)$ to satisfy

$$c_1\gamma + c_2 A < \mu_2 + \gamma, \tag{5}$$

$$\beta S_{n+1} - c_2 - c_3 \beta S_{n+1} \left(S_{n+1} - S^0 \right) < 0. \tag{6}$$

Then, (6) holds if the following inequality is true:

$$\beta \left(1 + c_3 S^0 \right)^2 < 4 c_2 c_3. \tag{7}$$

Since $\sigma < 1$, which implies that $\beta A S^0 < \mu_2 + \gamma$, we choose $c_2 = \beta S^0 + \varepsilon$. Here ε is a small positive number such that $\beta A S^0 + A\varepsilon < \mu_2 + \gamma$. Since $\beta S^0 - 2c_2 < 0$ and $(\beta S^0 - 2c_2)^2 > (\beta S^0)^2$, we can choose $c_3 > 0$ to satisfy (7). We may further choose $c_1 > 0$ to satisfy (5). Therefore, ΔV is negative definite and is equal to zero if and only if $S_{n+1} = S^0$, $I_{n+1} = 0$ and $R_{n+1} = 0$. The proof is complete. □

4. Permanence of the system (2)

The system (2) is said to be permanent if there are positive constants m and M such that

$$m \leqslant \liminf_{n\to+\infty} S_n \leqslant \limsup_{n\to+\infty} S_n \leqslant M$$

holds for any sequence S_n of the system (2), and for I_n and R_n, there also exist positive constants m and M. In each class S_n, I_n and R_n, m and M are independent of initial conditions.

Applying Wang's technique in [17], we prove the following main result in this paper.

Theorem 5. *If $\sigma > 1$, then the system* (2) *is permanent for any initial conditions* (3).

Proof. Firstly, from (2) and Lemma 2, for any $\varepsilon_0 > 0$, there exists sufficiently large $n_0 > 0$ such that $I_n \leqslant \lambda/\mu_1 + \varepsilon_0$ as $n \geqslant n_0$. Then, we have

$$S_n = \frac{\lambda + S_n + \delta R_{n+1}}{1 + \mu_1 + \beta(I_n)\sum_{k=0}^{\omega} I_{n-k}\eta_k} > \frac{\lambda}{1 + \mu_1 + \beta(I_n)\sum_{k=0}^{\omega} I_{n-k}\eta_k}.$$

Set $\beta^M(\varepsilon_0) = \max_{I\in[0,\lambda/\mu_1+\varepsilon_0]}\beta(I)$. Thus, we have

$$S_n \geqslant \frac{\lambda}{1 + \mu_1 + \beta^M \sum_{k=0}^{\omega} I_{n-k}\eta_k} \geqslant \frac{\lambda}{1 + \mu_1 + \beta^M(\lambda/\mu_1 + \varepsilon_0)A}.$$

Notice that ε_0 can be arbitrarily small. Then, we have

$$\liminf_{n\to+\infty} S_n \geqslant m_S \equiv \frac{\lambda}{1 + \mu_1 + \beta^M A\lambda/\mu_1}, \quad \beta^M = \max_{I\in[0,\lambda/\mu_1]}\beta(I).$$

Next, let us consider the positive sequences $\{S_n\}$ and $\{I_n\}$ of (2). According to these sequences, we define

$$V_n \equiv I_n + \frac{\mu_2 + \gamma}{A}\sum_{k=0}^{\omega}\sum_{l=n-k}^{n} I_l\eta_k. \tag{8}$$

Then, for $n \geqslant 0$ we obtain

$$\Delta V = V_{n+1} - V_n = I_{n+1} - I_n + \frac{\mu_2 + \gamma}{A}\sum_{k=0}^{\omega}(I_{n+1}\eta_k - I_{n-k}\eta_k) = \left(\beta(I_n)S_{n+1} - \frac{\mu_2 + \gamma}{A}\right)\sum_{k=0}^{\omega} I_{n-k}\eta_k.$$

