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Sections, selections and Prohorov's theorem

Valentin Gutev [a,1], Vesko Valov [b,*,2]

[a] School of Mathematical Sciences, University of KwaZulu-Natal, Westville Campus, Private Bag X54001, Durban 4000, South Africa
[b] Department of Computer Science and Mathematics, Nipissing University, 100 College Drive, PO Box 5002, North Bay, ON P1B 8L7, Canada

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ABSTRACT

The famous Prohorov theorem for Radon probability measures is generalized in terms of usco mappings. In the case of completely metrizable spaces this is achieved by applying a classical Michael result on the existence of usco selections for l.s.c. mappings. A similar approach works when sieve-complete spaces are considered.

1. Introduction

All spaces in this paper are assumed to be completely regular and Hausdorff. For a space X, let $\mathscr{B}(X)$ be the Borel σ-algebra associated to X, i.e. the smallest σ-algebra that contains all closed subsets of X. Thus, $\mathscr{B}(X)$ is closed with respect to complements and countable unions, its elements are often called *Borel* subsets of X.

A countably additive function $\mu : \mathscr{B}(X) \to [0, +\infty]$ is called a *Radon measure* on X if

$$\mu(B) = \sup\{\mu(K)\colon\ K \subset B \text{ and } K \text{ is compact}\}, \quad B \in \mathscr{B}(X). \tag{1.1}$$

A *Radon probability measure* is a Radon measure μ, with $\mu(X) = 1$. In the sequel, we will denote by $\mathscr{P}(X)$ the set of all Radon probability measures on X. Every measure $\mu \in \mathscr{P}(X)$ uniquely defines a positive linear functional $\mu(g) = \int g\, d\mu$, where g runs over the bounded continuous functions on X. As a topological space, we consider $\mathscr{P}(X)$ endowed with the weakest topology with respect to which all these functionals are continuous. Thus, a net $\{\mu_\alpha\} \subset \mathscr{P}(X)$ converges to $\mu \in \mathscr{P}(X)$ if and only if $\{\mu_\alpha(g)\}$ converges to $\mu(g)$ for every bounded continuous function $g : X \to \mathbb{R}$. With respect to this topology, for every closed $F \subset X$ and $\varepsilon > 0$,

$$\{\mu \in \mathscr{P}(X)\colon\ \mu(F) < \varepsilon\} \quad \text{is open in } \mathscr{P}(X). \tag{1.2}$$

The famous Prohorov theorem [13] states that if X is a Polish space (i.e., a completely metrizable separable space), then for every compact $T \subset \mathscr{P}(X)$ and every $\varepsilon > 0$ there exists a compact $K \subset X$, with $\mu(X \setminus K) < \varepsilon$ for all $\mu \in T$. Spaces having this property, called *Prohorov spaces*, are widely investigated in the literature.

* Corresponding author.
E-mail addresses: gutev@ukzn.ac.za (V. Gutev), veskov@nipissingu.ca (V. Valov).

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In this paper, we give a simple proof that all sieve-complete spaces are Prohorov (Theorem 3.1). In the special case of completely metrizable spaces, this result follows by the Michael theorem on the existence of usco selections for l.s.c. mappings, [10, Theorem 1.1]. The general case of arbitrary sieve-complete spaces follows by a selection-like result [5, Corollary 7.2] which utilizes “usco sections” instead of “usco selections”.

The idea to use some selection theorem for the proof of Prohorov's theorem goes back to a question of Bouziad [2]. In fact, our approach provides a natural generalization of Prohorov's theorem in which the compact subset $T \subset \mathscr{P}(X)$ is replaced by a paracompact one $Z \subset \mathscr{P}(X)$, and the compact $K \subset X$—by an usco mapping from Z into the compact subsets of X. This gives a solution to another problem of Bouziad [2] whether there is a “continuous” version of Prohorov's theorem, see Corollary 3.2.

The paper is organized as follows. Section 2 is devoted to the main ingredient of our approach which is a construction of l.s.c. mappings generated by Radon probability measures (Proposition 2.1). Section 3 contains the proof of Theorem 3.1 which is preceded by that one for the special case of completely metrizable spaces.

