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Fuzzy Sets and Models of Decision Making

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Abstract—Results of research into the use of fuzzy sets for handling various forms of uncertainty in optimization problems related to the design and control of complex systems are presented. Much attention is given to considering the uncertainty of goals that is associated with a multicriteria character of many optimization problems. The application of a multicriteria approach is needed to solve

(1) problems in which solution consequences cannot be estimated on the basis of a single criterion, that involves the necessity of analyzing a vector of criteria, and
(2) problems that may be considered on the basis of a single criterion but their unique solutions are not achieved because the uncertainty of information produces so-called decision uncertainty regions, and the application of additional criteria can serve as a convincing means to contract these regions.

According to this, two classes of models ($\langle X, M\rangle$ and $\langle X, R\rangle$ models) are considered with applying the Bellman-Zadeh approach and techniques of fuzzy preference relations to their analysis. The consideration of $\langle X, R\rangle$ models is associated with a general approach to solving a wide class of optimization problems with fuzzy coefficients. This approach consists in formulating and analyzing one and the same problem within the framework of interrelated models with constructing equivalent analogs with fuzzy coefficients in objective functions alone. It allows one to maximally cut off dominated alternatives. The subsequent contraction of the decision uncertainty region is associated with reduction of the problem to multicriteria decision making in a fuzzy environment with its analysis applying one of two techniques based on fuzzy preference relations. The results of the paper are of a universal character and are already being used to solve problems of power engineering.

Keywords—Uncertainty factor, Multicriteria optimization problems, Bellman-Zadeh approach, Fuzzy coefficients, Fuzzy preference relations.

1. INTRODUCTION

In the process of posing and solving a wide range of problems related to the design and control of complex systems, one inevitably encounters diverse kinds of uncertainty. Taking into account the uncertainty factor in shaping the mathematical models serves as a means for increasing their adequacy and, as a result, the credibility and factual efficiency of decisions based on their analysis.

Investigations of recent years show the utility of applying fuzzy set theory [1] for considering diverse kinds of uncertainty. Its use in problems of optimization character offers advantages of both fundamental nature (the possibility of validly obtaining more effective, less "cautious solutions") and computational character [2].

The uncertainty of goals is the notable kind of uncertainty that is associated with a multicriteria character of many optimization problems. It is possible to classify two types of problems, which

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need the use of a multicriteria approach [3]:

- problems in which solution consequences cannot be estimated on the basis of a single criterion: these problems are associated with the analysis of models including economic as well as natural indices (when alternatives cannot be reduced to the comparable form) and also by the need to consider indices whose cost estimates are hampered or impossible,
- problems that, from the substantial point of view, may be solved on the basis of a single criterion; however, if the uncertainty of information does not allow one to obtain a unique solution, it is possible to reduce these problems to multicriteria decision making; the use of additional criteria (including the criteria of qualitative character) can serve as a convincing means to contract the decision uncertainty regions [4].

In accordance with these types of problems, two classes of models (so-called $\langle X, M\rangle$ and $\langle X, R\rangle$ models) may be constructed. Their analysis is associated with the application of the Bellman-Zadeh approach and techniques of fuzzy preference relations, which are the main subject of the present paper.

2. BELLMAN-ZADEH APPROACH AND MULTICRITERIA OPTIMIZATION PROBLEMS

When analyzing $\langle X, M\rangle$ models, a vector of objective functions $F(X) = \{F_1(X), \dots, F_q(X)\}$ is considered, and the problem consists in simultaneous optimizing all objective functions (local criteria), i.e.,

$$F_p(X) \to \underset{X\in L}{\text{extr}}, \qquad p = 1, \dots, q, \tag{1}$$

where L is a feasible region in R^n.

The first step in solving problem (1) is associated [5] with determining a set of Pareto optimal solutions $\Omega \subset L$, which is to include the solution X^0 of the problem. The construction of $\Omega \subset L$ is useful for reducing a number of alternatives. However, it does not permit one to obtain unique solutions. It is necessary to choice a particular Pareto optimal solution on the basis of additional information of a decision-maker (DM). It is possible to classify three approaches to using this information [6]: *a priori*, *a posteriori*, and adaptive.

When analyzing multicriteria optimization problems, it is necessary to solve some questions related to normalizing criteria, selecting principles of optimality and considering priorities of the local crciteria. The solution of these questions and, therefore, developing multicriteria methods is carried out in the following directions [5,6]: scalarization techniques, imposing constraints on criteria, utility function method, goal programming, and using the principle of guarantee result. Without discussion of these directions, it is necessary to point out the validity and advisability of using the direction based on the principle of guarantee result [7].

At present much attention is given to rational using additional information of DM and developing interactive systems to solve multicriteria problems. When applying the adaptive approach, a procedure of improving the solution is realized as a result of transition from $X^0_\alpha \in \Omega \subset L$ to $X^0_{\alpha+1} \in \Omega \subset L$ with considering information I_α of DM.

