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Poisson algebras associated to quasi-Hopf algebras

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Abstract

We define admissible quasi-Hopf quantized universal enveloping (QHQUE) algebras by $\hbar$-adic valuation conditions. We show that any QHQUE algebra is twist-equivalent to an admissible one. We prove a related statement: any associator is twist-equivalent to a Lie associator. We attach a quantized formal series algebra to each admissible QHQUE algebra and study the resulting Poisson algebras.
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0. Introduction

In [WX], Weinstein and Xu introduced a geometric counterpart of the R-matrix of a quasi-triangular quantum group: they proved that if $(\mathfrak{g}, r)$ is a finite dimensional quasi-triangular Lie bialgebra, then the dual group G^* is equipped with a braiding $\mathscr{R}_{\mathrm{WX}} \in \mathrm{Aut}((G^*)^2)$ with properties analogous to those of quantum R-matrices (in particular, it is a set-theoretic solution of the quantum Yang–Baxter Equation). An explicit relation to the theory of quantum groups was later given in [GH,EH,EGH]: to a quasi-triangular QUE algebra $(U_\hbar(\mathfrak{g}), m, R)$ quantizing $(\mathfrak{g}, r)$, one associates its quantized formal series algebra (QFSA) $U_\hbar(\mathfrak{g})' \subset U_\hbar(\mathfrak{g})$; $U_\hbar(\mathfrak{g})'$ is a flat deformation of the Hopf–Poisson algebra $\mathcal{O}_{G^*} = (U(\mathfrak{g}^*))^*$ of formal functions of G^*. Then one proves that $\mathrm{Ad}(R)$ preserves $U_\hbar(\mathfrak{g})'^{\bar{\otimes} 2}$, and $\mathrm{Ad}(R)|_{\hbar=0}$ coincides with the

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automorphism $\mathscr{R}_{\text{WX}}$ of $\mathcal{O}_{G^*}^{\bar{\otimes}2}$; moreover, $\rho = \hbar\log(R)|_{\hbar=0}$ is a function of $\mathcal{O}_{G^*}^{\bar{\otimes}2}$, independent of a quantization of $\mathfrak{g}^*$, which may be expressed universally in terms of r, and $\mathscr{R}_{\text{WX}}$ coincides with the "time one automorphism" of the Hamiltonian vector field generated by ρ.

In this paper, we study the analogous problem in the case of quasi-quantum groups (quasi-Hopf QUE algebras). The classical limit of a QHQUE algebra is a Lie quasi-bialgebra (LQBA). V. Drinfeld proposed to attach Poisson–Lie "quasi-groups" to each LQBA ([Dr4]). Axioms for Poisson–Lie quasi-groups are the quasi-Hopf analogues of the Weinstein–Xu axioms.

A *Poisson–Lie quasi-group* is a Poisson manifold X, together with a "product" Poisson map $X^2 \xrightarrow{m_X} X$, a unit for this product $e \in X$, and Poisson automorphisms $\Phi_X \in \text{Aut}(X^3)$, $\Phi_X^{12,3,4}$, $\Phi_X^{1,23,4}$ and $\Phi_X^{1,2,34} \in \text{Aut}(X^4)$, such that

$$m_X \circ (\text{id} \times m_X) = m_X \circ (m_X \times \text{id}) \circ \Phi_X,$$

$$(m_X \times \text{id} \times \text{id}) \circ \Phi_X^{12,3,4} = \Phi_X \circ (m_X \times \text{id} \times \text{id}),$$

$$(\text{id} \times m_X \times \text{id}) \circ \Phi_X^{1,23,4} = \Phi_X \circ (\text{id} \times m_X \times \text{id}), \text{ etc.}$$

$$\text{and } \Phi_X^{1,2,34} \circ \Phi_X^{12,3,4} = (\text{id} \times \Phi_X) \circ \Phi_X^{1,23,4} \circ (\Phi_X \times \text{id}).$$

A *twistor* for the quasi-group (X, m_X, Φ_X) is a collection of Poisson automorphisms $F_X \in \text{Aut}(X^2)$, $F_X^{12,3}$, $F_X^{1,23} \in \text{Aut}(X^3)$, $F_X^{(12)3,4}$, $F_X^{1(23),4}$, $F_X^{12,34}$, $F_X^{1(23),4}$, $F_X^{1,(23)4} \in \text{Aut}(X^4)$ such that

$$(m_X \times \text{id}) \circ F_X^{12,3} = F_X \circ (m_X \times \text{id}),$$

$$((m_X \circ (\text{id} \times m_X)) \times \text{id}) \circ F_X^{1(23),4} = F_X \circ ((m_X \circ (\text{id} \times m_X)) \times \text{id}),$$

$$F_X^{(12)3,4} = (\Phi_X \times \text{id}) \circ F_X^{1(23),4} \circ (\Phi_X \times \text{id})^{-1}, \text{ etc.}$$

A twistor replaces the quasi-group (X, m_X, Φ_X) by (X, m'_X, Φ'_X) with $m'_X = m_X \circ F_X$ and $\Phi'_X = (F_X^{1,23})^{-1} \circ (F_X \times \text{id})^{-1} \circ \Phi_X \circ F_X^{1,23} \circ (\text{id} \times F_X)$.

It is useful to further require that the automorphisms Φ_X, F_X are given by Lagrangian bisections of a Karasev–Weinstein groupoid associated with X^3, X^2. Other axioms for Poisson–Lie quasi-groups were proposed in a differential-geometric language in [Ban,KS].

We do not know a "geometric" construction of a twist-equivalence class of (X, m_X, Φ_X) associated to each Lie quasi-bialgebra, in the spirit of [WX]. Instead we generalize the "construction of a QFS algebra and passage to Poisson geometry" part of the above discussion, and we derive from there a construction of triples (X, m_X, Φ_X), in the case of Lie quasi-bialgebras with vanishing cobracket.

Let us describe the generalization of the "construction of a QFS algebra" part (precise statements are in Section 1). We introduce the notion of an *admissible* quasi-Hopf QUE algebra, and we associate a QFSA to such a QHQUE algebra. Each QHQUE algebra can be made admissible after a suitable twist.

We generalize the "passage to Poisson geometry" part as follows. The reduction modulo $\hbar$ of the obtained QFS algebra is a quintuple $(A, m, P, \Delta, \tilde{\varphi})$ satisfying certain axioms; in particular $\exp(V_{\tilde{\varphi}})$ is an automorphism of $A^{\widehat{\otimes} 3}$, and $(A, m, \exp(V_{\tilde{\varphi}}))$ satisfies the axioms dual to those of (X, m_X, Φ_X).

When the Lie quasi-bialgebra arises from a metrized Lie algebra, admissible QHQUE algebras quantizing it are given by Lie associators, and we obtain a quasi-group (X, m_X, Φ_X) using our construction. We also prove that its twist-equivalence class does not depend on the choice of an associator.

Finally, we prove a related result: any associator is twist-equivalent to a unique Lie associator.

1. Outline of results

Let $\mathbb{K}$ be a field of characteristic 0. Let (U, m) be a topologically free $\mathbb{K}[[\hbar]]$-algebra equipped with algebra morphisms

$$\Delta : U \to U \widehat{\otimes} U, \quad \text{and} \quad \varepsilon : U \to \mathbb{K}[[\hbar]]$$

$$\text{with } (\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$$

such that the reduction of (U, m, Δ) modulo $\hbar$ is a universal enveloping algebra. Set

$$U' = \{x \in U | \text{ for any tree } P,\ \delta^{(P)}(x) \in \hbar^{|P|} U^{\widehat{\otimes} |P|}\}$$

(see the definitions of a tree, $\delta^{(P)}$, and $|P|$ in Section 2). We prove:

Theorem 1.1. *U' is a topologically free $\mathbb{K}[[\hbar]]$-algebra. It is equipped with a complete decreasing algebra filtration*

$$(U')^{(n)} = \{x \in U | \textit{ for any tree } P,\ \delta^{(P)}(x) \in \hbar^n U^{\widehat{\otimes} |P|}\}.$$

U' is stable under the multiplication m and the map $\Delta : U \to U^{\widehat{\otimes} 2}$ induces a continuous algebra morphism

$$\Delta_{U'} : U' \to U'^{\bar{\otimes} 2} = \lim_{\leftarrow n} \left(U'^{\widehat{\otimes} 2} \Big/ \sum_{p,q | p+q=n} U'^{(p)} \otimes U'^{(q)} \right).$$

Set $\mathcal{O} := U'/\hbar U'$. Then $\mathcal{O}$ is a complete commutative local ring and the reduction modulo $\hbar$ of $\Delta_{U'}$ is a continuous ring morphism

$$\Delta_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}^{\bar{\otimes} 2} = \lim_{\substack{\longleftarrow \\ n}} \left(\mathcal{O}^{\otimes 2} \Big/ \sum_{p,q|p+q=n} \mathcal{O}^{(p)} \otimes \mathcal{O}^{(q)} \right),$$

where $\mathcal{O}^{(p)} = U'^{(p)}/(\hbar U \cap U'^{(p)})$.

Theorem 1.2. *Let (U, m, Δ, Φ) be a quasi-Hopf QUE algebra. Let $\mathfrak{g}$ be the Lie algebra of primitive elements of $U/\hbar U$, so $U/\hbar U = U(\mathfrak{g})$. Assume that*

$$\hbar \log(\Phi) \in (U')^{\bar{\otimes} 3}. \tag{1.1}$$

Then there is a noncanonical isomorphism of filtered algebras $U'/\hbar U' \to \widehat{S}^{\cdot}(\mathfrak{g})$, where $\widehat{S}^{\cdot}(\mathfrak{g})$ is the formal series completion of the symmetric algebra $S^{\cdot}(\mathfrak{g})$.

When (U, m, Δ, Φ) satisfies the hypothesis (1.1), we say that it is *admissible*. In that case, we say that U' is the quantized formal series algebra (QFSA) corresponding to (U, m, Δ, Φ). Let us recall the notion of a *twist* of a quasi-Hopf QUE algebra (U, m, Δ, Φ). This is an element $F \in (U^{\widehat{\otimes 2}})^{\times}$, such that $(\varepsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \varepsilon)(F) = 1$. It transforms (U, m, Δ, Φ) into the quasi-Hopf algebra $(U, m, {}^F\Delta, {}^F\Phi)$, where

$${}^F\Delta = \mathrm{Ad}(F) \circ \Delta, \quad \text{and} \quad {}^F\Phi = (1 \otimes F)(\mathrm{id} \otimes \Delta)(F)\Phi(\Delta \otimes \mathrm{id})(F)^{-1}(F \otimes 1)^{-1}.$$

Theorem 1.3.

(1) *Let (U, m, Δ, Φ) be an admissible quasi-Hopf QUE algebra. Let us say that a twist F of U is admissible if $\hbar \log(F) \in U'^{\bar{\otimes} 2}$. Then the twisted quasi–Hopf algebra $(U, m, {}^F\Delta, {}^F\Phi)$ is also admissible, and its QFSA coincides with U'.*
(2) *Let (U, m, Δ, Φ) be an arbitrary quasi-Hopf QUE algebra. There exists a twist F_0 of U such that the twisted quasi–Hopf algebra $(U, m, {}^{F_0}\Delta, {}^{F_0}\Phi)$ is admissible.*

Theorem 1.3 can be interpreted as follows. Let (U, m) be a formal deformation of a universal enveloping algebra. The set of twists of U is a subgroup $\mathcal{T}$ of $(U^{\widehat{\otimes 2}})^{\times}$. Denote by $\mathcal{Q}$ the set of all quasi-Hopf structures on (U, m), and by $\mathcal{Q}_{\mathrm{adm}}$ the subset of admissible structures. If $\mathcal{Q}$ is nonempty, then $\mathcal{Q}_{\mathrm{adm}}$ is also nonempty, and all its elements give rise to the same subalgebra $U' \subset U$ (Theorem 1.3, (1)). Using U', we then define the subgroup $\mathcal{T}_{\mathrm{adm}} \subset \mathcal{T}$ of admissible twists. We have a natural action of $\mathcal{T}$ on $\mathcal{Q}$, which restricts to an action of $\mathcal{T}_{\mathrm{adm}}$ on $\mathcal{Q}_{\mathrm{adm}}$. Theorem 1.3 (2) says that the natural map

$$\mathcal{Q}_{\mathrm{adm}}/\mathcal{T}_{\mathrm{adm}} \to \mathcal{Q}/\mathcal{T}$$

is surjective. Let us explain why it is not injective in general. Any QUE Hopf algebra (U, m, Δ) is admissible as a quasi-Hopf algebra. If $u \in U^{\times}$ and $F = (u \otimes u)\Delta(u)^{-1}$, then $(U, m, {}^F\Delta)$ is a Hopf algebra. So (U, m, Δ) and $(U, m, {}^F\Delta)$ are in the same class of $\mathcal{Q}/\mathcal{T}$. These are also two elements of $\mathcal{Q}_{\mathrm{adm}}$; the corresponding QFS algebras are U' and $\mathrm{Ad}(u)(U')$. In general, these algebras do not coincide, so (U, m, Δ) and $(U, m, {}^F\Delta)$ are not in the same class of $\mathcal{Q}_{\mathrm{adm}}/\mathcal{T}_{\mathrm{adm}}$.

The following result is a refinement of Proposition 3.10 of [Dr2]. Let $(\mathfrak{g}, \mu, \varphi)$ be a pair of a Lie algebra $(\mathfrak{g}, \mu)$ and $\varphi \in \bigwedge^3(\mathfrak{g})^{\mathfrak{g}}$. Then $(\mathfrak{g}, \delta = 0, \varphi)$ is a Lie bialgebra.

Proposition 1.4. *There exists a series $\mathscr{E}(\varphi) \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$, expressed in terms of (μ, φ) by universal acyclic expressions, such that $(U(\mathfrak{g})[[\hbar]], m_0, \Delta_0, \mathscr{E}(\varphi))$ is an admissible quantization of $(\mathfrak{g}, \mu, \delta = 0, \varphi)$.*

This proposition is proved in Section 6.

Recall that the main axioms for a quasi-Hopf algebra (A, m, Δ, Φ) are that (a) Φ measures the noncoassociativity of Δ, and (b) Φ satisfies the pentagon equation. By analogy, we set:

Definition 1.5. A quasi-Hopf Poisson algebra is a quintuple $(A, m_0, P, \Delta, \tilde{\varphi})$, where

- (A, m_0) is a formal series algebra,
- P is a Poisson structure on A "vanishing at the origin" (i.e., such that $(P) \subset \mathfrak{m}_A$, where $\mathfrak{m}_A$ is the maximal ideal of A),
- $\Delta : A \to A \widehat{\otimes} A$ is a continuous Poisson algebra morphism, such that $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$, where $\varepsilon : A \to A/\mathfrak{m}_A = \mathbb{K}$ is the natural projection,
- $\tilde{\varphi} \in (\mathfrak{m}_A)^{\widehat{\otimes} 3}$ satisfies
$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \tilde{\varphi} \bigstar (\Delta \otimes \mathrm{id})(\Delta(a)) \bigstar (-\tilde{\varphi}), \ \ a \in A,$$
$$\tilde{\varphi}^{1,2,34} \bigstar \ \tilde{\varphi}^{12,3,4} = \tilde{\varphi}^{2,3,4} \bigstar \tilde{\varphi}^{1,23,4} \bigstar \tilde{\varphi}^{1,2,3},$$
where we set $f \bigstar g = f + g + \frac{1}{2} P(f, g) + \cdots$, the Campbell–Baker–Hausdorff (CBH) series of the Lie algebra (A, P).

Such a structure is the function algebra of a "formal Poisson–Lie quasi-group".

If $\tilde{f} \in \mathfrak{m}_A^{\widehat{\otimes} 2}$, we define the twist of the quasi-Hopf Poisson algebra $(A, m_0, P, \Delta, \tilde{\varphi})$ by $\tilde{f}$ as the algebra $(A, m_0, P, {}^{\tilde{f}}\Delta, {}^{\tilde{f}}\tilde{\varphi})$, where

$${}^{\tilde{f}}\Delta(a) = \tilde{f} \bigstar \Delta(a) \bigstar (-\tilde{f}), \quad \text{and}$$

$${}^{\tilde{f}}\tilde{\varphi} = \tilde{f}^{2,3} \bigstar \tilde{f}^{1,23} \bigstar \tilde{\varphi} \bigstar (-\tilde{f}^{12,3}) \bigstar (-\tilde{f}^{1,2});$$

then $(A, m_0, P, {}^{\tilde{f}}\Delta, {}^{\tilde{f}}\tilde{\varphi})$ is again a quasi-Hopf Poisson algebra.

