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Probabilistic representation and uniqueness results for measure-valued solutions of transport equations

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Abstract

The Cauchy problem for a multidimensional linear non-homogeneous transport equation in divergence form is investigated. An explicit and an implicit representation formulas for the unique solution of this transport equation in the case of a regular vector field v are proved. Then, together with a regularizing argument, these formulas are used to obtain a very general *probabilistic* representation for measure-valued solutions in the case when the initial datum is a measure and the involved vector field is no more regular, but satisfies suitable summability assumptions w.r.t. the solution. Finally, uniqueness results for solutions of the initial-value problem are derived from the uniqueness of the characteristic curves associated to v through the theory of the probabilistic representation previously developed.

Résumé

Dans cet article on s'intéresse au problème de Cauchy associé à une équation de transport linéaire non homogène multidimensionnelle sous forme divergence. Nous démontrons tout d'abord deux formules de représentation, l'une explicite, la seconde implicite, de l'unique solution de cette équation de transport dans le cas d'un champ de vecteurs régulier v. Puis dans un deuxième temps, nous utilisons ces formules et un argument de régularisation pour obtenir une représentation *probabiliste* très générale des solutions de type mesure de cette équation dans le cas où la condition initiale est une mesure et pour un champ de vecteurs qui peut être non régulier mais qui satisfait par rapport à la solution à des hypothèses de sommabilité convenables. Enfin, l'unicité des courbes caractéristiques associées au champ de vecteurs v obtenue par la théorie de la représentation probabiliste que nous développons, nous permet d'obtenir des résultats d'unicité pour le problème de Cauchy.

Keywords: Linear non-homogeneous transport equation; Probabilistic representation; Uniqueness results

1. Introduction

One of the aims of this paper is to provide a *probabilistic representation* of positive measure-valued solutions of the forward non-homogeneous transport equation in divergence-form (also called conservative-form). This representation briefly tells us that any *positive* solution of our PDE is a superposition of solutions of the first order associated ODE. The ultimate purpose of the paper, as we will see in the *applications* in the last section, is then to use this representation

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as a tool to get *uniqueness results* for the solutions of the transport equation, first of all in the class of positive solutions, and then also in other classes, as for example in the class of renormalized solutions or in the class of bounded solutions, but considering the homogeneous equation and under additional hypotheses on the involved vector field.

The non-homogeneity we consider in the PDE is given by a linear term, i.e. we consider:

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = c_t \mu_t \quad \text{in } (0,T) \times \mathbb{R}^d, \tag{1.1}$$

where $\{\mu_t\}_{t\in(0,T)}$ is a Borel family of finite positive measures on $\mathbb{R}^d$, c is a Borel bounded function and furthermore we suppose that there is a suitable integrability bound on v with respect to μ. The ODE that will play a fundamental role in our discussion is:

$$X(s,s,x) = x, \quad \frac{\mathrm{d}}{\mathrm{d}t} X(t,s,x) = v_t\big(X(t,s,x)\big) \tag{1.2}$$

(and for the meaning of the notation $X(t,s,x)$ see Definition 3.3).

Observe that, if we suppose that the distributional divergence of v is a measure absolutely continuous with respect to the Lebesgue measure and with bounded density, then our results can be applied also to the non-conservative form of the transport equation (because in this case the distribution $v_t \cdot \nabla_x \mu_t$ is well defined as $D_x \cdot (v_t \mu_t) - \operatorname{div}_x v_t(x) \mu_t$) in the case when we consider solutions μ_t of the form $\mu_t = \omega_t \mathcal{L}^d$ with $\omega \in L^1([0,T]; L^1(\mathbb{R}^d))$.

A deep investigation of the homogeneous conservative case is carried out in Chapter 8 of [7]. Moreover it is worth noticing that, since we are considering measure-valued solutions, a non-linear generalization of the right-hand side of (1.1) seems not possible.

When the velocity field v is sufficiently regular and there exists a unique solution of (1.2), the classical *method of characteristics* provides the explicit expression of the unique solution of the homogeneous continuity equation (the reader can see, for instance, the counterexamples provided in [19]-IV to understand which kind of regularity assumptions must be imposed on v indeed to have uniqueness of solutions of (1.2)). When we take a locally bounded vector field v, an initial datum ω_0 and functions c and g such that they all are Lipschitz continuous in their domain of definition, it is well known that the explicit formula of the locally Lipschitz continuous solution of,

$$\partial_t \omega_t + v_t \cdot \nabla_x \omega_t = c_t \omega_t + g_t \quad \text{in } (0,T) \times \mathbb{R}^d,$$

is given by

$$\omega_t(x) = \omega_0\big(X(0,t,x)\big) \mathrm{e}^{\int_0^t c_\tau(X(\tau,t,x))\,\mathrm{d}\tau} + \int_0^t g_s\big(X(s,t,x)\big) \mathrm{e}^{\int_s^t c_\tau(X(\tau,t,x))\,\mathrm{d}\tau}\,\mathrm{d}s. \tag{1.3}$$

As shown in [16], this results still holds if we relax the hypothesis on v, i.e. if instead of the Lipschitz regularity we suppose that there exists a non-negative function $\alpha(t)$, integrable in the interval $[0,T]$, such that for almost every $t \in (0,T)$ the inequality $\langle v_t(x) - v_t(y), x - y\rangle \geqslant -\alpha(t)|x-y|^2$ is valid for all $x, y \in \mathbb{R}^d$.

If we take an initial datum that is a measure, and so we look for distributional solutions of our equation, under the assumption,

$$v \in L^1\big([0,T]; W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)\big) \tag{1.4}$$

it is well known (see for example [2,19,10]) that the explicit formula for the unique measure-valued solution of

$$\partial_t \vartheta_t + D_x \cdot (v_t \vartheta_t) = 0, \quad \vartheta_0 = \bar{\vartheta}, \quad t \in [0,T], \tag{1.5}$$

is given by

$$\vartheta_t := X(t,0,\cdot)_\# \bar{\vartheta}. \tag{1.6}$$

What we want to do first in this paper is to give the analogous result in the non-homogeneous case (1.1) (assuming v and c sufficiently regular). As we will recall in Proposition 3.6, generalizing in the obvious way the validity of formula (1.3) to the measure-valued case, also in the non-homogeneous case (1.1), but still assuming (1.4), we can explicitly express the solution through the characteristics by the well known formula:

$$\mu_t = X(t,0,\cdot)_\# \big(\mu_0 \mathrm{e}^{\int_0^t c_s(X(s,0,\cdot))\,\mathrm{d}s}\big) \quad \forall t \in [0,T].$$

Since in the previous formula the dependence upon the characteristics is not linear, to get, in a certain way, this linearity property we will also prove the following *implicit representation formula*:

$$\mu_t = X(t,0,\cdot)_\#\mu_0 + \int_0^t X(t,s,\cdot)_\#(c_s\mu_s)\,\mathrm{d}s \quad \forall t \in [0,T]. \tag{1.7}$$

The linear dependence upon the characteristics in the previous formula will play an essential role in the proof of the probabilistic representation for solutions of the transport equation in the case when we relax the hypotheses on v.

The whole theory of characteristics fails if v is not smooth: the solution of (1.2) is no longer uniquely defined, and the notion of solution to (1.1) and (1.5) has to be reinvestigated.

In this connection it is worth to mention that Filippov's theory (see [20]) gives a generalized definition of solutions of (1.2) for a merely bounded vector field v; furthermore in this context uniqueness and stability results are ensured under the so-called *one-sided Lipschitz condition* (OSLC). Let us recall that a vector field v satisfies the (OSLC) if there exists $\alpha \in L^1_+(0,T)$ such that for $\mathscr{L}^1$-a.e. $t \in (0,T)$:

$$\langle v_t(y) - v_t(x), y - x\rangle \leqslant \alpha(t)|y-x|^2 \quad \text{for } \mathscr{L}^d\text{-a.e. } x, y \in \mathbb{R}^d.$$

Filippov's theory is used to solve differential equations and inclusions (see for example [8,13,14]) and also to study the multidimensional (i.e. for $x \in \mathbb{R}^d$) transport equation for instance by Poupaud and Rascle in [24] and by Petrova and Popov in [22] and [23].

The study of solutions of the homogeneous transport equation for a vector field v satisfying the (OSLC) has been deeply carried out by Bouchut and James in [11] for the one dimensional case (i.e. for $x \in \mathbb{R}$), and by Bouchut, James and Mancini in [12] for the multidimensional case (here the non-divergence form of the equation is considered): in these works, among other things, provided v satisfies a (OSLC), existence, uniqueness and weak stability of solutions are obtained through the notion of *duality solution*.

As we previously announced, in this paper we want to study solutions of (1.1) assuming on v, to give a distributional sense to the PDE, the integrability bound w.r.t. μ_t,

$$\int_0^T \int_{\mathbb{R}^d} |v_t(x)|\,\mathrm{d}\mu_t(x)\,\mathrm{d}t < +\infty.$$

This condition is not enough to guarantee uniqueness of solutions of (1.2): our aim is to show the relationship that holds in this general case between solutions of (1.1) and solutions of (1.2).

We denote by Γ_T the space $C([0,T];\mathbb{R}^d)$ of continuous maps from $[0,T]$ into $\mathbb{R}^d$ endowed with the sup norm, then we introduce the *evaluation maps*:

$$e_t : (x,\gamma) \in \mathbb{R}^d \times \Gamma_T \mapsto \gamma(t) \in \mathbb{R}^d \quad \text{for } t \in [0,T]. \tag{1.8}$$

We take into account that characteristics are not unique considering suitable probability measures in $\mathbb{R}^d \times \Gamma_T$.

The probabilistic representation for solutions of $\partial_t\mu_t + D_x \cdot (v_t\mu_t) = 0$ is provided by Theorem 8.2.1 in [7], and for the convenience of the reader we recall the precise statement of this result (that gave us the initial hint for our discussion).