A drawback of existing interactive systems is associated with their attachment to the sole form of additional information representation. In many cases, DM has more spacious information reflecting his or her preferences and reducing time of the solution search. Thus, the development of adaptive interactive decision making systems (AIDMS) allowing to perceive information on a limited language of DM is important.

The lack of clarity in the concept of "optimal solution" is the basic methodological complexity in solving multicriteria problems. When applying the Bellman-Zadeh approach [8] for analyzing $\langle X, M\rangle$ models, this concept is defined with reasonable validity: the maximum degree of implementing all goals serves as a criterion of optimality. This conforms to the principle of guarantee result and provides a constructive line in obtaining harmonious solutions [9]. Furthermore, the

Bellman-Zadeh approach permits one to realize an effective (from the computational standpoint) as well as rigorous (from the standpoint of obtaining solutions $X^0 \in \Omega \subset L$) method of analyzing multicriteria optimization models [9,10]. Finally, its use allows one to preserve natural measure of uncertainty in decision making and to take into account indices, criteria, and constraints of qualitative (contextual) character.

When using the Bellman-Zadeh approach, each of objective functions $F_p(X), X \in L,\ p=1,\dots,q$ of problem (1) is replaced by a fuzzy objective function or a fuzzy set

$$A_p = \{X, \mu_{A_p}(X)\}, \qquad X \in L, \quad p = 1, \dots, q, \tag{2}$$

where $\mu_{A_p}(X)$ is a membership function of A_p [1].

As it is shown in [8], a fuzzy solution D with setting up the fuzzy sets (2) is turned out as a result of the intersection $D = \bigcap_{p=1}^{q} A_p$ with a membership function

$$\mu_D(X) = \bigwedge_{p=1}^{q} \mu_{A_p}(X) = \min_{p=1,\dots,q} \mu_{A_p}(X), \qquad X \in L. \tag{3}$$

Using (3), it is possible to obtain the solution X^0, which provides us with the maximum degree of belonging

$$\max \mu_D(X) = \max_{X \in L} \min_{p=1,\dots,q} \mu_{A_p}(X) \tag{4}$$

to D, and problem (1) is reduced to

$$X^0 = \arg\max_{X \in L} \min_{p=1,\dots,q} \mu_{A_p}(X). \tag{5}$$

To obtain the solution (5), it is necessary to build the membership functions $\mu_{A_p}(X)$, $p = 1, \dots, q$. These membership functions may be considered as acceptable if they can convincingly reflect a degree of achievement of the "own" optimums by the corresponding $F_p(X)$, $X \in L$, $p = 1, \dots, q$. This is satisfied by the use of the membership functions

$$\mu_{A_p}(X) = \left[\frac{F_p(X) - \min\limits_{X \in L} F_p(X)}{\max\limits_{X \in L} F_p(X) - \min\limits_{X \in L} F_p(X)} \right]^{\lambda_p} \tag{6}$$

for objective functions, which must be maximized, or by the use of the membership functions

$$\mu_{A_p}(X) = \left[\frac{\max\limits_{x \in L} F_p(X) - F_p(X)}{\max\limits_{X \in L} F_p(X) - \min\limits_{X \in L} F_p(X)} \right]^{\lambda_p} \tag{7}$$

for objective functions, which must be minimized. Other types of acceptable membership functions are considered in [9].

In (6) and (7), λ_p, $p = 1, \dots, q$ are importance factors for the corresponding objective functions. The construction of (6) or (7) demands to solve the following problems:

$$F_p(X) \to \min_{X \in L}, \tag{8}$$

$$F_p(X) \to \max_{X \in L}, \tag{9}$$

providing $X_p^0 = \arg\min_{X \in L} F_p(X)$ and $X_p^{00} = \arg\max_{X \in L} F_p(X)$, respectively.

Thus, the solution of problem (1) on the basis of the Bellman-Zadeh approach demands analysis of $2q + 1$ monocriteria problems (8), (9), and (4), respectively.

As indicated above, the solution of problem (1) must belong to the Pareto set $\Omega \subset L$. In other words, it is necessary (from the formal standpoint) to consider the membership function

$$\bar{\mu}_D(X) = \min\left\{\min_{p=1,\dots,q} \mu_{A_p}(X), \mu_\pi(X)\right\}, \tag{10}$$

where

$$\mu_\pi(X) = \begin{cases} 1, & \text{if } X \in \Omega, \\ 0, & \text{if } X \notin \Omega. \end{cases}$$

Taking this into account, it should be noted that the corresponding construction of procedures for solving problem (4) provides the line of obtaining $X^0 \in \Omega \subset L$ in accordance with (10). Besides, there are theoretical bases [10] indicating that the fuzzy solution D accords well with the set $\Omega \subset L$ for a wide class of membership functions $\mu_{A_p}(X)$. Thus, it can be said about equivalence of $\bar{\mu}_D(X)$ and $\mu_D(X)$, that makes it possible to give up the necessity of implementing a cumbersome procedure of determining the set $\Omega \subset L$.