Since $\sigma = \beta(0)A\lambda/\mu_1(\mu_2 + \gamma) > 1$, there exist $0 < \alpha < I_\beta$ and $\rho > 0$ such that

$$\frac{A\beta(0)}{\mu_2 + \gamma} \times \frac{\lambda}{\mu_1 + \alpha\beta(\alpha)A}\left\{1 - \left(\frac{1}{1 + \mu_1 + \alpha\beta(\alpha)A}\right)^{\rho\omega}\right\} > 1.$$

Note that

$$S^\Delta \equiv \frac{\lambda}{\mu_1 + \alpha\beta(\alpha)A}\left\{1 - \left(\frac{1}{1 + \mu_1 + \alpha\beta(\alpha)A}\right)^{\rho\omega}\right\}.$$

We claim that it is impossible that $I_n \leqslant \alpha$ for all $n \geqslant n_1 \geqslant \lceil\rho\omega\rceil$. The function $\lceil x\rceil$ gives the smallest integer not less than x. Suppose the contrary, then for $n \geqslant n_1 + \omega$,

$$\begin{aligned} S_{n+1} &= \frac{\lambda + S_n + \delta R_{n+1}}{1 + \mu_1 + \beta(I_n)\sum_{k=0}^{\omega} I_{n-k}\eta_k} > \frac{\lambda + S_n}{1 + \mu_1 + \alpha\beta(\alpha)A} \\ &> \frac{\lambda}{1 + \mu_1 + \alpha\beta(\alpha)A}\left\{1 + \frac{1}{1 + \mu_1 + \alpha\beta(\alpha)A} + \cdots + \left(\frac{1}{1 + \mu_1 + \alpha\beta(\alpha)A}\right)^{n-n_1-\omega-1}\right\} \\ &\quad + \left(\frac{1}{1 + \mu_1 + \alpha\beta(\alpha)A}\right)^{n-n_1-\omega} S_{n_1+\omega+1}. \end{aligned}$$

From Lemma 2, S_n satisfies

$$S_{n+1} > \frac{\lambda}{\mu_1 + \alpha\beta(\alpha)A}\left\{1 - \left(\frac{1}{1 + \mu_1 + \alpha\beta(\alpha)A}\right)^{n-n_1-\omega}\right\},$$

and we have that, for $n \geqslant n_1 + \omega + \lceil\rho\omega\rceil$,

$$S_{n+1} > \frac{\lambda}{\mu_1 + \alpha\beta(\alpha)A}\left\{1 - \left(\frac{1}{1 + \mu_1 + \alpha\beta(\alpha)A}\right)^{\lceil\rho\omega\rceil}\right\} \geqslant \frac{\lambda}{\mu_1 + \alpha\beta(\alpha)A}\left\{1 - \left(\frac{1}{1 + \mu_1 + \alpha\beta(\alpha)A}\right)^{\rho\omega}\right\} = S^\Delta.$$

Hence, for $n \geqslant n_1 + \omega + \lceil\rho\omega\rceil$, we have

$$\Delta V > \left(\beta(0)S^\Delta - \frac{\mu_2 + \gamma}{A}\right)\sum_{k=0}^{\omega} I_{n-k}\eta_k = \frac{\mu_2 + \gamma}{A}\left(\frac{A\beta(0)S^\Delta}{\mu_2 + \gamma} - 1\right)\sum_{k=0}^{\omega} I_{n-k}\eta_k.$$

Set

$$\varepsilon = \min_{\theta} I_{n_1+\lceil\rho\omega\rceil+2\omega+\theta}, \quad \theta = -\omega, -\omega+1, \dots, 0.$$

Now, we show that $I_n \geqslant \varepsilon$ for $n \geqslant n_1 + \lceil\rho\omega\rceil + \omega$. In fact, if there is an integer $\bar{n} \geqslant 0$ such that

$$I_n \geqslant \varepsilon \quad \text{for } n_1 + \lceil\rho\omega\rceil + \omega \leqslant n \leqslant n_1 + \lceil\rho\omega\rceil + 2\omega + \bar{n},$$
$$I_{n+1} < \varepsilon \quad \text{for } n = n_1 + \lceil\rho\omega\rceil + 2\omega + \bar{n}.$$