2. A construction of l.s.c. mappings

For a space X, let 2^X be the family of all nonempty subsets of X, and let $\mathscr{C}(X)$ be the subfamily of 2^X which consists of all compact members of 2^X. A part of our considerations will involve $\mathscr{C}(X)$ endowed with the *Vietoris topology* τ_V. Recall that τ_V is generated by all collections of the form

$$\langle \mathscr{V} \rangle = \left\{ S \in \mathscr{C}(X):\ S \subset \bigcup \mathscr{V} \text{ and } S \cap V \neq \varnothing,\ \text{whenever } V \in \mathscr{V} \right\},$$

where $\mathscr{V}$ runs over the finite families of open subsets of X. For convenience, for an open subset $V \subset X$, we write $\langle V \rangle$ rather than $\langle \{V\} \rangle$.

Another topology on $\mathscr{C}(X)$ that will play an important role in this paper is the *upper Vietoris topology* τ_V^+, i.e. the topology generated by the family

$$\left\{ \langle V \rangle:\ V \subset X \text{ is open} \right\}.$$

Clearly, τ_V^+ is a coarser topology than the Vietoris one τ_V, i.e. $\tau_V^+ \subset \tau_V$. In this regard, let us make the explicit agreement that if τ is a topology on $\mathscr{C}(X)$, then the prefix “τ-” will be used to express properties related to the topology τ, say τ-open sets, τ-closure, etc.

Finally, let us recall that a set-valued mapping $\Phi : Z \to 2^Y$ is *lower semi-continuous*, or l.s.c., if the set

$$\Phi^{-1}(U) = \left\{ z \in Z:\ \Phi(z) \cap U \neq \varnothing \right\}$$

is open in Z for every open $U \subset Y$.

Proposition 2.1. *Let X be a space, and let $\varepsilon \in (0,1)$. Define a set-valued mapping $\Psi_\varepsilon : \mathscr{P}(X) \to 2^{\mathscr{C}(X)}$ by*

$$\Psi_\varepsilon(\mu) = \left\{ K \in \mathscr{C}(X):\ \mu(X \setminus K) < \varepsilon \right\}, \quad \mu \in \mathscr{P}(X).$$

Then, Ψ_ε is a nonempty-valued τ_V-l.s.c. mapping.

Proof. Take $\mu \in \mathscr{P}(X)$. Since $\mu(X) = 1 > 1 - \varepsilon$, by (1.1), there is $K \in \mathscr{C}(X)$ such that $\mu(K) > 1 - \varepsilon$, so $\Psi_\varepsilon(\mu) \neq \varnothing$. Let $K \in \Psi_\varepsilon(\mu)$ and let $\mathscr{V}$ be a finite family of open subsets of X, with $K \in \langle \mathscr{V} \rangle$. Then, $X \setminus \bigcup \mathscr{V} \subset X \setminus K$, it is closed in X and $\mu(X \setminus \bigcup \mathscr{V}) < \varepsilon$. Hence, by (1.2), there exists a neighbourhood U of μ such that $\nu(X \setminus \bigcup \mathscr{V}) < \varepsilon$ for every $\nu \in U$. If $\nu \in U$, then $\nu(\bigcup \mathscr{V}) > 1 - \varepsilon$ and, by (1.1), there is a compact subset $H \subset \bigcup \mathscr{V}$, with $\nu(H) > 1 - \varepsilon$. We now have that $H \cup K \in \langle \mathscr{V} \rangle$, while $H \cup K \in \Psi_\varepsilon(\nu)$ because $\nu(X \setminus (H \cup K)) \leqslant \nu(X \setminus H) < \varepsilon$. $\square$

Proposition 2.2. *Let X be a space, $\varepsilon \in (0,1)$, $\Psi_\varepsilon : \mathscr{P}(X) \to 2^{\mathscr{C}(X)}$ be defined as in Proposition 2.1, and let $\Phi_\varepsilon(\mu)$ be the τ_V^+-closure of $\Psi_\varepsilon(\mu)$, for each $\mu \in \mathscr{P}(X)$. Then, $\mu(X \setminus K) \leqslant \varepsilon$ for every $K \in \Phi_\varepsilon(\mu)$ and $\mu \in \mathscr{P}(X)$.*