Remark 1.6. If Λ is any Artinian local $\mathbb{K}$-ring with residue field $\mathbb{K}$, set $X = \mathrm{Hom}_{\mathbb{K}}(A, \Lambda)$. Then X is the "Poisson–Lie quasi-group", in the sense of the Introduction. Namely, Δ induces a product $m_X : X \times X \to X$, and $\exp(V_{\tilde{\varphi}})$, $\exp(V_{\tilde{\varphi}^{12,3,4}})$, etc., induce automorphisms Φ_X, $\Phi_X^{12,3,4}$, etc., of X, that satisfy the quasi-group axioms (we denote by V_f the Hamiltonian derivation of $A^{\widehat{\otimes} k}$ induced by $f \in A^{\widehat{\otimes} k}$). Moreover, if $\tilde{f}$ is a twist of A, then $\exp(V_{\tilde{f}})$, $\exp(V_{\tilde{f}^{12,3}})$, $\exp(V_{\tilde{f}^{(12)3,4}})$, etc., define a twistor $(F_X, F_X^{12,3}, F_X^{(12)3,4}, \ldots)$ of (X, m_X, Φ_X). Twisting A by $\tilde{f}$ corresponds to twisting (X, m_X, Φ_X) by $(F_X, F_X^{12,3}, \ldots)$.

Lemma 1.7. *If $(A, m_0, P, \Delta, \tilde{\varphi})$ is a quasi-Hopf Poisson algebra, set $\mathfrak{g} = \mathfrak{m}_A/(\mathfrak{m}_A)^2$; then P induces a Lie bracket μ on $\mathfrak{g}$, the map $\Delta - \Delta^{1,2}$ induces a linear map $\delta : \mathfrak{g} \to \Lambda^2(\mathfrak{g})$, and the reduction of $\mathrm{Alt}(\tilde{\varphi})$ is an element φ of $\Lambda^3(\mathfrak{g})$. Then $(\mathfrak{g}, \mu, \delta, \varphi)$ is a Lie quasi-bialgebra. Moreover, twisting $(A, m_0, P, \Delta, \tilde{\varphi})$ by $\tilde{f}$ corresponds to twisting $(\mathfrak{g}, \mu, \delta, \varphi)$ by*

$$f := (\mathrm{Alt}(\tilde{f}) \bmod (\mathfrak{m}_A)^2 \otimes \mathfrak{m}_A + \mathfrak{m}_A \otimes (\mathfrak{m}_A)^2) \in \Lambda^2(\mathfrak{g}).$$

Taking the reduction modulo $\hbar$ of a QUE algebra over $\mathfrak{g}$ induces a natural map

$$\mathcal{Q}_{\mathrm{adm}}/\mathcal{T}_{\mathrm{adm}} \to \{\text{quasi-Hopf Poisson algebra structures on } \widehat{S}^{\cdot}(\mathfrak{g})\}/\text{twists}.$$

To summarize, we have a diagram

$$\begin{array}{ccccc} \mathcal{Q}/\mathcal{T} & \leftarrow & \mathcal{Q}_{\mathrm{adm}}/\mathcal{T}_{\mathrm{adm}} & \rightarrow & \left\{ \begin{array}{c} \text{quasi-Hopf poisson algebra} \\ \text{structures on } \widehat{S}^{\cdot}(\mathfrak{g}) \end{array} \right\} /\text{twists} \\ \mathit{class} \downarrow & & & & \downarrow \mathit{red} \\ & & \{\text{Lie quasi-bialgebra structures on } (\mathfrak{g}, \mu)\} / \text{twists}, & & \end{array}$$

where *class* is the classical limit map described in [Dr2], and *red* is the map described in Lemma 1.7. It is easy to see that this diagram commutes.

When U is a Hopf QUE algebra, it can be viewed as a quasi-Hopf algebra with $\Phi = 1$, which is then admissible. The corresponding quasi-Hopf Poisson algebra is the Hopf-Poisson structure on $\mathcal{O}_{G^*} = (U(\mathfrak{g}^*))^*$, and $\tilde{\varphi} = 0$.

Let $(\mathfrak{g}, \mu, \delta, \varphi)$ be a Lie quasi-bialgebra. A *lift of* $(\mathfrak{g}, \mu, \delta, \varphi)$ is a quasi-Hopf Poisson algebra, whose reduction is $(\mathfrak{g}, \mu, \delta, \varphi)$. A general problem is to construct a lift for any Lie quasi-bialgebra. We will not solve this problem, but we will give partial existence and uniqueness results.

Assume that $\delta = 0$. A Lie quasi-bialgebra is then the same as a triple $(\mathfrak{g}, \mu, \varphi)$ of a Lie algebra $(\mathfrak{g}, \mu)$ and $\varphi \in \bigwedge^3(\mathfrak{g})^{\mathfrak{g}}$.

Theorem 1.8. (1) *There exists a lift*

$$(\widehat{S}^{\cdot}(\mathfrak{g}), m_0, P_{\mathfrak{g}^*}, \Delta_0, \tilde{\varphi}) \tag{1.2}$$

of $(\mathfrak{g}, \mu, \delta = 0, \varphi)$. *Here* $P_{\mathfrak{g}^*}$ *is the Kostant–Kirillov Poisson structure on* $\mathfrak{g}^*$ *and* Δ_0 *is the coproduct for which the elements of* $\mathfrak{g}$ *are primitive.*

(2) *Any two lifts of* $(\mathfrak{g}, \mu, \delta = 0, \varphi)$ *of the form* (1.2) *are related by a* $\mathfrak{g}$*-invariant twist.*

Examples of Lie quasi-bialgebras with $\delta = 0$ arise from *metrized Lie algebras*, i.e., pairs $(\mathfrak{g}, t_{\mathfrak{g}})$ of a Lie algebra $\mathfrak{g}$ and $t_{\mathfrak{g}} \in S^2(\mathfrak{g})^{\mathfrak{g}}$. Then $\varphi = [t_{\mathfrak{g}}^{1,2}, t_{\mathfrak{g}}^{2,3}]$. Recall that a *Lie associator* is a noncommutative formal series $\Phi(A, B)$, such that $\log \Phi(A, B)$ is a Lie series $[A, B]$+higher degree terms, satisfying the pentagon and hexagon identities (see [Dr3]).

Proposition 1.9. *If* Φ *is a Lie associator, we may set* $\varphi = \log(\Phi)(\bar{t}_{\mathfrak{g}}^{1,2}, \bar{t}_{\mathfrak{g}}^{2,3})$, *where* $\bar{t}_{\mathfrak{g}}^{i,j}$ *is the image of* $t_{\mathfrak{g}}^{i,j}$ *in* $\widehat{S}^{\cdot}(\mathfrak{g})^{\widehat{\otimes} 3}$, *and we use the Poisson bracket of* $\widehat{S}^{\cdot}(\mathfrak{g})^{\widehat{\otimes} 3}$ *in the expression of* $\log(\Phi)(\bar{t}_{\mathfrak{g}}^{1,2}, \bar{t}_{\mathfrak{g}}^{2,3})$.

We prove these results in Section 6. If now Φ is a general (non–Lie) associator, $(U(\mathfrak{g})[[\hbar]], m_0, \Delta_0, \Phi(\hbar t_{\mathfrak{g}}^{1,2}, \hbar t_{\mathfrak{g}}^{2,3}))$ is a quasi-Hopf QUE algebra, but it is admissible only when Φ is Lie (for general $\mathfrak{g}$). According to Theorem 1.3 (2), it is twist-equivalent to an admissible quasi-Hopf QUE algebra. We prove

Theorem 1.10. *Any* (*non-Lie*) *associator is twist-equivalent to a unique Lie associator.*

So the "concrete" version of the twist of Theorem 1.10 is an example of the twist F_0 of Theorem 1.3, (2).

2. Definition and properties of U'

In this section, we prove Theorem 1.1. We first introduce the material for the definition of U': trees (a); the map $\delta^{(P)}$ (b); then we prove Theorem 1.1 in (c) and (d).

2.1. *Binary complete planar rooted trees*

Definition 2.1. An n-binary complete planar rooted tree (n-tree for short) is a set of vertices and oriented edges satisfying the following conditions:

- each edge carries one of the labels $\{l, r\}$.
- if we set:

$$\text{valency of a vertex} = (\text{card(incoming edges)}, \ \text{card(outgoing edges)}),$$

  we have
  - there exists exactly one vertex with valency $(0, 2)$ (the root)
  - there exists exactly n vertices with valency $(1, 0)$ (the leaves)

- all other vertices have valency $(1,2)$
- if a vertex has valency $(x,2)$, then one of its outgoing edges has label l and the other has label r.

Let us denote, for $n \geqslant 2$,

$$\mathrm{Tree}_n = \{n\text{-binary complete planar rooted trees}\}.$$

By definition, Tree_1 consists of one element (the tree with a root and one nonmarked edge) and Tree_0 consists of one element (the tree with a root and no edge). We will write $|P| = n$ if P is a tree in Tree_n.

Definition 2.2 (Extracted trees). Let P be a binary complete planar rooted tree. Let L be the set of its leaves and let L' be a subset of L. We define the extracted subtree $P_{L'}$ as follows:

(1) $\tilde{P}_{L'}$ is the set of all edges connecting the root with an element of L',
(2) the vertices of $\tilde{P}_{L'}$ all have valency $(0,2)$, $(1,0)$, $(1,2)$ or $(1,1)$;
(3) $P_{L'}$ is obtained from $\tilde{P}_{L'}$ by replacing each maximal sequence of edges related by a $(1,1)$ vertex, by a single edge whose label is the label of the first edge of the sequence.

Then $P_{L'}$ is a $|L'|$-binary complete planar rooted tree.

Definition 2.3 (Descendants of a tree). If we cut the tree P by removing its root and the related edges, we get two trees P' and P'', its *left and right descendants*.

In the same way, we define the left and right descendants of a vertex of P.

If P is a n-tree, there exists a unique bijection of the set of leaves with $\{1, \dots, n\}$, such that for each vertex, the number attached to any leaf of its left descendant is smaller than the number attached to any leaf of its right descendant.

2.2. Definition of $\Delta^{(P)}, \delta^{(P)} : U \to U^{\hat{\otimes} n}$

Let us place ourselves in the hypothesis of Theorem 1.1. For P_0 (resp., P_1, P_2) the only tree of Tree_0 (resp., $\mathrm{Tree}_1, \mathrm{Tree}_2$), we set $\Delta^{(P_0)} = \varepsilon$ (resp., $\Delta^{(P_1)} = \mathrm{id}$, $\Delta^{(P_2)} = \Delta$). When P is a n-tree with descendants P' and P'', we set

$$\Delta^{(P)} = (\Delta^{(P')} \otimes \Delta^{(P'')}) \circ \Delta,$$

so $\Delta^{(P)}$ is a linear map $U \to U^{\hat{\otimes} n}$.

We set $\delta^{(P)} = (\mathrm{id} - \eta \circ \varepsilon)^{\otimes |P|} \circ \Delta^{(P)}$, so $\delta^{(P)}$ is a linear map $U \to U^{\hat{\otimes} n}$.

In particular, $\delta^{(P_0)}(x) = \varepsilon(x)$, $\delta^{(P_1)}(x) = x - \varepsilon(x)1$, and $\delta^{(P_2)}(x) = \Delta(x) - x \otimes 1 - 1 \otimes x + \varepsilon(x) 1 \otimes 1$.

We use the notation $\delta^{(i)} = \Delta^{(P_i)}$ for $i = 0, 1, 2$, and $\delta = \delta^{(2)}$.

We have also

$$\delta^{(P)} = (\delta^{(P')} \otimes \delta^{(P'')}) \circ \delta.$$

2.3. Behavior of $\delta^{(P)}$ with respect to multiplication

If $\Sigma = \{i_1, \dots, i_k\}$ is a subset of $\{1, \dots, n\}$, where $i_1 < i_2 < \dots < i_k$, the map $x \mapsto x^\Sigma$ is the linear map $U^{\widehat{\otimes} k} \to U^{\widehat{\otimes} n}$, defined by

$$x_1 \otimes \dots \otimes x_k \mapsto 1^{\otimes i_1 - 1} \otimes x_1 \otimes 1^{\otimes i_2 - i_1 - 1} \otimes x_2 \otimes \dots \otimes 1^{\otimes i_k - i_{k-1} - 1} \otimes x_k \otimes 1^{\otimes n - i_k - 1}.$$

If $\Sigma = \emptyset$, $x \mapsto x^\Sigma$ is the map $\mathbb{K} \to U^{\widehat{\otimes} n}$, $1 \mapsto 1^{\otimes n}$.

Proposition 2.4. *For $P \in \mathrm{Tree}_n$, we have the identity*

$$\delta^{(P)}(xy) = \sum_{\substack{\Sigma', \Sigma'' \subset \{1, \dots, n\} | \\ \Sigma' \cup \Sigma'' = \{1, \dots, n\}}} (\delta^{(\Sigma')}(x))^{\Sigma'} (\delta^{(\Sigma'')}(y))^{\Sigma''},$$

for any $x, y \in U$.

This proposition is proved in Section 5.

2.4. Construction of U'

Let us set

$$U' = \{x \in U | \text{ for any tree } P,\ \delta^{(P)}(x) \in \hbar^{|P|} U^{\widehat{\otimes} |P|}\}.$$

Then U' is a topologically free $\mathbb{K}[[\hbar]]$-submodule of U. Moreover, if $x, y \in U'$, and P is a tree, then

$$\delta^{(P)}([x, y]) = \sum_{\substack{\Sigma, \Sigma' \subset \{1, \dots, |P|\} \\ \Sigma \cup \Sigma' = \{1, \dots, |P|\}}} [\delta^{(P_\Sigma)}(x)^\Sigma,\ \delta^{(P_{\Sigma'})}(y)^{\Sigma'}];$$

the summand corresponding to a pair (Σ, Σ') with $\Sigma \cap \Sigma' = \emptyset$ is zero, and the $\hbar$-adic valuation of the other summands is $\geqslant |\Sigma| + |\Sigma'| \geqslant |P| + 1$; so $\delta^{(P)}([x, y]) \in \hbar^{|P|+1} U^{\widehat{\otimes} |P|}$. On the other hand, there exists $z \in U$ such that $[x, y] = \hbar z$, so $\delta^{(P)}(z) \in \hbar^{|P|} U^{\widehat{\otimes} |P|}$; so $z \in U'$ and we get $[x, y] \in \hbar U'$. It follows that $U'/\hbar U'$ is commutative. Let us set

$$U'^{(n)} = U' \cap \hbar^n U. \tag{2.3}$$

We have a decreasing filtration

$$U' = U'^{(0)} \supset U'^{(1)} \supset U'^{(2)} \supset \cdots;$$

we have $U'^{(n)} \subset \hbar^n U$, so U' is complete for the topology induced by this filtration. This is an algebra filtration, i.e., $U'^{(i)} U'^{(j)} \subset U'^{(i+j)}$. It induces an algebra filtration on $U'/\hbar U'$,

$$U'/\hbar U' \supset \cdots \supset U'^{(i)}/(U'^{(i)} \cap \hbar U') \supset \cdots,$$

for which $U'/\hbar U'$ is complete. Moreover, the completed tensor product

$$U' \bar{\otimes} U' = \lim_{\substack{\longleftarrow \\ n}} \left(U' \widehat{\otimes} U' \Big/ \sum_{p,q|p+q=n} U'^{(p)} \widehat{\otimes} U'^{(q)} \right)$$

identifies with

$$\lim_{\substack{\longleftarrow \\ n}} (\{x \in U \widehat{\otimes} U | \forall P, \ Q, \ (\delta^{(P)} \otimes \delta^{(Q)})(x) \in \hbar^{|P|+|Q|} U^{\widehat{\otimes} |P|+|Q|}\}/$$

$$\{x \in U \widehat{\otimes} U | \forall P, \ Q, \ (\delta^{(P)} \otimes \delta^{(Q)})(x) \in \hbar^{\max(n,|P|+|Q|)} U^{\widehat{\otimes} |P|+|Q|}\}).$$

If $x \in U'$, and P, Q are trees, with $|P|, |Q| \neq 0$, then since $\delta^{(P)}(1) = \delta^{(Q)}(1) = 0$, we have

$$(\delta^{(P)} \otimes \delta^{(Q)})(\Delta(x)) = (\delta^{(P)} \otimes \delta^{(Q)})(\delta(x)) = \delta^{(R)}(x) \in \hbar^{|R|} U^{\widehat{\otimes} |R|}$$

$$= \hbar^{|P|+|Q|} U^{\widehat{\otimes} |P|+|Q|},$$

where R is the tree whose left and right descendants are P and Q; so $|R| = |P| + |Q|$. On the other hand,

$$(\delta^{(P)} \otimes \varepsilon)(\Delta(x)) = \delta^{(P)}(x) \otimes 1 \in \hbar^{|P|} U^{\otimes |P|}.$$

$$(\varepsilon \otimes \delta^{(P)})(\Delta(x)) = 1 \otimes \delta^{(P)}(x) \in \hbar^{|P|} U^{\otimes |P|},$$

so $\Delta(x)$ satisfies $(\delta^{(P)} \otimes \delta^{(Q)})(\Delta(x)) \in \hbar^{|P|+|Q|} U^{\widehat{\otimes} |P|+|Q|}$ for any pair of trees (P, Q). $\Delta : U \to U \widehat{\otimes} U$ therefore induces an algebra morphism $\Delta_{U'} : U' \to U'^{\bar{\otimes} 2}$, whose reduction modulo $\hbar$ is a morphism of complete local rings

$$\mathcal{O} \to \mathcal{O}^{\bar{\otimes} 2} = \lim_{\substack{\longleftarrow \\ n}} \left(\mathcal{O}^{\otimes 2} \Big/ \sum_{p,q|p+q=n} \mathcal{O}^{(p)} \otimes \mathcal{O}^{(q)} \right),$$

where $\mathcal{O} = U'/\hbar U'$ and $\mathcal{O}^{(p)} = U'^{(p)}/(U'^{(p)} \cap \hbar U')$.