Theorem 1.1 *(Probabilistic representation in the homogeneous case). Let $\mu_t : [0,T] \to \mathscr{P}(\mathbb{R}^d)$ be a narrowly continuous solution of the continuity equation $\partial_t\mu_t + D_x \cdot (v_t\mu_t) = 0$ for a suitable Borel vector field $v(t,x) = v_t(x)$ satisfying*:

$$\int_0^T \int_{\mathbb{R}^d} |v_t(x)|^p\,\mathrm{d}\mu_t(x)\,\mathrm{d}t < +\infty \quad \textit{for some } p > 1.$$

Then there exists a probability measure η in $\mathbb{R}^d \times \Gamma_T$ such that

(i) *η is concentrated on the set of pairs (x,γ) such that $\gamma \in AC^p(0,T;\mathbb{R}^d)$ is a solution of the ODE $\dot\gamma(t) = v_t(\gamma(t))$ for $\mathscr{L}^1$-a.e. $t \in (0,T)$, with $\gamma(0) = x$;*

(ii) $\mu_t = \mu_t^\eta$ *for any* $t \in [0,T]$, *with* μ_t^η *defined as*

$$\int_{\mathbb{R}^d} \varphi \, \mathrm{d}\mu_t^\eta = \int_{\mathbb{R}^d \times \Gamma_T} \varphi\big(e_t(\gamma)\big) \, \mathrm{d}\eta(x,\gamma) \quad \forall \varphi \in C_b^0\big(\mathbb{R}^d\big), \ \forall t \in [0,T]. \tag{1.9}$$

Conversely, any η *satisfying* (i) *and*

$$\int_0^T \int_{\mathbb{R}^d \times \Gamma_T} \big|v_t\big(\gamma(t)\big)\big| \, \mathrm{d}\eta(x,\gamma) \, \mathrm{d}t < +\infty,$$

induces, via (1.9)*, a solution of the continuity equation, with* $\mu_0 = \gamma(0)_\# \eta$.

(Observe that the definition of the evaluation maps e_t does not contain x, and so also formula (1.9) defining $\mu_t^\eta = e_{t\#}\eta$ does not contain x: for a discussion of this matter see Remark 11 in [5].)

The implicit representation formula (1.7) and the probabilistic representation in the homogeneous case gave us the suggestion that a similar result must hold for solutions of (1.1). Indeed, under the same assumptions on v of the previous theorem we will prove (see Theorem 4.1 for the precise statement) that if μ_t is a positive narrowly continuous solution of $\partial_t \mu_t + D_x \cdot (v_t \mu_t) = c_t \mu_t$ then still there exists a positive finite measure η in $\mathbb{R}^d \times \Gamma_T$ such that (i) of Theorem 1.1 holds, but condition (ii) now must be replaced by the following conditions:

(ii′) $\mu_s \ll \mu_s^\eta$ for every s in $[0,T]$, and the measure μ_s^η is given by the disintegration theorem, i.e.

$$\eta = \int_{\mathbb{R}^d} \eta_x^s \, \mathrm{d}\mu_s^\eta(x) \quad \text{with } \mu_s^\eta = (e_s)_\# \eta; \tag{1.10}$$

(ii″) for every $t \in [0,T]$, μ_t satisfies the following equality:

$$\langle \mu_t, \varphi \rangle = \big\langle \mu_t^\eta, \varphi \big\rangle + \int_0^t \bigg(\int_{\mathbb{R}^d} \int_{\Gamma_T} \varphi\big(\gamma(t)\big) \, \mathrm{d}\eta_x^s(\gamma) c_s(x) \, \mathrm{d}\mu_s(x) \bigg) \, \mathrm{d}s, \tag{1.11}$$

for any test function $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Observe that the explicit representation formula $\mu_t = (e_t)_\# \eta$ given in part (ii) of Theorem 1.1 has been replaced in the non-homogeneous case by the implicit formula (1.11) in (ii″), and furthermore the new condition (ii′) appears (since η_x^s are well defined only up to μ_s^η-negligible sets, formula (1.11) makes sense only if $\mu_s \ll \mu_s^\eta$ for every s in $[0,T]$).

The importance of such a probabilistic representation is that, under suitable hypothesis on the vector field v, it gives uniqueness results for *positive* solutions of the transport equation once we know, in some way, uniqueness of the characteristic curves associated to the vector field v. This happens for example in the well known case of Lipschitz (w.r.t. the space variable) vector fields, or of vector fields satisfying an Osgood type condition. So in this paper we investigate in a non-classical setting the relationship between the transport equation (a PDE) and the associated first order ODE, and the final aim of the work is to *get uniqueness of the PDE from uniqueness of the ODE*. In [19,3,2] the converse implication is also studied about uniqueness results of the PDE versus uniqueness results of the ODE (i.e. to what extent (1.2) has a unique solution for $\mathscr{L}^d$-almost every initial condition x), but here we will not consider this aspect of the problem.

The reader must take care of the fact that all the results stated by now are true for *positive* solutions only, and in fact we have considered a family $\{\mu_t\}_{t \in [0,T]}$ of positive finite measures, while for possibly signed solutions nothing can be said without additional regularity assumptions on v: in the last section we investigate what we can say making further assumptions on v. In particular, in the last section we first give the precise definition of what we intend for *probabilistic solution*, then we recall the definition of *renormalized solution* so to illustrate the relationship between the two kind of solutions of,

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d,$$

when both notions make sense. Then we state and prove a *uniqueness theorem* (Theorem 5.6) for renormalized solutions, and an *existence and uniqueness theorem* (Theorem 5.9) for bounded solutions of,

$$\partial_t \omega_t + D_x \cdot (v_t \omega_t) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \quad \omega(0,\cdot) = \omega_0,$$

and in these theorems we take in consideration also "possibly signed" solutions, but of course we have to impose additional regularity assumptions on v and on the class in which we look for our solutions.

The plan of the paper is the following: first in Section 2 we recall our notation and some preliminary results, then in Section 3 we provide the two representation formulas (the explicit formula and the implicit one) for the solutions of our equation. In Section 4 we prove the theorem about the probabilistic representation of solutions, and finally in Section 5 we use it in some applications so to understand, as previously announced, why our results can be of interest in the theory of PDEs.

2. Preliminaries and notation

In this section first we introduce our main notation and then we recall some basic tools that we will need in the following.

We will denote by $\mathcal{L}^d$ the Lebesgue measure in $\mathbb{R}^d$.

Given a map $f(t,x)$ depending on time and space, we will use the notation f_t for the map $x \mapsto f(t,x)$, while a derivative with respect to t will be denoted by $\frac{\mathrm{d}}{\mathrm{d}t}$ or by $\dot{}$ in the case of ODEs and by ∂_t in the case of PDEs. The least Lipschitz constant of a Lipschitz function f will be denoted by $\operatorname{Lip} f$.

If $I \subset \mathbb{R}$ and $f : I \times \mathbb{R}^m \to \mathbb{R}^k$ is a Borel function, and if X is a normed space of $\mathbb{R}^k$-valued functions defined on $\mathbb{R}^d$ (e.g. L^p or a Sobolev space), we say that $f \in L^1(I; X)$ if $\int_I \|f(t,\cdot)\| \,\mathrm{d}t < +\infty$. Analogously, if X is a local space, we say that $f \in L^1(I; X_{\mathrm{loc}})$ if $\int_I \|f(t,\cdot)\|_{B_R} \,\mathrm{d}t < +\infty$ for any $R > 0$.

Given a complete separable metric space X we will denote by $\mathcal{B}(X)$ the family of the Borel subsets of X, by $\mathcal{M}$ the set of all Radon measures on X, by $\mathcal{M}^+(X)$ the set of all positive Radon measures on X, by $\mathcal{M}_b^+(X)$ the set of all positive finite Radon measures on X and by $\mathcal{P} \subset \mathcal{M}_b^+(X)$ the subset of all probability measures.

We are going now to describe a construction that is quite natural in geometric measure theory and that will be widely used in the paper.

Push-forward. If X, Y are complete separable metric spaces, $\mu \in \mathcal{M}$ and $f : X \to Y$ is a Borel (or, more generally, μ-measurable) function, we denote by $f_\# \mu \in \mathcal{M}$ the *push-forward of μ through f*, defined by:

$$f_\# \mu(B) := \mu\big(f^{-1}(B)\big) \quad \forall B \in \mathcal{B}(Y).$$

An approximation by simple functions shows the chain rule,

$$\int_X \varphi\big(f(x)\big) \,\mathrm{d}\mu(x) = \int_Y \varphi(y) \,\mathrm{d}(f_\# \mu)(y),$$

for every bounded (or $f_\# \mu$ integrable) Borel function $\varphi : Y \to \mathbb{R}$.

Let us go on giving the definition of some properties of measures that will be needed in the following.

Narrow convergence. We say that a bounded sequence $(\mu_n) \subset \mathcal{M}$ is *narrowly convergent* to $\mu \in \mathcal{M}$ as $n \to \infty$ if,

$$\lim_{n\to\infty} \int_X \varphi(x) \,\mathrm{d}\mu_n(x) = \int_X \varphi(x) \,\mathrm{d}\mu(x),$$

for every function $\varphi \in C_b^0(X)$, the space of all continuous bounded real functions defined on X (but it is sufficient to check the previous equality on any subset $\mathcal{C}$ of continuous bounded functions whose linear envelope is dense in $C_b^0(X)$ with respect to the uniform topology induced by the "sup" norm).

Tightness. We say that a set $\mathcal{K} \subset \mathcal{M}$ is *tight* if,

$$\forall \varepsilon > 0 \ \exists K_\varepsilon \text{ compact in } X \text{ such that } |\mu|(X \setminus K_\varepsilon) \leqslant \varepsilon \ \forall \mu \in \mathcal{K}.$$

For a complete treatment of the relationships between narrow convergence and tightness we refer to Chapter 5 of [7] and Chapter III of [17]. Anyway, for the convenience of the reader, we want to recall an *integral condition for the* *tightness* that will be useful in the sequel. It is easy to check that the tightness condition is equivalent to the following one: there exists a function $\phi : X \to [0, +\infty]$, whose sublevels $\{x \in X\colon \phi(x) \leqslant c\}$ are compact in X, such that

$$\sup_{\mu \in \mathcal{K}} \int_X \phi(x)\,\mathrm{d}|\mu|(x) < +\infty.$$

We will also use the following *tightness criterion*: let X, X_1, X_2 be separable metric spaces and let $r^i : X \to X_i$ be continuous maps such that the product map $r := r^1 \times r^2 : X \to X_1 \times X_2$ is proper. Let $\mathcal{K} \subset \mathcal{P}(X)$ be such that $\mathcal{K}_i := r^i_\#(\mathcal{K})$ is tight in $\mathcal{P}(X_i)$ for $i = 1, 2$. Then also $\mathcal{K}$ is tight in $\mathcal{P}(X)$.

