Solution of Physical Problems by Decomposition

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Abstract—Recent generalizations are discussed and results are presented for the theory and applications of the decomposition method. Application is made to the Duffing equation with an error of 0.0001% in only four terms and less than $10^{-16}$ in thirteen terms of the decomposition series. Application is also made to a dissipative wave equation, a matrix Riccati equation, and advection-diffusion nonlinear transport.

INTRODUCTION

The decomposition method has gradually developed since 1961 and now can be reasonably viewed as a general theory of dynamical systems which is quantitative rather than merely qualitative, analytic, requiring no linearization or perturbation, and continuous with no resort to discretization and consequent computer-intensive calculations.

It solves linear or nonlinear, ordinary or partial differential equations (as well as stochastic systems and many algebraic equations, delay and integral equations). It solves Navier-Stokes equations and preliminary work shows we can expect significantly enhanced computational efficiency over techniques of computational fluid dynamics. Extensive comparisons are planned.

Several papers have appeared showing the (rapid) convergence and successful applications have been shown by many others; some are listed here. Mathematical examples by the author and others have shown highly accurate solutions in a few terms. A brief summary is included as an appendix.

The ability to solve nonlinear equations by an analytic method is important because linearization changes the problem being analyzed to a different problem and perturbation is only reasonable when nonlinear or stochastic effects are very small. Hierarchy methods which use closure approximations were shown two decades ago to have the same disadvantages as perturbation.

To illustrate the importance of nonlinearity let's consider the simple equation

$$\frac{dy}{dx} = (y - 1)^2,$$

$$y(0) = 1,$$

for which the solution is, of course, $y = 1$. If we make a 1% change in the initial condition to $y(0) = 1.01$, we get a hyperbola

$$y = 1 - \frac{1}{x - 100},$$

with one vertical asymptote. If we write

$$\frac{dy}{dx} = (y - 1)^2 + .01,$$
again a 1% change, we get a periodic solution

$$y = 1 + 0.1 \tan \left( \frac{x}{10} \right),$$

with vertical asymptotes at $(2k + 1)5\pi$, $k = 0, \neq 1, \neq 2, \ldots$. Thus, we see how sensitive a nonlinear equation can be and the nonsense we can get if very small changes are introduced randomly by ignored stochasticity or round-off errors in iterative computer solutions so that the solution changes randomly between the different solutions. In a strongly nonlinear and stochastic system such as a general Navier Stokes flow, we get turbulence which cannot be analyzed except by methodology which is able to treat strongly nonlinear, strongly stochastic equations without resort to nonphysical processes.

The decomposition series converges for nonlinear partial differential equations and special cases such as ordinary differential equations and, of course, linear equations. Convergence originally shown by Cherruault [4,5] is extended to partial differential equations by treating operators other than the one we are solving for as the $R$ operator in $Lu + Ru + Nu = g$. The $Ru$ term is a special (linear) case of $Nu$. The $Nu$ is represented as $\sum A_n$, and $\sum A_n$ is shown to be a generalized Taylor series about a function $u_0(x)$ rather than a point. Further the series approaches zero very rapidly. For an $n^{th}$ order $L$, the $m^{th}$ term of the series decreases as $1/(mn)!$. Thus we get rapidly converging accurate solutions. Numerically, one can see the solution stabilizing to more decimal places as more terms are calculated. Finally since a complicated excitation term $g$ in a nonlinear equation can cause difficult integrations and proliferation of terms, we can expand $g$ in convenient series which are truncated since we only need accuracy to a certain number of decimal places in a physics or engineering problem. We can illustrate this by the example of a dissipative wave equation or the example of a Duffing equation.