Proof. Take $\mu \in \mathscr{P}(X)$ and $K \in \mathscr{C}(X)$ such that $\mu(X \setminus K) > \varepsilon$. By (1.1), there exists a compact subset $H \subset X \setminus K$, with $\mu(H) > \varepsilon$. Let $V = X \setminus H$. We now have that $K \in \langle V \rangle$, while $\varepsilon < \mu(H) = \mu(X \setminus V) \leqslant \mu(X \setminus S)$ for every $S \in \langle V \rangle$. Consequently, $K \notin \Phi_\varepsilon(\mu)$ because $\Psi_\varepsilon(\mu) \subset \mathscr{C}(X) \setminus \langle V \rangle$. $\square$

We conclude this section with a well-known property of compact sets in the upper Vietoris topology.

Proposition 2.3. *Let $\mathscr{K} \subset \mathscr{C}(X)$ be a τ_V^+-compact set. Then, $\bigcup \mathscr{K}$ is compact in X.*

Proof. Take an open in X cover $\mathscr{U}$ of $\bigcup \mathscr{K}$. Then, $\Omega = \{\langle \bigcup \mathscr{E} \rangle:\ \mathscr{E} \subset \mathscr{U} \text{ is finite}\}$ is a τ_V^+-open cover of $\mathscr{K}$. Hence, Ω contains a finite subcover of $\mathscr{K}$, so there exists a finite $\mathscr{V} \subset \mathscr{U}$, with $\mathscr{K} \subset \bigcup \{\langle \bigcup \mathscr{E} \rangle:\ \mathscr{E} \subset \mathscr{V} \text{ is finite}\}$. This $\mathscr{V}$ is a finite cover of $\bigcup \mathscr{K}$. $\square$

3. Usco mappings and Prohorov's theorem

Recall that a set-valued mapping $\psi : Z \to 2^X$ is *upper semi-continuous*, or u.s.c., if the set

$$\psi^\#(U) = \{z \in Z: \ \psi(z) \subset U\}$$

is open in Z for every open $U \subset X$. We say that $\psi : Z \to 2^X$ is *usco* if it is u.s.c. and compact-valued. Let us explicitly mention that if $\psi : Z \to \mathscr{C}(X)$ is usco, then $\psi(T) = \bigcup\{\psi(z): \ z \in T\}$ is compact for every compact $T \subset Z$.

A space X is *sieve-complete* [3] if it has an open complete sieve. Every Čech-complete space is sieve-complete, and it was shown in [3] (see, also, [11]) that the two concepts are equivalent in the presence of paracompactness.

Theorem 3.1. *Let X be a sieve-complete space, and let $Z \subset \mathscr{P}(X)$ be paracompact. Then, for every $\varepsilon > 0$ there is an usco mapping $\varphi : Z \to \mathscr{C}(X)$ such that $\mu(X \setminus \varphi(\mu)) < \varepsilon$ for every $\mu \in Z$.*

Turning to the proof of Theorem 3.1, let us first demonstrate the special case of a completely metrizable X. In this case, let $\Psi_\varepsilon : \mathscr{P}(X) \to 2^{\mathscr{C}(X)}$ be defined as in Proposition 2.1, and let $\Phi(\mu)$ be the τ_V-closure of $\Psi_\varepsilon(\mu)$, for each $\mu \in \mathscr{P}(X)$. By Proposition 2.1 and [9, Proposition 2.3], $\Phi : \mathscr{P}(X) \to 2^{\mathscr{C}(X)}$ is τ_V-l.s.c. Also, $(\mathscr{C}(X), \tau_V)$ is completely metrizable because so is X, [6–8]. Hence, by [10, Theorem 1.1], $\Phi \restriction Z$ has a τ_V-usco selection $\theta : Z \to 2^{\mathscr{C}(X)}$. That is, θ is a τ_V-usco mapping such that $\theta(\mu) \subset \Phi(\mu)$ for every $\mu \in Z$. Then, define $\varphi : Z \to \mathscr{C}(X)$ by letting $\varphi(\mu) = \bigcup \theta(\mu)$, $\mu \in Z$. This φ is as required. Indeed, each $\theta(\mu)$, $\mu \in Z$, is τ_V-compact, hence τ_V^+-compact as well, and, by Proposition 2.3, each $\varphi(\mu)$, $\mu \in Z$, is a compact subset of X. If V is a neighbourhood of $\varphi(\mu)$ for some $\mu \in Z$, then $\langle V \rangle$ is a neighbourhood of $\theta(\mu)$. This implies that φ is u.s.c. Finally, take $\mu \in Z$ and $K \in \theta(\mu) \subset \Phi(\mu)$. Since $\tau_V^+ \subset \tau_V$, we have that $\Phi(\mu)$ is a subset of the τ_V^+-closure of $\Psi_\varepsilon(\mu)$. Therefore, by Proposition 2.2, $\mu(X \setminus \varphi(\mu)) \leqslant \mu(X \setminus K) \leqslant \varepsilon$ because $K \subset \varphi(\mu)$.