Disintegration of measures. Now we give the statement of the *disintegration theorem*, that will be an essential tool in providing a probabilistic representation for solutions of our problem.

Let X, Y be separable metric spaces and let $X \ni x \mapsto \eta_x \in \mathcal{P}(Y)$ be a measure-valued map. We say that η is a Borel map if $X \ni x \mapsto \eta_x(B)$ is a Borel map for any Borel set $B \subset Y$ (or equivalently if this property holds for any open set $A \subset Y$). Then the monotone class theorem gives that

$$X \ni x \mapsto \int_Y f(x, y)\,\mathrm{d}\eta_x(y) \tag{2.1}$$

is Borel for every bounded (or non-negative) Borel function $f : X \times Y \to \mathbb{R}$.

By (2.1) the formula,

$$\eta(f) = \int_X \left(\int_Y f(x, y)\,\mathrm{d}\eta_x(y) \right) \mathrm{d}\nu,$$

defines for any $\nu \in \mathcal{P}^+(X)$ a unique measure $\eta \in \mathcal{P}^+(X \times Y)$, that will be denoted by $\int_X \eta_x\,\mathrm{d}\nu(x)$. Actually the disintegration theorem implies that any $\eta \in \mathcal{P}^+(X \times Y)$ whose first marginal is ν can be represented in this way.

Theorem 2.1 *(Disintegration). Let X, Y be complete separable metric spaces, $\eta \in \mathcal{P}^+(X \times Y)$, let $\pi : X \times Y \to X$ be a Borel map and let $\mu^\eta_\pi = \pi_\# \eta \in \mathcal{P}^+(X)$. Then there exists a μ^η_π-a.e. uniquely determined Borel family of probability measures $\{\eta^\pi_x\}_{x \in X} \subset \mathcal{P}(Y)$ such that*

$$\eta = \int_X \eta^\pi_x\,\mathrm{d}\mu^\eta_\pi(x),$$

i.e.

$$\int_{X \times Y} f(x, y)\,\mathrm{d}\eta(x, y) = \int_X \left(\int_Y f(x, y)\,\mathrm{d}\eta^\pi_x(y) \right) \mathrm{d}\mu^\eta_\pi(x)$$

for every Borel map $f : X \times Y \to [0, +\infty]$.

This is not the more general statement of the disintegration theorem, but is the one useful for our purposes. The reader can find the more general statement in the context of Radon spaces for instance in Chapter 5 of [7], or in [17] (III-70).

3. Representation formulas for the non-homogeneous continuity equation

In this section we collect some technical results on the continuity equation that will be fundamental in the proof of the main theorem of the paper.

We are going to consider the non-homogeneous continuity equation,

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = c_t \mu_t \quad \text{in } (0, T) \times \mathbb{R}^d, \tag{3.1}$$

and we intend to extend the results, true for the homogeneous equation, concerning the representation formula of the solutions through the characteristic curves of the vector field v. Here μ_t is a Borel family of finite positive measures on $\mathbb{R}^d$ defined for t in the open interval $I := (0, T)$, $c : (x, t) \mapsto c_t(x) \in \mathbb{R}$ is a Borel bounded scalar function and $v : (x, t) \mapsto v_t(x) \in \mathbb{R}^d$ is a Borel vector field such that

$$\int_0^T \int_{\mathbb{R}^d} \left|v_t(x)\right| \mathrm{d}\mu_t(x)\, \mathrm{d}t < +\infty. \tag{3.2}$$

We suppose that (3.1) holds in the sense of distributions, i.e.

$$\int_0^T \int_{\mathbb{R}^d} \big(\partial_t \varphi(t, x) + \big\langle v_t(x), \nabla_x \varphi(t, x)\big\rangle + c_t(x)\varphi(t, x)\big)\, \mathrm{d}\mu_t(x)\, \mathrm{d}t = 0, \tag{3.3}$$

for any test function $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^d)$.

Remark 3.1. As observed in [7] (see Remark 8.1.1) for the homogeneous case, also in (3.3) more general test functions can be used. By a regularization argument via convolution, it can be shown that (3.3) holds if $\varphi \in C_c^1((0, T) \times \mathbb{R}^d)$ as well, and then, under condition (3.2), one can choose bounded test functions φ with bounded gradient and whose support has a compact projection in $(0, T)$ (that is, the support in x needs not be compact).

The following technical lemma allows us to consider $t \mapsto \mu_t$ as a measurable map with values in the space of narrowly continuous maps (provided we change the values of the map over an $\mathcal{L}^1$-negligible set of t's), and this is convenient because in this way for any test function φ we have a well defined map $t \mapsto \langle \mu_t, \varphi\rangle$.

Lemma 3.2 *(Narrowly continuous representative). Let μ_t be a Borel family of finite positive measures satisfying* (3.3) *for Borel functions v_t and c_t satisfying* (3.2). *Then there exists a* narrowly continuous *curve* $[0, T] \ni t \mapsto \tilde{\mu}_t \in \mathcal{M}_b^+(\mathbb{R}^d)$ *such that $\mu_t = \tilde{\mu}_t$ for $\mathcal{L}^1$-a.e. $t \in (0, T)$. Moreover, if $\varphi \in C_c^1([0, T] \times \mathbb{R}^d)$ and $t_1 \leqslant t_2 \in [0, T]$ we have*:

$$\int_{\mathbb{R}^d} \varphi(t_2, x)\, \mathrm{d}\tilde{\mu}_{t_2}(x) - \int_{\mathbb{R}^d} \varphi(t_1, x)\, \mathrm{d}\tilde{\mu}_{t_1}(x) = \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \big(\partial_t \varphi(t, x) + \big\langle \nabla\varphi(t, x), v_t(x)\big\rangle + \varphi(t, x)c_t(x)\big)\, \mathrm{d}\mu_t(x)\, \mathrm{d}t.$$

This is a classical result, whose proof can be obtained as in the homogeneous case (see for example Lemma 8.1.2 in [7]) with minor changes.

Since the ODE (1.2) will play, as announced before, a fundamental role in our discussion, first we recall some definitions and elementary results of the theory of ordinary differential equations.

Definition 3.3 *(Characteristic curves).* We say that a function $X_t(s, x) = X(t, s, x)$ is a *characteristic curve* for a vector field v if it is absolutely continuous w.r.t. the first variable in an interval I of $\mathbb{R}$, with values in $\mathbb{R}^d$, and is an integral solution of the ODE (1.2), i.e.

$$X_t(s, x) = x + \int_s^t v_\tau\big(X_\tau(s, x)\big)\, \mathrm{d}\tau, \quad 0 \leqslant s \leqslant t \leqslant T.$$

So our notation $X_t(s, x)$ means that we are considering the integral solution of $\dot{\gamma}(r) = v_r(\gamma(r))$ evaluated at the time t and that at the time s has value x.

By the classical theory of ODEs we know that if $v \in L^1([0, T]; W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d))$, i.e.

$$S := \int_0^T \Big(\sup_{\mathbb{R}^d} |v_t| + \mathrm{Lip}\big(v_t, \mathbb{R}^d\big)\Big)\, \mathrm{d}t < +\infty,$$

then solutions of the ODE are unique and stable. Furthermore we have also the following quantitative estimates for the map $X_t(s,x)$:

$$\sup_{s\in[0,T]}\int_0^T \sup_{x\in\mathbb{R}^d}\big|\partial_t X_t(s,x)\big|\,\mathrm{d}t \leqslant S, \qquad \sup_{t,s\in[0,T]} \operatorname{Lip}\big(X_t(s,\cdot),\mathbb{R}^d\big) \leqslant e^S.$$

Remark 3.4. Since it is a result that we will use in the sequel, we recall that if v is a locally bounded and Lipschitz continuous vector field then for any $t\in[0,T]$ the characteristic curves $X(t,s,x)$ are solutions of the following transport equation:

$$\partial_s X(t,s,x) + v_s(x)\cdot\nabla_x X(t,s,x) = 0. \tag{3.4}$$

Indeed, from the definition of X and from the uniqueness in the Cauchy problem, we immediately get that

$$\forall t_1,t_2,t_3\in[0,T],\ \forall x\in\mathbb{R}^d \quad X\big(t_3,t_2,X(t_2,t_1,x)\big) = X(t_3,t_1,x), \tag{3.5}$$

i.e. the flow satisfies the so-called *semigroup property*. If we differentiate the previous relation with respect to t_2 we have:

$$\partial_s X\big(t_3,t_2,X(t_2,t_1,x)\big) + \nabla_x X\big(t_3,t_2,X(t_2,t_1,x)\big)\partial_t X(t_2,t_1,x) = 0,$$

and using that $\partial_t X(t,s,x) = v_t(X(t,s,x))$ we get (3.4), after an obvious change of variable, which is possible thanks to the fact that for all $t,s\in[0,T]$ the map $x\mapsto X(t,s,x)$ is a diffeomorphism of $\mathbb{R}^d$ with inverse $X(s,t,\cdot)$ (this property is simply deduced by (3.5) taking $t_1=t_3$).

We refer the reader to [10] for a brief self-contained exposition of the *theory of characteristics* for the transport equation in the case of a *regular* vector field.

The next lemma is a uniqueness result for the continuity equation (the first step in order to prove the representation formulas), and although its proof is a standard result (see for example [19,1,7]) we recall it for the convenience of the reader.