**DUFFING EQUATION**

$$u'' + \alpha u' + \beta u + \gamma u^3 = g(t),$$

where $\alpha$, $\beta$, $\gamma$ are constants, $u(0) = c_0$, $u'(0) = c_1$, and $g(t) = \sum_{n=0}^{\infty} g_n t^n$. The solution can be given as a rapidly converging series. In the usual decomposition format, let

$$L = \frac{d^2}{dt^2}, \quad R = \alpha \left( \frac{d}{dt} \right) + \beta, \quad Nu = \gamma u^3,$$

to write $Lu + Ru + Nu = g$. Let $u = \sum_{n=0}^{\infty} u_n$ and $Nu = \sum_{n=0}^{\infty} A_n$ where the $A_n$ are Adomian polynomials generated, in this case, for $Nu = u^3$. $L^{-1}$ is now the two-fold definite integration from 0 to $t$. The components of $u$ are given by calculating the components through $\varphi_6 = \sum_{n=0}^{5} u_n$ and arranging the terms in ascending powers of $t$, we can write $\varphi_6$ as the approximation to the solution:

$$u = \sum_{n=0}^{\infty} c_n t^n,$$

with

$$c_0 = u(0),$$
$$c_1 = u'(0),$$
$$c_2 = \frac{1}{2!} (g_0 - \alpha c_1 - \beta c_0 - \gamma c_0^3),$$
$$c_3 = \frac{1}{3!} (g_1 - \alpha g_0 - \beta c_1 - 3\gamma c_0^2 c_1 + \alpha^2 c_1 + \alpha \beta c_0 + \alpha \gamma c_0^3),$$
$$c_4 = \frac{1}{4!} (2g_2 - \alpha g_1 - \beta g_0 - 3\gamma c_0^2 g_0 - 6\gamma c_0 c_1^2 + 2\alpha^2 c_1 + 6\alpha \gamma c_0 c_1 + \beta^2 c_0 + 4\beta \gamma c_0^3 + 3c_0^2 \gamma^2 - \alpha^3 c_1 - \alpha^2 \beta c_0 - \alpha^2 \gamma c_0^3).$$
\[ c_5 = \frac{1}{6!} (6g_3 - 2\alpha g_2 - \beta g_1 - 3\gamma c_0^2 g_1 - 18\gamma c_0 c_1 g_0 - 6\gamma c_1^2 + \alpha^2 g_1 + 2\alpha \beta g_0 + 6\alpha \gamma c_0^2 g_0 \\
+ 24\alpha \gamma c_0 c_1^2 + \beta^2 c_1 + 24\beta \gamma c_0^2 c_1 + 27c_0^4 c_1 \gamma^2 - \alpha^2 g_0 - 3\alpha^2 \beta c_1 \\
- 9\alpha^2 c_0^2 c_1 - 2\alpha \beta^2 c_0 - 8\alpha \beta \gamma c_0^2 - 6\alpha^2 c_0^2 \gamma^2 + 4\alpha^4 c_1 + \alpha^3 \beta c_0 + \alpha^3 \gamma c_0^3), \]

\[ c_6 = \frac{1}{6!} (66\beta \gamma c_0 c_1^3 - 6\gamma c_0^3 g_2 - 36\gamma c_0 c_1^2 + 126\gamma^2 c_0^2 c_1^2 - 18\gamma c_0 g_0^2 + 24g_4 \\
+ 45\gamma^2 c_0^2 g_0 + 2\alpha \beta g_1 + 42\alpha \gamma c_1^3 + 12\alpha^2 \beta \gamma c_0^2 + 4\alpha^3 \beta c_1 + 9\alpha^2 c_0^2 \gamma^2 \\
+ 3\alpha^2 \beta^2 c_0 + 12\alpha^3 \gamma c_0^2 c_1 - 6\alpha g_0 + \alpha^4 g_0 - 2\beta g_2 - 24\gamma c_0 c_1 g_1 + 42\beta \gamma c_0^2 g_0 \\
- \alpha^5 c_1 - \alpha^4 \beta c_0 - \alpha^4 \gamma c_0^3 - 27c_0^4 \gamma^3 + 2\alpha^2 g_2 + \beta^2 g_0 + 6\alpha \gamma c_0^2 g_1 \\
+ 78\alpha \gamma c_0 c_1 g_1 - 3\alpha^2 g_0 - 3\alpha \beta^2 c_1 - 25\beta^2 \gamma c_0^2 - 51\beta c_0^5 \gamma^2 - \alpha^2 g_1 - \beta^3 c_0 \\
- 9\alpha^2 \gamma c_0^3 g_0 - 66\alpha^2 \gamma c_0 c_1^2 - 96\alpha \beta \gamma c_0^2 c_1 - 105\alpha c_0^3 c_1 \gamma^2, \]

