Electronic Notes in Theoretical Computer Science **20** (1999)
URL: http://www.elsevier.nl/locate/entcs/volume20.html 11 pages

Contractible Coherence Spaces and Maximal Maps

Hongde Hu

Department of Mathematics
University of Pennsylvania
Philadelphia, PA 19104, USA

Abstract

A coherence space is contractible if it can be generated from the singleton space under products and coproducts. We give an intrinsic characterization of contractible coherence spaces in connection with totality in [1, 2, 15]. We extend contractible coherence spaces with an arbitrary category **C**, and prove that the category of non-empty **C**-valued contractible coherence spaces and **C**-valued maximal maps is exactly the free bicompletion of **C** under non-empty products and coproducts.

Introduction

Totality has been recently considered in the semantics and logic of computation [1, 2, 11, 15, 16]. Based on a notion of total object, Loader [15] introduced totality spaces as models of linear logic, and proved a full completeness theorem of linear logic. Totality spaces are related to the game semantics recently studied by Abramsky *et. al* [1, 2, 4], retaining the fact that a strategy (total object) may be played against a counter-strategy (co-total object) to produce a result (their intersection). In this extended abstract we establish the relation between totality and *contractible coherence spaces*, and present a canonical construction of the free bicompletion of a category **C** under non-empty products and coproducts.

Contractible coherence spaces were introduced in [8] in order to establish the connection between coherence spaces and free bicompletions of categories. In [5] Ehrhard studied contractible coherence spaces in relation to serial-parallel hypercoherence spaces. A non-empty contractible coherence spaces A can be viewed as a totality space whose total objects are *maximal cliques* of A, and its co-total objects are *maximal anti-cliques* of A. We call a coherence space A is *total* if for every *maximal clique* u and *maximal anti-clique* v of A, the intersection $u \cap v$ is non-empty. Every non-empty induced subspace of a contractible coherence space is total. However the converse of the statement above is not true: let $Q_\omega = (X, \sim)$ be an infinite coherence space with $X = \{a_1, \ldots, a_n, \ldots\}$ an infinite set, and the reflexive and symmetric relation $\sim$ on X defined by $a_m \sim a_{2n+1}$ for $m \leq 2n+1$. Q_ω is not contractible although every non-empty induced subspace of Q_ω is total. We prove that a non-empty coherence space A is contractible if and only if every non-empty induced subspace of A is total and A is Q_ω-free, i.e., A does not contain Q_ω as an induced subspace. We also have that a finitely non-empty coherence spaces A is contractible if and only if every non-empty induced subspace of A is total, or equivalently, A does not contain the four vertex path P_4 as induced subspace.

Enriched softness between products and coproducts of coherence spaces has been recently studied in [8, 9]. In [8] we viewed the category of coherence spaces and linear maps as a category enriched over the category of pointed sets. With this approach we showed that the category of contractible coherence spaces and linear maps is exactly the free bicompletion of the singleton under the zero object and products and coproducts. The present paper describes a genuine softness between products and coproducts in the free bicompletion of a category. For non-empty contractible coherence spaces A and B, we consider those linear maps which take maximal cliques of A into maximal cliques of B. We call them *maximal maps*. Indeed, we show that maximal maps are exactly the maximal cliques of the linear implication $A \multimap B$. Let CSM be the category of non-empty contractible coherence spaces and maximal maps. The category CSM forms the free bicompletion of the singleton under non-empty products and coproducts. Furthermore, for a category $\mathbf{C}$, we construct a category $CSM(\mathbf{C})$ whose objects consist of non-empty contractible coherence spaces together with families of objects in $\mathbf{C}$, called $\mathbf{C}$-valued contractible coherence spaces, and whose morphisms are maximal maps together with families of arrows in $\mathbf{C}$, called $\mathbf{C}$-valued maximal maps. $CSM(\mathbf{C})$ has softness between non-empty products and coproducts, and $CSM(\mathbf{C})$ is the free bicompletion of $\mathbf{C}$ under the non-empty products and coproducts.

1 Totality and contractible coherence spaces

Recall from [7] that a *coherence space* is a symmetric and reflexive graph. More precisely, it is a pair $A = (|A|, \sim_A)$, where $|A|$ is a set and $\sim_A$ is a symmetric and reflexive relation on $|A|$. A subset a of $|A|$ is said to be a *clique* (coherence subset) if for all $x, y \in a$, $x \sim_A y$. The set of cliques of A ordered under inclusion

$$A_c = \{a \subset A | \forall x, y \in a \ (x \sim_A y)\}.$$

is called a *Girard domain* (see [18]). We denote the relation $\sim_A$ on $|A|$ mostly by $\sim$, when that is convenient.