Moreover, we can choose a positive integer j such that

$$I_j = \varepsilon, \quad n_1 + \lceil\rho\omega\rceil + \omega \leqslant n \leqslant n_1 + \lceil\rho\omega\rceil + 2\omega + \bar{n}.$$

However, for $n = n_1 + \lceil\rho\omega\rceil + 2\omega + \bar{n}$, we have

$$\begin{aligned} I_{n+1} - I_j &= \frac{\beta(I_n)S_{n+1}\sum_{k=0}^{\omega} I_{n-k}\eta_k + I_n}{1+\mu_2+\gamma} - \frac{1+\mu_2+\gamma}{1+\mu_2+\gamma} I_j \geqslant \frac{\beta(0)S^{\Delta}A - (\mu_2+\gamma)}{1+\mu_2+\gamma} I_j \\ &= \frac{\mu_2+\gamma}{1+\mu_2+\gamma}\left\{\frac{A\beta(0)S^{\Delta}}{\mu_2+\gamma} - 1\right\}\varepsilon > 0, \end{aligned}$$

which is a contradiction. Thus, $I_n \geqslant \varepsilon$ for $n \geqslant n_1 + \lceil\rho\omega\rceil + \omega$. Therefore, for $n \geqslant n_1 + \lceil\rho\omega\rceil + \omega$,

$$\Delta V > \frac{\mu_2+\gamma}{1+\mu_2+\gamma}\left\{\frac{A\beta(0)S^{\Delta}}{\mu_2+\gamma} - 1\right\}\varepsilon > 0,$$

which implies that $V_n \to +\infty$ as $n \to +\infty$. But, from Theorem 3 and (8), there exists a sufficiently large integer $n_1' > 0$ such that, for $n > n_1'$,

$$V_n \leqslant \frac{\lambda}{\mu_1} + \frac{\mu_2+\gamma}{A}\sum_{k=0}^{\omega}\left(\sum_{l=n-k}^{n}\frac{\lambda}{\mu_1}\right)\eta_k \leqslant \frac{\lambda}{\mu_1}\big\{1 + (\mu_2+\gamma)(\omega+1)\big\},$$

which is a contradiction. Hence, the claim is proved.

In the rest, we are left to consider the following two cases;

i. $I_n \geqslant \alpha$ for all large n.
ii. I_n oscillates about α for all large n.

We show that $I_n \geqslant m_I$ for all large n, where $0 < m_I \leqslant \alpha$ is a constant which will be given later. Clearly, we only need to consider case ii. Let positive integers n_1 and n_2 be sufficiently large that

$$I_{n_1} \geqslant \alpha, \qquad I_{n_2} \geqslant \alpha, \qquad I_n < \alpha \quad \text{for } n_1 < n < n_2.$$

If $n_2 - n_1 \geqslant \omega + \lceil\rho\omega\rceil$, since

$$I_n = \frac{\beta(I_{n-1})S_n\sum_{k=0}^{\omega} I_{n-k-1}\eta_k + I_{n-1}}{1+\mu_2+\gamma} \geqslant \frac{I_{n-1}}{1+\mu_2+\gamma},$$

we have

$$I_n \geqslant \left(\frac{1}{1+\mu_2+\gamma}\right)^{n-n_1} I_{n_1} > \left(\frac{1}{1+\mu_2+\gamma}\right)^{n_2-n_1} I_{n_1} > m_I \equiv \left(\frac{1}{1+\mu_2+\gamma}\right)^{\omega+\lceil\rho\omega\rceil}\alpha.$$

Hence $I_n > m_I$ for $n \in [n_1, n_2]$.