3. Classical limit of U'

We will prove Theorem 1.2 as follows. We first compare the various $\delta^{(P)}$, where P is a n-tree (Proposition 3.1). Relations found between the $\delta^{(P)}$ imply that they have $\hbar$-adic valuation properties close to those of the Hopf case (Proposition 3.2). We then prove Theorem 1.2.

3.1. Comparison of the various $\delta^{(P)}$

Let P and P_0 be n-trees. There exists an element $\Phi^{P,P_0} \in U^{\widehat{\otimes} n}$, such that $\Delta^{(P)} = \mathrm{Ad}(\Phi^{P,P_0}) \circ \Delta^{(P_0)}$. The element Φ^{P,P_0} is a product of images of Φ and Φ^{-1} by the various maps $U^{\widehat{\otimes} 3} \to U^{\widehat{\otimes} n}$ obtained by iteration of Δ. We have

$$\Phi^{P',P_0} = \Phi^{P',P}\Phi^{P,P_0} \tag{3.4}$$

for any n-trees P_0, P, P'. For example,

$$(\mathrm{id} \otimes \Delta) \circ \Delta = \mathrm{Ad}(\Phi) \circ ((\Delta \otimes \mathrm{id}) \circ \Delta),$$

$$(\Delta \otimes \Delta) \circ \Delta = \mathrm{Ad}(\Phi^{12,3,4}) \circ ((\Delta \otimes \mathrm{id}^{\otimes 2}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta), \text{ etc.}$$

Proposition 3.1. *Assume that* $\hbar \log(\Phi) \in (U')^{\bar{\otimes} 3}$. *Then there exists a sequence of elements*

$$F^{PP_0R\Sigma v} = \sum_\alpha F_{1,\alpha}^{PP_0R\Sigma v} \otimes \cdots \otimes F_{v,\alpha}^{PP_0R\Sigma v} \in (U'^{\bar{\otimes} n})^{\bar{\otimes} v},$$

indexed by the triples (R, Σ, v), *where* R *is a tree such that* $|R| < n$, Σ *is a subset of* $\{1, \ldots, n\}$ *with* $\mathrm{card}(\Sigma) = |R|$, *and* v *is an integer* $\geqslant 1$, *such that the equality*

$$\delta^{(P)} = \mathrm{Ad}(\Phi^{P,P_0}) \circ \delta^{(P_0)} + \sum_{k | k<n} \sum_{R \text{ a } k-\text{tree}} \sum_{\substack{\Sigma \subset \{1,\ldots,n\},\\ \mathrm{card}(\Sigma)=k}} \sum_{v \geqslant 1} \sum_\alpha \mathrm{ad}_\hbar(F_{1,\alpha}^{PP_0R\Sigma v}) \circ \cdots \circ \mathrm{ad}_\hbar(F_{v,\alpha}^{PP_0R\Sigma v}) \circ (\delta^{(R)})^\Sigma \tag{3.5}$$

holds. Here $\mathrm{ad}_\hbar(x)(y) = \frac{1}{\hbar}[x, y]$.

Proof. Let us prove this statement by induction on n. When $n = 3$, we find

$$\delta^{(1(23))} = \mathrm{Ad}(\Phi)\delta^{((12)3)} + (\mathrm{Ad}(\Phi) - 1)(\delta^{1,2} + \delta^{1,3} + \delta^{2,3} + \delta^{(1)1} + \delta^{(1)2} + \delta^{(1)3}),$$

so the identity holds with $F^{PP_0R\Sigma v} = \frac{1}{v!}(\hbar \log \Phi)^{\bar{\otimes} v}$ for all choices of (R, Σ, v), except when $|R| = 0$, in which case $F^{PP_0R\Sigma v} = 0$. Assume that the statement holds for any pair of k-trees, $k\leqslant n$, and let us prove it for a pair (P,P_0) of $(n+1)$-trees. For k any integer, let $P_{\rm left}(k)$ be the k-tree corresponding to

$$\delta^{(P_{\rm left}(k))}=(\delta\otimes{\rm id}^{\otimes k-2})\circ\cdots\circ\delta.$$

Thanks to (3.4), we may assume that $P_0=P_{\rm left}(n+1)$ and P is arbitrary. Let P' and P'' be the subtrees of P, such that $|P'|+|P''|=n+1$, and $\delta^{(P)}=(\delta^{(P')}\otimes\delta^{(P'')})\circ\delta$. Let P_1 and P_2 be the n-trees such that

$$\delta^{(P_1)}=(\delta^{(P_{\rm left}(k'))}\otimes\delta^{(P'')})\circ\delta\ \text{ and }\ \delta^{(P_2)}=(\delta^{(P_{\rm left}(k'))}\otimes\delta^{(P_{\rm left}(k''))})\circ\delta$$

Assume that $|P_1|\neq 1$. Using (3.4), we reduce the proof of (3.5) to the case of the pairs $(P,P_1),(P_1,P_2)$ and (P_2,P_0). Then the induction hypothesis applied to the pair $(P',P_{\rm left}(k'))$, together with $\Phi^{P,P_1}=\Phi^{P',P_{\rm left}(k')}\otimes 1^{\otimes k''}$, implies

$$\begin{aligned}\delta^{(P)}={}&{\rm Ad}(\Phi^{P,P_1})\circ\delta^{(P_1)}+\sum_{k|k<k'}\ \sum_{R\ \text{a}\ k-\text{tree}}\ \sum_{\substack{\Sigma\subset\{1,\dots,k'\},\\ {\rm card}(\Sigma)=k}}\\ &\sum_{v\geqslant 1}\sum_{\alpha}{\rm Ad}(\Phi^{P,P_1})\circ{\rm ad}_\hbar(F_{1,\alpha}^{P'P_{\rm left}(k')\Sigma v}\otimes 1^{\otimes k''})\cdots{\rm ad}_\hbar(F_{v,\alpha}^{P'P_{\rm left}(k')\Sigma v}\otimes 1^{\otimes k''})\\ &\circ((\delta^{(R)}\otimes\delta^{(P'')})\circ\delta)^{\Sigma,k'+1,\dots,n+1},\end{aligned}$$

which is (3.5) for (P,P_1). In the same way, one proves a similar identity relating P_1 and P_2. Let us now prove the identity relating P_2 and P_0. We have $\delta^{(P_2)}=(\delta\otimes{\rm id}^{\otimes n-1})\circ\delta^{(P_2')}$ and $\delta^{(P_0)}=(\delta\otimes{\rm id}^{\otimes n-1})\circ\delta^{(P_0')}$, where P_2' and P_0' are n-trees. We have

$$\Phi^{P_2,P_0}=(\Delta\otimes{\rm id}^{\otimes n-2})\circ\Phi^{P_2',P_0'}$$

so we get

$$\begin{aligned}\delta^{(P_2)}={}&{\rm Ad}(\Phi^{P_2,P_0})\circ\delta^{(P_0)}\\ &+({\rm Ad}(\Phi^{P_2,P_0})-{\rm Ad}((\Phi^{P_2',P_0'})^{1,3,\dots,n+1}))\circ(\delta^{(P_0')})^{1,3,\dots,n+1}\\ &+({\rm Ad}(\Phi^{P_2,P_0})-{\rm Ad}((\Phi^{P_2',P_0'})^{2,3,\dots,n+1}))\circ(\delta^{(P_0')})^{2,3,\dots,n+1}\\ &+(\delta\otimes{\rm id}^{\otimes n-1})\left(\sum_{k\leqslant n}\ \sum_{R\ \text{a}\ k-\text{tree}}\ \sum_{\substack{\Sigma\subset\{1,\dots,n\},\\ {\rm card}(\Sigma)=k}}\ \sum_{v\geqslant 1}\right.\\ &\left.\times\sum_{\alpha}{\rm ad}_\hbar(F_{1,\alpha}^{P_2'P_0'\Sigma v})\cdots{\rm ad}_\hbar(F_{v,\alpha}^{P_2'P_0'\Sigma v})\circ(\delta^{(R)})^{\Sigma}\right).\end{aligned}$$

We have $\hbar \log \Phi^{P_2,P_0} \in U'^{\bar{\otimes} n+1}$ and $\hbar \log \Phi^{P'_2,P'_0} \in U'^{\bar{\otimes} n}$; this fact and the relations

$$(\delta \otimes \mathrm{id}^{\otimes n-1})(\mathrm{ad}_\hbar(x_1)\cdots \mathrm{ad}_\hbar(x_v) \circ (\delta^{(R)})^{\Sigma})$$
$$= (\mathrm{ad}_\hbar(x_1^{12,\dots,n+1}) \circ \cdots \circ \mathrm{ad}_\hbar(x_v^{12,\dots,n+1}) - \mathrm{ad}_\hbar(x_1^{1,3,\dots,n+1}) \circ \cdots \circ \mathrm{ad}_\hbar(x_v^{1,3,\dots,n+1})$$
$$- \mathrm{ad}_\hbar(x_1^{2,3,\dots,n+1}) \circ \cdots \circ \mathrm{ad}_\hbar(x_v^{2,3,\dots,n+1})) \circ (\delta^{(R)})^{\Sigma+1}$$

if $1 \notin \Sigma$, and

$$(\delta \otimes \mathrm{id}^{\otimes n-1})(\mathrm{ad}_\hbar(x_1)\cdots \mathrm{ad}_\hbar(x_v) \circ (\delta^{(R)})^{\Sigma})$$
$$= \mathrm{ad}_\hbar(x_1^{12,\dots,n+1}) \circ \cdots \circ \mathrm{ad}_\hbar(x_v^{12,\dots,n+1}) \circ ((\delta \otimes \mathrm{id}^{\otimes n-1}) \circ \delta^{(R)})^{1,2,\Sigma'+1}$$
$$+ (\mathrm{ad}_\hbar(x_1^{12,\dots,n+1}) \circ \cdots \circ \mathrm{ad}_\hbar(x_v^{12,\dots,n+1}) - \mathrm{ad}_\hbar(x_1^{1,3,\dots,n+1}) \circ \cdots \circ \mathrm{ad}_\hbar(x_v^{1,3,\dots,n+1}))$$
$$\circ (\delta^{(R)})^{1,\Sigma'+1} + (\mathrm{ad}_\hbar(x_1^{12,\dots,n+1}) \circ \cdots \circ \mathrm{ad}_\hbar(x_v^{12,\dots,n+1}) - \mathrm{ad}_\hbar(x_1^{2,3,\dots,n+1}) \circ \cdots \circ \mathrm{ad}_\hbar(x_v^{2,3,\dots,n+1}))$$
$$\circ (\delta^{(R)})^{2,\Sigma'+1}$$

if $\Sigma = \Sigma' \cup \{1\}$, where $1 \notin \Sigma'$, imply that $\delta^{(P_2)} - \mathrm{Ad}(\Phi^{P_2,P_0}) \circ \delta^{(P_0)}$ has the desired form.

Let us now treat the case $|P_1| = 1$. For this, we introduce the trees P_3 and P_4, such that:

$$\delta^{(P_3)} = (\mathrm{id}^{\otimes n-1} \otimes \delta) \circ (\mathrm{id}^{\otimes n-2} \otimes \delta) \circ \cdots \circ \delta,$$

$$\delta^{(P_4)} = (\mathrm{id}^{\otimes n-1} \otimes \delta) \circ (\delta \otimes \mathrm{id}^{\otimes n-2}) \circ (\delta \otimes \mathrm{id}^{\otimes n-3}) \circ \cdots \circ (\delta \otimes \mathrm{id}) \circ \delta.$$

We then prove the relation for the pair (P, P_3) in the same way as for (P_1, P_2) (only the right branch of the tree is changed); the relation for (P_3, P_4) in the same way as for (P_2, P_3) (instead of composing a known relation by $\delta \otimes \mathrm{id}^{\otimes n-1}$, we compose it with $\mathrm{id}^{\otimes n-1} \otimes \delta$); and using the identity

$$\delta^{(P_4)} = (\delta \otimes \mathrm{id}^{\otimes n-1}) \circ (\mathrm{id}^{\otimes n-2} \otimes \delta) \circ (\delta \otimes \mathrm{id}^{\otimes n-3}) \circ \cdots \circ \delta,$$

we prove the relation for (P_4, P) in the same way as for (P_2, P_3) (composing a known relation by $\delta \otimes \mathrm{id}^{\otimes n-1}$). □

3.2. *Properties of $\delta^{(P)}$*

Proposition 3.2. *Let n be an integer and $x \in U$.*

(1) *Assume that for any tree R, such that $|R| < n$, we have $\delta^{(R)}(x) \in \hbar^{|R|} U^{\widehat{\otimes} |R|}$. Then the conditions*

$$\delta^{(P)}(x) \in \hbar^n U^{\widehat{\otimes} n}, \tag{3.6}$$

where P is an n-tree, are all equivalent.

(2) *Assume that for any tree R, such that* $|R|<n$, *we have* $\delta^{(R)}(x)\in\hbar^{|R|+1}U^{\widehat{\otimes}|R|}$. *Then the elements*

$$\left(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar\right)\in U(\mathfrak{g})^{\otimes n},$$

where P is an n-tree, are all equal and belong to $(\mathfrak{g}^{\otimes n})^{\mathfrak{S}_n}=S^n(\mathfrak{g})$.

Proof. Let us prove (1). We have $\delta^{(P)}=(\mathrm{id}-\eta\circ\varepsilon)^{\otimes|P|}\circ\delta^{(P)}$, where $\eta:\mathbb{K}[[\hbar]]\to U$ is the unit map of U, so

$$\delta^{(P)}=\mathrm{Ad}(\Phi^{P,P_0})\circ\delta^{(P_0)}+\sum_{k|k<n}\sum_{R\text{ a }k-\text{tree}}\sum_{\substack{\Sigma\subset\{1,\dots,n\},\\ \mathrm{card}(\Sigma)=k}}\sum_{v\geqslant 1}\sum_{\alpha}$$
$$(\mathrm{id}-\eta\circ\varepsilon)^{\otimes n}\circ\mathrm{ad}_\hbar(F_{1,\alpha}^{PP_0R\Sigma v})\circ\cdots\circ\mathrm{ad}_\hbar(F_{v,\alpha}^{PP_0R\Sigma v})\circ(\delta^{(R)})^{\Sigma}.$$

Then (1) follows from:

Lemma 3.3. *Let* Σ *be a subset of* $\{1,\dots,n\}$ (*we will write* $|\Sigma|$ *instead of* $\mathrm{card}(\Sigma)$) *and let* U_0 *be the kernel of the counit of U. Let* $x\in\hbar^{|\Sigma|}(U_0)^{\widehat{\otimes}|\Sigma|}$ *and* $F_1,\dots,F_v$ *be elements of* $(U')^{\bar{\otimes}n}$. *Then*

$$(\mathrm{id}-\eta\circ\varepsilon)^{\otimes n}(\mathrm{ad}_\hbar(F_1)\cdots\mathrm{ad}_\hbar(F_v)(x^{\Sigma}))\in\hbar^n(U_0)^{\widehat{\otimes}n}.$$

Proof of Lemma. Each element $F\in(U')^{\bar{\otimes}n}$ is uniquely expressed as a sum $F=\sum_{\Sigma\in\mathscr{P}(\{1,\dots,n\})}F_\Sigma$, where F_Σ belongs to the image of

$$(U'_0)^{\bar{\otimes}|\Sigma|}\to(U')^{\bar{\otimes}n},$$

$$f\mapsto f^{\Sigma},$$

$\mathscr{P}(\{1,\dots,n\})$ is the set of subsets of $\{1,\dots,n\}$, and U'_0 is the kernel of the counit of U'. Then

$$(\mathrm{id}-\eta\circ\varepsilon)^{\otimes n}(\mathrm{ad}_\hbar(F_1)\cdots\mathrm{ad}_\hbar(F_v)(x^{\Sigma}))$$
$$=\sum_{\Sigma_1,\dots,\Sigma_v\in\mathscr{P}(\{1,\dots,n\})}(\mathrm{id}-\eta\circ\varepsilon)^{\otimes n}(\mathrm{ad}_\hbar((F_1)_{\Sigma_1})\cdots\mathrm{ad}_\hbar((F_v)_{\Sigma_v})(x^{\Sigma})).$$

The summands corresponding to $(\Sigma_1,\dots,\Sigma_v)$ such that $\Sigma_1\cup\cdots\Sigma_v\cup\Sigma\neq\{1,\dots,n\}$ are all zero. Moreover, each $(F_\alpha)_{\Sigma_\alpha}$ can be expressed as $(f_\alpha)^{\Sigma_\alpha}$, where $f_\alpha\in\hbar^{|\Sigma_\alpha|}(U_0)^{\widehat{\otimes}|\Sigma_\alpha|}$. The lemma then follows from the statement:

Statement 3.4. *If* $\Sigma, \Sigma' \subset \{1, \dots, n\}$, $x \in \hbar^{|\Sigma|}(U_0)^{\widehat{\otimes}|\Sigma|}$, $y \in \hbar^{|\Sigma'|}(U_0)^{\widehat{\otimes}|\Sigma'|}$, *then* $\frac{1}{\hbar}[x, y]$ *can be expressed as* $z^{\Sigma \cup \Sigma'}$, *where* $z \in \hbar^{|\Sigma \cup \Sigma'|}(U_0)^{\widehat{\otimes}|\Sigma \cup \Sigma'|}$.