Definition 1.1 ([7]) (i) Let A and B be coherence spaces. The linear implication $A \multimap B$ from A into B is defined as a coherence space $(|A| \times |B|, \sim_{\multimap})$ such that

$$(x, y) \sim_{\multimap} (x', y') \iff$$
$$x \sim_A x' \Rightarrow (y \sim_B y' \text{ and } (y = y' \Rightarrow x = x')).$$

(ii) A linear map from A into B is defined by a clique of $A \multimap B$, i.e., a set $X \subset |A| \times |B|$ such that

(a) if $(x, y), (x', y') \in X$, then $x \sim_A x' \Rightarrow y \sim_B y'$; and
(b) if $(x, y), (x', y) \in X$, then $x \sim_A x' \Rightarrow x = x'$.

The negation of A is the coherence space $\neg(A) = (|A|, \sim_\neg)$ where for x, y of $|A|$ and $x \neq y$, $x \sim_\neg y$ is defined by that $x \not\sim y$ in A. We call cliques of $\neg(A)$ anti-cliques of A. For more information on coherence spaces and denotational semantics of linear logic, we refer to [7, 18].

Coherence spaces and linear maps forms a category Coh. The set $id_A = \{(x, x) | x \in |A|\}$ is a linear map on A, i.e., the identity on A. For linear maps $f : A \to B$ and $g : B \to C$, the composite $g \circ f$ is defined by

$$g \circ f = \{(x, z) | \exists y \in |B| \, ((x, y) \in f \ and \ (y, z) \in g)\},$$

which is just the composition of graphs (relations). Notice that if $(x, z) \in g \circ f$ then there is a unique y such that $(x, y) \in f$ and $(y, z) \in g$. Indeed, suppose that there is another y' such that $(x, y') \in f$ and $(y', z) \in g$. That $y \sim_B y'$ follows from (ii)(a) of 1.1 as $(x, y) \in f$ and $(x, y') \in f$. But $(y, z) \in g$ and $(y', z) \in g$, $y = y'$ therefore follows from (ii)(b) of 1.1. And it is easy to see that the composition is associative.

Products, coproducts and the zero object in Coh are described as follows. The zero object is just the coherence space 0 with the empty set and the empty relation. For a family of coherence spaces $(X_i)_{i \in I}$, the product of (X_i) is determined by

$$\sqcap_{i \in I} X_i = (\bigcup_{i \in I} |X_i|, \sim);$$

here $\bigcup_{i \in I} |X_i|$ is the disjoint union of all $|X_i|$, and we represent it as $\bigcup(\{i\} \times |X_i|)$. The relation $\sim$ on $\bigcup_{i \in I} |X_i|$ is defined by

$$(i) \ (i, x) \sim (i, x') \ iff \ x \sim_{X_i} x' \ \ for \ i \in I; \ and$$

$$(ii) \ (i, x) \sim (j, y) \ for \ i \neq j, \ x \in |X_i| \ and \ y \in |X_j|.$$

The projection $p_{X_i} : \sqcap X_i \to X_i$ is given by the set

$$\{((i, x), x) | x \in |X_i|\}.$$

Dually, we have coproduct

$$\sqcup X_i = \neg(\sqcap \neg(X_i)).$$

Definition 1.2 (i) A coherence space is contractible if it can be generated from the singleton space under products and coproducts by finite steps.

(ii) A coherence space A is total if for every maximal clique u and maximal anti-clique v of A, the intersection $u \cap v$ is not empty.

Remark 1.3 (1) Recall from [15] that a totality space $A = (|A|, A_\top, A_\perp)$ consists of a set $|A|$ with subsets $A_\top$ and $A_\perp$ of the power set of $|A|$ such that: (i) $A_\top$ is the set of those $s \subset |A|$ such that $s \cap t$ is the singleton whenever $t \in A_\perp$; (ii) $A_\top$ and $A_\perp$ are interchanged for the condition (i); (iii) $|A| = \bigcup A_\top = \bigcup A_\perp$. For a total coherence space A, $t(A) = (|A|, A_c, \neg(A)_c)$ is a totality spaces.