Finally, the existence of additional conditions of qualitative character reduces (5) to

$$X^0 = \arg\max_{X \in L} \min_{p=1,\dots,q+s} \mu_{A_p}(X), \tag{11}$$

where $\mu_{A_p}(X)$, $X \in L$, $p = q+1, \dots, s$ are the membership functions of fuzzy values of linguistic variables [1], which reflect additional conditions (indices, criteria and/or constraints of qualitative character).

There are attempts to use the product operation as well as other aggregation operators in place of the min operation in (5) and (11). However, investigations (for example, [11]) show the validity of applying min operation. Besides, our computing practice shows that its use provides the most harmonious solutions [3].

The use of the Bellman-Zadeh approach in multicriteria optimization has found wide applications in solving power engineering problems [3,12]. For example, it is possible to distinguish a problem of power and energy shortage (natural or associated with the advisability of load management) allocation. This problem is applicable to regulated and deregulated (conditions of energy markets) environments and is to be analyzed as the technical, economical, ecological, and social problem. In addition, when solving the problem, it is necessary to account for considerations of creating incentive influences for consumers. From these points of view, existing methods of its solution have drawbacks [9]. Their overcoming is possible on the basis of formulating the problem within the framework of the multicriteria model (1) (that includes linear, fractional and/or quadratic functions [9]) with

$$L = \left\{X \in R^n \mid 0 \le x_i \le A_i,\ \sum_{i=1}^{n} x_i = A\right\}, \tag{12}$$

where $X = (x_1, \dots, x_n)$ is the sought for a vector of limitations, A_i is the permissible value of limitation for the i^{th} consumer, A is a total value of limitatios for all customers considered in planning or control.

The AIDMS has been developed to solve problem (1),(12). Its calculating kernel destined for obtaining X^0 on the basis of (5) or (11) is associated with a nonlocal search that is a modification of the Gelfand's and Tsetlin's "long valley" method [13]. The AIDMS includes a procedure for constructing a term-set [1] and membership functions of the linguistic variable Q—*Limitation for Consumer* (the initial available term-set is $T(Q) = \langle$*Near, Approximately, Slightly Less, Considerably Less, Slightly More, Considerably More*$\rangle$) to provide DM with the possibility to take into account conditions, which are difficult to formalize. Furthermore, the AIDMS includes diverse procedures for forming and correcting the vector $\lambda = (\lambda_1, \dots, \lambda_q)$ of importance factors.

These procedures are oriented to the individual DM as well as to the group DM. In particular, one of the procedures is associated with processing of the results of paired qualitative comparisons of the importance for different goals (objective functions). The use of this type of information is rational because psychological experiments [14] show that DM is faced with difficulties in direct estimating the importance factors. In accordance with [14], DM is to indicate which among two goals is more important and to estimate his or her perception of distinction using a rank scale. This scale includes the following ranks: *Identical Significance, Weak Superiority, Strong Superiority, Evident Superiority*, and *Absolute Superiority*. The comparisons allow one to construct the matrix $[b_{pt}]$, $p, t = 1, \dots, q$. The components of the eigenvector corresponding to the maximum eigennumber of the matrix (normalized in a certain way, if necessary) can serve as estimates for λ_p, $p = 1, \dots, q$.

Among other applications of the Bellman-Zadeh approach, it is possible to indicate the following. As it is shown in [15], there is a deep connection between production rules used in fuzzy control technology and multicriteria optimization models. This opens up new ways for tuning fuzzy models applying the Bellman-Zadeh approach.

3. OPTIMIZATION PROBLEMS WITH FUZZY COEFFICIENTS

Numerous problems related to the design and control of complex systems [4,16] may be formulated as follows

$$\text{maximize}\, \tilde{F}(x_1, \dots, x_n), \tag{13}$$

subject to the constraints

$$\tilde{g}_j(x_1, \dots, x_n) \subseteq \tilde{B}_j, \qquad j = 1, \dots, m. \tag{14}$$

The objective function (13) and constraints (14) include fuzzy coefficients, as indicated by the $\sim$ symbol.

Given the maximization problem (13),(14), we can state the following problem:

$$\text{minimize}\, \tilde{F}(x_1, \dots, x_n), \tag{15}$$

subject to the constraints (14).

A possible approach to handling constraints of form (14) is proposed in [4]. This approach involves approximate replacement of each of the constraints of form (14) by a finite set of deterministic (nonfuzzy) constraints, represented in the form of inequalities; these can be formulated readily, but with considerable increase in the dimension of the problem being solved. However, the principle of explicit domination [4], realized using the method of normalized functions [17], substantially reduces the dimensionality of the resulting equivalent nonfuzzy analog before solution of the problem commences. According to the physical essence of the problem solved, we may go over the constraints with fuzzy coefficients (14) to constraints

$$g_j(x_1, \dots, x_n) \le b_j, \qquad j = 1, \dots, d' \ge m, \tag{16}$$

or to constraints

$$g_j(x_1, \dots, x_n) \ge b_j, \qquad j = 1, \dots, d'' \ge m. \tag{17}$$

The solution of problems with fuzzy coefficients in the objective functions alone is possible by a modification of traditional mathematical programming methods [2,4]. In particular, it is possible to solve problem (13) with satisfying the constraints (16) as well as problem (15) with satisfying the constraints (17). As an example of these problems, we can consider the analysis of fuzzy discrete optimization models.