we can continue farther of course using a symbolic program such as Maple, however rapid convergence makes it unnecessary.

For special cases, i.e., for explicit values of the parameters, we will have easily calculated terms. Use of constant parameters is not a limitation to the method. Time-varying and even stochastic cases can be calculated if necessary. The algorithms for calculation to a ten-term approximant, as well as discussion of generic oscillators with any nonlinearity have been derived in the 1994 book referenced on the last page of the appendix.

Consider an example using first an approximation of terms in \( g \) to order \( t^3 \) only. We then expect an approximation to \( u \) of the same order.

\[
\begin{align*}
    u'' + 3u - 2u^3 & = g(t) = \cos t \sin 2t, \\
    u(0) & = 0, \\
    u'(0) & = 1, \\
    \cos t & = 1 - \frac{t^2}{2}, \\
    \sin 2t & = 2t - \frac{8t^3}{3!}.
\end{align*}
\]

In decomposition form, the equation is

\[
Lu = g - 3u + 2u^3.
\]

Let \( u = \sum_{n=0}^{\infty} u_n \), \( u^3 = \sum_{n=0}^{\infty} A_n \), where the \( A_n \) are the appropriate Adomian polynomials generated for this nonlinearity, and identifying \( u_0 = u(0) + t u'(0) + L^{-1} g \), we find

\[
\begin{align*}
    u_0 & = t + \frac{t^3}{3}, \\
    u_1 & = -3L^{-1} u_0 + 2L^{-1} u_0^3 = \frac{t^3}{3!},
\end{align*}
\]

to order \( t^3 \), thus, the two-term approximation to \( u \) is given by

\[
\varphi_2 = t - \frac{t^3}{3!},
\]

or \( \sin t \) to this approximation. We can get to a higher-order or simply substitute to verify that \( u = \sin t \) is the solution. Using the parameters of the example and calculating through \( t^3 \), we have

\[
\begin{align*}
    c_0 & = 0, & c_1 & = 1, & g_0 & = 0, & g_1 & = 2, & g_2 & = 0, & g_3 & = \frac{7}{3}, & \alpha & = 0, & \beta & = 3, & \gamma & = -2.
\end{align*}
\]
We find
\[ c_2 = 0, \quad c_3 = -\frac{1}{3!}, \quad c_4 = 0, \quad c_5 = \frac{1}{5!}, \]
so that
\[ u = t - \frac{t^3}{3!} + \frac{t^5}{5!}, \]
or \( \sin t \) to a three-term approximation of the decomposition. This result used no computers or even calculators but can be carried as far as necessary by computer algorithms already developed.

**DISSIPATIVE WAVE EQUATION**

\[ u_{tt} - u_{xx} + \left( \frac{\partial}{\partial t} \right) f(u) = g(x, t). \]

Letting \( L_t = \frac{\partial^2}{\partial t^2} \) and \( L_t^{-1} \int_0^t \int_0^t (\cdot) \, dt \, dt \), we can write the operator equation
\[ L_t u = g - \left( \frac{\partial}{\partial t} \right) f(u) + \left( \frac{\partial^2}{\partial x^2} \right) u. \]

The nonlinear function \( f(u) \) must be given explicitly as well as the excitation or forcing function \( g \) and the conditions. We will use \( f(u) = u u_x \) and \( g = 2e^{-t} \sin x - 2e^{-t} \sin x \cos x \) and the conditions
\[ u(x, 0) = \sin x, \]
\[ u_t(x, 0) = -\sin x, \]
\[ u(0, t) = u(\pi, t) = 0. \]

(We realize from the conditions \( x = 0 \) and \( x = \pi \) that we cannot use the \( L_x \) operator equation since the \( u_0 \) term would vanish so all following terms also vanish.) Solving the above equation is similar to solving an ordinary differential equation of the form \( Lu + Ru + Nu = g \), i.e., the \( \frac{\partial^2}{\partial x^2} \) is treated like the \( R \) operator.