If $n_2 - n_1 > \omega + \lceil\rho\omega\rceil$, we can easily obtain that $I_n > m_I$ for $n \in [n_1, n_1 + \omega + \lceil\rho\omega\rceil]$. Assume that there exists an integer $\hat{n} \geqslant 0$ such that

$$I_n \geqslant m_I \quad \text{for } n_1 + \omega + \lceil\rho\omega\rceil \leqslant n \leqslant n_1 + \omega + \lceil\rho\omega\rceil + \hat{n},$$
$$I_{n+1} < m_I \quad \text{for } n = n_1 + \omega + \lceil\rho\omega\rceil + \hat{n}.$$

Moreover, we can choose a positive integer j such that

$$I_j = \min_{n} I_n \geqslant m_I \quad \text{for } n_1 \leqslant n \leqslant n_1 + \omega + \lceil\rho\omega\rceil + \hat{n}.$$

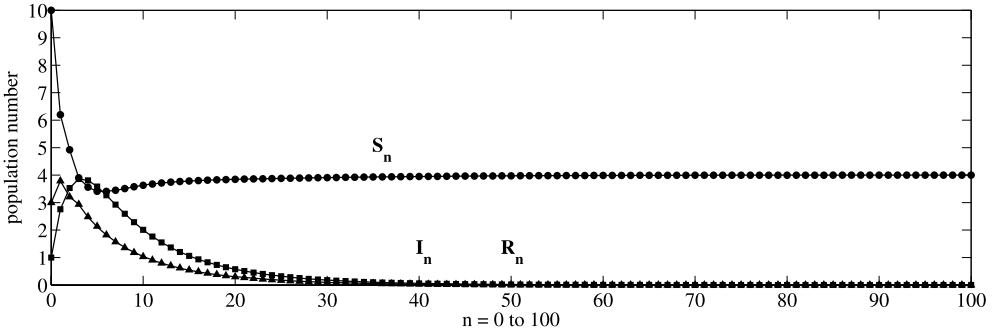

Fig. 1. Numerical solution with $\lambda = 0.4$, $\mu_1 = 0.1$, $\mu_2 = 0.2$, $\mu_3 = 0.1$, $\beta = 0.2$, $\gamma = 0.9$, $\delta = 0.5$ and $\sigma \approx 0.72 < 1$.

Then, for $n = n_1 + \omega + \lceil \rho\omega \rceil + \hat{n}$,

$$I_{n+1} - I_j > \frac{\mu_2 + \gamma}{1 + \mu_2 + \gamma} \left\{ \frac{A\beta(0)S^{\Delta}}{\mu_2 + \gamma} - 1 \right\} m_I > 0.$$

This is a contradiction to the proposition that $I_{n+1} < m_I$. Therefore, $I_n \geqslant m_I$ for $n \in [n_1, n_2]$. Since these positive integers n_1 and n_2 are chosen in an arbitrary way, we conclude that $I_n \geqslant m_I$ for all large n in case ii. Hence, $\liminf_{n\to+\infty} I_n \geqslant m_I$.

Note that from the third equation of (2), we have

$$\liminf_{n\to+\infty} R_n \geqslant m_R \equiv \frac{\gamma}{\mu_3 + \delta} m_I.$$

From Theorem 3 and the above discussion, we have

$$m_S \leqslant \liminf_{n\to+\infty} S_n \leqslant \limsup_{n\to+\infty} S_n \leqslant S^0,$$
$$m_I \leqslant \liminf_{n\to+\infty} I_n \leqslant \limsup_{n\to+\infty} I_n \leqslant S^0,$$
$$m_R \leqslant \liminf_{n\to+\infty} R_n \leqslant \limsup_{n\to+\infty} R_n \leqslant S^0.$$

Hence, the proof is complete. □

5. Numerical example

For the system (2), Theorem 4 implies that the disease dies out if $\sigma < 1$, and Theorem 5 implies that the disease persists if $\sigma > 1$. The global properties of the solution in the system (2) are the same as those in the original continuous model (1). In order to confirm the validity of our results, we consider the following SIRS epidemic model with a discrete time delay:

$$\begin{cases} S_{n+1} - S_n = \lambda - \mu_1 S_{n+1} - \beta S_{n+1} I_{n-\omega} + \delta R_{n+1}, \\ I_{n+1} - I_n = \beta S_{n+1} I_{n-\omega} - (\mu_2 + \gamma) I_{n+1}, \\ R_{n+1} - R_n = \gamma I_{n+1} - (\mu_3 + \delta) R_{n+1}. \end{cases} \tag{9}$$