The desirability of allowing for constraints on the discreteness of variables in the form of discrete sequences

$$x_{s_i}, \rho_{s_i}, \tau_{s_i}, \dots, \qquad s_i = 1, \dots, r_i, \tag{18}$$

has been validated in [18,19]; here $\rho_{s_i}, \tau_{s_i}, \dots$ are characteristics required for forming objective functions, constraints, and their increments that correspond to the s^{th} standard value of the variable x_i.

Taking this into account, the maximization problem (13), (16) may be formulated as follows. Assume, we are given discrete sequences of type (18) (increasing or decreasing, depending on the formulation of the problem). From these sequences it is necessary to choose elements such that the objective

$$\text{maximize}\, \tilde{F}(x_{s_1}, \rho_{s_1}, \tau_{s_1}, \dots, x_{s_n}, \rho_{s_n}, \tau_{s_n}, \dots) \tag{19}$$

is met while satisfying the constraints

$$g_j(x_{s_1}, \rho_{s_1}, \tau_{s_1}, \dots, x_{s_n}, \rho_{s_n}, \tau_{s_n}, \dots) \le b_j, \qquad j = 1, \dots, d'. \tag{20}$$

Given the maximization problem (18)–(20), we can state a problem of minimization as follows

$$\text{minimize}\, \tilde{F}(x_{s_1}, \rho_{s_1}, \tau_{s_1}, \dots, x_{s_n}, \rho_{s_n}, \tau_{s_n}, \dots), \tag{21}$$

while satisfying the constraints

$$g_j(x_{s_1}, \rho_{s_1}, \tau_{s_1}, \dots, x_{s_n}, \rho_{s_n}, \tau_{s_n}, \dots) \ge b_j, \qquad j = 1, \dots, d''. \tag{22}$$

Generalized algorithms of discrete optimization have been proposed in [18,19]. These algorithms are based on the method of normalized functions [17] and belong to the class of greedy algorithms [20]. They allow one to obtain quasioptimal solutions after a small number of steps, thus overcoming the NP-completeness of discrete optimization problems. The algorithms do not require the objective functions and constraints to be in analytical form. They may be tabular or algorithmic, ensuring flexibility and the possibility to solve complex problems for which adequate analytical descriptions are difficult. Considering this, we shall describe the algorithm next on the basis of results of [18,19].

Assume that at step t, variable x_i is at its discrete level $x_{s_i}^{(t)}$ and its associated parameters $\rho_i, \tau_i, \dots$ are at the respective levels $\rho_{s_i}^{(t)}, \tau_{s_i}^{(t)}, \dots$. These can be gathered in what we introduce as the set $\varphi_{s_i}^{(t)} = \{x_{s_i}^{(t)}, \rho_{s_i}^{(t)}, \tau_{s_i}^{(t)}, \dots\}$. Then, the algorithm of solving problem (18)–(20) can be written as follows.

(1) The components of the constraint increment vector $\{\Delta G_i^{(t)}\}$ are calculated

$$\Delta G_i^{(t)} = \max_j \Delta g_{ji}^{(t)}, \qquad i \in I^{(t)}, \quad j = 1, \dots, d', \tag{23}$$

where t is the index number of the optimization step, $I^{(t)}$ is the set of variables at the t^{th} step, which, at their present values, satisfy all constraints.

In (23), $\Delta g_{ji}^{(t)}$ is the increment in the j^{th} constraint when $x_{s_i}^{(t)}$ undergoes a step change from the current level s_i to the level $s_i + 1$ while all the other $x_{s_k}^{(t)}$, $k \ne i$, remain at their current levels s_i

$$\Delta g_{ji}^{(t)} = \left[g_j\left(\varphi_{s_1}^{(t)}, \dots, \varphi_{s_i+1}^{(t)}, \dots, \varphi_{s_n}^{(t)}\right) - g_j\left(\varphi_{s_1}^{(t)}, \dots, \varphi_{s_i}^{(t)}, \dots, \varphi_{s_n}^{(t)}\right)\right] \frac{b}{b_j^{(t-1)}},$$
$$j = 1, \dots, d', \qquad i \in I^{(t)},$$

where b is a normalizing factor (an arbitrary positive number), which may be interpreted as unit resource. For the first step ($t = 1$), we have $i \in I_n$ (I_n is the initial set of variables), $b_j^{(t-1)} = b_j^{(0)} = b_j$.

(2) If $I^{(t)} = \{i \mid \Delta G_i^{(t)} \le b,\ i \in I^{(t)}\} \neq \emptyset$, then go to Operation 3, otherwise go to Operation 9.