While we can use decomposition in a straightforward manner, the complicated \( g \) means a complicated \( L_t^{-1} g \) and resulting \( u_0 \) so calculation becomes tedious. Also, we note that calculation past an appropriate accuracy for the application is unnecessary. Hence, we will approximate the functions and can improve our approximation as necessary until we see and verify a solution.

Writing \( u = \sum_{n=0}^{\infty} u_n \) and \( f(u) = \sum_{n=0}^{\infty} A_n \{ f(u) \} \), we obtain the components
\[ u_0 = u(x, 0) + tu_t(x, 0) + L_t^{-1} g, \]
\[ u_{m+1} = -L_t^{-1} \left( \frac{\partial}{\partial t} \right) A_m + L_t^{-1} \left( \frac{\partial^2}{\partial x^2} \right) u_m, \]
for \( m > 0 \). Then since \( \sum_{n=0}^{\infty} u_n \) is a (rapidly) converging series, the partial sum \( \phi_m = \sum_{i=0}^{m-1} u_i \) is our approximant to the solution.

We can calculate the above terms \( u_0, u_1, \ldots, u_m \) as given. However, since we calculate approximants, we can simplify the integrations by approximating \( g \) by a few terms of its double Maclaurin series representation. Thus, we will drop terms involving \( t^3 \) and \( x^3 \) and higher terms. Then
\[ e^{-t} = 1 - t + \frac{t^2}{2}, \]
\[ \sin x = x, \]
\[ \cos x = 1 - \frac{x^2}{2}, \]
so that \( L^{-1} g \approx 0 \) and \( u_0 = x - xt = (1 - t) x \).

\[
\begin{align*}
u_1 &= -L_i^{-1} \left( \frac{\partial}{\partial t} \right) A_0 - L_i^{-1} \left( \frac{\partial^2}{\partial x^2} \right) u_0, \\
&= -L_i^{-1} \left( \frac{\partial}{\partial t} \right) u_0 \left( \frac{\partial}{\partial x} \right) u_0 - L_i^{-1} \left( \frac{\partial^2}{\partial x^2} \right) u_0, \\
&= L_t^{-1} \left( \frac{\partial}{\partial t} \right) (x - xt - xt^2) \left( \frac{\partial}{\partial x} \right) (x - xt - xt^2) - L_i^{-1} \left( \frac{\partial^2}{\partial x^2} \right) (x - xt - xt^2), \\
&= \frac{x t^2}{2}.
\end{align*}
\]

Our two-term approximant is \( \phi_2 = u_0 + u_1 = (1 - t) x \) which approximates \( u = e^{-t} \sin x \) to order \( t^2 \). If needed to recognize the solution, we can carry the series to a higher approximation to see the clear convergence to \( e^{-t} \sin x \). Once we guess the solution may be \( e^{-t} \sin x \), we can verify it by substitution, or substitute \( e^{-t} \sin x + N \) and show that \( N = 0 \).