Lemma 3.5 *(Comparison principle). Let σ_t be a narrowly continuous family of signed measures solving $\partial_t\sigma_t + D_x\cdot(v_t\sigma_t) = c_t\sigma_t$ in $(0,T)\times\mathbb{R}^d$, with $\sigma_0\leqslant 0$, c a Borel bounded scalar function in $(0,T)\times\mathbb{R}^d$ such that*

$$\int_0^T\int_{\mathbb{R}^d}\big(\big|v_t(x)\big|+\big|c_t(x)\big|\big)\,\mathrm{d}|\sigma_t|(x)\,\mathrm{d}t < +\infty$$

and

$$\int_0^T\Big(|\sigma_t|(B) + \sup_B|v_t| + \operatorname{Lip}(v_t,B)\Big)\,\mathrm{d}t < +\infty,$$

$$\int_0^T\Big(|\sigma_t|(B) + \sup_B|c_t| + \operatorname{Lip}(c_t,B)\Big)\,\mathrm{d}t < +\infty,$$

for any bounded closed set $B\subset\mathbb{R}^d$. Then $\sigma_t\leqslant 0$ for any $t\in[0,T]$.

Proof. The proof of this lemma strictly follows the scheme of the proof of the analogous result in the homogeneous case.

Fix $\psi\in C_c^\infty((0,T)\times\mathbb{R}^d)$ with $0\leqslant\psi\leqslant 1$, $R>0$ and a smooth cut-off function $\chi_R(\cdot)=\chi(\cdot/R)\in C_c^\infty(\mathbb{R}^d)$ such that

$$0\leqslant\chi_R\leqslant 1, \quad |\nabla\chi_R|\leqslant 2/R, \qquad \chi_R\equiv 1 \quad \text{on } B_R(0), \qquad \chi_R\equiv 0 \quad \text{on } \mathbb{R}^d\setminus B_{2R}(0).$$

We define w_t and d_t so that

$$w_t = v_t \quad \text{and} \quad d_t = c_t \quad \text{on } B_{2R}(0) \times (0,T), \qquad w_t = 0 = d_t \quad \text{if } t \notin [0,T],$$

and

$$\sup_{\mathbb{R}^d} |w_t| + \mathrm{Lip}\big(w_t, \mathbb{R}^d\big) \leqslant \sup_{B_{2R}(0)} |v_t| + \mathrm{Lip}\big(v_t, B_{2R}(0)\big) \quad \forall t \in [0,T],$$
$$\sup_{\mathbb{R}^d} |d_t| + \mathrm{Lip}\big(d_t, \mathbb{R}^d\big) \leqslant \sup_{B_{2R}(0)} |c_t| + \mathrm{Lip}\big(c_t, B_{2R}(0)\big) \quad \forall t \in [0,T].$$

Let w_t^ε and d_t^ε be obtained from w_t and d_t by a double mollification with respect to the space and time variables. Notice that w_t^ε are equibounded with respect to ε in the space $L^1([0,T]; W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$, and the same holds for d_t^ε in $L^1([0,T]; W^{1,\infty}(\mathbb{R}^d))$, i.e. they satisfy the following estimates:

$$\sup_{\varepsilon\in(0,1)} \int_0^T \Big(\sup_{\mathbb{R}^d}\big|w_t^\varepsilon\big| + \mathrm{Lip}\big(w_t^\varepsilon, \mathbb{R}^d\big)\Big)\,\mathrm{d}t < +\infty, \tag{3.6}$$

$$\sup_{\varepsilon\in(0,1)} \int_0^T \Big(\sup_{\mathbb{R}^d}\big|d_t^\varepsilon\big| + \mathrm{Lip}\big(d_t^\varepsilon, \mathbb{R}^d\big)\Big)\,\mathrm{d}t < +\infty. \tag{3.7}$$

We now build (using the classical method of characteristics) a smooth solution $\varphi^\varepsilon : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ of the backward problem:

$$\frac{\partial\varphi^\varepsilon}{\partial t} + \big\langle w_t^\varepsilon, \nabla\varphi^\varepsilon\big\rangle = \psi - \chi_R d_t^\varepsilon \varphi^\varepsilon \quad \text{in } [0,T]\times\mathbb{R}^d,\ \varphi^\varepsilon(T,x) = 0,\ x\in\mathbb{R}^d.$$

If we denote by $X^\varepsilon(s,t,x)$ the continuous (w.r.t. the first variable) solution of the ODE,

$$X_t^\varepsilon(t,x) = x, \qquad \frac{\mathrm{d}}{\mathrm{d}s} X_s^\varepsilon(t,x) = w_s^\varepsilon\big(X_s(t,x)\big),$$

the explicit formula for the solution of the backward problem is given by:

$$\varphi^\varepsilon(t,x) = -\int_t^T \psi\big(s, X^\varepsilon(s,t,x)\big) \mathrm{e}^{-\int_t^s \chi_R(X^\varepsilon(\tau,t,x)) d^\varepsilon(\tau, X^\varepsilon(\tau,t,x))\,\mathrm{d}\tau}\,\mathrm{d}s, \tag{3.8}$$

and this can be verified directly by simple calculations, using Leibniz's formula and Remark 3.4.

By (3.8) we immediately obtain that $\varphi^\varepsilon \leqslant 0$, and that $|\varphi^\varepsilon|$ and $|\nabla\varphi^\varepsilon|$ are uniformly bounded w.r.t. ε for $(t,x)\in[0,T]\times\mathbb{R}^d$. Now we can insert the test function $\varphi^\varepsilon\chi_R$ in the continuity equation and take into account that $\sigma_0 \leqslant 0$ and $\varphi^\varepsilon \leqslant 0$ to obtain:

$$\begin{aligned}
0 \geqslant -\int_{\mathbb{R}^d} \varphi^\varepsilon \chi_R\,\mathrm{d}\sigma_0 &= \int_0^T\int_{\mathbb{R}^d} \Big(\chi_R \frac{\partial\varphi^\varepsilon}{\partial t} + \big\langle v_t, \chi_R\nabla\varphi^\varepsilon + \varphi^\varepsilon\nabla\chi_R\big\rangle + c_t\varphi^\varepsilon\chi_R\Big)\,\mathrm{d}\sigma_t\,\mathrm{d}t \\
&= \int_0^T\int_{\mathbb{R}^d} \chi_R\big(\psi + \big\langle v_t - w_t^\varepsilon, \nabla\varphi^\varepsilon\big\rangle + \big(c_t - d_t^\varepsilon\big)\varphi^\varepsilon\big)\,\mathrm{d}\sigma_t\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^d} \varphi^\varepsilon\langle\nabla\chi_R, v_t\rangle\,\mathrm{d}\sigma_t\,\mathrm{d}t \\
&\geqslant \int_0^T\int_{\mathbb{R}^d} \chi_R\big(\psi + \big\langle v_t - w_t^\varepsilon, \nabla\varphi^\varepsilon\big\rangle + \big(c_t - d_t^\varepsilon\varphi^\varepsilon\big)\big)\,\mathrm{d}\sigma_t\,\mathrm{d}t - \sup_{x\in B_{2R}(0),\ 0<\varepsilon<1}\big|\varphi^\varepsilon(t,x)\big| \int_0^T\int_{\mathbb{R}^d} |\nabla\chi_R||v_t|\,\mathrm{d}|\sigma_t|\,\mathrm{d}t.
\end{aligned}$$

When we pass to the limit for $\varepsilon\downarrow 0$, using the uniform bounds on $|\varphi^\varepsilon|$ and $|\nabla\varphi^\varepsilon|$, and the fact that $w_t = v_t$ and $d_t = c_t$ on $[0,T]\times\operatorname{supp}\chi_R$, we get:

$$\int_0^T\int_{\mathbb{R}^d} \chi_R\psi\,\mathrm{d}\sigma_t\,\mathrm{d}t \leqslant C\int_0^T\int_{\mathbb{R}^d} |\nabla\chi_R||v_t|\,\mathrm{d}|\sigma_t|\,\mathrm{d}t \leqslant \frac{2C}{R}\int_0^T \int_{R\leqslant|x|\leqslant 2R} |v_t|\,\mathrm{d}|\sigma_t|\,\mathrm{d}t.$$

Eventually letting $R \to \infty$ we obtain $\int_0^T \int_{\mathbb{R}^d} \psi \, \mathrm{d}\sigma_t \, \mathrm{d}t \leqslant 0$. By the arbitrariness of ψ we get the thesis. $\square$

Now we are ready to state and prove the representation formulas for measure-valued solutions of (1.1) through the solutions of (1.2) in the case when v is sufficiently regular.

Proposition 3.6 *(Existence and uniqueness). Let v be a Borel vector field in $L^1([0,T]; W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$, c a Borel bounded and locally Lipschitz continuous (w.r.t. the space variable) scalar function and $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. Then there exists a unique μ_t, $t \in [0,T]$, narrowly continuous family of Borel finite positive measures solving (in the distributional sense) the initial value problem*

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = c_t \mu_t \quad \text{in } (0,T) \times \mathbb{R}^d, \quad \mu(0,\cdot) = \mu_0, \tag{3.9}$$

and it is given by the explicit *formula*:

$$\mu_t = X(t,0,\cdot)_\#\big(\mu_0 \mathrm{e}^{\int_0^t c_s(X(s,0,\cdot))\,\mathrm{d}s}\big) \quad \forall t \in [0,T]. \tag{3.10}$$

Moreover

$$\mu_t = X(t,0,\cdot)_\# \mu_0 + \int_0^t X(t,s,\cdot)_\#(c_s \mu_s)\,\mathrm{d}s \quad \forall t \in [0,T] \tag{3.11}$$

provides an implicit *formula for μ_t.*

Proof. In this proof we focus only on the existence and the representation issue: in a first step we show that the measure μ_t defined by formula (3.10) is a solution of (3.1), then in a second step we prove that it also satisfies the integral equality (3.11). At this point Lemma 3.5 guarantees that for any initial datum μ_0 there exists a unique solution of (3.1), and so it must be necessarily expressed by formula (3.10) and must necessarily satisfy Eq. (3.11) by the first two steps of the proof.