Proof. If $\Sigma \cap \Sigma' = \emptyset$, then $[x, y] = 0$, so the statement holds. If $\Sigma \cap \Sigma' \neq \emptyset$, then the $\hbar$-adic valuation of $\frac{1}{\hbar}[x, y]$ is $\geqslant -1 + |\Sigma| + |\Sigma'| \geqslant |\Sigma| + |\Sigma'| - |\Sigma \cap \Sigma'| = |\Sigma \cup \Sigma'|$. □

Let us now prove property (2). The above arguments immediately imply that the $(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar)$, $|P| = n$, are all equal. This defines an element $S_n(x) \in U(\mathfrak{g})^{\otimes n}$. If $|P| = n$, we have $(\mathrm{id}^{\otimes k} \otimes \delta \otimes \mathrm{id}^{\otimes n-k-1}) \circ \delta^{(P)}(x) \in \hbar^{n+1} U^{\widehat{\otimes} n+1}$, so if $\delta_0 : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is defined by $\delta_0(x) = \Delta_0(x) - x \otimes 1 - 1 \otimes x + \varepsilon(x) 1 \otimes 1$, Δ_0 being the coproduct of $U(\mathfrak{g})$, then $(\mathrm{id}^{\otimes k} \otimes \delta_0 \otimes \mathrm{id}^{\otimes n-k-1})(S_n(x)) = 0$, so

$$S_n(x) \in \mathfrak{g}^{\otimes n}. \tag{3.7}$$

Let us denote by $\sigma_{i,i+1}$ the permutation of the factors i and $i+1$ in a tensor power. For $i = 1, \dots, n-1$, let us compute $(\sigma_{i,i+1} - \mathrm{id})(S_n(x))$. Let P' be a $(n-1)$-tree and let P be the n-tree such that $\delta^{(P)} = (\mathrm{id}^{\otimes i-1} \otimes \delta \otimes \mathrm{id}^{\otimes n-i-1}) \circ \delta^{(P')}$. Then

$$(\sigma_{i,i+1} - \mathrm{id})(S_n(x)) = \left[\frac{1}{\hbar}(\mathrm{id}^{\otimes i-1} \otimes (\delta^{2,1} - \delta) \otimes \mathrm{id}^{\otimes n-i-1}) \circ \delta^{(P')}(x) \bmod \hbar\right].$$

By assumption, $\delta^{(P')}(x) \in \hbar^n U^{\widehat{\otimes} n-1}$; moreover, $\delta^{2,1} - \delta = \Delta^{2,1} - \Delta$, so $(\delta^{2,1} - \delta)(U) \subset \hbar(U \widehat{\otimes} U)$; therefore

$$(\mathrm{id}^{\otimes i-1} \otimes (\delta^{2,1} - \delta) \otimes \mathrm{id}^{\otimes n-i-1}) \circ \delta^{(P')}(x) \in \hbar^{n+1} U^{\widehat{\otimes} n};$$

it follows that $(\sigma_{i,i+1} - \mathrm{id})(S_n(x)) = 0$, therefore $S_n(x)$ is a symmetric tensor of $U(\mathfrak{g})^{\otimes n}$. Together with (3.7), this gives $S_n(x) \in (\mathfrak{g}^{\otimes n})^{\mathfrak{S}_n}$. This ends the proof of Proposition 3.2. □

3.3. *Flatness of U' (proof of Theorem 1.2)*

Let us set

$$U''^{(n)} = \{x \in U' | \delta^{(P)}(x) \in \hbar^{|P|+1} U^{\widehat{\otimes}|P|} \quad \text{if } |P| \leqslant n-1\}.$$

Then by Proposition 2.4, we have a decreasing algebra filtration

$$U' = U''^{(0)} \supset U''^{(1)} \supset U''^{(2)} \supset \cdots \supset \hbar U'. \tag{3.8}$$

We have $U''^{(n)} \supset U'^{(n)} + \hbar U'$ (we will see later that this is an equality). We derive from (3.8) a decreasing filtration

$$\mathcal{O} = \mathcal{O}''^{(0)} \supset \mathcal{O}''^{(1)} \supset \mathcal{O}''^{(2)} \supset \cdots,$$

where $\mathcal{O} = U'/\hbar U'$ and $\mathcal{O}''^{(n)} = U''^{(n)}/\hbar U'$. We have

$$\bigcap_{n\geqslant 0} \mathcal{O}''^{(n)} = \{0\};$$

this means that $\bigcap_{n\geqslant 0} U''^{(n)} = \hbar U'$, which is proved as follows: if x belongs to $\cap_{n\geqslant 0} U''^{(n)}$, then $\varepsilon(x) = O(\hbar)$, $x - \varepsilon(x)1 = O(\hbar^2)$, so $x = \hbar y$, where $y \in U$. Moreover, $\delta^{|P|}(y) = O(\hbar^{|P|})$ for any P, so $y \in U'$.

The fact that $\mathcal{O}$ is complete for this filtration will follow from its identification with the filtration $\mathcal{O} \supset \mathcal{O}'^{(1)} \supset \cdots$ (this will be established in Proposition 3.6), where $\mathcal{O}'^{(i)} = U'^{(i)}/\hbar U' \cap U'^{(i)}$ and $U'^{(i)}$ is defined in (2.3). We first prove:

Proposition 3.5. *Set* $\widehat{\mathrm{gr}}''(\mathcal{O}) = \widehat{\bigoplus}_{n\geqslant 0} \mathcal{O}''^{(n)}/\mathcal{O}''^{(n+1)}$. *Then there is a unique linear map* $\lambda_n : \mathrm{gr}''_n(\mathcal{O}) \to S^n(\mathfrak{g})$, *taking the class of* x *to the common value of all* $\frac{1}{n!}(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar)$, *where* P *is a* n*-tree. The resulting map* $\lambda : \widehat{\mathrm{gr}}''(\mathcal{O}) \to \widehat{S}^{\cdot}(\mathfrak{g})$ *is an isomorphism of graded complete algebras.*

Proof. In Proposition 3.2, we constructed a map $U''^{(n)} \to S^n(\mathfrak{g})$, by $x \mapsto$ common value of $\frac{1}{n!}(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar)$ for all n-trees P. The subspace $U''^{(n+1)} \subset U''^{(n)}$ is clearly contained in the kernel of this map, so we obtain a map

$$\lambda_n : U''^{(n)}/U''^{(n+1)} = \mathcal{O}''^{(n)}/\mathcal{O}''^{(n+1)} \to S^n(\mathfrak{g}).$$

Let us prove that $\lambda = \widehat{\bigoplus}_{n\geqslant 1} \lambda_n$ is a morphism of algebras. If $x \in U''^{(n)}$ and $y \in U''^{(m)}$, Proposition 2.4 implies that if R is any $(n+m)$-tree, we have

$$\delta^{(P)}(xy) = \sum_{\substack{\Sigma', \Sigma'' \subset \{1,\dots,n+m\} | \\ \Sigma' \cup \Sigma'' = \{1,\dots,n+m\}}} \delta^{(R_{\Sigma'})}(x)^{\Sigma'} \delta^{(R_{\Sigma''})}(y)^{\Sigma''}.$$

The $\hbar$-adic valuation of the term corresponding to (Σ', Σ'') is $\geqslant |\Sigma'| + |\Sigma''|$ if $|\Sigma'| \geqslant n$ and $|\Sigma''| \geqslant m$, and $\geqslant |\Sigma'| + |\Sigma''| + 1$ otherwise, so the only contributions to $(\frac{1}{\hbar^{n+m}}\delta^{(R)}(xy) \bmod \hbar)$ are those of the pairs (Σ', Σ'') such that $\Sigma' \cap \Sigma'' = \emptyset$. Then:

$$\begin{aligned}
&\left(\frac{1}{\hbar^{n+m}}\delta^{(R)}(xy) \bmod \hbar\right) \\
&= \sum_{\substack{\Sigma', \Sigma'' \subset \{1,\dots,n+m\} | \\ |\Sigma'| = n, |\Sigma''| = m, \\ \Sigma' \cap \Sigma'' = \emptyset}} \left(\frac{1}{\hbar^n}\delta^{(R_{\Sigma'})}(x) \bmod \hbar\right)\left(\frac{1}{\hbar^m}\delta^{(R_{\Sigma''})}(y) \bmod \hbar\right) \\
&= \sum_{\substack{\Sigma', \Sigma'' \subset \{1,\dots,n+m\} | \\ |\Sigma'| = n, |\Sigma''| = m, \\ \Sigma' \cap \Sigma'' = \emptyset}} (n!\lambda_n(x)^{\Sigma'})(m!\lambda_m(y)^{\Sigma''}) \\
&= (n+m)!\lambda_n(x)\lambda_m(y),
\end{aligned}$$

because the map

$$S^{\cdot}(\mathfrak{g}) \to (T(\mathfrak{g}), \text{ shuffle product}),$$

$$x_1 \cdots x_n \mapsto \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$$

is an algebra morphism. Therefore $\lambda_{n+m}(xy) = \lambda_n(x)\lambda_m(y)$. Let us prove that λ_n is injective. If $x \in U''^{(n)}$ is such that $(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar) = 0$ for any n-tree P, then $x \in U''^{(n+1)}$, so its class in $\mathcal{O}''^{(n)}/\mathcal{O}''^{(n+1)} = U''^{(n)}/U''^{(n+1)}$ is zero. So each λ_n is injective, so λ is injective.

To prove that λ is surjective, it suffices to prove that λ_1 is surjective. Let us fix $x \in \mathfrak{g}$. We will construct a sequence $x_n \in U$, $n \geqslant 0$ such that $\varepsilon(x_n) = 0$, $(\frac{1}{\hbar}x_n \bmod \hbar) = x$, $x_{n+1} \in x_n + \hbar^n U$ for any $n \geqslant 1$, and if P is any tree such that $|P| \leqslant n$, $\delta^{(P)}(x_n) \in \hbar^{|P|} U^{\widehat{\otimes}|P|}$ (this last condition implies that $\delta^{(Q)}(x_n) \in \hbar^n U^{\widehat{\otimes}|Q|}$ for $|Q| \geqslant n$). Then the limit $\tilde{x} = \lim_{n\to\infty}(x_n)$ exists, belongs to U', satisfies $\varepsilon(\tilde{x}) = 0$ and $(\frac{1}{\hbar}\delta_1(\tilde{x}) \bmod \hbar) = x$, so its class in $U''^{(1)}/U''^{(2)}$ is a preimage of x.

Let us now construct the sequence $(x_n)_{n\geqslant 0}$. We fix a linear map $\mathfrak{g} \to \{y \in U | \varepsilon(y) = 0\}$, $y \mapsto \bar{y}$, such that for any $y \in \mathfrak{g}$, $(\bar{y} \bmod \hbar) = y$. We set $x_1 = \hbar\bar{x}$. Let us construct x_{n+1} knowing x_n. By Proposition 3.2, if Q is any $(n+1)$-tree, $\delta^{(Q)}(x_n) \in \hbar^n U^{\widehat{\otimes} n+1}$, and $(\frac{1}{\hbar^n}\delta^{(Q)}(x_n) \bmod \hbar)$ is an element of $S^{n+1}(\mathfrak{g})$, independent of Q. Let us write this element as

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} \sum_\alpha y^\alpha_{\sigma(1)} \cdots y^\alpha_{\sigma(n+1)}, \quad \text{where } \sum_\alpha y^\alpha_1 \otimes \cdots \otimes y^\alpha_{n+1} \in \mathfrak{g}^{\otimes n+1}.$$

Then we set

$$x_{n+1} = x_n - \frac{\hbar^n}{(n+1)!} \sum_{\sigma \in \mathfrak{S}_{n+1}} \sum_\alpha \bar{y}^\alpha_{\sigma(1)} \cdots \bar{y}^\alpha_{\sigma(n+1)}. \qquad \square$$

We now prove:

Proposition 3.6. (1) *For any* $n \geqslant 0$, $U''^{(n)} = U'^{(n)} + \hbar U'$;

(2) *The filtrations* $\mathcal{O} = \mathcal{O}'^{(0)} \supset \mathcal{O}'^{(1)} \supset \cdots$ *and* $\mathcal{O} = \mathcal{O}''^{(0)} \supset \mathcal{O}''^{(1)} \supset \cdots$ *coincide, and* $\mathcal{O}$ *is complete and separated for this filtration.*

Proof. Let us prove (1). We have to show that $U''^{(n)} \subset U'^{(n)} + \hbar U'$. Let $x \in U''^{(n)}$. We have $\delta^{(P)}(x) \in \hbar^{|P|+1} U^{\widehat{\otimes}|P|}$ for $|P| \leqslant n-1$, and for P an n-tree, $(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar) \in S^n(\mathfrak{g})$ and is independent of P. Write this element of $S^n(\mathfrak{g})$ as $\sum_{\sigma \in \mathfrak{S}_n} \sum_\alpha y^\alpha_{\sigma(1)} \otimes \cdots y^\alpha_{\sigma(n)}$.

In Proposition 3.5, we construct a linear map $\mathfrak{g} \to U' \cap \hbar U$, $x \mapsto \tilde{x}$, such that $\varepsilon(\tilde{x}) = 0$ and $(\frac{1}{\hbar}\tilde{x} \bmod \hbar) = x$.

Set $f_n = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sum_\alpha \tilde{y}^\alpha_{\sigma(1)}\cdots\tilde{y}^\alpha_{\sigma(n)}$. Then each $\tilde{y}^\alpha_i$ belongs to $U'\cap\hbar U$, so $f_n\in U'\cap\hbar^n U = U'^{(n)}$. Moreover, $x-f_n$ belongs to $U''^{(n+1)}$. Iterating this procedure, we construct elements $f_{n+1}, f_{n+2}, \ldots$, where each f_k belongs to $U'^{(k)}$. The series $\sum_{k\geqslant n} f_k$ converges in U'; denote by f its sum, then $x-f$ belongs to $\bigcap_{k\geqslant n} U''^{(k)} = \hbar U'$. So $U''^{(n)}\subset U'^{(n)}+\hbar U'$. The inverse inclusion is obvious. This proves (1). Then (1) immediately implies that for any n, $\mathcal{O}'^{(n)} = \mathcal{O}''^{(n)}$. We already know that $\mathcal{O}$ is complete and separated for $\mathcal{O} = \mathcal{O}'^{(0)}\supset\mathcal{O}'^{(1)}\supset\cdots$, which proves (2). □

Proof of Theorem 1.2 (End). $\mathcal{O}$ is a complete local ring, and we have a ring isomorphism $\widehat{\mathrm{gr}}(\mathcal{O})\to\widehat{S}^\cdot(\mathfrak{g})$. Then any lift $\mathfrak{g}\to\mathcal{O}'^{(1)}$ of $\mathcal{O}'^{(1)}\to\mathcal{O}'^{(1)}/\mathcal{O}'^{(2)} = \mathfrak{g}$ yields a continuous ring morphism $\mu:\widehat{S}^\cdot(\mathfrak{g})\to\mathcal{O}$. The associated graded of μ is the identity, so μ is an isomorphism. So $\mathcal{O}$ is noncanonically isomorphic to $\widehat{S}^\cdot(\mathfrak{g})$. □

Remark 3.7. When U is Hopf and $\mathfrak{g}$ is finite-dimensional, $U'/\hbar U'$ identifies canonically with $\mathcal{O}_{G^*} = (U(\mathfrak{g}^*))^*$, where $\mathfrak{g}^*$ is the dual Lie bialgebra of $\mathfrak{g}$ (see [Dr1,Ga]). The natural projection $T(\mathfrak{g}^*)\to U(\mathfrak{g}^*)$ and the identification $T(\mathfrak{g}^*)^* = \widehat{T}(\mathfrak{g})$ (where $\widehat{T}(\mathfrak{g})$ means the degree completion) induce an injection $U'/\hbar U' = \mathcal{O}_{G^*} = (U(\mathfrak{g}^*))^*\hookrightarrow\widehat{T}(\mathfrak{g})$. The map $U'/\hbar U'\hookrightarrow\widehat{T}(\mathfrak{g})$ can be interpreted simply as follows. For any $x\in U'$, we have $(\frac{1}{\hbar^n}\delta_n(x) \bmod \hbar)\in\mathfrak{g}^{\otimes n}$. Then $U'/\hbar U'\hookrightarrow\widehat{T}(\mathfrak{g})$ takes the class of $x\in U'$ to the sequence $(\frac{1}{\hbar^n}\delta_n(x) \bmod \hbar)_{n\geqslant 0}$.