**MATRIX RICCATI EQUATION**

Consider the matrix Riccati equation which appears in invariant imbedding [1]

\[
R'(x) = B(x) + D(x) R(x) + R(x) D(x) + R(x) B(x) R(x),
\]

\( R(0) = 0 \),

where \( B, D, R \) are continuous \( n \times n \) nonnegative matrices. Suppressing the argument \( x \), we have

\[
R' = B + DR + RD + RBR.
\]

If \( L = \frac{d}{dx} \)

\[
LR = B + HR + NR,
\]

where \( LR = R' \), \( HR = DR + RD \), and \( NR = RBR \), represents a nonlinear operator on \( R \). Since \( R(0) = 0 \), operation with \( L^{-1} \) on both sides yields

\[
R = L^{-1} B + L^{-1} HR + L^{-1} NR.
\]

Let \( R \) and \( NR \) be written in terms of the Adomian \( A_n \) polynomials. For \( R \), this is equivalent to writing \( R = \sum_{n=0}^{\infty} A_n \). For \( NR \) we write \( \sum_{n=0}^{\infty} A_n \). Identify \( R_0 = L^{-1} B \), then

\[
\begin{align*}
R_0 &= L^{-1} B, \\
R_1 &= L^{-1} HR_0 + L^{-1} A_0, \\
R_2 &= L^{-1} HR_1 + L^{-1} A_1, \\
&\vdots \\
R_n &= L^{-1} HR_{n-1} + L^{-1} A_n-1,
\end{align*}
\]

for \( n \geq 1 \). The \( A_n \) for \( NR \) are given by

\[
\begin{align*}
A_0 &= R_0 BR_0, \\
A_1 &= R_0 BR_1 + R_1 BR_0, \\
A_2 &= R_1 BR_1 + R_0 BR_2 + R_2 BR_0, \\
A_3 &= R_0 BR_3 + R_3 DR_0 + R_1 BR_2 + R_2 BR_1, \\
A_4 &= R_2 BR_3 + R_3 BR_4 + R_4 BR_0 + R_1 BR_3 + R_3 BR_1, \\
&\vdots \\
\end{align*}
\]
so that

\[ R_0 = L^{-1} B, \]
\[ R_1 = L^{-1} H R_0 + L^{-1} R_0 B R_0, \]
\[ R_2 = L^{-1} H R_1 + L^{-1}\{ R_0 B R_1 + R_1 B R_0 \}, \]
\[ R_3 = L^{-1} H R_2 + L^{-1}\{ R_1 B R_1 + R_0 B R_2 + R_2 B R_0 \}, \]
\[ \vdots \]

Finally, since \( H R = D R + R D \),

\[ R_0 = L^{-1} B, \]
\[ R_1 = L^{-1} (D R_0 + R_0 D) + L^{-1} (R_0 B R_0), \]
\[ R_2 = L^{-1} (D R_1 + R_1 D) + L^{-1} (R_1 B R_1 + R_1 B R_0), \]
\[ R_3 = L^{-1} (D R_2 + R_2 D) + L^{-1} (R_2 B R_1 + R_2 B R_2 + R_2 B R_0), \]
\[ \vdots \]

An n-term approximant is \( \phi_n = \sum_{n=0}^{n-1} R_n \) which approaches \( R \) as \( n \to \infty \). Thus, given \( B, D \), a specific \( R \) can be calculated to a desired approximation. Accuracy has been demonstrated in an application to neutron transport [1].

**ADVECTION-DIFFUSION**

Let \( \xi(x, y, z, t) \) represent concentration [2,3]. Let the fluid velocity be \( \bar{u} \) with components \( u, v, w \) in \( \mathbb{R}^3 \) and assume an incompressible fluid

\[ \frac{\partial \xi}{\partial t} + \bar{u} \cdot \nabla \xi = D \nabla^2 \xi, \]

where \( D \) is the diffusion constant (which is a constant for a particular fluid or contaminant, temperature, and pressure), \( \xi(x, y, z, 0) \) is a given initial condition and various boundary conditions are possible, e.g., \( \xi \to 0 \) as \( x, y, z \to \infty \), or \( \xi(t) \) is specified on a boundary \( \Gamma \), or, we have a preassigned flux at \( \Gamma \). We have

\[ \frac{\partial \xi}{\partial t} = D \left\{ \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} \right\} - u \frac{\partial \xi}{\partial x} - v \frac{\partial \xi}{\partial y} - w \frac{\partial \xi}{\partial z}. \]