(2) Every non-empty contractible coherence space A is total. Since an induced subspace (subgraphs) of a contractible coherence space is contractible, every non-empty induced subspace of A is total.

(3) The four-vertex path P_4: $a \sim b \sim c \sim d$ is not total. Hence every contractible coherence space is P_4-free, that is, it does not contain P_4 as an induced subspace.

The following example shows that infinitely P_4-free total coherence space may not be contractible.

Example 1.4 Let Q_ω be the coherence space with $|Q_\omega| = \{a_1, \ldots, a_k, \ldots\}$ an infinite set and the relation on $|Q_\omega|$ defined by that

$$a_m \sim a_{(2n+1)}$$

for every m and n with $m \leq (2n+1)$. Q_ω can be written as

$$(\ldots(\ldots(a_3 \sqcap (a_2 \sqcup a_1) \ldots)$$

where we use a_m for the singleton space with one vertex a_m. We can see that every induced subspace of Q_ω is P_4-free and total. However, Q_ω is not contractible: Q_ω is not generated from the singleton space products and coproducts by finite steps.

A coherence space A is said to be a product if $A = A' \sqcap A''$, and A is a coproduct if $A = A' \sqcup A''$ for some A' and A''.

Proposition 1.5 ([10]) *Let A be a P_4-free coherence space, we have*

(i) *If A is not a coproduct then for any a and b of $|A|$ with $a \not\sim b$, there is c of $|A|$ such that $c \sim a$ and $c \sim b$;*

(ii) *If A is not a product then for any a and b of $|A|$ with $a \sim b$, there is c of $|A|$ such that $c \not\sim a$ and $c \not\sim b$.* □

Lemma 1.6 *Let A be a P_4-free non-empty coherence space. If A is neither a product nor a coproduct then Q_ω is an induced subspace of A.*

Proof. Let Q_k be the coherence space with $|Q_k| = \{a_1, \dots a_k\}$ and the relation on $|Q_k|$ is defined by that

$$a_m \sim a_{2n+1}$$

for every m and n with $m \leq (2n+1) \leq k$. To show that Q_ω is an induced subspace of A, it is suffice to show that for each k, Q_k is an induced subspace of A. We show that Q_k is an induced subspace of A as follows.

Since A is not a product, there are a_1 and a_2 of $|A|$ with $a_1 \not\sim a_2$. But A is also not a coproduct, from 1.5, there is a_3 of $|A|$ such that $a_3 \sim a_1$ and $a_3 \sim a_2$. This shows that Q_3 is an induced subspace of A.

Suppose that A has an induced subspace Q_m. If $m = 2n$, we want to show that Q_{2n+1} is an induced subspace of A. Since $a_{2n-1} \not\sim a_{2n}$, and A is not a coproduct, from 1.5 (i) there is a a_{2n+1} of $|A|$ such that $a_{2n+1} \sim a_{2n-1}$ and $a_{2n+1} \sim a_{2n}$. We will see that $a_{2n+1} \sim a_t$ for all $t < (2n+1)$. Indeed, suppose that there is a_t of $|Q_{2n}|$ so that $a_t \not\sim a_{2n+1}$. t must not be $(2n-1)$ and $2n$. But $a_{2n} \not\sim a_t$ and $a_{2n-1} \sim a_t$, A has an induced subspace $P_4 : a_{2n} \sim a_{2n+1} \sim s_{2n-1} \sim a_t$, which is contrary to the assumption: P_4 is not an induced subspace of A. This shows that for any $t < (2n+1)$, $a_t \sim a_{2n+1}$. So Q_{2n+1} is an induced subspace of A.

When $m = 2n+1$, we can similarly show that Q_{m+1} is an induced subspace of A. □

Theorem 1.7 *A non-empty coherence space A is contractible if and only if A is Q_ω-free and every induced subspace of A is total.*

Proof. From 1.3 and 1.4, a contractible coherence space is Q_ω-free and its non-empty induced subspaces are total.

Conversely, if A is not contractible and its non-empty induced subspaces are total, then A does not contain P_4 as an induced subspace. We want show either that Q_ω is an induced subspace of A or that some induced subspace of A is not total.

Case 1: Every induced subspace of A is a products or a coproduct.