On the quasi-Hopf case, we have no canonical embedding $U'/\hbar U'\hookrightarrow\widehat{T}(\mathfrak{g})$ because the various $(\frac{1}{\hbar^n}\delta^{(P)}(x) \bmod \hbar)$ do not necessarily coincide for all the n-trees P. This is related to the fact that one cannot expect a Hopf pairing $U(\mathfrak{g}^*)\otimes(U'/\hbar U')\to\mathbb{K}$ since $\mathfrak{g}^*$ is no longer a Lie algebra, so $U(\mathfrak{g}^*)$ does not make sense.

On the other hand, Theorem 1.2 can be interpreted as follows: in the Hopf case, the exponential induces an isomorphism of formal schemes $\mathfrak{g}^*\to G^*$, so $U'/\hbar U'$ identifies noncanonically with $\mathcal{O}_{\mathfrak{g}^*} = \widehat{S}^\cdot(\mathfrak{g})$. In the quasi-Hopf case, although there is no formal group G^*, we still have an isomorphism $U'/\hbar U'\xrightarrow{\sim}\widehat{S}^\cdot(\mathfrak{g})$.

4. Twists

4.1. Admissible twists

If (U, m, Δ, Φ) is an arbitrary QHQUE algebra, we will call a twist $F\in(U^{\widehat{\otimes}2})^\times$ *admissible* if $\hbar\log(F)\in(U')^{\bar{\otimes}2}$.

Proposition 4.1. *Let (U, m, Δ, Φ) be an admissible quasi-Hopf algebra and F an admissible twist. Then the twisted quasi-Hopf algebra $(U, m, {}^F\Delta, {}^F\Phi)$ is admissible.*

Proof. Let $\varepsilon_0 : U' \to \mathbb{K}$ be the composed map $U' \xrightarrow{\varepsilon} \mathbb{K}[[\hbar]] \xrightarrow{\text{mod } \hbar} \mathbb{K}$, where ε is the counit map. Let $\mathfrak{m}_\hbar = \mathrm{Ker}(\varepsilon_0)$. We set $\mathfrak{m}_\hbar^{(3)} = \mathrm{Ker}(\varepsilon^{\otimes 3})$. We have $\mathfrak{m}_\hbar^{(3)} = \mathfrak{m}_\hbar \bar{\otimes} (U')^{\bar{\otimes} 2} + U' \bar{\otimes} \mathfrak{m}_\hbar \bar{\otimes} U' + (U')^{\bar{\otimes} 2} \bar{\otimes} \mathfrak{m}_\hbar$.

When a, b are in $(\mathfrak{m}_\hbar^{(3)})^2$, the CBH series $a \bigstar b = a + b + [a,b]_\hbar + \cdots$ converges in $(U')^{\bar{\otimes} 3}$, where $[-,-]_\hbar = \frac{1}{\hbar}[-,-]$. Indeed, $[\mathfrak{m}_\hbar^{(3)}, \mathfrak{m}_\hbar^{(3)}]_\hbar \subset \mathfrak{m}_\hbar^{(3)}$, so

$$[(\mathfrak{m}_\hbar^{(3)})^2, \ [\ldots, (\mathfrak{m}_\hbar^{(3)})^2]_\hbar,]_\hbar \subset (\mathfrak{m}_\hbar^{(3)})^{n+2},$$

where n is the number of $[-,-]_\hbar$ involved. Finally, a series $\sum_{n \geqslant 0} f_n$, where $f_n \in (\mathfrak{m}_\hbar^{(3)})^n$, converges in $(U')^{\bar{\otimes} 3}$: indeed, $\mathfrak{m}_\hbar^{(3)} \subset \hbar (U')^{\bar{\otimes} 3}$, so $(\mathfrak{m}_\hbar^{(3)})^n \subset \hbar^n (U')^{\bar{\otimes} 3}$.

Both $f := \hbar \log(F)$ and $\hbar \log(\Phi)$ belong to $(\mathfrak{m}_\hbar^{(3)})^2$. Then we have

$$\hbar \log({}^F\Phi) = f^{1,2} \bigstar f^{12,3} \bigstar (\hbar \log(\Phi)) \bigstar (-f^{1,23}) \bigstar (-f^{2,3}).$$

Since $U'^{\bar{\otimes} 3}$ is stable under $\bigstar$, we have $\hbar \log({}^F\Phi) \in U'^{\bar{\otimes} 3}$. So $(U, m, {}^F\Delta, {}^F\Phi)$ is admissible. $\square$

Let us now prove

Proposition 4.2. *Under the hypothesis of Proposition* 4.1, *the QFS algebra* U'_F *corresponding to* $(U, m, {}^F\Delta, {}^F\Phi)$ *coincides with the QFS algebra* U' *corresponding to* (U, m, Δ, Φ).

We will first prove the following lemma:

Lemma 4.3. *Let P be an n-tree. Then*

$$\delta_F^{(P)} = \delta^{(P)} + \sum_{k \leqslant n} \sum_{R \text{ a } k-\text{tree}} \sum_{\substack{\Sigma \subset \{1,\ldots,n\} | \\ \mathrm{card}(\Sigma) = k}} \sum_{v \geqslant 1} \sum_{\alpha} \mathrm{ad}_\hbar(f_{1,\alpha}^{\Sigma,P}) \circ \cdots \circ \mathrm{ad}_\hbar(f_{v,\alpha}^{\Sigma,P}) \circ (\delta^{(R)})^{\Sigma}, \tag{4.9}$$

where for each v, $\sum_\alpha f_{1,\alpha}^{\Sigma,P} \otimes \cdots \otimes f_{v,\alpha}^{\Sigma,P} \in (U'^{\bar{\otimes} n})^{\bar{\otimes} v}$.

Remark 4.4. One can prove that in the right-hand side of (4.9), the contribution of all terms with $k = n$ is $(\mathrm{Ad}(F^{(P)}) - \mathrm{id}) \circ \delta^{(P)}$ where $F^{(P)}$ is the product of $F^{I,J}$ (I, J subsets of $\{1, \ldots, n\}$, such that $\max(I) < \min(J)$) and their inverses such that

$$\Delta_F^{(P)} = \mathrm{Ad}(F^{(P)}) \circ \Delta^{(P)}.$$

Proof of the Lemma. Eq. (4.9) may be proved by induction on $|P|$. Let us prove it for the unique tree P such that $|P| = 2$:

$$\delta_F^{(2)} = \delta^{(2)} + \sum_{v \geqslant 1} \frac{1}{v!} \mathrm{ad}_\hbar(f)^v (\delta^{(2)}(x) + \delta^{(1)}(x)^1 + \delta^{(1)}(x)^2),$$

where (1) and (2) are the 1- and 2-trees. Assume that (4.9) is proved when $|P| = n$. Let P' be an $(n+1)$-tree. Then for some $i \in \{1, \dots, n\}$, we have

$$\delta_F^{(P)} = (\mathrm{id}^{\otimes i-1} \otimes \delta_F^{(2)} \otimes \mathrm{id}^{\otimes n-i}) \circ \delta_F^{(P')},$$

where $|P'| = n$. Then:

$$\begin{aligned}
\delta_F^{(P)} &= (\mathrm{id}^{\otimes i-1} \otimes \Delta_F \otimes \mathrm{id}^{\otimes n-i}) \circ \delta_F^{(P')} - (\delta_F^{(P')})^{1,\dots,\widehat{i},\dots,n+1} - (\delta_F^{(P')})^{1,\dots,\widehat{i+1},\dots,n+1} \\
&= (\mathrm{id}^{\otimes i-1} \otimes \Delta_F \otimes \mathrm{id}^{\otimes n-i}) \circ \Bigg(\delta^{(P')} + \sum_{k \leqslant n} \sum_{R \text{ a } k-\text{tree}} \sum_{\substack{\Sigma \subset \{1,\dots,n\} | \\ \mathrm{card}(\Sigma) = k}} \\
&\quad \times \sum_{v \geqslant 1} \sum_{\alpha} \mathrm{ad}_\hbar(f_{1,\alpha}^{\Sigma,P'}) \circ \cdots \circ \mathrm{ad}_\hbar(f_{v,\alpha}^{\Sigma,P'}) \circ (\delta^{(R)})^{\Sigma} \Bigg) \\
&\quad - (\cdots)^{1 \cdots \widehat{i},\dots,n+1} - (\cdots)^{1,\dots,\widehat{i+1},\dots,n+1} \\
&= \mathrm{Ad}(F^{i,i+1}) \circ \Big(\delta^{(P)} + (\delta^{(P')})^{1,\dots,\widehat{i},\dots,n+1} + (\delta^{(P')})^{1,\dots,\widehat{i+1},\dots,n+1} \\
&\quad + \sum_{k \leqslant n} \sum_{R \text{ a } k-\text{tree}} \sum_{\substack{\Sigma \subset \{1,\dots,n\} | \\ \mathrm{card}(\Sigma) = k}} \sum_{v \geqslant 1} \sum_{\alpha} \mathrm{ad}_\hbar((f_{1,\alpha}^{\Sigma,P'})^{1,\dots,\{i,i+1\},\dots,n+1}) \\
&\quad \circ \mathrm{ad}_\hbar((f_{v,\alpha}^{\Sigma,P'})^{1,\dots,\{i,i+1\},\dots,n+1}) \circ (1^{\otimes i-1} \otimes \Delta \otimes 1^{\otimes n-i}) \circ (\delta^{(R)})^{\Sigma} \Big) \\
&\quad - (\cdots)^{1,\dots,\widehat{i},\dots,n+1} - (\cdots)^{1,\dots,\widehat{i+1},\dots,n+1};
\end{aligned}$$

this has the desired form because:

$$\begin{aligned}
&(\mathrm{Ad}(F^{i,i+1}) - 1) \circ (\delta^{(P)} + (\delta^{(P')})^{1,\dots,\widehat{i},\dots,n+1} + (\delta^{(P')})^{1,\dots,\widehat{i+1},\dots,n+1}) \\
&\quad = \sum_{v \geqslant 1} \frac{1}{v!} \mathrm{ad}_\hbar(f^{i,i+1})^v (\delta^{(P)} + (\delta^{(P')})^{1,\dots,\widehat{i},\dots,n+1} + (\delta^{(P')})^{1,\dots,\widehat{i+1},\dots,n+1}).
\end{aligned}$$

This proves (4.9). □

Proof of Proposition 4.2 (End). One repeats the proof of Proposition 3.2 to prove that if $x \in U'$, then we have $\delta_F^{(P)}(x) \in \hbar^{|P|} U^{\widehat{\otimes}|P|}$ for any tree P. So $U' \subset U'_F$. Since (U, m, Δ, Φ) is the twist by F^{-1} of $(U, m, {}^F\Delta, {}^F\Phi)$, and $\hbar \log(F^{-1}) = -\hbar \log(F) \in (U')^{\bar{\otimes}2} \subset (U'_F)^{\bar{\otimes}2}$, F^{-1} is admissible for $(U, m, {}^F\Delta, {}^F\Phi)$, so we have also $U'_F \subset U'$, so $U'_F = U'$. $\square$

4.2. Twisting any algebra into an admissible algebra

Proposition 4.5. *Let (U, m, Δ, Φ) be a quasi-Hopf algebra. There exists a twist F_0 such that the twisted quasi-Hopf algebra $(U, m, {}^{F_0}\Delta, {}^{F_0}\Phi)$ is admissible.*

Proof. We construct F_0 as a convergent infinite product $F_0 = \cdots F_n \cdots F_2$, where $F_n \in 1 + \hbar^{n-1} U^{\widehat{\otimes}2}$, and the F_n's have the following property: if $\bar{F}_n = F_n F_{n-1} \cdots F_2$, if $\Phi_n = {}^{\bar{F}_n}\Phi$, and $\delta_n^{(P)} : U \to U^{\widehat{\otimes}|P|}$ is the map corresponding to a tree P and to $\Delta_n = \mathrm{Ad}(\bar{F}_n) \circ \Delta$, then we have

$$(\delta_n^{(P)} \otimes \delta_n^{(Q)} \otimes \delta_n^{(R)})(\hbar \log(\Phi_n)) \in \hbar^{|P|+|Q|+|R|} U^{\widehat{\otimes}|P|+|Q|+|R|}$$

for any trees P, Q, R such that $|P| + |Q| + |R| \leqslant n$.

Assume that we have constructed $F_1, \ldots, F_n$, and let us construct F_{n+1}. The argument of Proposition 3.2 shows that for any integers (n_1, n_2, n_3) such that $n_1 + n_2 + n_3 = n + 1$, and any trees P, Q, R such that $|P| = n_1$, $|Q| = n_2$, $|R| = n_3$,

$$\left(\frac{1}{\hbar^n} (\delta_n^{(P)} \otimes \delta_n^{(Q)} \otimes \delta_n^{(R)})(\hbar \log(\Phi_n)) \bmod \hbar\right) \in S^{n_1}(\mathfrak{g}) \otimes S^{n_2}(\mathfrak{g}) \otimes S^{n_3}(\mathfrak{g}),$$

and is independent of the trees P, Q, R. The direct sum of these elements is an element $\bar{\varphi}_n$ of $S^{\cdot}(\mathfrak{g})^{\otimes 3}$, homogeneous of degree $n+1$. Since Φ_n satisfies the pentagon equation

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_n)(\Phi_n)^{-1}(1 \otimes \Phi_n)(\mathrm{id} \otimes \Delta_n \otimes \mathrm{id})(\Phi_n)(\Phi_n \otimes 1)(\Delta_n \otimes \mathrm{id} \otimes \mathrm{id})(\Phi_n)^{-1} = 1,$$

$\varphi_n^\hbar := \hbar \log(\Phi_n)$ satisfies the equation

$$\begin{aligned}&(-(\mathrm{id} \otimes \mathrm{id} \otimes \Delta_n)(\varphi_n^\hbar)) \bigstar (1 \otimes \varphi_n^\hbar) \bigstar ((\mathrm{id} \otimes \Delta_n \otimes \mathrm{id})(\varphi_n^\hbar)) \bigstar \\ &(\varphi_n^\hbar \otimes 1) \bigstar (-(\Delta_n \otimes \mathrm{id} \otimes \mathrm{id})(\varphi_n^\hbar)) = 0,\end{aligned} \tag{4.10}$$

where we set

$$a \bigstar b = a + b + \tfrac{1}{2}[a, b]_\hbar + \cdots$$

(the CBH series for the Lie bracket $[-,-]_\hbar$). The left-hand side of (4.10) is equal to

$$(-\Delta_n\otimes\mathrm{id}\otimes\mathrm{id}+\mathrm{id}\otimes\Delta_n\otimes\mathrm{id}-\mathrm{id}\otimes\mathrm{id}\otimes\Delta_n)(\varphi_n^\hbar) + (1\otimes\varphi_n^\hbar)+(\varphi_n^\hbar\otimes 1)+\text{brackets}. \quad (4.11)$$

Let (n_1,n_2,n_3,n_4) be integers such that $n_1+\cdots+n_4=n+1$. Let P,Q,R,S be trees such that $|P|=n_1,\dots,|S|=n_4$. Let us apply $\delta_n^{(P)}\otimes\cdots\otimes\delta_n^{(S)}$ to (4.11). On the one hand,

$$(\delta_n^{(P)}\otimes\delta_n^{(Q)}\otimes\delta_n^{(R)}\otimes\delta_n^{(S)})(\Delta_n\otimes\mathrm{id}\otimes\mathrm{id})(\varphi_n^\hbar)=(\delta_n^{(P\cup Q)}\otimes\delta_n^{(R)}\otimes\delta_n^{(S)})(\varphi_n^\hbar),$$

where $P\cup Q$ is the tree with left descendant P and right descendant Q. Therefore

$$\left(\frac{1}{\hbar^n}(\delta_n^{(P)}\otimes\delta_n^{(Q)}\otimes\delta_n^{(R)}\otimes\delta_n^{(S)})(\Delta_n\otimes\mathrm{id}\otimes\mathrm{id})(\varphi_n^\hbar)\bmod\hbar\right)=(\Delta_0\otimes\mathrm{id}\otimes\mathrm{id})(\bar\varphi_n)_{n_1,n_2,n_3,n_4},$$

where the index $(n_1,\dots,n_4)$ means the component in $\otimes_{i=1}^4 S^{n_i}(\mathfrak g)$. In the same way,

$$(\delta_n^{(P)}\otimes\delta_n^{(Q)}\otimes\delta_n^{(R)}\otimes\delta_n^{(S)})((4.11)\ \text{without brackets})=\mathrm d(\bar\varphi_n)_{n_1,n_2,n_3,n_4},$$

where $\mathrm d: S^\cdot(\mathfrak g)^{\otimes 2}\to S^\cdot(\mathfrak g)^{\otimes 3}$ is the co-Hochschild cohomology differential.