Notice that in both cases (explicit and implicit formulas) we need to check (3.3) only on test functions of the form $\psi(t)\varphi(x)$, i.e.

$$\int_{\mathbb{R}} \psi'(t)\langle \mu_t, \varphi\rangle \,\mathrm{d}t + \int_{\mathbb{R}} \psi(t) \int_{\mathbb{R}^d} \big(\langle v_t, \nabla\varphi\rangle + c_t\varphi\big)\,\mathrm{d}\mu_t\,\mathrm{d}t = 0, \tag{3.12}$$

so we have simply to show that the map $t \mapsto \langle \mu_t, \varphi\rangle$ belongs to $W^{1,1}(0,T)$ for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, and that its time derivative is $\int_{\mathbb{R}^d}(\langle v_t, \nabla\varphi\rangle + c_t\varphi)\,\mathrm{d}\mu_t$.

Step 1 (*Explicit representation formula*). We are going first to prove that

$$t \mapsto \mu_t := X(t,0,\cdot)_\#\big(\mu_0 \mathrm{e}^{\int_0^t c_s(X(s,0,\cdot))\,\mathrm{d}s}\big)$$

is absolutely continuous, and in particular $W^{1,1}(0,T)$. For every choice of finitely many pairwise disjoint intervals $(a_i,b_i) \subset (0,T)$, we have:

$$\langle \mu_{b_i} - \mu_{a_i}, \varphi\rangle = \int_{\mathbb{R}^d} \big(\varphi\big(X(b_i,0,x)\big)\mathrm{e}^{\int_{a_i}^{b_i} c_s(X(s,0,x))\,\mathrm{d}s} - \varphi\big(X(a_i,0,x)\big)\big)\mathrm{e}^{\int_0^{a_i} c_s(X(s,0,x))\,\mathrm{d}s}\,\mathrm{d}\mu_0(x).$$

Now we want to take the modulus of $\langle \mu_{b_i} - \mu_{a_i}, \varphi\rangle$ and then the sum over all intervals (a_i,b_i). Let us denote by $\|c\|$ the positive constant such that $-\|c\| \leqslant c \leqslant \|c\|$. Since,

$$\begin{aligned}
&\sum_{i=1}^n \big|\varphi\big(X(b_i,0,x)\big)\mathrm{e}^{\int_{a_i}^{b_i} c_s(X(s,0,x))\,\mathrm{d}s} - \varphi\big(X(a_i,0,x)\big)\big| \\
&\quad\leqslant \mathrm{e}^{T\|c\|}\|\nabla\varphi\|_\infty \int_{\bigcup_i (a_i,b_i)} \big|\dot{X}(t,0,x)\big|\,\mathrm{d}t + \|c\|\|\varphi\|_\infty \sum_i (b_i - a_i) \\
&\quad\leqslant \mathrm{e}^{T\|c\|}\|\nabla\varphi\|_\infty \int_{\bigcup_i (a_i,b_i)} \|v_t\|_\infty\,\mathrm{d}t + \|c\|\|\varphi\|_\infty \sum_i (b_i - a_i),
\end{aligned}$$

an integration with respect to $(\mathrm{e}^{\int_0^{a_i} c_s(X(s,0,x))\,\mathrm{d}s}\mu_0)$ gives:

$$\int_{\mathbb{R}^d}\sum_{i=1}^{n}\bigl|\varphi\bigl(X(b_i,0,x)\bigr)\mathrm{e}^{\int_{a_i}^{b_i} c_s(X(s,0,x))\,\mathrm{d}s}-\varphi\bigl(X(a_i,0,x)\bigr)\bigr|\mathrm{e}^{\int_0^{a_i} c_s(X(s,0,x))\,\mathrm{d}s}\,\mathrm{d}\mu_0(x)$$
$$\leqslant \mathrm{e}^{2T\|c\|}\|\nabla\varphi\|_\infty\int_{\bigcup_i(a_i,b_i)}\|v_t\|_\infty\,\mathrm{d}t+\|c\|\mathrm{e}^{T\|c\|}\|\varphi\|_\infty\sum_i(b_i-a_i),$$

so the absolute continuity of the integral shows that the right-hand side can be made small when $\sum_{i=1}^n(b_i-a_i)$ is small. We have shown the absolute continuity of the map, and in particular its continuity, then to check the distributional equality it is sufficient to prove the pointwise one. Indeed, we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\varphi(x)\,\mathrm{d}\mu_t(x)&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\varphi(x)\,\mathrm{d}\bigl(X(t,0,x)_\#\bigl(\mu_0(x)\mathrm{e}^{\int_0^t c_s(X(s,0,x))\,\mathrm{d}s}\bigr)\bigr)\\
&=\int_{\mathbb{R}^d}\mathrm{e}^{\int_0^t c_s(X(s,0,x))\,\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\varphi\bigl(X(t,0,x)\bigr)\,\mathrm{d}\mu_0(x)\\
&\quad+\int_{\mathbb{R}^d}\varphi\bigl(X(t,0,x)\bigr)\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\int_0^t c_s(X(s,0,x))\,\mathrm{d}s}\,\mathrm{d}\mu_0(x)\\
&=\int_{\mathbb{R}^d}\bigl\langle\nabla_x\varphi\bigl(X(t,0,x)\bigr),\dot{X}(t,0,x)\bigr\rangle\mathrm{e}^{\int_0^t c_s(X(s,0,x))\,\mathrm{d}s}\,\mathrm{d}\mu_0(x)\\
&\quad+\int_{\mathbb{R}^d}\varphi\bigl(X(t,0,x)\bigr)\mathrm{e}^{\int_0^t c_s(X(s,0,x))\,\mathrm{d}s}c_t\bigl(X(t,0,x)\bigr)\,\mathrm{d}\mu_0(x)\\
&=\int_{\mathbb{R}^d}\bigl\langle\nabla_x\varphi(x),v_t(x)\bigr\rangle\,\mathrm{d}\mu_t(x)+\int_{\mathbb{R}^d}\varphi(x)c_t(x)\,\mathrm{d}\mu_t(x).
\end{aligned}$$

Step 2 (*Implicit representation formula*). Let us focus now on the implicit representation and let us show that the measure μ_t defined by (3.10) satisfies (3.11).
It suffices to show that for any test function $\varphi\in C_c^\infty(\mathbb{R}^d)$ the following equality holds:

$$\int_{\mathbb{R}^d}\varphi\bigl(X(t,0,x)\bigr)\mathrm{e}^{\int_0^t c_\tau(X(\tau,0,x))\,\mathrm{d}\tau}\,\mathrm{d}\mu_0(x)-\int_{\mathbb{R}^d}\varphi\bigl(X(t,0,x)\bigr)\,\mathrm{d}\mu_0(x)$$
$$=\int_{\mathbb{R}^d}\int_0^t\varphi\bigl(X\bigl(t,s,X(s,0,x)\bigr)\bigr)c_s\bigl(X(s,0,x)\bigr)\mathrm{e}^{\int_0^s c_\tau(X(\tau,0,x))\,\mathrm{d}\tau}\,\mathrm{d}s\,\mathrm{d}\mu_0(x).$$

Since $X(t,s,X(s,0,x))=X(t,0,x)$ for any $0\leqslant s\leqslant t\leqslant T$, the previous equality reduces to the following one:

$$\int_{\mathbb{R}^d}\varphi\bigl(X(t,0,x)\bigr)\left(\mathrm{e}^{\int_0^t c_\tau(X(\tau,0,x))\,\mathrm{d}\tau}-1-\int_0^t c_s\bigl(X(s,0,x)\bigr)\mathrm{e}^{\int_0^s c_\tau(X(\tau,0,x))\,\mathrm{d}\tau}\,\mathrm{d}s\right)\mathrm{d}\mu_0(x)=0. \tag{3.13}$$

Once we set $F(s):=\mathrm{e}^{\int_0^s c_\tau(X(\tau,0,x))\,\mathrm{d}\tau}$, we have that $F(0)=1$ and, by the Leibniz formula:

$$\frac{\mathrm{d}}{\mathrm{d}s}F(s)=F(s)c_s\bigl(X(s,0,x)\bigr).$$

So, by the fundamental theorem of the integral calculus, (3.13) holds true, and this proves the implicit representation formula. □

Remark 3.7. Observe that the proof of Step 1 in the previous proposition simply uses the definition of characteristic curve. So, given a Borel vector field v, if we know in some way that there exists at least one characteristic curve X associated to v, using formula (3.10) we can immediately construct a solution of the initial value problem (3.9).

Notice also that the proof of Step 2 requires in addition the semigroup property of X, that is true when we have uniqueness of the solutions of the ODE (1.2).

The implicit representation formula for solutions μ_t of (3.1) will be our starting point for a probabilistic representation of μ_t under weaker regularity assumptions on the vector field v (i.e. when the characteristics of v are not unique).