In that case we will see that there is an induced subspace of A which is not total (contradiction!). We build the representation tree of A as follows. Start from the root A, say $A = \sqcap A_i$ where each A_i is either a coproduct or a 1(an atom). Since A is not contractible, there is i such that A_i is not 1. Let $A_i = \sqcup_{j \in J_i} A_{ij}$. By the same reason, we can find some j so that A_{ij} is not 1. Continuing this process, we then build an infinite induced subspace

$$R_\omega = (a_1 \sqcap (a_2 \sqcup a_3(\sqcap \dots)$$

of A where $|R_\omega| = \{a_1, \dots, a_n, \dots\}$ is an infinite set, the relation on $|R_\omega|$ is defined by that $a_{2n+1} \sim a_m$ for every m and n with $(2n+1) \leq m$. R_ω is not total because the intersection of the maximal clique $u = \{a_1, \ldots, a_{2n+1}, \ldots\}$ and the maximal anti-clique $v = \{a_2, \ldots, a_{2n}, \ldots\}$ is empty.

Case 2. There is an induced subgraph B of A so that B is neither a product nor a coproduct.

From 1.6, Q_ω is an induced subgraph of B. So Q_ω is an induced subgraph of A. □

Corollary 1.8 *A finitely non-empty coherence space A is contractible if and only if every induced subspace of A is total, and if and only if it is P_4-free.* □

2 Maximal maps and softness

Definition 2.1 A linear map $f : A \to B$ between non-empty contractible coherence spaces is said to be maximal if f takes maximal cliques of A into maximal cliques of B, i.e., if u is maximal of A_c then

$$f(u) = \{b|(a, b) \in f for some a \in u\}$$

is maximal of B_c.

Non-empty contractible coherence spaces and maximal maps form a category, denoted it by CSM. Since the projections of products and injections of coproducts in section 1 are maximal maps, non-empty products and coproducts in CSM are exactly those in Coh.

Proposition 2.2 *A linear map $f : A \to B$ between non-empty contractible coherence spaces A and B is maximal if and only if f is a maximal clique in $A \multimap B$.*

Proof. A maximal linear map f is clearly a maximal clique of $A \multimap B$. To show that a maximal clique of $A \multimap B$ is a maximal map from A into B, it suffices to show that $A \multimap B$ is total for contractible coherence spaces A and B. □

Recall from [15] that for totality spaces A and B, the tensor $A \otimes B$ is defined by

$$|A \otimes B| = |A| \times |B|$$
$$(A \otimes B)_\top = \{s \times t | s \in A_\top, t \in B_\top\}$$

The linear implication is the totality space

$$A \multimap B = \neg(A \otimes \neg(B))$$

Where $\neg(A) = (|A|, A_\perp, A_\top)$. A linear map $f : A \to B$ is defined by $f \in (A \multimap B)_\top$. Let TOT be the category of non-empty totality spaces and linear maps.

Proposition 2.3 *CSM is a full subcategory of TOT which is closed under non-empty products and coproducts.*

Proof. For a contractible coherence space A, $i(A) = (|A|, A_c, \neg(A)_c)$ is a totality space. Also, $i(A \multimap B)$ as a totality space is the same as $i(A) \multimap i(B)$. □

For a category **C**, we start to describe the category $CSM(\mathbf{C})$ of non-empty **C**-valued contractible coherence spaces and **C**-maximal maps as follows.

(i) Objects of $CSM(\mathbf{C})$ are defined by $A_I = (|I|, \sim, \{A_i\}_{i\in|I|})$, here $I = (|I|, \sim)$ are non-empty contractible coherence spaces, and A_i are objects of **C**.

(ii) Arrows of $CSM(\mathbf{C})$: **C**-valued maximal maps from $(|I|, \sim, \{A_i\}_{i\in|I|})$ into $(|J|, \sim, \{B_j\}_{j\in|J|})$, are defined by pairs

$$f = (t, \{f_{i,j}\}_{(i,j)\in t}) : A_I \to B_J,$$

where $t : I \to J$ are maximal maps of CSM and $f_{(i,j)} : A_i \to B_j$ are arrows of **C**, for all $(i, j) \in t$.

