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Two Addition Theorems*

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ABSTRACT

The following theorems are proved:

(1) *Let* $A \oplus B = A \cup B \cup (A + B)$. *If G is a finite Abelian group and* $A_1 + \cdots + A_k$ *subsets of G with* $|A_1| + \cdots + |A_k| \geqslant |G|$ *then either* $A_1 \oplus \cdots \oplus A_k = G$ or $0 \in A_2 + \cdots + A_k$. *For* $k = 2$ *this statement is true for any group.*

(2) *Let* $a_1, ..., a_{p+k-1}$ *be a sequence of* $p + k - 1$ *integers. Then it is possible to select k distinct indices* $i_1, ..., i_k$ *such that*

$$a_{i_1} + \cdots + a_{i_k} \equiv 0 \pmod{p}.$$

By means of (2), the proof of a theorem of Erdös, Ginzburg, and Ziv can be considerably simplified.

Let G be a group and A, B subsets of G. We shall use the notation of [2] and define

$$A \oplus B = (A + B) \cup A \cup B. \tag{1}$$

The operation $\oplus$ is clearly associative.

We shall prove the following theorem.

THEOREM 1. *If* $|A| + |B| \geqslant |G|$ *then either* $0 \in A$, *and* $0 \in B$ *or* $A \oplus B = G$.

PROOF: Suppose $c \notin A \oplus B$ then $(c - B) \cap A = \emptyset$. Since

$$|A| + |B| \geqslant |G|$$

it follows that every element of G is either in A or in $c - B$. The element c itself is not in A hence is in $c - B$. Therefore $0 \in B$. Similarly we find $0 \in A$ and Theorem 1 is proved.

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P. Scherk [3] proved: If G is an Abelian group A, B finite subsets of G such that $0 \in A$, $0 \in B$ and $a + b = 0$ with $a \in A$ and $b \in B$ only if $a = 0$, $b = 0$, then

$$| A + B | \geqslant | A | + | B | - 1. \tag{2}$$

The inequality (2) implies: If $0 \notin A + B$ then

$$| A \oplus B | \geqslant | A | + | B |. \tag{3}$$

To prove (3) let $A_0 = A \cup 0$, $B_0 = B \cup 0$. If $A \ni 0$, $B \not\ni 0$ then

$$| A \oplus B | = | A_0 + B_0 | \geqslant | A_0 | + | B_0 | - 1 = | A | + | B | .$$

If $A \not\ni 0$, $B \not\ni 0$ then

$$| A \oplus B | = | A_0 + B_0 | - 1 \geqslant | A_0 | + | B_0 | - 2 = | A | + | B | .$$

Together with Theorem 1 Scherk's result gives the following corollary.

COROLLARY TO THEOREM 1. *Let G be a finite Abelian group, $A_1 ,..., A_k$ subsets of G, and let $| A_1 | + \cdots + | A_k | \geqslant | G |$. Then either*

$$A_1 \oplus A_2 \oplus \cdots \oplus A_k = G \tag{4}$$

or

$$0 \in A_2 \oplus \cdots \oplus A_k . \tag{5}$$

By applying the D transform of [2, p. 5] one can refine the corollary in various ways. We can for instance set

$$A_2^* = A_1 \cap A_2 ,..., A_j^* = (A_1 \cup A_2 \cdots \cup A_{j-1}) \cap A_j .$$

Then either (4) holds or

$$0 \in A_2^* \oplus \cdots \oplus A_k^* .$$

THEOREM 2. *Let G be a group of prime order p and let $a_1 ,..., a_{p+k-1}$ be a sequence of $p + k - 1$ elements of G such that no element is repeated more than k times. Let b be any element of G. Then we can find $a_{i_1} ,..., a_{i_k}$ such that $i_1 < \cdots < i_k$ and*

$$a_{i_1} + \cdots + a_{i_k} = b.$$

PROOF: We partition the elements $a_1 ,..., a_{p+k-1}$ into k non-empty sets $A_1 ,..., A_k$. By the theorem of Cauchy-Davenport [2, p. 3] we have

$$| A_1 + \cdots + A_k | \geqslant \sum_{j=1}^{k} | A_j | - (k - 1) = p$$

which proves Theorem 2.

Erdös, Ginzburg, and Ziv [1] proved the following theorem: *If G is a solvable group (written additively),* $|G| = v$ *and* $a_1, ..., a_{2v-1}$ *is a sequence of* $2v - 1$ *elements of G then one can always find v distinct indices* $i_1, ..., i_v$ *such that*

$$a_{i_1} + \cdots + a_{i_v} = 0.$$

The major portion of [1] is devoted to the proof for the case $v = p$ a prime, while the induction to all solvable groups is comparatively easy. The theorem of Erdös, Ginzburg, and Ziv for a group of prime order p is however an immediate consequence of Theorem 2 with $k = p$.

References

1. P. Erdös and A. Ginzburg, A. ZIV, *Bull. Res. Council Israel,* **10** (Aug. 1961),
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3. P. Scherk, *Amer. Math. Monthly* **62** (1955), 46–47.