Then, rearrangement to the system (9) yields the following explicit form:

$$S_{n+1} = \frac{(\lambda + S_n)(1 + \mu_2 + \gamma)(1 + \mu_3 + \delta) + \gamma\delta I_n + \delta(1 + \mu_2 + \gamma) R_n}{(1 + \mu_1)(1 + \mu_2 + \gamma)(1 + \mu_3 + \delta) + \beta\{(1 + \mu_2)(1 + \mu_3) + \delta(1 + \mu_2) + \gamma(1 + \mu_3)\} I_{n-\omega}},$$
$$I_{n+1} = \frac{I_n + \beta S_{n+1} I_{n-\omega}}{1 + \mu_2 + \gamma},$$
$$R_{n+1} = \frac{R_n + \gamma I_{n+1}}{1 + \mu_3 + \delta}.$$

Now, we present a numerical example. For simplicity, we choose the initial conditions $\omega = 10$, $\psi_n^{(1)} = 10$, $\psi_n^{(2)} = 0.1n + 3$, $\psi_n^{(3)} = 1$ for $n \in [-\omega, 0]$ with respect to the system (2). In Figs. 1 and 2, the numbers of susceptible, infective and recovered individuals (on the vertical axis) are plotted versus the time steps n (on the horizontal axis). Fig. 1 shows the solution of the system (9) when $\sigma < 1$. We can see that the disease free equilibrium E^0 of the system (9) is globally asymptotically stable. On the other hand, Fig. 2 shows the solution when $\sigma > 1$, and indicates that the system (9) is permanent.

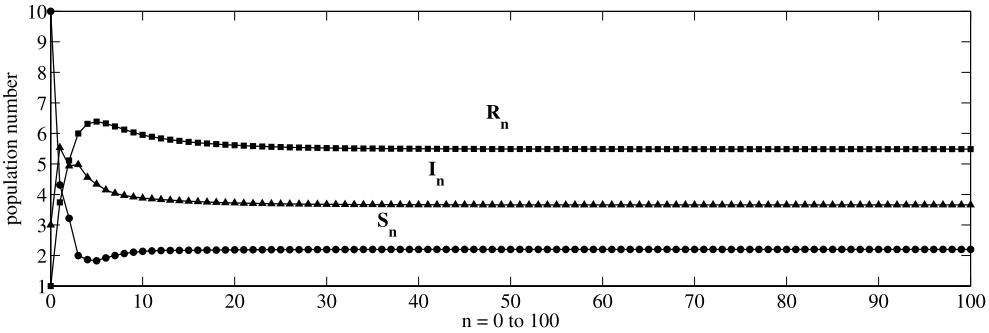

Fig. 2. Numerical solution with $\lambda = 1.5$, $\mu_1 = 0.1$, $\mu_2 = 0.2$, $\mu_3 = 0.1$, $\beta = 0.5$, $\gamma = 0.9$, $\delta = 0.5$ and $\sigma \approx 6.8 > 1$.

6. Discussion

In this paper, we obtain a discretized SIRS epidemic model with time delay (2) and show the sufficient conditions for global behaviors of the solution. Our sufficient conditions are corresponding to those of the original continuous model (1).

For continuous epidemic models with pulse vaccination and time delay, it is well known that the global attractivity of the infection-free periodic solution and the permanence of the models (see for example, [4,11]). The sufficient conditions for the global properties and the proofs are different from those in continuous epidemic models without pulse vaccination. In a separate paper, we will prove that in a discretized epidemic model with pulse vaccination and time delay, sufficient conditions for global properties correspond to those in the original continuous model.

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