Let $f = (t, \{f_{i,j}\}_{(i,j)\in t}) : A_I \to B_J$ and $g = (s, \{g_{j,k}\}_{(j,k)\in s}) : B_J \to C_K$ be **C**-valued maximal maps. We define the composition $g \circ f$ by the set

$$g \circ f = (s \circ t, \{g_{j,k} \circ f_{i,j}\})$$

where the family $\{g_{j,k} \circ f_{i,j}\}$ is the collection of all composition $g_{j,k} \circ f_{i,j}$ with $(i, j) \in t$ and $(j, k) \in s$. The uniqueness of j above ensures that $CSM(\mathbf{C})$ forms a category. Also, it is clearly that CSM is isomorphic to $CSM(\{*\})$; here $\{*\}$ is the category with one object $*$ and one (identity) arrow.

Proposition 2.4 *$CSM(\boldsymbol{C})$ has non-empty products and coproducts.*

Proof. For **C**-valued contractible coherence spaces $A_{X_i} = (|X_i|, \sim_{X_i}, \{A_x\}_{x\in|X_i|})$ with $i \in I$, the product of $\{A_{X_i}\}$ is given by

$$(\bigcup_{i\in I} |X_i|, \sim, \bigcup_{i\in I} \{A_x\}_{x\in|X_i|}).$$

And the projection

$$p_{A_{X_i}} = (p_{X_i}, \{id_{A_x}\}_{x\in|X_i|}) : \sqcap A_{X_i} \to X_i;$$

here p_{X_i} is the projection of the product of X_i and id_{A_x} is the identity on A_x for each $x \in |X_i|$. The coproduct of $\{A_{X_i}\}$ is given by

$$(\bigcup_{i\in I} |X_i|, \sim_{\bigcup}, \bigcup_{i\in I} \{A_x\}_{x\in|X_i|})$$

with the injections $q_{A_{X_i}} = (q_{X_i}, \{id_{A_x}\}_{x\in|X_i|})$ for all $i \in I$. □

We now discuss with softness of $CSM(\mathbf{C})$. Recall from [6] that a lattice L is soft, i.e., it satisfies Whitman's condition, if for any a, b, c and d in L,

$$a \wedge b \leq c \vee d \Leftrightarrow$$

$$a \leq c \vee d, \ \ or \ \ b \leq c \vee d, \ \ or \ \ a \wedge b \leq c, \ \ or \ \ a \wedge b \leq d.$$

For a poset X, let F be a lattice and the inclusion $X \subset F$ closed under the order relation on F. F is said to be freely generated by X if X generates F and every order-preserving map from X into any lattice L extends to a lattice homomorphism of F into L. Whitman showed in [19] that the free lattice generated by X is soft. The categorical generalization of Whitman's condition is described as follows.

Definition 2.5

(i) Let $\mathbf{B}$ be a category with non-empty products and coproducts. $\mathbf{B}$ is said to be soft (or, has softness between non-empty products and coproducts) if for any pair of discrete diagrams $D : I \to \mathbf{B}$ and $E : J \to \mathbf{B}$ the commutative square of canonical maps

$$\begin{array}{ccc} colim\mathbf{B}(D,E) & \longrightarrow & colim\mathbf{B}(D, colimE) \\ \downarrow & & \downarrow \\ colim\mathbf{B}(limD,E) & \longrightarrow & \mathbf{B}(limD, colimE) \end{array}$$

is a pushout in **Set**.

(ii) An object A of $\mathbf{B}$ is σ-atomic if the functor $\mathbf{B}(A,-) : \mathbf{B} \to \mathbf{Set}$ preserves non-empty coproducts. A is π-atomic if it is σ-atomic in $\mathbf{B}^{op}$. A is atomic if it is both σ- and π-atomic.

In $CSM(\mathbf{C})$, we will use C for the singleton $\mathbf{C}$-valued contractible coherence space, i.e., $C = (1, \{C\})$.

Proposition 2.6 *Let* $\boldsymbol{C}$ *be a category. Then*

(i) *For* $C \in \boldsymbol{C}$, C *are only atoms in* $CSM(\boldsymbol{C})$ *under non-empty products and coproducts.*

(ii) $CSM(\boldsymbol{C})$ *has softness between non-empty products and coproducts.*

Proof. To show that C is an atom, let $\{A_{I_m}\}_{m \in M}$ be a non-empty family of objects of $CSM(\mathbf{C})$. For a $\mathbf{C}$-valued maximal map

$$f = (t, \{f_{*,u}\}_{u \in \coprod I_m}) : C \to \coprod A_{I_m},$$

if $(*, a)\,, (*, b) \in t$ then $a \sim b$. Hence there is a unique $m \in M$ such that $\{a, b\} \in |I_m|$. Consequently, C factors uniquely through the injection $q_m : A_{I_m} \to \coprod A_{I_m}$. Dually, we can show that C is π-atomic.