(3) The components of the increment vector of the objective function $\{\Delta\tilde{F}_i^{(t)}\}$ are calculated as

$$\Delta\tilde{F}_i^{(t)} = \tilde{F}\left(\varphi_{s_1}^{(t)}, \dots, \varphi_{s_i+1}^{(t)}, \dots, \varphi_{s_n}^{(t)}\right) - \tilde{F}\left(\varphi_{s_1}^{(t)}, \dots, \varphi_{s_i}^{(t)}, \dots, \varphi_{s_n}^{(t)}\right), \qquad i \in I^{(t)}. \tag{24}$$

(4) If $I^{(t)} = \{i \mid \Delta\tilde{F}_i^{(t)} > 0,\ i \in I^{(t)}\} \neq \emptyset$, then go to Operation 5, otherwise go to Operation 9.

(5) The components of a vector $\{\tilde{V}_i^{(t)}\}$ are calculated as

$$\tilde{V}_i^{(t)} = \frac{\Delta\tilde{F}_i^{(t)}}{\Delta G_i^{(t)}}, \qquad i \in I^{(t)}. \tag{25}$$

(6) The index $i = l$ of the incremented variable is determined from

$$\tilde{V}_l^{(t)} = \max_i \tilde{V}_i^{(t)}, \qquad i \in I^{(t)}. \tag{26}$$

(7) We recalculate the current values of the quantities

$$x_{s_i}^{(t)} = \begin{cases} x_{s_i}^{(t)}, & \text{if } i \neq l,\ i \in I^{(t)}, \\ x_{s_i+1}^{(t)}, & \text{if } i = l, \end{cases}$$

$$b_j^{(t)} = b_j^{(t-1)} - \Delta g_{jl}^{(t)} \frac{b_j^{(t-1)}}{b}, \qquad j = 1, \dots, d'.$$

(8) If $I^{(t)} = \{i \mid s_i < r_i,\ i \in I^{(t)}\} \neq \emptyset$, then go to Operation 1, taking $t := t + 1$, otherwise go to Operation 9.

(9) The calculations are completed.

The algorithm of solving the minimization problem (18),(21),(22) can be written in an analog manner on the basis of results in references [18,19].

When characterizing the algorithm given above, it is necessary to point out that the execution of algebraic operations by means of (24) and (25) is accomplished on the basis of algorithms given in [21] with taking into account the results of [22,23].

To compare alternatives on the basis of (26) (in essence, the comparison or ranking of fuzzy numbers $\tilde{V}_i^{(t)}$, $i \in I^{(t)}$ on the basis of magnitude in order to choose the largest) it is necessary to use the corresponding methods, which are analyzed in [23,24]. In particular, the authors of [24] classify four groups of methods related to comparing alternatives in a fuzzy environment. One of the groups is based on the construction of fuzzy preference relations, that provides [25] the most justified and practical way to compare alternatives. Taking this into account, it is necessary to distinguish the choice function or fuzzy number ranking index introduced by Orlovski [26]. It is based on the conception of a membership function of a generalized preference relation.

If the membership functions corresponding to the natural or relative (as in (25)) values $\tilde{F}_1$ and $\tilde{F}_2$ of the objective function to be maximized are $\mu(f_1)$ and $\mu(f_2)$, the quantity $\eta\{\mu(f_1), \mu(f_2)\}$ is the degree of preference $\mu(f_1) \succeq \mu(f_2)$, while $\eta\{\mu(f_2), \mu(f_1)\}$ is the degree of preference $\mu(f_2) \succeq \mu(f_1)$. Then, the membership functions of the generalized preference relations $\eta\{\mu(f_1), \mu(f_2)\}$ and $\eta\{\mu(f_2), \mu(f_1)\}$ take the following form:

$$\eta\{\mu(f_1), \mu(f_2)\} = \sup_{f_1, f_2 \in F} \min\{\mu(f_1), \mu(f_2), \mu_R(f_1, f_2)\}, \tag{27}$$

$$\eta\{\mu(f_2), \mu(f_1)\} = \sup_{f_1, f_2 \in F} \min\{\mu(f_2), \mu(f_1), \mu_R(f_2, f_1)\}, \tag{28}$$

where $\mu_R(f_1, f_2)$ and $\mu_R(f_2, f_1)$ are the membership functions of the corresponding fuzzy preference relations.

If F is the numerical axis on which the values of the maximized objective function are plotted, and R is the natural order ($\geq$) along F, then (27) and (28) reduce to the following expressions:

$$\eta\{\mu(f_1), \mu(f_2)\} = \sup_{\substack{f_1, f_2 \in F \\ f_1 \geq f_2}} \min\{\mu(f_1), \mu(f_2)\}, \tag{29}$$

$$\eta\{\mu(f_2), \mu(f_1)\} = \sup_{\substack{f_1, f_2 \in F \\ f_2 \geq f_1}} \min\{\mu(f_1), \mu(f_2)\}, \tag{30}$$

that agree with the Baas-Kwakernaak [27], Baldwin-Guild [28], and one of the Dubois-Prade [29] fuzzy number ranking indices.