By decomposition, using \( L = \frac{\partial}{\partial t} \) and \( L^{-1} = \int_0^t (\cdot) \, dt \)

\[ \xi = \xi(t = 0) + DL^{-1} \left( \frac{\partial^2}{\partial x^2} \right) \sum_{n=0}^{\infty} \xi_n + DL^{-1} \left( \frac{\partial^2}{\partial y^2} \right) \sum_{n=0}^{\infty} \xi_n + DL^{-1} \left( \frac{\partial^2}{\partial z^2} \right) \sum_{n=0}^{\infty} \xi_n \]
\[ - L^{-1} \left( \frac{\partial}{\partial x} \right) \sum_{n=0}^{\infty} \xi_n - L^{-1} \left( \frac{\partial}{\partial y} \right) \sum_{n=0}^{\infty} \xi_n - L^{-1} \left( \frac{\partial}{\partial z} \right) \sum_{n=0}^{\infty} \xi_n. \]

Thus,

\[ \xi_0 = \xi(t = 0) = f(x, y, z), \]
\[ \xi_{m+1} = DL^{-1} \nabla^2 \xi_m + L^{-1} \bar{u} \cdot \nabla \xi_m, \]

for \( m > 0 \). Now all components are determined and we can write \( \phi_N(\xi) = \sum_{m=0}^{N-1} \xi_m \), as an approximation to \( \xi \) improving as \( N \) increases.
NONLINEAR TRANSPORT

Let’s consider the equation \( L \xi + R \xi + N \xi = g \), where

\[
L = \frac{\partial}{\partial t}, \quad N \xi = f(\xi), \quad R = \bar{u} \cdot \nabla - D\nabla^2.
\]

Let \( \xi = \sum_{n=0}^{\infty} \xi_n \) and \( N \xi = \sum_{n=0}^{\infty} A_n \). Then

\[
\xi = L_t^{-1} g - L_t^{-1} \sum_{n=0}^{\infty} R \xi_n - L_t^{-1} \sum_{n=0}^{\infty} A_n,
\]

where

\[
\xi_0 = L_t^{-1} g,
\]
\[
\xi_{m+1} = -L_t^{-1} R \xi_m - L_t^{-1} A_m,
\]

for \( m > 0 \). Then \( \phi_{m+1} = \sum_{n=0}^{m} \xi_n \) which converges to \( \xi = \sum_{n=0}^{\infty} \xi_n \). Further generalizations are straightforward. We can, for example, consider \( F \xi = g \) where

\[
F \xi = L_t \xi u + L_x \xi u + L_y \xi u + L_z \xi u + R \xi u + N \xi = g,
\]

and solve for \( L_t \xi u, L_x \xi u, L_y \xi u \) or \( L_z \xi u \) which would simply treat the other operator terms as the remainder operator \( R \) and would require the appropriate given boundary conditions.

The case of stochastic \( \xi \) or stochastic processes in the \( R \) term leads to a stochastic \( \phi_N \) which can be averaged or from which expectations and covariances can be found.

Convergence has been proven by Cherruault [4,5]. The solutions are verifiable by checking that the original equation and the given conditions are satisfied.

Since the concern here is solution of physical systems, unbounded or unrealistic inputs and conditions are of no interest. If the model equation and the conditions are physically correct and consistent, a solution is obtained which is unique and accurate. If numerical results are calculated, one sees the approach to a stable solution for the desired number of decimal places. If conditions on one variable are better known than the others, we consider the appropriate operator equation which can yield the solution.

REFERENCES


APPENDIX

DISCUSSION OF THE BASIC DECOMPOSITION METHOD

INTRODUCTION

To hope to attain the objective of solving frontier physical problems, we must be prepared to deal with dynamical systems modelled by differential, delay-differential, integro-differential, and partial differential equations which may be strongly nonlinear and stochastic in parameters, inputs, or specified conditions. The modelling of such systems is necessarily a compromise between
a realistic representation of the attributes of interest and a model which will be tractable by available mathematics. Thus, simplifying assumptions are generally necessary. Linearization and perturbation are common assumptions which have been very useful. However, their use may mean that the problem being solved is no longer a proper representation of the physical problem whose solution is desired. Thus, the result, however elegant with well-stated theorems, may not be physically realistic. That is, the mathematical results may deviate seriously from the actual results.