For the uniqueness part of (i), let $A_I \neq C$ for all $C \in \mathbf{C}$. There are two cases: (a) A_I is a coproduct of A_i, then id_{A_I} cannot factor through any coproduct injection; (b) A_I is a product of A_i, then id_{A_I} cannot factor through any product projection.

For (ii), Let

$$f = (t, \{f_{i,j}\}) : \prod_{m \in M} A_{I_m} \to \coprod_{n \in N} B_{J_n}$$

be a $\mathbf{C}$-valued maximal map. There are only three possibilities:

Case 1. There are $m \neq m'$, $x \in |I_m|$ and $x' \in |I_{m'}|$ such that $(x, y), (x', y') \in t$.

In $\prod I_m$ we have that $x \sim x'$, hence $y \sim y'$ is true in $\coprod J_n$. This implies that there is a unique n such that $y, y' \in |J_n|$. Consider arbitrary $(z, z') \in t$. (a) If $z \in |I_m|$ then $z \sim y$. So $z' \sim y'$ is true in $\coprod J_n$. Consequently, $z' \in |J_n|$. (b) If $z \notin |I_m|$ then $z \sim x$. So $z' \sim x'$ is true in $\coprod J_n$. Again, we have $z' \in |J_n|$. This shows that f uniquely factors through the injection $B_{J_n} \to \coprod B_{J_n}$.

Case 2. There is unique m such that if $(x, x') \in t$ then $x \in |I_m|$, and there are $n \neq n'$, $y \in J_n$ and $y' \in J_{n'}$ such that $(x, y), (x', y') \in t$.

This implies that f factors uniquely through the projection $\prod A_{I_m} \to A_{I_m}$.

Case 3. f factors through some B_{J_n} and A_m.

By Case 1 and Case 2, m and n must be unique. Thus we have completed the proof of (ii). □

Proposition 2.7 *For any* $\boldsymbol{C}$*, we have*

(i) *The only* π*-atoms of* $CSM(\boldsymbol{C})$ *are coproducts of families of objects of the forms* C*.*

(ii) *The only* σ*-atoms of* $CSM(\boldsymbol{C})$ *are products of families of objects of the forms* C*.*

Proof. The proof is straightforward. □

We now discuss the freeness of $CSM(\mathbf{C})$. For a category $\mathbf{C}$, we say that the free bicompletion of $\mathbf{C}$ under non-empty products and coproducts is a pair $(i, \Lambda(\mathbf{C}))$ where $\Lambda(\mathbf{C})$ has non-empty products and coproducts and $i : \mathbf{C} \to \Lambda(\mathbf{C})$ is a functor such that:

Existence: for any functor $F : \mathbf{C} \to \mathbf{B}$ with $\mathbf{B}$ having non-empty products and coproducts there is a functor $F' : \Lambda(\mathbf{C}) \to \mathbf{B}$ preserving non-empty products and coproducts such that $F = F' \circ i$;

Uniqueness: If $F', F'' : \Lambda(\mathbf{C}) \to \mathbf{B}$ are functors which satisfy (i) then there is a unique isomorphism $u : F' \to F''$ such that $u \circ i = id_F$.

The following characterization theorem of $\Lambda(\mathbf{C})$ is just a special case of a more general result presented in [12].

Theorem 2.8 *For a category* $\boldsymbol{C}$*, the free bicompletion* $i : \boldsymbol{C} \to \Lambda(\boldsymbol{C})$ *under non-empty products and coproducts has the following properties:*

(i) $\Lambda(\boldsymbol{C})$ *is soft.*

(ii) $i(A)$ *is atomic for any* $A \in \boldsymbol{C}$.