On the basis of the relations between (29) and (30), it is possible to judge the preference (and the degree of preference) of any of the alternatives compared. Utilization of this approach is justified, that is confirmed by the results of [30]. However, experience shows that in many cases the membership functions of the alternatives $\mu(f_1)$ and $\mu(f_2)$ compared form flat apices (for example, [2,3]), i.e., they are so-called flat fuzzy numbers [21]. In view of this, using (29) and (30), we can say that the alternatives $\tilde{F}_1$ and $\tilde{F}_2$ are indistinguishable if

$$\eta\{\mu(f_1), \mu(f_2)\} = \eta\{\mu(f_2), \mu(f_1)\}. \tag{31}$$

In such situations the algorithm given above does not allow one to obtain a unique solution because it "stops" when conditions like (31) arise. This occurs also with other modifications of traditional mathematical programming methods because combination of the uncertainty and the relative stability of optimal solutions can be produce these so-called decision uncertainty regions. In this connection, other choice functions or indices (for example, [23,25,31–34]) may be used as additional means for the ranking of fuzzy numbers. However, these indices occasionally result in choices which appear inconsistent with intuition [4,23], and their application does not permit one to close the question of constructing an order on a set of fuzzy numbers [4]. Besides, from the substantial point of view, these indices have been proposed with the aspiration for obligatory distinguishing the alternatives, that is not natural because the uncertainty of information creates the decision uncertainty regions. There actually is another approach that is better validated and natural for the practice of decision making. This approach is associated with transition to multicriteria choosing alternatives in a fuzzy environment because the application of additional criteria can serve as convincing means to contract the decision uncertainty regions.

4. MULTICRITERIA CHOICE PROCEDURES IN A FUZZY ENVIRONMENT

Before starting to discuss multicriteria decision making in a fuzzy environment, it is necessary to note that considerable contraction of the decision uncertainty regions may be obtained by formulating and solving one and the same problem within the framework of mutually interrelated models:

(a) the model of maximization (13) with satisfaction of the constraints (16) interpreted as convex down,
(b) the model of minimization (15) with satisfaction of the constraints (17) interpreted as convex up.

For example, we can solve problem (19),(20) with the discrete sequences (18) given as decreasing (incresing) and problem (21),(22) with the discrete sequences (18) given as increasing (decreasing). In this case, solutions dominated by the initial objective function are cut off from below as well as from above to the greatest degree [4]. It should be stressed that this is a universal approach and may also be used in solving continuous problems, for example, by modifying the zero order optimization methods.

Assume we are given a set X of alternatives, which are to be examined by q criteria (of quantitative and/or qualitative nature) to make a choice among alternatives. The problem of decision making is presented by a pair $\langle X, R\rangle$ where $R = \{R_1, \ldots, R_q\}$ is a vector fuzzy preference relation [33,35]. In this case, we have

$$R_p = [X \times X, \mu_{R_p}(X_k, X_l)], \qquad p = 1, \ldots, q, \quad X_k, X_l \in X,$$

where $\mu_{R_p}(X_k, X_l)$ is a membership function of fuzzy preference relation.

It is supposed in [33,35], that the matrices R_p, $p = 1, \ldots, q$ are directly given as expert's estimates. However, there is another, more convincing, approach to obtaining these matrices. In particular, the availability of fuzzy or linguistic estimates of alternatives $\tilde{F}_p(X_k)$, $p = 1, \ldots, q$, $X_k \in X$ (constructed on the basis of expert estimation or on the basis of aggregating information arriving from different sources of both formal and informal character [2]) with the membership funtions $\mu[f_p(X_k)]$, $p = 1, \ldots, q$, $X_k \in X$, permits one to construct the matrices R_p, $p = 1, \ldots, q$ with the use of (29) and (30) as follows

$$\mu_{R_p}(X_k, X_l) = \sup_{\substack{X_k, X_l \in X \\ f_p(X_k) \geq f_p(X_l)}} \min\{\mu[f_p(X_k)], \mu[f_p(X_l)]\}, \tag{32}$$

$$\mu_{R_p}(X_l, X_k) = \sup_{\substack{X_k, X_l \in X \\ f_p(X_l) \geq f_p(X_k)}} \min\{\mu[f_p(X_k)], \mu[f_p(X_l)]\}. \tag{33}$$

If the p^{th} criterion is associated with minimization, then (32) and (33) are written for regions $f_p(X_k) \leq f_p(X_l)$ and $f_p(X_l) \leq f_p(X_k)$, respectively.