On the other hand, if a_1 and $a_2\in U^{\hat\otimes 4}$ are such that

$$(\delta_n^{(P)}\otimes\cdots\otimes\delta_n^{(S)})(a_i)\in\hbar^{\inf(|P|+\cdots+|S|,n)}U^{\hat\otimes 4}$$

for any trees $(P,\dots,S)$, then if $(P,\dots,S)$ are such that $|P|+\cdots+|S|=n$, we have

$$(\delta_n^{(P)}\otimes\cdots\otimes\delta_n^{(S)})\left(\frac{1}{\hbar}[a_1,a_2]\right)\in\hbar^{n+1}U^{\hat\otimes n};$$

one proves this in the same way as the commutativity of $U'/\hbar U'$ (see Theorem 1.1). Then the relation $\frac{1}{\hbar^n}(\delta_n^{(P)}\otimes\cdots\otimes\delta_n^{(S)})(4.11)|_{\hbar=0}=0$ yields $\mathrm d(\bar\varphi_n)=0$.

This relation implies that

$$\bar\varphi_n=\mathrm d(\bar f_n)+\lambda_n,$$

where $\bar f_n\in S^\cdot(\mathfrak g)^{\otimes 2}$ and $\lambda_n\in\Lambda^3(\mathfrak g)$. Moreover, f_n and λ_n both have degree $n+1$. This implies that $\lambda_n=0$. Let $f_n\in(U(\mathfrak g)^{\otimes 2})_{\leqslant n+1}$ be a preimage of $\bar f_n$ by the projection

$$(U(\mathfrak g)^{\otimes 2})_{\leqslant n+1}\to(U(\mathfrak g)^{\otimes 2})_{\leqslant n+1}/(U(\mathfrak g)^{\otimes 2})_{\leqslant n}=(S^\cdot(\mathfrak g)^{\otimes 2})_{n+1}$$

(where the indices n and $\leqslant n$ mean "homogeneous part of degree n" and "part of degree $\leqslant n$"). Let $f_n^\hbar\in U^{\hat\otimes 2}$ be a preimage of f_n by the projection $U^{\hat\otimes 2}\to U^{\hat\otimes 2}/\hbar U^{\hat\otimes 2}=U(\mathfrak g)^{\otimes 2}$. Set $F_{n+1}=\exp(\hbar^{n-1}f_n)$. We may assume that $\hbar^n f_n\in(U(\bar F_n)')^{\bar\otimes 2}$, where $U(\bar F_n)'=\{x\in U|\delta_n^{(P)}(x)\in\hbar^{\inf(n,|P|)}U^{\hat\otimes|P|}\}$. Then $\Phi_{n+1}={}^{F_{n+1}}\Phi_n$.

If P, Q, R are such that $|P| + |Q| + |R| = n + 1$, then

$$(\delta_n^{(P)} \otimes \delta_n^{(Q)} \otimes \delta_n^{(R)})(\hbar \log(\Phi_{n+1})) \in \hbar^{n+1} U^{\widehat{\otimes} n+1}.$$

Then according to Lemma 4.3,

$$(\delta_{n+1}^{(P)} \otimes \delta_{n+1}^{(Q)} \otimes \delta_{n+1}^{(R)} - \delta_n^{(P)} \otimes \delta_n^{(Q)} \otimes \delta_n^{(R)})(\hbar\log(\Phi_{n+1}))$$

has $\hbar$-adic valuation $> |P| + |Q| + |R|$ when $|P| + |Q| + |R| \leqslant n + 1$. So $(\delta_{n+1}^{(P)} \otimes \delta_{n+1}^{(Q)} \otimes \delta_{n+1}^{(R)})(\hbar \log(\Phi_{n+1})) \in \hbar^{|P|+|Q|+|R|} U^{\widehat{\otimes} |P|+|Q|+|R|}$ whenever $|P| + |Q| + |R| \leqslant n + 1$. □

5. Proof of Proposition 2.4

We work by induction on n. The statement is obvious when $n = 0, 1$. For $n = 2$, we get

$$\begin{aligned}\delta^{(2)}(xy) &= \delta^{(2)}(x)\delta^{(2)}(y) + \delta^{(2)}(x)(\delta^{(1)}(y)^1 + \delta^{(1)}(y)^2 + \delta^{(0)}(y)^{\emptyset}) \\ &\quad + (\delta^{(1)}(x)^1 + \delta^{(1)}(y)^1 + \delta^{(0)}(y)^{\emptyset})\delta^{(2)}(y) \\ &\quad + \delta^{(1)}(x)^1\delta^{(2)}(y)^2 + \delta^{(1)}(x)^2\delta^{(2)}(y)^1, \end{aligned} \tag{5.12}$$

so the statement also holds.

Assume that the statement is proved when P is a n-tree. Let $\bar{P}$ be a $(n+1)$-tree. There exists an integer $k \in \{0, \dots, n-1\}$, such that $\bar{P}$ may be viewed as the glueing of the 2-tree on the kth leaf of a n-tree P. Then we have

$$\delta^{(\bar{P})} = (\mathrm{id}^{\otimes k} \otimes \delta^{(2)} \otimes \mathrm{id}^{\otimes n-k-1}) \circ \delta^{(P)}.$$

Let us assume, for instance, that $k = n - 1$. If v is an integer, set

$$S_v = \{(\Sigma', \Sigma'') | \Sigma', \Sigma'' \subset \{1, \dots, v\} \quad \text{and} \quad \Sigma' \cup \Sigma'' = \{1, \dots, v\}\}.$$

Then

$$S_n = f_{\{n\},\emptyset}(S_{n-1}) \cup f_{\emptyset,\{n\}}(S_{n-1}) \cup f_{\{n\},\{n\}}(S_{n-1}) (\text{disjoint union}),$$

where $f_{\alpha,\beta}(\Sigma', \Sigma'') = (\Sigma' \cup \alpha, \Sigma'' \cup \beta)$. By hypothesis, we have

$$\delta^{(P)}(xy) = \sum_{(\Sigma_1,\Sigma_2) \in S_n} \delta^{(P_{\Sigma_1})}(x)^{\Sigma_1} \delta^{(P_{\Sigma_2})}(y)^{\Sigma_2},$$

therefore

$$\delta^{(P)}(xy) = \sum_{(\Sigma',\Sigma'')\in S_{n-1}} \delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}}\delta^{(P_{\Sigma''})}(y)^{\Sigma''} + \delta^{(P_{\Sigma'})}(x)^{\Sigma'}\delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}} + \delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}}\delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}}.$$

Applying $\mathrm{id}^{\otimes n-1}\otimes\delta^{(2)}$ to this identity and using (5.12) and the identities

$$(\mathrm{id}^{\otimes k}\otimes\delta^{(1)}\otimes\mathrm{id}^{\otimes|P|-k-1})\circ\delta^{(P)} = \delta^{(P)},$$

$$(\mathrm{id}^{\otimes k}\otimes\delta^{(0)}\otimes\mathrm{id}^{\otimes|P|-k-1})\circ\delta^{(P)} = 0,$$

we get $\delta^{(\bar P)}(xy)$

$$\begin{aligned}
&= \sum_{(\Sigma',\Sigma'')\in S_{n-1}} (((\mathrm{id}^{\otimes|\Sigma'|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma'\cup\{n\}})})(x)^{\Sigma'\cup\{n,n+1\}}\delta^{(P_{\Sigma''})}(y)^{\Sigma''}\\
&+ \delta^{(P_{\Sigma'})}(x)^{\Sigma'}((\mathrm{id}^{\otimes|\Sigma''|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma''\cup\{n\}})})(y)^{\Sigma''\cup\{n,n+1\}}\\
&+ ((\mathrm{id}^{\otimes|\Sigma'|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma'\cup\{n\}})})(x)^{\Sigma'\cup\{n,n+1\}}((\mathrm{id}^{\otimes|\Sigma''|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma''\cup\{n\}})})(y)^{\Sigma''\cup\{n,n+1\}}\\
&+((\mathrm{id}^{\otimes|\Sigma'|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma'\cup\{n\}})})(x)^{\Sigma'\cup\{n,n+1\}}(\delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}} + \delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n+1\}})\\
&+(\delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}} + \delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n+1\}})((\mathrm{id}^{\otimes|\Sigma''|}\otimes\delta^{(2)})\circ\delta^{(P_{\Sigma''\cup\{n\}})})(y)^{\Sigma''\cup\{n,n+1\}}\\
&+ \delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}}\delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n+1\}} + \delta^{(P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n+1\}}\delta^{(P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}}).
\end{aligned}$$

So we get $\delta^{(\bar P)}(xy)$

$$\begin{aligned}
&= \sum_{(\Sigma',\Sigma'')\in S_{n-1}} (\delta^{(\bar P_{\Sigma'\cup\{n,n+1\}})}(x)^{\Sigma'\cup\{n,n+1\}}\delta^{(\bar P_{\Sigma''})}(y)^{\Sigma''}\\
&+ \delta^{(\bar P_{\Sigma'})}(x)^{\Sigma'}\delta^{(\bar P_{\Sigma''\cup\{n,n+1\}})}(y)^{\Sigma''\cup\{n,n+1\}}\\
&+ \delta^{(\bar P_{\Sigma'\cup\{n,n+1\}})}(x)^{\Sigma'\cup\{n,n+1\}}\delta^{(\bar P_{\Sigma''\cup\{n,n+1\}})}(y)^{\Sigma''\cup\{n,n+1\}}\\
&+ \delta^{(\bar P_{\Sigma'\cup\{n,n+1\}})}(x)^{\Sigma'\cup\{n,n+1\}}(\delta^{(\bar P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}} + \delta^{(\bar P_{\Sigma''\cup\{n+1\}})}(y)^{\Sigma''\cup\{n+1\}})\\
&+ (\delta^{(\bar P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}} + \delta^{(\bar P_{\Sigma'\cup\{n+1\}})}(x)^{\Sigma'\cup\{n+1\}})\delta^{(\bar P_{\Sigma''\cup\{n,n+1\}})}(y)^{\Sigma''\cup\{n,n+1\}}\\
&+\delta^{(\bar P_{\Sigma'\cup\{n\}})}(x)^{\Sigma'\cup\{n\}}\delta^{(\bar P_{\Sigma''\cup\{n+1\}})}(y)^{\Sigma''\cup\{n+1\}} + \delta^{(\bar P_{\Sigma'\cup\{n+1\}})}(x)^{\Sigma'\cup\{n+1\}}\delta^{(\bar P_{\Sigma''\cup\{n\}})}(y)^{\Sigma''\cup\{n\}}).
\end{aligned}$$

We have

$$\begin{aligned} S_{n+1} =& f_{\{n,n+1\},\{n,n+1\}}(S_{n-1}) \cup f_{\{n,n+1\},\{n\}}(S_{n-1}) \cup f_{\{n,n+1\},\{n+1\}}(S_{n-1}) \\ & \cup f_{\{n,n+1\},\emptyset}(S_{n-1}) \cup f_{\{n\},\{n,n+1\}}(S_{n-1}) \cup f_{\{n+1\},\{n,n+1\}}(S_{n-1}) \\ & \cup f_{\emptyset,\{n,n+1\}}(S_{n-1}) \cup f_{\{n\},\{n+1\}}(S_{n-1}) \cup f_{\{n+1\},\{n\}}(S_{n-1}) \text{ (disjoint union)}, \end{aligned}$$

where we recall that $f_{\alpha,\beta}(\Sigma', \Sigma'') = (\Sigma' \cup \alpha, \Sigma'' \cup \beta)$. So we get

$$\delta^{(\bar{P})}(xy) = \sum_{(\bar{\Sigma}', \bar{\Sigma}'') \in S_{n+1}} \delta^{(P_{\bar{\Sigma}'})}(x)^{|\bar{\Sigma}'|} \delta^{(P_{\bar{\Sigma}''})}(y)^{|\bar{\Sigma}''|}.$$

The proof is the same for a general $k \in \{0, \dots, n-1\}$. This establishes the induction.

6. Proofs of Proposition 1.4, Theorem 1.8 and Proposition 1.9

Proof of Proposition 1.4. According to [Dr2], Proposition 3.10, there exists a series $\mathscr{E}'(\varphi) \in U(\mathfrak{g})^{\otimes 3}[[\hbar]]$, expressed in terms of (μ, φ) by universal acyclic expressions (and therefore invariant), such that $\mathscr{E}'(\varphi) = 1 + O(\hbar^2)$, and $\mathscr{E}'(\varphi)$ satisfies the pentagon identity. Then $(U(\mathfrak{g})[[\hbar]], m_0, \Delta_0, \mathscr{E}'(\varphi))$ is a quasi-Hopf algebra. By Theorem 1.3(2), there exists a twist $F \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]^{\times}$, such that $(U(\mathfrak{g})[[\hbar]], m_0, {}^F\Delta_0, {}^F\mathscr{E}'(\varphi))$ is admissible.

$\mathscr{E}'(\varphi)$ gives rise to a collection of invariant elements $\mathscr{E}'(\varphi)_{p_1,p_2,p_3,n} \in \otimes_{i=1}^{2} S^{p_i}(\mathfrak{g})$, defined by the condition that the image of $\mathscr{E}'(\varphi)$ by the symmetrization map $U(\mathfrak{g})^{\otimes 3}[[\hbar]] \to S^{\cdot}(\mathfrak{g})[[\hbar]]$ is $\sum_{n \geqslant 0, p_1,p_2,p_3 \geqslant 0} \hbar^n \mathscr{E}'(\varphi)_{p_1,p_2,p_3,n}$. F is then expressed using only the $\mathscr{E}'_{p_1,p_2,p_3,n}$, the Lie bracket and the symmetric group operations on the $\mathfrak{g}^{\otimes n}$. So F is invariant and defined by universal acyclic expressions. Therefore ${}^F\Delta_0 = \Delta_0$. $\mathscr{E}(\varphi) := {}^F\mathscr{E}'(\varphi)$ is then expressed by universal acyclic expressions, and defines an admissible quantization of $(\mathfrak{g}, \mu, \delta = 0, \varphi)$. □

Proof of Theorem 1.8(1). We have then $\mathscr{E}(\varphi) \in (U(\mathfrak{g})[[\hbar]]')^{\bar{\otimes} 3}$. Since the coproduct is Δ_0, $U(\mathfrak{g})[[\hbar]]'$ is the complete subalgebra of $U(\mathfrak{g})[[\hbar]]$ generated by $\hbar\mathfrak{g}$, so it is a flat deformation of $\widehat{S}^{\cdot}(\mathfrak{g})$ with Kostant–Kirillov Poisson structure. We then set $\tilde{\varphi} := \mathscr{E}(\varphi)$ modulo $\hbar$. □

Proof of Theorem 1.8(2). Let $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ be the elements of $\widehat{S}^{\cdot}(\mathfrak{g})^{\bar{\otimes} 3}$ such that

$$(\widehat{S}^{\cdot}(\mathfrak{g}), m_0, P_{\mathfrak{g}^*}, \Delta_0, \tilde{\varphi}_i)$$

are quasi-Hopf Poisson algebras. Let C be the lowest degree component of $\tilde{\varphi}_1 - \tilde{\varphi}_2$. Then the degree k of C is $\geqslant 4$. Taking the degree k part of the difference of the pentagon identities for $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$, we find $\mathrm{d}(C) = 0$, where $\mathrm{d} : S^{\cdot}(\mathfrak{g})^{\otimes 3} \to S^{\cdot}(\mathfrak{g})^{\otimes 4}$ is the co-Hochschild differential. So $\mathrm{Alt}(C) \in \Lambda^3(\mathfrak{g})$, and since $\mathrm{Alt}(C)$ also has degree $\geqslant 4$, $\mathrm{Alt}(C) = 0$. If C_{p_1,p_2,p_3} is the component of C in $\otimes_{i=1}^3 S^{p_i}(\mathfrak{g})$ then we may define inductively $B \in (S^{\cdot}(\mathfrak{g})^{\otimes 2})^{\mathfrak{g}}$, homogeneous of degree k, such that $\mathrm{d}(B) = C$, as follows. We set $B_{0,k} = B_{1,k-1} = 0$, $B_{2,k-2} = \frac{1}{2}(\mathrm{id} \otimes m)(C_{1,1,k-2})$, and

$$B_{i+1,k-i-1} = \frac{1}{i+1}(\mathrm{id} \otimes m)[C_{i,1,k-i-1} + ((\mathrm{id} \otimes \mathrm{d})(B_{i,k-i}))_{i,1,k-i-1}],$$

where $B_{i,j}$ is the component of B in $S^i(\mathfrak{g}) \otimes S^j(\mathfrak{g})$ and m is the product of $S^{\cdot}(\mathfrak{g})$. Applying the twist B to the quasi-Hopf Poisson algebra $(\widehat{S}^{\cdot}(\mathfrak{g}), m_0, P_{\mathfrak{g}^*}, \Delta_0, \tilde{\varphi}_1)$ amounts to replacing $\tilde{\varphi}_1$ by $\tilde{\varphi}_1'$, such that $\tilde{\varphi}_1' - \tilde{\varphi}_2$ has valuation $\geqslant k+1$. Applying successive twists, we obtain the result. □

Proof of Proposition 1.9. According to [Dr3],

$$(U(\mathfrak{g})[[\hbar]],\ m_0,\ \Delta_0,\ e^{\hbar t_{\mathfrak{g}}/2},\ \Phi(\hbar t_{\mathfrak{g}}^{1,2},\ \hbar t_{\mathfrak{g}}^{2,3}))$$

is a quasi-triangular quasi-Hopf algebra. One checks that since Φ is Lie, it is admissible; then the reduction modulo $\hbar$ of the corresponding QFS algebra is the quasi-Hopf Poisson algebra of Proposition 1.9. □

Remark 6.1. In the proof of Theorem 1.8(2), we cannot use Theorem A of [Dr2] because we do not know that the twist constructed there is admissible.