(iii) *The functor* i *is full and faithful.*

(iv) $\Lambda(\boldsymbol{C})$ *is generated from* $i(C)$ *under non-empty products and coproducts.*

Moreover, these properties characterize the pair $(i, \Lambda(\boldsymbol{C}))$ *up to an equivalence of categories.* □

Let

$$i_{\mathbf{C}} : \mathbf{C} \to CMS(\mathbf{C})$$
$$C \mapsto (1, \{C\})$$

be the inclusion. We have

Theorem 2.9 *The pair* $(i, CSM(\boldsymbol{C}))$ *is the free bicompletion of* $\boldsymbol{C}$ *under non-empty products and coproducts.*

Proof. From 1.2 and 2.6. □

Corollary 2.10 CSM *is exactly the free bicompletion of* $\{*\}$ *under non-empty products and coproducts.* □

For categories $\mathbf{B}$ and $\mathbf{C}$, let $F : \mathbf{C} \to \mathbf{B}$ a functor, and $i_{\mathbf{B}} : \mathbf{B} \to CSM(\mathbf{B})$ be the canonical functor. For $G = i_{\mathbf{B}} \circ F$, 2.10 gives us a functor

$$CSM(F) : CSM(\mathbf{C}) \to CSM(\mathbf{B})$$

which preserves non-empty products and coproducts such that $i_{\mathbf{B}} \circ F = CSM(F) \circ i_{\mathbf{C}}$.

Corollary 2.11 *Up to isomorphism,* $CSM(F)$ *is the unique extension of* F *which preserves non-empty products and coproducts.* □

Remark 2.12 It is worth mentioning that there is softness between non-empty products and coproducts in TOT.

Acknowledgements

This work was partly done while I was visiting Prof. Vaughan Pratt's group at Stanford University. I am indebted to A. Joyal, G. Plotkin, V. Pratt and especially A. Scedrov for stimulating discussions concerning various aspects of this work.

References

[1] S. Abramsky and R. Jagadeesan, *Games and full completion for multiplicative linear logic,* J. of Symbolic Logic **59** (1994), 543-574

[2] S. Abramsky and G. McCusker, *Games and full abstraction for the Lazy λ-calculus,* Proceedings of LICS 1995, 234-243.

[3] M. Barr, *$*$-autonomous categories and linear logic,* Math. Structures and Comp. Sci. **1** (1991), 159-179.

[4] A. Blass, *A game semantics for linear logic,* Ann. Pure & Applied Logic **56** (1992), 183-220.

[5] T. Ehrhard, *Parallal and serial hypercoherence,* preprint (1997).

[6] R. Freese, J. Jezek and J. B. Nation, "Free Lattices," Math. Surveys and Monographs **42**, AMS, 1995.

[7] J. -Y. Girard, *Linear logic,* Theor. Comp. Sci. **50** (1987), 1-102.

[8] H. Hu and A. Joyal, *Coherence completions and enriched softness,* Electronic Notes in Theoretical Computer Science **6** (1997), URL: `http://www.elsevier/nl/locate/entcs/volume6.html`.

[9] H. Hu and A. Joyal, *Coherence completions of categories,* Theoretical Computer Science, to appear.

[10] H. Hu, *Contractible graphs and the maximum number of edges in P_4-free graphs of bounded degree,* preprint (1998).

[11] M. Hyland, *Game Semantics,* in A. M. Pitts and P. Dybjer, editors, Semantics and Logics of Computation, Cambridge University Press, 1997.

[12] A. Joyal, *Free bicomplete categories,* Math. Reports **XVII** (1995), Aca. Sci. Canada, 219-225.

[13] A. Joyal, *Free lattices, money games and communication,* Proceedings of the 10th Int. Congress of Logic, Methodology and Philosophy of Sci., Firenze, August 1995.

[14] L. Kristiansen and D. Normann, *Interpreting higher Computations as types with totality,* Technical report, Institute of Mathematics, University of Oslo (1992)

[15] R. Loader, *Linear logic, Totality and Full Completeness,* Proceedings of LICS 1994.

[16] G. Plotkin, *λ-definability in the full type hierarchy,* in J. P. Seldin and J. R. Hindley, editors, "Combinatory Logic, Lambda Calculus and Formalism," Academic Press, 1980.

[17] A. Scedrov, *Linear logic and computation: a survey,* Proceedings of Marktoberdorf Summer School 1993, NATO Adv. Sci. Inst., Series F, Springer-Verlag, 1994, 281-298.

[18] A. S. Troelstra, "Lectures on Linear Logic," CSLI Lecture Notes **29**, Center for the Study of Language and Information, Stanford Univ., 1992.

[19] P. M. Whitman, *Free lattices,* Ann. of Math. **42** (1941), 325-330.