When we have the homogeneous equation, and we denote by ϑ_t its solutions, from the explicit representation $\vartheta_t(x) = X(t,0,x)_{\#}\mu_0(x)$ and using the *area formula* (see [2]), we can see that

$$\text{if } \vartheta_0 = \rho\mathcal{L}^d \quad \text{then } \vartheta_t = \frac{\rho(\cdot)}{JX(t,0,\cdot)} \circ X^{-1}(t,0,\cdot)\mathcal{L}^d,$$

where $JX = |\det \nabla X|$. If we look again at (3.10) we can immediately conclude that an analogous result holds in the non-homogeneous case, i.e.

$$\text{if } \mu_0 = \rho\mathcal{L}^d \quad \text{then } \mu_t = \frac{\rho(\cdot)\mathrm{e}^{\int_0^t c(s,X(s,0,\cdot))\,\mathrm{d}s}}{JX(t,0,\cdot)} \circ X^{-1}(t,0,\cdot)\mathcal{L}^d. \tag{3.14}$$

Remark 3.8. Observe that for an initial datum $\omega_0 \in L^\infty(\mathbb{R}^d)$ (resp. $\omega_0 \in L^1(\mathbb{R}^d)$) and a bounded Borel vector field $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ that satisfies also,

$$D_x \cdot v_t = \operatorname{div}_x v_t \mathcal{L}^d \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (0,T) \text{ with } [\operatorname{div}_x v]^- \in L^\infty\big((0,T)\times\mathbb{R}^d\big),$$

the existence of a solution to

$$\partial_t\omega_t + D_x \cdot (v_t\omega_t) = c_t\omega_t \quad \text{in } (0,T)\times\mathbb{R}^d, \quad \omega(0,\cdot) = \omega_0, \tag{3.15}$$

such that $\omega_t \in L^\infty(\mathbb{R}^d)$ (resp. $\omega_t \in L^1(\mathbb{R}^d)$) for $\mathcal{L}^1$-a.e. $t \in (0,T)$ is not a problem and easily follows by a regularization argument. Indeed, in the case of a Lipschitz vector field the solution ω_t is given by formula (3.14), where we take $\rho(x) = \omega_0(x)$, and observe that from the same formula (3.14) we get the following estimate on ω_t:

$$\omega_t \leqslant |\omega_0|\big(\mathrm{e}^{\int_0^t \|c_s^+\|_\infty \,\mathrm{d}s}\big)\big(\mathrm{e}^{\int_0^t \|[\operatorname{div} b_s]^-\|_\infty \,\mathrm{d}s}\big)$$

(we refer the reader to [5] where a detailed explanation of the estimate on JX is made). In the case of a bounded vector field we can find a suitable ω_t as a weak limit in the following way: we take a sequence $(v_t^n)_{n\in\mathbb{N}}$ of locally Lipschitz (in the space variable) functions such that v_t^n strongly converges to v_t in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$ as $n \to +\infty$. Then we denote by ω_t^n the unique solution of,

$$\partial_t\omega_t^n + D_x \cdot \big(v_t^n\omega_t^n\big) = c_t\omega_t^n \quad \text{in } (0,T)\times\mathbb{R}^d, \quad \omega^n(0,\cdot) = \omega_0,$$

that by (3.14) can be explicitly written as

$$\omega_n^t(x) = \frac{\omega_0(\cdot)\mathrm{e}^{\int_0^t c_s(X_n(s,0,\cdot))\,\mathrm{d}s}}{JX(t,0,\cdot)} \circ X_n^{-1}(t,0,x),$$

where X_n is the unique characteristic associated to v^n. Finally we choose ω_t to be the weak limit in the sense of distributions of the sequence (ω_t^n) as $n \to +\infty$.

In the remaining part of this section we state some technical lemmas that will be used in the sequel: Lemma 3.9 is a generalization of Lemma 8.1.10 in [7], while Lemma 3.10 is a suitable and not difficult generalization of Lemma 8.1.9 in [7]. In Lemma 3.9 the generalization consists in taking a convex non-decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ instead of the function $|\cdot|^p$, so we show how to adapt the proof, while we skip the proof of Lemma 3.10 because in this case the generalization, besides considering the non-homogeneous case, is the same one of Lemma 3.9.

Lemma 3.9. *Let $p \geqslant 1$, $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a non-decreasing convex function, $\mu \in \mathcal{M}_b^+(\mathbb{R}^d)$ and let E be a $\mathbb{R}^m$-valued measure in $\mathbb{R}^d$ with finite total variation and absolutely continuous with respect to μ. Then for any convolution kernel ρ the following inequality holds*:

$$\int_{\mathbb{R}^d} \psi\left(\left|\frac{E * \rho}{\mu * \rho}\right|\right) \mu * \rho \, \mathrm{d}x \leqslant \int_{\mathbb{R}^d} \psi\left(\left|\frac{E}{\mu}\right|\right) \mathrm{d}\mu,$$

where in the right-hand side of the inequality $\frac{E}{\mu}$ denotes the density of the measure E w.r.t. μ.

Proof. We use Jensen's inequality in the form:

$$\Phi\left(\int_{\mathbb{R}^d} \varphi(x) \, \mathrm{d}\vartheta(x)\right) \leqslant \int_{\mathbb{R}^d} \Phi(\varphi(x)) \, \mathrm{d}\vartheta(x), \tag{3.16}$$

where $\Phi : \mathbb{R}^{m+1} \to [0, +\infty]$ is convex, l.s.c. and positively 1-homogeneous, $\varphi : \mathbb{R}^d \to \mathbb{R}^{m+1}$ is a Borel map and ϑ is a finite measure in $\mathbb{R}^d$. We fix $x \in \mathbb{R}^d$, then we choose $\varphi := (\frac{E}{\mu}, 1)$, $\vartheta := \rho(x - \cdot)\mu$ and

$$\Phi(x, t) := \begin{cases} \psi(\frac{|x|}{t})t & \text{if } t > 0, \\ 0 & \text{if } (z, t) = (0, 0), \\ +\infty & \text{if either } t < 0 \text{ or } t = 0, z \neq 0. \end{cases}$$

If we apply (3.16) with φ, ϑ and Φ as above we get the thesis (see Lemma 8.1.10 in [7] for details), and for this purpose we have only to verify that Φ is convex. This holds true, indeed for $z_0 > 0$ we can write:

$$\psi(z_0) = \sup_i \bigl\{a_i z_0 + b_i \colon a_i, b_i \in \mathbb{R},\ a_i z + b_i \leqslant \psi(z)\ \forall z > 0\bigr\}, \tag{3.17}$$

and consequently,

$$\psi\left(\frac{|x|}{t}\right)t = \sup_i \bigl\{a_i|x| + b_i t\bigr\}, \tag{3.18}$$

where the sup is taken over the same families of a_i and b_i that appear in (3.17). Since ψ is non-decreasing we have that a_i must be positive, and so the function given by (3.18) is convex in the variable (x, t). $\square$

Lemma 3.10 *(Approximation by regular curves). Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing convex function, and let μ_t be a time-continuous solution of* (3.1) *w.r.t. a velocity field v_t and a scalar function c_t satisfying the integrability condition*:

$$\int_0^T \int_{\mathbb{R}^d} \psi\bigl(\bigl|v_t(x)\bigr|\bigr) \, \mathrm{d}\mu_t(x) \, \mathrm{d}t < +\infty$$

and

$$\int_0^T \int_{\mathbb{R}^d} \psi\bigl(\bigl|c_t(x)\bigr|\bigr) \, \mathrm{d}\mu_t(x) \, \mathrm{d}t < +\infty.$$

Let $(\rho_\varepsilon) \subset C_c^\infty(\mathbb{R}^d)$ be a family of strictly positive *mollifiers in the x-variable, and set*:

$$\mu_t^\varepsilon := \mu_t * \rho_\varepsilon, \qquad v_t^\varepsilon := \frac{(v_t\mu_t) * \rho_\varepsilon}{\mu_t * \rho_\varepsilon}, \qquad c_t^\varepsilon := \frac{(c_t\mu_t) * \rho_\varepsilon}{\mu_t * \rho_\varepsilon}.$$

Then μ_t^ε is a continuous solution of (3.1) *w.r.t. v_t^ε and c_t^ε, which satisfy,*

$$\int_0^T \Bigl(\sup_B |v_t^\varepsilon| + \mathrm{Lip}\bigl(v_t^\varepsilon, B\bigr)\Bigr) \, \mathrm{d}t < +\infty, \qquad \int_0^T \Bigl(\sup_B |c_t^\varepsilon| + \mathrm{Lip}\bigl(c_t^\varepsilon, B\bigr)\Bigr) \, \mathrm{d}t < +\infty,$$

for every compact set $B \subset \mathbb{R}^d$ *and the uniform integrability bounds*:

$$\int_{\mathbb{R}^d} \psi\big(\big|v_t^\varepsilon(x)\big|\big)\,\mathrm{d}\mu_t^\varepsilon(x) \leqslant \int_{\mathbb{R}^d} \psi\big(\big|v_t(x)\big|\big)\,\mathrm{d}\mu_t(x) \quad \forall t \in (0,T),$$

$$\int_{\mathbb{R}^d} \psi\big(\big|c_t^\varepsilon(x)\big|\big)\,\mathrm{d}\mu_t^\varepsilon(x) \leqslant \int_{\mathbb{R}^d} \psi\big(\big|c_t(x)\big|\big)\,\mathrm{d}\mu_t(x) \quad \forall t \in (0,T).$$

Moreover,

$$(v_t\mu_t) * \rho_\varepsilon \to v_t\mu_t, \quad (c_t\mu_t) * \rho_\varepsilon \to c_t\mu_t \quad \textit{narrowly},$$

and

$$\lim_{\varepsilon\downarrow 0}\big\|\psi\big(\big|v_t^\varepsilon\big|\big)\big\|_{L^1(\mu_t^\varepsilon;\mathbb{R}^d)} = \big\|\psi\big(|v_t|\big)\big\|_{L^1(\mu_t;\mathbb{R}^d)} \quad \forall t \in (0,T),$$

$$\lim_{\varepsilon\downarrow 0}\big\|\psi\big(\big|c_t^\varepsilon\big|\big)\big\|_{L^1(\mu_t^\varepsilon;\mathbb{R}^d)} = \big\|\psi\big(|c_t|\big)\big\|_{L^1(\mu_t;\mathbb{R}^d)} \quad \forall t \in (0,T).$$

4. The probabilistic representation

The aim of the theorem proved in this section is to extend the probabilistic representation of the solutions of the homogeneous continuity equation (in divergence form) to the non-homogeneous one. This representation is of interest because it tells us to what extent formula (3.11) is still valid when the vector field v fails to satisfy the regularity condition:

$$\int_0^T \Big(\sup_B |v_t| + \mathrm{Lip}(v_t, B)\Big)\,\mathrm{d}t < +\infty.$$

It is well known that in this case, without further information on v, nothing guarantees uniqueness of characteristics, so it is useful to consider suitable probability measures in the space $C([0,T];\mathbb{R}^d)$ endowed with the sup norm, that we denote by Γ_T. We recalled in the Introduction the precise statement of the theorem that provides the probabilistic representation for positive narrowly continuous solutions of $\partial_t\mu_t + D_x\cdot(v_t\mu_t) = 0$, so now we are going to state and prove the analogous theorem for positive narrowly continuous solutions of $\partial_t\mu_t + D_x\cdot(v_t\mu_t) = c_t\mu_t$.