There are other considerations as well. Closed-form analytical results are considered ideal and series solutions are considered approximations. However, all modeling is approximation and a closed-form solution to a linearized version, or one in which stochastic behavior is replaced by delta-correlated processes, is not to be preferred over a series solution of the actual problem—especially if the convergence is rapid.

The decomposition method does not change the problem into a convenient one for the use of linear theory. It therefore provides more realistic solutions. It provides series solutions which generally converge very rapidly in real physical problems. When solutions are computed numerically, the rapid convergence is obvious. The method makes unnecessary the massive computation arising in the use of discretized methods for solution of partial differential equations. No linearization or perturbation is required. It provides an effective procedure for analytical solution of a wide and general class of dynamical systems representing real physical problems (as opposed to pathological mathematical systems). There are some quite significant advantages over methods which must assume linearity, "smallness," deterministic behavior, stationarity, restricted kinds of stochastic behavior, uncoupled boundary conditions, etc. The method has features in common with many other methods, but it is distinctly different on close examination and one should not be misled by apparent simplicity into superficial conclusions.

**BASIC METHOD**

Beginning with a deterministic equation \( F(u(t)) = g(t) \), where \( F \) represents a general nonlinear ordinary differential operator involving both linear and nonlinear terms, the linear term is decomposed into \( L + R \), where \( L \) is easily invertible and \( R \) is the remainder of the linear operator. \( L \) is taken as the highest order derivative avoiding the difficult integrations which result when complicated Green’s functions are involved. Thus, the equation is written:

\[
Lu + Ru + Nu = g,
\]

where \( Nu \) represents the nonlinear terms. Solving for \( Lu \),

\[
Lu = g - Ru - Nu.
\]

Because \( L \) is invertible, an equivalent expression is

\[
L^{-1}Lu = L^{-1}g + L^{-1}Ru - L^{-1}Nu.
\]

If this corresponds to an initial-value problem the integral operator \( L^{-1} \) may be regarded as definite integrations from 0 to \( t \). If \( L \) is a second-order operator \( L^{-1} \) is a two-fold integration and \( L^{-1}Lu = u - u(0) - t u'(0) \). For boundary value problems (and, if desired, for initial value problems as well), indefinite integrations are used and the constants are evaluated from the given conditions. Solving for \( u \) yields

\[
u = A + Bt + L^{-1}g - L^{-1}Ru - L^{-1}Nu.
\]

The nonlinear term \( Nu \) assumed to be an analytic function \( f(u) \), will be equated to \( \sum_{n=0}^{\infty} \Delta_n \), where the \( \Delta_n \) are special polynomials defined by Adomian. The solution \( u \) is decomposed into \( \sum_{n=0}^{\infty} u_n \) with \( u_0 \) identified as \( A + Bt + L^{-1}g \). (We note that \( \sum_{n=0}^{\infty} \Delta_n \) reduces to \( \sum_{n=0}^{\infty} u_n \) for \( f(u) = u \) so that we are actually expanding \( u \) and \( f(u) \) in the Adomian polynomials.)

\[
\sum_{n=0}^{\infty} u_n = u_0 - L^{-1}R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} \Delta_n.
\]
Thus, we can identify
\[ u_1 = -L^{-1} Ru_0 - L^{-1} A_0, \]
\[ u_2 = -L^{-1} Ru_1 - L^{-1} A_1, \]
\[ \vdots \]
\[ u_{n+1} = -L^{-1} Ru_n - L^{-1} A_n. \]

Thus, all components of \( u \) can be calculated once the \( A_n \) are given for \( n = 0, 1, 2, \ldots \). We then define the \( n \)-term approximant to the solution, \( u \) by \( \phi_n[u] \) or simply \( \phi_n \) by \( \sum_{i=0}^{n-1} u_i \) with \( \lim_{n \to \infty} \phi_n[u] = u \).