7. Associators and Lie associators

In this section, we state precisely and prove Theorem 1.10.

7.1. Statement of the result

Let $\mathcal{T}_n$, $n \geqslant 2$, be the algebra with generators $t^{i,j}$, $1 \leqslant 1 \neq j \leqslant n$, and relations $t^{j,i} = t^{i,j}$,

$$[t^{i,j} + t^{i,k}, t^{j,k}] = 0 \quad \text{when } i, j, k \text{ are all distinct},$$

$$[t^{i,j}, t^{k,l}] = 0 \quad \text{when } i, j, k, l \text{ are all distinct}.$$

$\mathfrak{t}_n$ is defined as the Lie algebra with the same generators and relations. Then $\mathcal{T}_n = U(\mathfrak{t}_n)$. ($\mathfrak{t}_n$ is introduced in [Dr3]; $\mathcal{T}_n$ is called the "algebra of infinitesimal chord diagrams" in [BN].)

When $n \leqslant m$ and $(I_1, \dots, I_n)$ is a collection of disjoint subsets of $\{1, \dots, m\}$, there is a unique algebra morphism $\mathcal{T}_n \to \mathcal{T}_m$ taking $t^{i,j}$ to $\sum_{\alpha \in I_i, \beta \in I_j} t^{\alpha,\beta}$. We call it an insertion-coproduct morphism and denote it by $x \mapsto x^{I_1,\dots,I_n}$. In particular, we have an action of $\mathfrak{S}_n$ on $\mathcal{T}_n$. Let us attribute degree 1 to each generator $t^{i,j}$; this defines gradings on the algebra $\mathcal{T}_n$ and on the Lie algebra $\mathfrak{t}_n$. We denote by $\widehat{\mathcal{T}}_n$ and $\hat{\mathfrak{t}}_n$ their completions for this grading. Then $(\widehat{\mathcal{T}}_n)^\times$ is the preimage of $\mathbb{K}^\times$ by the natural projection $\widehat{\mathcal{T}}_n \to \mathbb{K}$, and the exponential is a bijection $(\widehat{\mathcal{T}}_n)_0 \to 1 + (\widehat{\mathcal{T}}_n)_0$ (where $(\widehat{\mathcal{T}}_n)_0 = \mathrm{Ker}(\widehat{\mathcal{T}}_n \to \mathbb{K})$). We have an exact sequence

$$1 \to 1 + (\widehat{\mathcal{T}}_n)_0 \to (\widehat{\mathcal{T}}_n)^\times \to \mathbb{K}^\times \to 1.$$

An *associator* is an element Φ of $1 + (\widehat{\mathcal{T}}_n)_0$, satisfying the pentagon equation

$$\Phi^{1,2,34}\Phi^{12,3,4} = \Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3}, \tag{7.13}$$

the hexagon equations

$$e^{\frac{t^{1,3}+t^{2,3}}{2}} = \Phi^{3,1,2} e^{\frac{t^{1,3}}{2}} (\Phi^{1,3,2})^{-1} e^{\frac{t^{2,3}}{2}} \Phi^{1,2,3}$$

and

$$e^{\frac{t^{1,2}+t^{1,3}}{2}} = (\Phi^{2,3,1})^{-1} e^{\frac{t^{1,3}}{2}} \Phi^{2,1,3} e^{\frac{t^{1,2}}{2}} (\Phi^{1,2,3})^{-1}$$

and $\mathrm{Alt}(\Phi) = \frac{1}{8}[t^{1,2}, t^{2,3}] +$ terms of degree >2. We denote by $\underline{\mathrm{Assoc}}$ the set of associators. If Φ satisfies the duality condition $\Phi^{3,2,1} = \Phi^{-1}$, then both hexagon equations are equivalent. We denote by $\underline{\mathrm{Assoc}}^0$ the subset of all $\Phi \in \underline{\mathrm{Assoc}}$ satisfying the duality condition. If $F \in 1 + (\widehat{\mathcal{T}}_2)_0$ and $\Phi \in 1 + (\widehat{\mathcal{T}}_3)_0$, the *twist of Φ by F* is

$${}^F\Phi = F^{2,3}F^{1,23}\Phi(F^{1,2}F^{12,3})^{-1}.$$

This defines an action of $1 + (\widehat{\mathcal{T}}_2)_0$ on $1 + (\widehat{\mathcal{T}}_3)_0$, which preserves $\underline{\mathrm{Pent}} = \{\Phi \in 1 + (\widehat{\mathcal{T}}_3)_0 | \Phi$ satisfies (7.13)$\}$, $\underline{\mathrm{Assoc}}$ and $\underline{\mathrm{Assoc}}^0$ ($\underline{\mathrm{Pent}}$ and $\underline{\mathrm{Assoc}}$ are preserved because F has the form $f(t^{1,2})$, $f \in 1 + t\mathbb{K}[[t]]$, so the "twisted R-matrix" ${}^F R = F^{2,1}RF^{-1} = f(t^{2,1})e^{t^{1,2}/2} f(t^{1,2})^{-1} = e^{t^{1,2}/2}$. $\underline{\mathrm{Assoc}}^0$ is preserved because each F is such that $F = F^{2,1}$.) We denote by $\underline{\mathrm{Assoc}}^0_{\mathrm{Lie}}$, $\underline{\mathrm{Assoc}}_{\mathrm{Lie}}$ and $\underline{\mathrm{Pent}}_{\mathrm{Lie}}$ the subsets of all Φ in $\underline{\mathrm{Assoc}}$, $\underline{\mathrm{Assoc}}^0$ and $\underline{\mathrm{Pent}}$, such that $\log(\Phi) \in \hat{\mathfrak{t}}_3$.

Theorem 7.1. *There is exactly one element of* $\underline{\mathrm{Pent}}_{\mathrm{Lie}}$ (*resp.*, $\underline{\mathrm{Assoc}}_{\mathrm{Lie}}$, $\underline{\mathrm{Assoc}}^0_{\mathrm{Lie}}$) *in each orbit of the action of* $1 + (\widehat{\mathcal{T}}_2)_0$ *on* $\underline{\mathrm{Pent}}$ (*resp.*, $\underline{\mathrm{Assoc}}$, $\underline{\mathrm{Assoc}}^0$). *The isotropy group of each element of* $\underline{\mathrm{Pent}}$ *is* $\{e^{\lambda t^{1,2}} | \lambda \in \mathbb{K}\} \subset 1 + (\widehat{\mathcal{T}}_2)_0$.

7.2. Proof of Theorem 7.1

The arguments are the same in all three cases, so we treat the case of $\underline{\mathrm{Assoc}}$.

Let Φ belong to $\underline{\mathrm{Assoc}}$. Set $\Phi = 1 + \sum_{i>0} \Phi_i$, where Φ_i is the degree i component of Φ. Let d be the co-Hochschild differential,

$$\mathrm{d} : \mathscr{T}_n \to \mathscr{T}_{n+1}$$

$$x \mapsto \sum_{i=1}^{n} (-1)^{i+1} x^{1,\dots,\{i,i+1\},\dots,n+1} - x^{2,3,\dots,n+1} + (-1)^n x^{1,2,\dots,n}.$$

Then $\mathrm{d}(\Phi_2) = 0$, and $\mathrm{Alt}(\Phi_2) = \frac{1}{8}[t^{1,2}, t^{2,3}]$. Computation shows that this implies that for some $\lambda \in \mathbb{K}$, we have $\Phi_2 = \frac{1}{8}[t^{1,2}, t^{2,3}] + \lambda \mathrm{d}((t^{1,2})^2)$. We construct $F \in 1 + (\widehat{\mathscr{T}})_0$, such that ${}^F\Phi \in \underline{\mathrm{Assoc}}_{\mathrm{Lie}}$, as an infinite product $F = \cdots F_n \cdots F_2$, where $F_i \in 1 + (\widehat{\mathscr{T}}_2)_{\geqslant i}$ (the index $\geqslant i$ means the part of degree $\geqslant i$). If we set $F_2 = 1 + \lambda (t^{1,2})^2$, then $\log({}^{F_2}\Phi) \in \mathfrak{t}_3 + (\widehat{\mathscr{T}}_3)_{\geqslant 3}$. Assume that we have found $F_3, \dots, F_{n-1}$, such that $\log({}^{\bar{F}_{n-1}}\Phi) \in \mathfrak{t}_3 + (\widehat{\mathscr{T}}_3)_{\geqslant n}$, where $\bar{F}_{n-1} = F_{n-1} \cdots F_2$. Then $\varphi^{(n-1)} := \log({}^{\bar{F}_{n-1}}\Phi)$ satisfies

$$(\varphi^{(n-1)})^{1,2,34} \bigstar (\varphi^{(n-1)})^{12,3,4} = (\varphi^{(n-1)})^{2,3,4} \bigstar (\varphi^{(n-1)})^{1,23,4} \bigstar (\varphi^{(n-1)})^{1,2,3},$$

where $\bigstar$ is the CBH product in $(\widehat{\mathscr{T}}_3)_0$. Let $\varphi_n^{(n-1)}$ be the degree n part of $\varphi^{(n-1)}$. Then we get $\mathrm{d}(\varphi_n^{(n-1)}) \in \mathfrak{t}_4$. We now use the following statement, which will be proved in the next subsection.

Proposition 7.2. *If $\gamma \in \mathscr{T}_3$ is such that $\mathrm{d}(\gamma) \in \mathfrak{t}_4$, then there exists $\beta \in \mathscr{T}_2$, such that $\gamma + \mathrm{d}(\beta) \in \mathfrak{t}_3$. If γ has degree n, one can choose β of degree n.*

It follows that there exists $\beta \in \mathscr{T}_2$ of degree n, such that $\varphi_n^{(n-1)} - \mathrm{d}(\beta) \in \mathfrak{t}_3$. Set $F_n = 1 + \beta$, then $\varphi^{(n)} = \log({}^{\bar{F}_n}\Phi)$ is such that $\varphi^{(n)} \in \varphi^{(n-1)} - \mathrm{d}(\beta) + (\widehat{\mathscr{T}}_3)_{\geqslant n+1}$, so $\varphi^{(n)} \in \mathfrak{t}_3 + (\widehat{\mathscr{T}}_3)_{\geqslant n+1}$. Moreover, the product $F = \cdots F_n \cdots F_2$ is convergent, and ${}^F\Phi$ then satisfies $\log({}^F\Phi) \in \hat{\mathfrak{t}}_3$. This proves the existence of F, such that ${}^F\Phi \in \underline{\mathrm{Assoc}}_{\mathrm{Lie}}$.

Let us now prove the uniqueness of an element of $\underline{\mathrm{Assoc}}_{\mathrm{Lie}}$, twist-equivalent to $\Phi \in \underline{\mathrm{Assoc}}$. This follows from:

Proposition 7.3. *Let Φ' and Φ'' be elements of $\underline{\mathrm{Assoc}}_{\mathrm{Lie}}$, and let F belong to $1 + (\widehat{\mathscr{T}}_2)_0$. Then ${}^F\Phi' = \Phi''$ if and only if there exists $\lambda \in \mathbb{K}$ such that $F = e^{\lambda t^{1,2}}$ and $\Phi'' = \Phi'$.*

Proof of Proposition 7.3. Since $t^{1,2} + t^{1,3} + t^{2,3}$ is central in $\widehat{\mathscr{T}}_3$, we have ${}^{F_\lambda}\Phi' = \Phi'$ when $F_\lambda = e^{\lambda t}$, for any $\lambda \in \mathbb{K}$. Conversely, let F_i be the degree i part of F. Then for some $\lambda_0 \in \mathbb{K}$, we have $F_1 = \lambda_0 t$. Replacing F by $F' = FF_{-\lambda_0}$, we get ${}^{F'}\Phi' = \Phi''$, and $F' - 1$ has valuation $\geqslant 2$ (for the degree in t). Assume that $F' - 1 \neq 0$ and let v be its valuation. Let F'_v be the degree v part of F'. Then $\mathrm{d}(F'_v) \in \mathfrak{t}_3$. On the other hand, $F'_v = \mu (t^{1,2})^v$, where $\mu \in \mathbb{K} - \{0\}$. Now $\mathrm{d}((t^{1,2})^v) \in \mathscr{T}_3 = U(\mathfrak{t}_3)$ has degree $\leqslant v$ for the filtration of $U(\mathfrak{t}_3)$, and its symbol in $S^v(\mathfrak{t}_3) = \mathrm{gr}_v(U(\mathfrak{t}_3))$ is

$\sum_{v'=1}^{v-1}\binom{v}{v'}(t^{1,3})^{v'}(t^{2,3})^{v-v'}-\sum_{v''=1}^{v-1}\binom{v}{v''}(t^{1,2})^{v''}(t^{1,3})^{v-v''}$: this is the image of a nonzero element in $S^v(\mathbb{K}t^{1,2}\oplus\mathbb{K}t^{1,3}\oplus\mathbb{K}t^{2,3})$ under the injection $S^v(\oplus_{1\leqslant i<j\leqslant 3}\mathbb{K}t^{i,j})\hookrightarrow S^v(\mathfrak{t}_3)$, so it is nonzero. So $F'\neq 1$ leads to a contradiction. So $F=F_{\lambda_0}$, therefore $\Phi''=\Phi'$. $\square$

Note that we have proved the analogue of Proposition 7.2, where the indices of $\mathscr{T}_3, \mathfrak{t}_4$, etc., are shifted by -1.

7.3. *Decomposition of* $\mathfrak{t}_3$ *and proof of Proposition 7.2*

To end the proof of the first part of Theorem 7.1, it remains to prove Proposition 7.2. For this, we construct a decomposition of $\mathfrak{t}_n$. For $i=1,\dots,n$, there is a unique algebra morphism $\varepsilon_i:\mathscr{T}_n\to\mathscr{T}_{n-1}$, taking $t_{i,j}$ to 0 for any $j\neq i$, and taking $t_{j,k}$ to $t_{j-\lambda_i(j),k-\lambda_i(k)}$ if $j,k\neq i$, where $\lambda_i(j)=0$ if $j<i$ and $=1$ if $j>i$. Then ε_i induces a Lie algebra morphism $\tilde{\varepsilon}_i:\mathfrak{t}_n\to\mathfrak{t}_{n-1}$. Set $\tilde{\mathfrak{t}}_n=\bigcap_{i=1}^n \mathrm{Ker}(\tilde{\varepsilon}_i)$. Then we have

Lemma 7.4.

$$\mathfrak{t}_n=\bigoplus_{k=0}^{n}\ \bigoplus_{I\in\mathscr{P}_k(\{1,\dots,n\})}(\tilde{\mathfrak{t}}_k)^I,$$

where $\mathscr{P}_k(\{1,\dots,n\})$ is the set of subsets of $\{1,\dots,n\}$ of cardinal k, and $(\tilde{\mathfrak{t}}_k)^I$ is the image of $\tilde{\mathfrak{t}}_k$ under $\mathfrak{t}_k\to\mathfrak{t}_n$, $x\mapsto x^{i_1,\dots,i_k}$, where $I=\{i_1,\dots,i_k\}$.