Theorem 4.1 *(Probabilistic representation in the non-homogeneous case). Let* $\mu_t : [0,T] \to \mathcal{M}_b^+(\mathbb{R}^d)$ *be a narrowly continuous solution of the continuity equation* (3.1) *for a suitable Borel vector field* $v_t(x) = v(t,x)$ *and a Borel bounded scalar function* $c_t(x) = c(t,x)$ *satisfying*:

$$\int_0^T \int_{\mathbb{R}^d} \big|v_t(x)\big|^p\,\mathrm{d}\mu_t(x)\,\mathrm{d}t < +\infty \quad \textit{for some } p \geqslant 1. \tag{4.1}$$

Then there exists a finite positive measure η *in* $\mathbb{R}^d \times \Gamma_T$ *such that*

(i) η *is concentrated on the set of* $\gamma \in AC^p(0,T;\mathbb{R}^d)$ *that are solutions of the ODE* $\dot\gamma(t) = v_t(\gamma(t))$ *for* $\mathcal{L}^1$*-a.e.* $t \in (0,T)$ *with* $\gamma(0) = x$;

(ii) $\mu_s \ll \mu_s^\eta$ *for every* s *in* $[0,T]$, *and the measure* μ_s^η *is given by the disintegration theorem, i.e.*

$$\eta = \int_{\mathbb{R}^d} \eta_x^s\,\mathrm{d}\mu_s^\eta(x) \quad \textit{with } \mu_s^\eta = (e_s)_\#\eta; \tag{4.2}$$

(iii) *for every* $t \in [0,T]$, μ_t *satisfies the following equality*:

$$\langle \mu_t, \varphi\rangle = \langle \mu_t^\eta, \varphi\rangle + \int_0^t \bigg(\int_{\mathbb{R}^d}\int_{\Gamma_T} \varphi\big(\gamma(t)\big)\,\mathrm{d}\eta_x^s(\gamma)c_s(x)\,\mathrm{d}\mu_s(x)\bigg)\,\mathrm{d}s \tag{4.3}$$

for any test function $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Conversely, any η *satisfying* (i), (ii) *and*

$$\int_0^T \int_{\mathbb{R}^d \times \Gamma_T} \left(\left|v_t\big(\gamma(t)\big)\right| + \left|c_t\big(\gamma(t)\big)\right|\right) \mathrm{d}\eta(x,\gamma)\,\mathrm{d}t < +\infty, \tag{4.4}$$

induces via (4.3) *a solution of* (3.1), *with* $\mu_0 = \gamma(0)_\#\eta$.

Proof. Let us first prove the *direct implication*.

Step 1 (*Regularization*). Let μ_t, v_t and c_t be given as in the statement of the theorem and let us denote by $\|c\|$ the positive constant such that $-\|c\| \leqslant c \leqslant \|c\|$. Let us apply the regularization Lemma 3.10, finding approximations μ_t^ε, v_t^ε and c_t^ε satisfying the continuity equation and the uniform integrability condition (3.2). Therefore we can apply Proposition 3.6, obtaining the representation formula:

$$\mu_t^\varepsilon = X^\varepsilon(t,0,\cdot)_\#\mu_0^\varepsilon + \int_0^t X^\varepsilon(t,s,\cdot)_\#\big(c_s^\varepsilon \mu_s^\varepsilon\big)\,\mathrm{d}s,$$

where $X^\varepsilon(t,s,x)$ is the maximal solution of

$$\dot{X}^\varepsilon(t,s,x) = v_t^\varepsilon\big(X^\varepsilon(t,s,x)\big), \quad X^\varepsilon(s,s,x) = x.$$

Thinking $X^\varepsilon(t,0,x)$ as a map from $\mathbb{R}^d$ to Γ_T, we can define:

$$\eta^\varepsilon(x,\gamma) := \big(\mathrm{id}(x) \times X^\varepsilon(t,0,x)\big)_\#\mu_0^\varepsilon(x) \in \mathcal{M}_b^+\big(\mathbb{R}^d \times \Gamma_T\big). \tag{4.5}$$

Now we claim that the family η^ε is tight as $\varepsilon \downarrow 0$ and that any limit point η fulfills (i), (ii) and (iii).

Step 2 (*Tightness*). The tightness of the family can be obtained by the *tightness criterion* recalled in the preliminaries. Choosing the maps r^1, r^2 defined in $\mathbb{R}^d \times \Gamma_T$ as

$$r^1 : (x,\gamma) \mapsto x \in \mathbb{R}^d, \qquad r^2 : (x,\gamma) \mapsto \gamma - x \in \Gamma_T,$$

observe that $r = r^1 \times r^2 : \mathbb{R}^d \times \Gamma_T \to \mathbb{R}^d \times \Gamma_T$ is proper. The family $r^1_\#\eta^\varepsilon$ is given by the first marginal μ_0^ε which are tight (indeed, they narrowly converge to μ_0). About $\beta^\varepsilon := r^2_\#\eta^\varepsilon$ we have to distinguish two cases to prove their tightness.

Case $p > 1$. We can write:

$$\begin{aligned}
\int_{\Gamma_T}\int_0^T |\dot\gamma|^p\,\mathrm{d}t\,\mathrm{d}\beta^\varepsilon &= \int_{\mathbb{R}^d}\int_0^T \left|\dot{X}^\varepsilon(t,0,x)\right|^p \mathrm{d}t\,\mathrm{d}\mu_0^\varepsilon(x) \\
&= \int_{\mathbb{R}^d}\int_0^T \left|v_t^\varepsilon\big(X^\varepsilon(t,0,x)\big)\right|^p \mathrm{d}t\,\mathrm{d}\mu_0^\varepsilon(x) \\
&= \int_0^T \int_{\mathbb{R}^d} \frac{|v_t^\varepsilon(x)|^p}{\mathrm{e}^{\int_0^t c^\varepsilon(s, X^\varepsilon(s,t,x))\,\mathrm{d}s}}\,\mathrm{d}\big(X^\varepsilon(t,0,x)_\#\big(\mathrm{e}^{\int_0^t c^\varepsilon(s,X^\varepsilon(s,0,x))\,\mathrm{d}s}\mu_0^\varepsilon(x)\big)\big)\,\mathrm{d}t \\
&\leqslant \mathrm{e}^{T\|c\|}\int_0^T\int_{\mathbb{R}^d} \left|v_t(x)\right|^p \mathrm{d}\mu_t(x)\,\mathrm{d}t < +\infty,
\end{aligned}$$

where in the last inequality we used Proposition 3.6 and Lemma 3.9 with $\psi(\cdot) = |\cdot|^p$. Since for $p > 1$ the functional $\gamma \mapsto \int_0^T |\dot\gamma|^p\,\mathrm{d}t$ (set to $+\infty$ if $\gamma \notin AC^p((0,T);\mathbb{R}^d)$ or $\gamma(0) \neq 0$) has compact sublevels in Γ_T, also β^ε is tight.

Case $p = 1$. When $p = 1$ the argument previously used fails because the functional $\gamma \mapsto \int_0^T |\dot\gamma|\,dt$ is not coercive in Γ_T, anyway this difficulty can be easily dropped.

If $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex superlinear function then the functional $\gamma \mapsto \int_0^T \psi(|\dot\gamma|)\,dt$ is coercive in Γ_T by the Dunford–Pettis theorem (see for instance Proposition 1.27 and Theorem 1.38 in [6]), then, using Lemma 3.9 and calculations analogous to the ones in the case $p > 1$, we have also:

$$\int_{\Gamma_T}\int_0^T \psi\big(|\dot\gamma|\big)\,dt\,d\beta^\varepsilon \leqslant e^{T\|c\|}\int_0^T\int_{\mathbb{R}^d} \psi\big(|v_t|\big)\,d\mu_t\,dt.$$

Now to conclude we have simply to show that if $\int_0^T\int_{\mathbb{R}^d}|v_t|\,d\mu_t\,dt < +\infty$ then there exists a convex superlinear function ψ such that $\int_0^T\int_{\mathbb{R}^d}\psi(|v_t|)\,d\mu_t\,dt < +\infty$, and this follows again by the Dunford–Pettis theorem. Indeed, once we set $\mu := \int_0^T \mu_t\,dt$, we can write $\int_0^T\int_{\mathbb{R}^d}|v_t|\,d\mu_t\,dt = \int_{\mathbb{R}^d}|v|\,d\mu$, and since the family $\mathcal{F} = \{v\}$ is a compact set, by the Dunford–Pettis theorem we deduce the existence of the useful ψ.

Step 3 (*Conditions* (ii) *and* (iii)). Let now η be a narrow limit point of η^ε along some infinitesimal sequence ε_i. Using that $\mu_t^{\eta^{\varepsilon_i}} = (e_t)_\#\eta^{\varepsilon_i}$, the definition of η^{ε_i} in (4.5) and the implicit representation (3.11), we can write:

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi(x)\,d\mu_t^{\eta^{\varepsilon_i}}(x) &= \int_{\mathbb{R}^d\times\Gamma_T}\varphi\big(\gamma(t)\big)\,d\eta^{\varepsilon_i}(x,\gamma) = \int_{\mathbb{R}^d}\varphi\big(X^{\varepsilon_i}(t,0,x)\big)\,d\mu_0^{\varepsilon_i}(x)\\ &= \int_{\mathbb{R}^d}\varphi(x)\,d\mu_t^{\varepsilon_i}(x) - \int_0^t\bigg(\int_{\mathbb{R}^d}\varphi\big(X^{\varepsilon_i}(t,s,x)\big)c_s^{\varepsilon_i}(x)\,d\mu_s^{\varepsilon_i}(x)\bigg)\,ds\end{aligned}$$

for any $\varphi \in C_b^0(\mathbb{R}^d)$. When we let $i \to \infty$ we just obtain (4.3), defining η_x^s and μ_s^η as in (4.2).