**THE \( A_n \) POLYNOMIALS**

The \( A_n \) are given by:
\[
A_0 = f(u_0), \\
A_1 = u_1 \left( \frac{d}{du_0} \right) f(u_0), \\
A_2 = u_2 \frac{d}{du_0} f(u_0) + \left( \frac{u_1^2}{2!} \right) \left( \frac{d^2}{du_0^2} f(u_0), \right) \\
A_3 = u_3 \frac{d}{du_0} f(u_0) + u_1 u_2 \left( \frac{d^2}{du_0^2} f(u_0) + \left( \frac{u_1^3}{3!} \right) \frac{d^3}{du_0^3} f(u_0), \right) \\
\vdots \\
\]
and can be found from the stated formula which is easily generalized to functions of several variables
\[
A_n = \sum_{v=1}^{n} c(v, n) f^{(v)}(u_0), 
\]
where the second index in the coefficient is the order of the polynomial and the first index progresses from 1 to \( n \) along with the order of the derivative. In the linear case \( f(u) = u \), and the \( A_n \) reduce to \( u_n \). Otherwise \( A_n - A_n(u_0, u_1, \ldots, u_n) \) as seen in [1-7]. For \( f(u) = u^3 \), for example
\[
A_0 = u_0^2, \\
A_1 = 2u_0 u_1, \\
A_2 = u_1^2 + 2u_0 u_2, \\
A_3 = 2u_1 u_2 + 2u_0 u_3, \\
\vdots \\
\]

It is to be noted that in this scheme, the sum of the subscripts in each term of the \( A_n \) are equal to \( n \). The \( c(v, n) \) are products (or sums of products) of \( u \) components of \( u \) whose subscripts sum to \( n \)—divided by the factorial of the number of repeated subscripts. Thus, \( c(1, 3) \) can only be \( u_3 \). \( c(2, 3) \) is \( u_1 u_2 \) and \( c(3, 3) = (1/3!) u_1^3 \). For a nonlinear equation in \( u \), one may express any given function \( f(u) \) in the \( A_n \) by \( f(u) = \sum_{n=0}^{\infty} A_n \). Other convenient algorithms have also been developed for much more complicated and multidimensional functions, trigonometric functions, fractional powers, and for some particular functions. The \( A_n \) for \( f(u) = u^3 \) symbolized by \( A_n[u^3] \) is also given by
\[
A_n[u^3] = \sum_{n=0}^{m} \sum_{u=0}^{n} u_{m-n} u_{n-u} u_v, 
\]
It is now established in the series \( \sum_{n=0}^{\infty} A_n \) for \( Nu \) is equal to a generalized Taylor series for \( f(u_0) \), that \( \sum_{n=0}^{\infty} u_n \) is equal to a generalized Taylor series about the function \( u_0 \), and that the series terms approach zero as \( 1/m! \) if \( m \) is the order of the highest linear differential operator. Convergence of the series as been established by Y. Cherruault of the University of Paris [3]. Since the series converges and does so very rapidly, the \( n \)-term partial sum \( \phi_n = \sum_{i=0}^{n-1} u_i \) can serve as a practical solution. The \( \lim_{n \to \infty} \phi_n = u \). We will see that \( n \) is generally very small.

A paper by Adomian and Rach [5] has shown that linear or nonlinear partial differential equations can be solved analogously with a single operator equation involving the highest ordered partial derivative in either \( x \), \( y \), \( z \), or \( t \) as the operator and using the other derivatives as an \( R \) term. Recent generalizations [7] have made the decomposition method a unified method of solving differential and partial differential equations which is rapidly demonstrating its applicability in the hands of investigators in a number of areas.

REFERENCES