Proof. Let $\mathfrak{F}$ be the free Lie algebra with generators $\tilde{t}_{i,j}$, where $1\leqslant i<j\leqslant n$. It is graded by $\Gamma:=\mathbb{N}^{\{(i,j)|1\leqslant i<j\leqslant n\}}$: the degree of $\tilde{t}_{i,j}$ is the vector $\mathbf{d}_{i,j}$, whose (i',j') coordinate is $\delta_{(i,j),(i',j')}$. For $\underline{k}\in\Gamma$, we denote by $\mathfrak{F}_{\underline{k}}$ the part of $\mathfrak{F}$ of degree $\underline{k}$. Let $\pi:\mathfrak{F}\to\mathfrak{t}_n$ be the canonical projection. Since the defining ideal of $\mathfrak{t}_n$ is graded, we have

$$\mathfrak{t}_n=\bigoplus_{\underline{k}\in\Gamma}\pi(\mathfrak{F}_{\underline{k}}). \tag{7.14}$$

On the other hand, one checks that $\tilde{\mathfrak{t}}_n=\oplus_{\underline{k}\in\tilde{\Gamma}}\pi(\mathfrak{F}_{\underline{k}})$, where $\tilde{\Gamma}$ is the set of maps $k:\{(i,j)|1\leqslant i<j\leqslant n\}\to\mathbb{N}$, such that for each i, $\sum_{j\,|\,j>i}k(i,j)+\sum_{j\,|\,j<i}k(j,i)\neq 0$. Define a map $\lambda:\Gamma\to\mathscr{P}(\{1,\dots,n\})$ as follows ($\mathscr{P}(\{1,\dots,n\})$ is the set of subsets of $\{1,\dots,n\}$): λ takes the map $k:\{(i,j)|1\leqslant i<j\leqslant n\}\to\mathbb{N}$ to $\{i|\sum_{j\,|\,j>i}k(i,j)+\sum_{j\,|\,j<i}k(j,i)\neq 0\}$. Then for each $I\in\mathscr{P}(\{1,\dots,n\})$, $(\tilde{\mathfrak{t}}_{|I|})^I$ identifies with $\oplus_{\underline{k}\in\lambda^{-1}(I)}\pi(\mathfrak{F}_{\underline{k}})$. Comparing with (7.14), we get

$$\mathfrak{t}_n=\bigoplus_{I\in\mathscr{P}(\{1,\dots,n\})}(\tilde{\mathfrak{t}}_{|I|})^I. \qquad\square$$

When $n = 3$, we get $\mathfrak{t}_3 = \mathbb{K}t^{1,2} \oplus \mathbb{K}t^{1,3} \oplus \mathbb{K}t^{2,3} \oplus \tilde{\mathfrak{t}}_3$. On the other hand, the fact that the insertion-coproduct maps take $\mathfrak{t}_n$ to $\mathfrak{t}_m$ implies that $\mathrm{d} : \mathscr{T}_n \to \mathscr{T}_{n+1}$ is compatible with the filtrations induced by the identification $\mathscr{T}_n = U(\mathfrak{t}_n)$, $\mathscr{T}_{n+1} = U(\mathfrak{t}_{n+1})$. The associated graded map is

$$\mathrm{gr}^{\cdot}(\mathrm{d}) : S^{\cdot}(\mathfrak{t}_n) \to S^{\cdot}(\mathfrak{t}_{n+1}).$$

Proposition 7.2 now follows from:

Lemma 7.5. *When $k \geqslant 2$, the cohomology of the complex*

$$S^k(\mathfrak{t}_2) \xrightarrow{\mathrm{gr}^k(\mathrm{d})} S^k(\mathfrak{t}_3) \xrightarrow{\mathrm{gr}^k(\mathrm{d})} S^k(\mathfrak{t}_4)$$

vanishes.

Proof. We have

$$S^k(\mathfrak{t}_3) = \bigoplus_{\alpha=0}^{k} S^{k-\alpha}\left(\bigoplus_{1 \leqslant i<j \leqslant 3} \mathbb{K}t^{i,j}\right) \otimes S^{\alpha}(\tilde{\mathfrak{t}}_3). \tag{7.15}$$

Let $x \in S^k(\mathfrak{t}_3)$, and let $(x_\alpha)_{\alpha=0,\dots,k}$ be its components in the decomposition (7.15). We have

$$S^{\cdot}(\mathfrak{t}_4) = S^{\cdot}(\tilde{\mathfrak{t}}_4) \otimes \bigotimes_{2 \leqslant i<j \leqslant 4} S^{\cdot}(\tilde{\mathfrak{t}}_3^{1,i,j}) \otimes \bigotimes_{i=2}^{4} S^{\cdot}(\tilde{\mathfrak{t}}_2^{1,i}) \otimes S^{\cdot}(\mathfrak{t}_3^{2,3,4}).$$

We denote by p the projection

$$p : S^{\cdot}(\mathfrak{t}_4) \to \tilde{\mathfrak{t}}_3^{1,3,4} \otimes S^{\cdot}(\mathfrak{t}_3^{2,3,4}),$$

which is the tensor product of: the identity on the last factor, the projection to degree 1 on the factor $S^{\cdot}(\tilde{\mathfrak{t}}_3^{1,3,4})$, and the projection to degree 0 in all other factors. We also denote by $m : \tilde{\mathfrak{t}}_3^{1,3,4} \otimes S^{\cdot}(\mathfrak{t}_3^{2,3,4}) \to S^{\cdot}(\mathfrak{t}_3)$ the map induced by the identifications $\tilde{\mathfrak{t}}_3^{1,3,4} \subset \mathfrak{t}_3^{1,3,4} \simeq \mathfrak{t}_3$, $\mathfrak{t}_3^{2,3,4} \simeq \mathfrak{t}_3$ followed by the product map in $S^{\cdot}(\mathfrak{t}_3)$. We denote by $\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3$ the maps $\mathscr{T}_3 \to \mathscr{T}_4$ defined by

$$\mathrm{d}_1(x) = x^{12,3,4} - x^{1,3,4} - x^{2,3,4},$$

$$\mathrm{d}_2(x) = x^{1,23,4} - x^{1,2,4} - x^{1,3,4},$$

$$\mathrm{d}_3(x) = x^{1,2,34} - x^{1,2,3} - x^{1,2,4},$$

so $\mathrm{d} = \mathrm{d}_1 - \mathrm{d}_2 + \mathrm{d}_3$. The maps d_i are compatible with the filtrations of $\mathscr{T}_3$ and $\mathscr{T}_4$; we denote by $\mathrm{gr}^k(\mathrm{d}_i)$ the corresponding graded maps, so

$\mathrm{gr}^k(\mathrm{d}) = \mathrm{gr}^k(\mathrm{d}_1) - \mathrm{gr}^k(\mathrm{d}_2) + \mathrm{gr}^k(\mathrm{d}_3)$. Then if we set

$$x_1 = \sum_{a,b,c|a+b+c=k-1} (t^{1,2})^a (t^{1,3})^b (t^{2,3})^c \otimes e_{a,b,c},$$

where $e_{a,b,c} \in \tilde{\mathfrak{t}}_3$, we have

$$m \circ p \circ \mathrm{gr}^k(\mathrm{d}_1)(x) = \left(\sum_{\alpha=0}^{k} \alpha x_\alpha \right) - (t^{2,3})^{k-1} e_{0,0,k-1}.$$

On the other hand, let us define the i-degree of an element of $(\tilde{\mathfrak{t}}_{|I|})^I$ to be 1 if $i \in I$ and 0 if $i \notin I$. Then the i-degree of $\otimes_{I \subset \{1,\dots,n\}} S^{\alpha_I}((\tilde{\mathfrak{t}}_{|I|})^I) \subset S^{\cdot}(\mathfrak{t}_n)$ is $\sum_{I|i\in I} \alpha_I$. If x is homogeneous for the 1-degree, then so is $\mathrm{gr}^k(\mathrm{d}_2)(x)$, and 1-degree$(\mathrm{gr}^k(\mathrm{d}_2)(x)) =$ 1-degree(x). On the other hand, the elements of $S^{\cdot}(\mathfrak{t}_4)$ whose 1-degree is $\neq 1$ are in the kernel of p. It follows that

$$m \circ p \circ \mathrm{gr}^k(\mathrm{d}_2)(x_\alpha) = 0 \quad \text{if } \alpha \neq 1,$$

and $p \circ \mathrm{gr}^k(\mathrm{d}_2)(x_1) = (e_{0,0,k-1})^{1,3,4}[(t^{2,4} + t^{3,4})^{k-1} - (t^{3,4})^{k-1}]$, so

$$m \circ p \circ \mathrm{gr}^k(\mathrm{d}_2)(x_1) = e_{0,0,k-1}[(t^{1,3} + t^{2,3})^{k-1} - (t^{2,3})^{k-1}].$$

Finally, $p \circ \mathrm{gr}^k(\mathrm{d}_3)(x) = 0$. If x is such that $\mathrm{gr}^k(\mathrm{d})(x) = 0$, we have $m \circ p \circ \mathrm{gr}^k(\mathrm{d})(x) = 0$, so

$$\sum_{\alpha \geqslant 0} \alpha x_\alpha = e_{0,0,k-1}(t^{1,3} + t^{2,3})^{k-1}.$$

Looking at degrees in the decomposition (7.15), we get $x_\alpha = 0$ for $\alpha \geqslant 2$, and $x_1 = e_{0,0,k-1}(t^{1,3} + t^{2,3})^{k-1}$. Using the projection $p' : S^{\cdot}(\mathfrak{t}_4) \to \tilde{\mathfrak{t}}_3^{1,2,4} \otimes S^{\cdot}(\mathfrak{t}_3^{1,2,3})$, we get in the same way $x_1 = e_{k-1,0,0}(t^{1,2} + t^{1,3})^{k-1}$. Now $e_{k-1,0,0}(t^{1,2} + t^{1,3})^{k-1} = e_{0,0,k-1}(t^{1,3} + t^{2,3})^{k-1}$ implies $e_{k-1,0,0} = e_{0,0,k-1} = 0$ so $x_1 = 0$. Therefore $x \in S^k(\oplus_{1 \leqslant i<j \leqslant 3} \mathbb{K} t^{i,j})$. Let us set $x = S(t^{1,2}, t^{1,3}, t^{2,3})$, where S is a homogeneous polynomial of degree k of $\mathbb{K}[u,v,w]$. Since $\mathrm{d}(x) = 0$, we have

$$\begin{aligned} &S(t^{1,3} + t^{2,3}, t^{1,4} + t^{2,4}, t^{3,4}) - S(t^{1,2} + t^{1,3}, t^{1,4}, t^{2,4} + t^{3,4}) \\ &\quad + S(t^{1,2}, t^{1,3} + t^{1,4}, t^{2,3} + t^{2,4}) = S(t^{2,3}, t^{2,4}, t^{3,4}) + S(t^{1,2}, t^{1,3}, t^{2,3}) \end{aligned}$$

(equality in $S^{\cdot}(\oplus_{1 \leqslant i<j \leqslant 4} \mathbb{K} t^{i,j})$).

Applying $\frac{\partial}{\partial t^{1,2}} \circ \frac{\partial}{\partial t^{3,4}}$ to this equality, we get

$$(\partial_u \partial_w S)(t^{1,2} + t^{1,3}, t^{1,4}, t^{2,4} + t^{3,4}) = 0,$$

therefore $\partial_u \partial_w S = 0$. We have therefore

$$S(u, v, w) = P(u, v) + Q(v, w),$$

where P and Q are homogeneous polynomials of degree k. Moreover, $\mathrm{d}(x) = 0$, so

$$\begin{aligned}
&[P(t^{1,2}, t^{1,3} + t^{1,4}) - P(t^{1,2} + t^{1,3}, t^{1,4}) - P(t^{1,2}, t^{1,3})] \\
&\quad + [Q(t^{1,4} + t^{2,4}, t^{3,4}) - Q(t^{1,4}, t^{2,4} + t^{3,4}) - Q(t^{2,4}, t^{3,4})] \\
&\quad + [P(t^{1,3} + t^{2,3}, t^{1,4} + t^{2,4}) + Q(t^{1,3} + t^{1,4}, t^{2,3} + t^{2,4}) \\
&\quad - P(t^{2,4}, t^{2,4}) - Q(t^{1,3}, t^{2,3})] = 0. \qquad (7.16)
\end{aligned}$$

Write this as an identity

$$B(t^{1,2}, t^{1,3}, t^{1,4}) + C(t^{1,4}, t^{2,4}, t^{3,4}) + A(t^{2,3}, t^{1,4}, t^{1,3}, t^{2,4}) = 0.$$

Then A (resp., B, C) is independent of $t^{2,3}$ (resp., $t^{1,2}$, $t^{3,4}$). Let us now determine P and Q. Since $B(t^{1,2}, t^{1,3}, t^{1,4}) = B(0, t^{1,3}, t^{1,4})$, we have $P(u, v + w) - P(u + v, w) - P(u, v) = P(0, v + w) - P(v, w) - P(0, v)$. Therefore $(\mathrm{d}\tilde{P})(u, v, w) = 0$, where $\tilde{P}(u, v) = P(u, v) - P(0, v)$ and d is the co-Hochschild differential of polynomials in one variable. The corresponding cohomology is zero, so there exists a polynomial $\bar{P}$, such that

$$P(u, v) - P(0, v) = \bar{P}(u + v) - \bar{P}(u) - \bar{P}(v).$$

We conclude that $P(u, v)$ has the form

$$P(u, v) = \bar{P}(u + v) - \bar{P}(u) - R(v) \qquad (7.17)$$

where $\bar{P}$ and R are polynomials in one variable of degree k; since $P(u, v)$ is homogeneous of degree k, we can assume that $\bar{P}$ and R are monomials of degree k. In the same way, since $C(t^{1,4}, t^{2,4}, t^{3,4}) = C(t^{1,4}, t^{2,4}, 0)$, we have $Q(u + v, w) - Q(u, v + w) - Q(v, w) = Q(u + v, 0) - Q(u, v) - Q(v, 0)$, so $(\mathrm{d}\tilde{Q})(u, v, w) = 0$, where $\tilde{Q}(u, v) = Q(u, v) - Q(u, 0)$. So $Q(u, v)$ has the form

$$Q(u, v) = \bar{Q}(u + v) - \bar{Q}(v) - S(u), \qquad (7.18)$$

where $\bar{Q}$ and S are polynomials in one variable of degree k, which can be assumed to be monomials of degree k. We have therefore

$$x = \bar{P}^{1,23} + \bar{Q}^{12,3} - \bar{P}^{1,2} - \bar{Q}^{2,3} - T^{1,3},$$

where $\bar{P} = \bar{P}(t^{1,2})$, $\bar{Q} = \bar{Q}(t^{1,2})$ and $T = (R + S)(t^{1,2})$. So $x = \mathrm{d}(\bar{Q}) + (\bar{P} + \bar{Q})^{1,23} - (\bar{P} + \bar{Q})^{1,2} - T^{1,3}$. Set $a = \bar{P} + \bar{Q}$; we have $\mathrm{d}(y) = 0$, where $y = a^{1,23} - a^{1,2} - T^{1,3}$; applying ε_1 to $\mathrm{d}(y) = 0$, we get $T^{2,3} - T^{2,4} = 0$, so $T = 0$. We then get $a^{12,34} -$

$a^{12,3} - a^{2,34} + a^{2,3} = 0$. Applying $\varepsilon_3 \circ \varepsilon_2$ to this identity, we get $a^{1,4} = 0$. Finally $\bar{P} = -\bar{Q}$, so $x = \mathrm{d}(\bar{Q})$, which proves the lemma. □

7.4. Isotropy groups

Proposition 7.3 can be generalized to the case of a pair of elements of $\underline{\mathrm{Pent}}_{\mathrm{Lie}}$, and it implies that the isotropy group of each element of $\underline{\mathrm{Pent}}_{\mathrm{Lie}}$ is the additive group $\{e^{\lambda t^{1,2}}, \lambda \in \mathbb{K}\}$. Let Φ be an element of $\underline{\mathrm{Pent}}$. There exists an element Φ_{Lie} of $\underline{\mathrm{Pent}}_{\mathrm{Lie}}$ in the orbit of Φ. So the isotropy groups of Φ and Φ_{Lie} are conjugate. Since $1 + (\widehat{\mathscr{T}}_2)_0$ is commutative, the isotropy group of Φ is $\{e^{\lambda t^{1,2}}, \lambda \in \mathbb{K}\}$.

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