Considering that fuzzy preference relations R_p, $p = 1, \ldots, q$ play a role identical to objective functions $F_p(X)$, $p = 1, \ldots, q$ in $\langle X, M\rangle$ models, it should be noted that the fuzzy preference relations may be introduced in the analysis of these models as well. For example, for $F_p(X)$, which is to be maximized, it is possible to construct

$$\mu_{R_p}(X_k, X_l) = \alpha \left[F_p(X_k) - F_p(X_l)\right] + \beta. \tag{34}$$

Following [36], it is possible to demand the fulfillment of the condition $\mu_{R_p}(X_k, X_k) = 0,5$ leading to $\beta = 0.5$ and $\mu_{R_p}(X_k, X_l) + \mu_{R_p}(X_l, X_k) = 1$. This permits one to write

$$\alpha \left[\max_{X \in L} F_p(X) - \min_{X \in L} F_p(X)\right] + 0.5 = 1,$$

to obtain

$$\alpha = \frac{1}{2\left[\max_{X \in L} F_p(X) - \min_{X \in L} F_p(X)\right]}.$$

Thus, correlation (34) may be presented as

$$\mu_{R_p}(X_k, X_l) = \frac{F_p(X_k) - F_p(X_l)}{2\left[\max_{X \in L} F_p(X) - \min_{X \in L} F_p(X)\right]} + 0.5,$$

providing $0 \leq \mu_{R_p}(X_k, X_l) \leq 1$.

Let us consider the situation of setting up a single preference relation R. In a nonfuzzy case, we may be given a nonstrict preference in one of the following forms [35]:

(a) $(X_k, X_l) \in R$ or $X_k \succeq X_l$ that means "X_k is not worse than X_l",
(b) $(X_l, X_k) \in R$ or $X_l \succeq X_k$ that means "X_l is not worse than X_k",
(c) $(X_k, X_l) \notin R$ or $(X_l, X_k) \notin R$ that means "X_k and X_l" are not comparable.

The nonstrict preference relation R can be represented by a strict preference relation R^s and indifferent relation R^I [35,36]. We can say that "X_k is strictly better than X_l" if $(X_k, X_l) \in R$ and $(X_l, X_k) \notin R$. The subset of all these pairs is the strict preference relation R^s, and it is possible to use the inverse relation R^{-1} ($(X_k, X_l) \in R^{-1}$ is equivalent to $(X_l, X_k) \in R$ [35]) to obtain

$$R^s = R \setminus R^{-1}. \tag{35}$$

If $(X_k, X_l) \in R^s$, then X_k dominates X_l, i.e., $X_k \succ X_l$. The alternative $X_k \in X$ is nondominated in $\langle X, R\rangle$ if $(X_k, X_l) \in R^s$ for any $X_l \in X$. It is necessary to find these alternatives.

If we have $\mu_R(X_k, X_l)$ as a nonstrict fuzzy preference relation, then the value $\mu_R(X_k, X_l)$ is the degree of preference $X_k \succeq X_l$ for any $X_k, X_l \in X$. The membership function, which corresponds to (35) (considering that $\mu_{R^{-1}}(X_k, X_l) = \mu_R(X_l, X_k)$ [35]) is the following:

$$\mu_R^s(X_k, X_l) = \max\{\mu_R(X_k, X_l) - \mu_R(X_l, X_k), 0\}. \tag{36}$$

The use of (36) permits one to carry out the choice of alternatives. In particular, $\mu_R^s(X_l, X_k)$ for any X_l describes a fuzzy set of alternatives, which are strictly dominated by X_l. Therefore, the complement of this fuzzy set by $1 - \mu_R^s(X_l, X_k)$ gives the fuzzy set of alternatives, which are not dominated by other alternatives from X. To choice the set of all alternatives, which are not dominated by other alternatives from X, it is necessary to find the intersection of all $1 - \mu_R^s(X_l, X_k)$, $X_k \in X$ on all $X_l \in X$ [35]. This intersection is a subset of nondominated alternatives and has a membership function

$$\mu_R^n(X_k) = \inf_{X_l \in X} [1 - \mu_R^s(X_l, X_k)] = 1 - \sup_{X_l \in X} \mu_R^s(X_l, X_k). \tag{37}$$

Because $\mu_R^n(X_k)$ is the degree of nondominance, it is natural to obtain alternatives providing

$$X^n = \left\{ X_k^n \mid X_k^n \in X,\ \mu_R^n(X_k^n) = \sup_{X_k \in X} \mu_R^n(X_k) \right\}. \tag{38}$$

If $\sup_{X_k \in X} \mu_R^n(X_k) = 1$, then alternatives $X^{nn} = \{X_k^{nn} \mid X_k^{nn} \in X,\ \mu_R^n(X_k^{nn}) = 1\}$ are nonfuzzy nondominated [35] and can be considered as the nonfuzzy solution of the fuzzy problem.

If the fuzzy preference relation R is transitive, then $X^{nn} \neq \emptyset$. Taking this into account, it should be noted that when $\tilde{F}_p(X_k)$ is quantitatively expressed, $X^{nn} \neq \emptyset$. With qualitative $\tilde{F}_p(X_k)$ it is possible to have $X^{nn} = \emptyset$ under intransitivity of R, that permits one to detect contradictions in an expert's estimates.