The proof of Theorem 3.1 for the general case of arbitrary sieve-complete spaces follows exactly the same idea but is now based on the upper Vietoris topology and another selection-like result for usco mappings.

Proof of Theorem 3.1. Let X and $Z \subset \mathscr{P}(X)$ be as in that theorem, and let $\varepsilon \in (0, 1)$. Also, for each $\mu \in \mathscr{P}(X)$, let $\Phi_\varepsilon(\mu)$ be the τ_V^+-closure of $\Psi_\varepsilon(\mu)$, where $\Psi_\varepsilon : \mathscr{P}(X) \to 2^{\mathscr{C}(X)}$ is defined as in Proposition 2.1. By Proposition 2.1 and [9, Proposition 2.3], $\Phi_\varepsilon : \mathscr{P}(X) \to 2^{\mathscr{C}(X)}$ is τ_V^+-l.s.c. because $\tau_V^+ \subset \tau_V$. By [12, Lemma 3.1], $(\mathscr{C}(X), \tau_V^+)$ is sieve-complete because so is X. Hence, by [5, Corollary 7.2], $\Phi_\varepsilon \restriction Z$ has a τ_V^+-usco section $\theta : Z \to 2^{\mathscr{C}(X)}$. That is, θ is a τ_V^+-usco mapping such that $\theta(\mu) \cap \Phi_\varepsilon(\mu) \neq \varnothing$ for every $\mu \in Z$. Finally, define the required $\varphi : Z \to \mathscr{C}(X)$ by $\varphi(\mu) = \bigcup \theta(\mu)$, $\mu \in Z$. By Proposition 2.3, each $\varphi(\mu)$, $\mu \in Z$, is a compact subset of X. Just like before φ is u.s.c. because if V is a neighbourhood of $\varphi(\mu)$ for some $\mu \in Z$, then $\langle V \rangle$ is a neighbourhood of $\theta(\mu)$. Finally, if $\mu \in Z$ and $K \in \theta(\mu) \cap \Phi_\varepsilon(\mu)$, then, by Proposition 2.2, $\mu(X \setminus \varphi(\mu)) \leqslant \mu(X \setminus K) \leqslant \varepsilon$ because $K \subset \varphi(\mu)$. The proof is completed. □

It is well known that $\mathscr{P}(X)$ is paracompact (and Čech-complete) whenever X is so [1,14,15], see also [4]. This gives the following immediate consequence.

Corollary 3.2. *Let X be a paracompact Čech-complete space, and $\varepsilon > 0$. Then, there is an usco mapping $\varphi : \mathscr{P}(X) \to \mathscr{C}(X)$ such that $\mu(X \setminus \varphi(\mu)) < \varepsilon$ for every $\mu \in \mathscr{P}(X)$. In particular, $\Phi(T) = \bigcup\{\varphi(\mu): \ \mu \in T\}$, $T \in \mathscr{C}(\mathscr{P}(X))$, defines a continuous map $\Phi : (\mathscr{C}(\mathscr{P}(X)), \tau_V^+) \to (\mathscr{C}(X), \tau_V^+)$ such that $\mu(X \setminus \Phi(T)) < \varepsilon$ for every $T \in \mathscr{C}(\mathscr{P}(X))$ and $\mu \in T$.*

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