Expressions (36)–(38) may be used to solve the choice problem as well as ranking problem [33] with the single preference relation. If we have the vector fuzzy preference relation, expessions (36)–(38) can serve as the basis for building a lexicographic procedure associated with step by step introduction of criteria for comparing alternatives. This procedure permits one to obtain a sequence $X^1, X^2, \dots, X^q$ so that $X \supseteq X^1 \supseteq X^2 \supseteq \cdots \supseteq X^q$ with the use of the following expressions:

$$\mu_{R_p}^n(X_k) = \inf_{X_l \in X^{p-1}} \left[1 - \mu_{R_p}^s(X_l, X_k)\right] = 1 - \sup_{X_l \in X^{p-1}} \mu_{R_p}^s(X_l, X_k), \qquad p = 1, \dots, q,$$

$$X^p = \left\{ X_k^{n,p} \mid X_k^{n,p} \in X^{p-1}, \mu_{R_p}^n(X_k^{n,p}) = \sup_{X_l \in X^{p-1}} \mu_{R_p}^n(X_k) \right\},$$

obtained on the basis of (37) and (38), respectively.

It should be noted that if R_p is transitive, we can bypass the pairwise comparison of alternatives at the p^{th} step. In this situation, the comparison can be done on a serial basis (the direct use of (32) and (33)) with memorizing the best alternatives.

It is natural that the lexicographic procedure is applicable if criteria can be arranged in order of their importance. If the construction of the uniquely determined order is difficult, it is possible to apply another choice procedure. In particular, expressions (36)–(38) are applicable if we take $R = \bigcap_{p=1}^{q} R_p$, i.e., $\mu_R(X_k, X_l) = \min_{1 \le p \le q} \mu_{R_p}(X_k, X_l), X_k, X_l \in X$. When using this procedure, the application of (36)–(38) leads to the set X^n that fulfills [35,36] the role of a Pareto set. Its contraction is possible on the basis of differentiating the importance of R_p, $p = 1, \dots, q$ with the use of the following convolution (aggregation of monocriteria fuzzy preference relations) [35]:

$$\mu_T(X_k, X_l) = \sum_{p=1}^{q} \lambda_p \mu_{R_p}(X_k, X_l), \qquad X_k, X_l \in X,$$

where λ_p, $p = 1, \dots, q$ are weights (importance factors) of the corresponding criteria ($\lambda_p > 0$, $p = 1, \dots, q$, $\sum_{p=1}^{q} \lambda_p = 1$).

The construction of $\mu_T(X_k, X_l)$, $X_k, X_l \in X$ allows one to obtain the corresponding membership function $\mu_T^n(X_k)$ of the subset of nondominated alternatives according to an expression similar to (37). The intersection of $\mu_R^n(X_k)$ and $\mu_T^n(X_k)$ defined as

$$\mu^n(X_k) = \min\{\mu_R^n(X_k), \mu_T^n(X_k)\}, \qquad X_k \in X,$$

provides us with

$$X^n = \left\{ X_k^n \mid X_k^n \in X,\ \mu^n\left(X_k^n\right) = \sup_{X_k \in X} \mu^n\left(X_k\right) \right\}.$$

The results of the present paper associated with the procedures of multicriteria decision making in a fuzzy environment have served as a basis for solving problems of power engineering, including substation planning in power systems [12] and optimization of reliability (optimization of reliability indices while meeting restrictions on resources or minimization of resource consumption while meeting restrictions on reliability levels) in distribution systems.

5. CONCLUSIONS

Since the application of a multicriteria approach is associated with the need to analyze (1) problems in which consequences of obtained solutions cannot be evaluated using a single criterion and (2) problems that may be considered on the basis of a single criterion but their unique solutions are not achieved because the uncertainty of information produces decision uncertainty regions, and the use of additional criteria can serve as a convincing means to contract these regions, two corresponding classes of models ($\langle X, M\rangle$ and $\langle X, R\rangle$ models) have been considered. The analysis of $\langle X, M\rangle$ models is based on the Bellman-Zadeh approach. The advantages of substantial (improving the efficiency of multicriteria decision making and taking into account different types of qualitative information) as well as computational character opened up by the application of the approach have been demonstrated. The use of $\langle X, R\rangle$ models, which allow one to combine the consideration of different types of uncertainty, is associated with applying a general approach to solving a wide class of optimization problems with fuzzy coefficients. This approach is based on a modification of traditional mathematical programming methods and consists in formulating and solving one and the same problem within the framework of interrelated models. The contraction of a decision uncertainty region is associated with reduction of the problem to multicriteria selecting alternatives in a fuzzy environment. Two corresponding techniques based on fuzzy preference relations have been considered. The first technique consists in step by step comparison of alternatives, that provides the sequential contraction of the decision uncertainty region. The second technique is associated with constructing and analyzing membership functions of a subset of nondominated alternatives obtained as a result of simultaneous considering all criteria. The results of the paper are of a universal character and can be applied to the design and control of system and processes of different nature as well as the enhancement of corresponding CAD/CAM systems and intelligent decision support systems. In particular, the results of the paper are already being used for solving diverse problems of power engineering.

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