Furthermore, from the explicit representation (3.10), we also have:

$$\begin{aligned}\int_{\mathbb{R}^d}\varphi(x)\,d\mu_t^{\varepsilon_i}(x) &= \int_{\mathbb{R}^d}\varphi\big(X^{\varepsilon_i}(t,0,x)\big)e^{\int_0^t c^{\varepsilon_i}(s,X^{\varepsilon_i}(s,0,x))}\,d\mu_0^{\varepsilon_i}(x)\\ &\leqslant e^{T\|c\|}\int_{\mathbb{R}^d}\varphi\big(X^{\varepsilon_i}(t,0,x)\big)\,d\mu_0^{\varepsilon_i}(x) = e^{T\|c\|}\int_{\mathbb{R}^d}\varphi(x)\,d\mu_t^{\eta^{\varepsilon_i}}(x),\end{aligned}$$

that is $\mu_t^{\varepsilon_i} \leqslant C\mu_t^{\eta_{\varepsilon_i}}$, and in the limit for $i \to \infty$ we get $\mu_t \ll \mu_t^\eta$.

Finally we have to check condition (i).

Step 4 (*An auxiliary inequality*). Let $w(t,x) = w_t(x)$ be a bounded uniformly continuous function, and let us prove the estimate:

$$\int_{\mathbb{R}^d\times\Gamma_T}\bigg|\gamma(t) - x - \int_0^t w_\tau\big(\gamma(\tau)\big)\,d\tau\bigg|^p\,d\eta(x,\gamma) \leqslant e^{T\|c\|}\int_0^T\int_{\mathbb{R}^d}|v_\tau - w_\tau|^p\,d\mu_\tau\,d\tau. \tag{4.6}$$

Indeed we have:

$$\begin{aligned}&\int_{\mathbb{R}^d\times\Gamma_T}\bigg|\gamma(t) - x - \int_0^t w_\tau\big(\gamma(\tau)\big)\,d\tau\bigg|^p\,d\eta^\varepsilon(x,\gamma)\\ &\quad= \int_{\mathbb{R}^d}\bigg|X^\varepsilon(t,0,x) - x - \int_0^t w_\tau\big(X^\varepsilon(\tau,0,x)\big)\,d\tau\bigg|^p\,d\mu_0^\varepsilon(x)\\ &\quad= \int_{\mathbb{R}^d}\bigg|\int_0^t\big(v_\tau^\varepsilon - w_\tau\big)\big(X^\varepsilon(\tau,0,x)\big)\,d\tau\bigg|^p\,d\mu_0^\varepsilon(x)\end{aligned}$$

$$\leqslant e^{T\|c\|}\int_0^t\int_{\mathbb{R}^d}\left|v_\tau^\varepsilon-w_\tau\right|^p d\mu_\tau^\varepsilon\,d\tau$$
$$\leqslant e^{T\|c\|}\int_0^t\int_{\mathbb{R}^d}\left|v_\tau^\varepsilon-w_\tau^\varepsilon\right|^p d\mu_\tau^\varepsilon\,d\tau+e^{T\|c\|}\int_0^t\int_{\mathbb{R}^d}\left|w_\tau^\varepsilon-w_\tau\right|^p d\mu_\tau^\varepsilon\,d\tau$$
$$\leqslant e^{T\|c\|}\int_0^T\int_{\mathbb{R}^d}\left|v_\tau-w_\tau\right|^p d\mu_\tau\,d\tau+e^{T\|c\|}\int_0^T\sup_{x\in\mathbb{R}^d}\left|w_\tau^\varepsilon(x)-w_\tau(x)\right|^p d\tau,$$

where in the last two inequalities we have added and subtracted $w_\tau^\varepsilon := w_\tau * \rho_\varepsilon$ and then used Proposition 3.6 and Lemma 3.9 with $\psi(\cdot)=|\cdot|^p$. Setting $\varepsilon=\varepsilon_i$ and passing to the limit as $i\to\infty$ we recover (4.6), since the function under integral is a continuous and non-negative test function in $\mathbb{R}^d\times\Gamma_T$.

Step 5 (*Condition* (i)). Now let $\mu := \int_0^T \mu_t\,d\mathscr{L}^1(t)$ be the Borel measure on $\mathbb{R}^d\times\Gamma_T$ whose disintegration with respect to $\mathscr{L}^1$ is $\{\mu_t\}_{t\in(0,T)}$, and let $w^n\in C_c^0((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ be continuous functions with compact support converging to v in $L^p(\mu;\mathbb{R}^d)$. Using the representation given in point (ii), we have:

$$\int_{\mathbb{R}^d\times\Gamma_T}\int_0^T\left|w_\tau^n\big(\gamma(\tau)\big)-v_\tau\big(\gamma(\tau)\big)\right|^p d\tau\,d\eta$$
$$=\int_0^T\int_{\mathbb{R}^d}\left|w_\tau^n-v_\tau\right|^p d\mu_\tau^\eta\,d\tau=\int_0^T\int_{\mathbb{R}^d}\left|w_\tau^n-v_\tau\right|^p d\mu_\tau\,d\tau$$
$$-\int_0^T\left(\int_0^t\int_{\mathbb{R}^d}\int_{\Gamma_T}\left|w_\tau^n\big(\gamma(t)\big)-v_\tau\big(\gamma(t)\big)\right|^p d\eta_x^s(\gamma)c_s(x)\,d\mu_s(x)\,ds\right)d\tau.$$

Observe that as $n\to\infty$ both terms in the right-hand side of the previous equality go to zero from the hypothesis on w^n (for the second term we use the dominated convergence theorem, once we recall that $\mu_s\ll\mu_s^\eta$ for any $s\in[0,T]$).

So, using the triangular inequality in $L^p(\eta)$, (4.6) with $w=w^n$ and the previous equality we obtain $\forall t\in[0,T]$:

$$\int_{\mathbb{R}^d\times\Gamma_T}\left|\gamma(t)-x-\int_0^t v_\tau\big(\gamma(t)\big)\,d\tau\right|^p d\eta(x,\gamma)$$
$$\leqslant\int_{\mathbb{R}^d\times\Gamma_T}\left|\gamma(t)-x-\int_0^t w_\tau^n\big(\gamma(\tau)\big)\,d\tau\right|^p d\eta(x,\gamma)$$
$$+\int_{\mathbb{R}^d\times\Gamma_T}\int_0^T\left|w_\tau^n\big(\gamma(\tau)\big)-v_\tau\big(\gamma(\tau)\big)\right|^p d\tau\,d\eta(x,\gamma),$$

and when we let $n\to\infty$,

$$\int_{\mathbb{R}^d\times\Gamma_T}\left|\gamma(t)-x-\int_0^t v_\tau\big(\gamma(\tau)\big)\,d\tau\right|^p d\eta(x,\gamma)=0\quad\forall t\in[0,T].$$

Therefore,

$$\gamma(t)-x-\int_0^t v_\tau\big(\gamma(\tau)\big)\,d\tau=0\quad\text{for }\eta\text{-a.e. }(x,\gamma),$$

for any $t \in [0, T]$. Choosing all t's in $(0, T) \cap \mathbb{Q}$ we obtain an exceptional η-negligible set that does not depend on t and use the continuity of γ to show that the identity is fulfilled for any $t \in [0, T]$.

Let us now consider the *converse implication*, whose proof is much simpler.

Notice that, due to assumption (i), the set F of all (t, x, γ) such that either $\dot\gamma(t)$ does not exist or it is different from $v_t(\gamma(t))$ is $\mathscr{L}^1 \times \eta$-negligible. As a consequence, we have:

$$\dot\gamma(t) = v_t\big(\gamma(t)\big) \quad \eta\text{-a.e., for } \mathscr{L}^1\text{-a.e. } t \in [0, T].$$

To achieve the thesis we have to prove the following facts:

(1) the map $t \mapsto \mu_t$ is narrowly continuous;
(2) $\forall \zeta \in C^1(\mathbb{R}^d)$ bounded with a bounded gradient, the map $t \mapsto \int \zeta \, \mathrm{d}\mu_t$ is absolutely continuous;
(3) the pointwise equality,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \zeta \, \mathrm{d}\mu_t = \int_{\mathbb{R}^d} \big(\langle v_t, \nabla\zeta\rangle + c_t\zeta\big) \, \mathrm{d}\mu_t,$$

holds for any ζ given as in (2).

With a straightforward calculation, since (4.4) holds, we can check that $t \mapsto \mu_t$ is narrowly continuous, so we get (1). It is also easy to check (2), indeed for $s < t$ in $(0, T)$ we have:

$$\left| \int_{\mathbb{R}^d} \zeta \, \mathrm{d}\mu_s - \int_{\mathbb{R}^d} \zeta \, \mathrm{d}\mu_t \right| \leqslant \|\nabla\zeta\|_\infty \int_s^t \int_{\mathbb{R}^d \times \Gamma_T} \big| v_\tau\big(\gamma(\tau)\big)\big| \, \mathrm{d}\eta \, \mathrm{d}\tau + 2\|\zeta\|_\infty \int_s^t \int_{\mathbb{R}^d} \eta_x^\tau(\Gamma_T) \big| c_\tau(x)\big| \, \mathrm{d}\mu_\tau(x) \, \mathrm{d}\tau,$$

where we have used that $\dot\gamma(\tau) = v_\tau(\gamma(\tau))$. Now, taking into account that $\mu_s \ll \mu_s^\eta$ and that for any $\tau \in [0, T]$ η_x^τ is a probability measure in Γ_T for μ_τ^η-a.e. $x \in \mathbb{R}^d$, we get (2).

Finally, let us show (3). Using Leibniz's formula for differentiating integral functions we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \zeta(x) \, \mathrm{d}\mu_t(x) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \zeta(x) \, \mathrm{d}\mu_t^\eta(x) + \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \int_{\mathbb{R}^d} \int_{\Gamma_T} \zeta\big(\gamma(t)\big) \, \mathrm{d}\eta_x^s(\gamma) c_s(x) \, \mathrm{d}\mu_s(x) \, \mathrm{d}s \\
&= \int_{\mathbb{R}^d \times \Gamma_T} \big\langle \nabla_x \zeta\big(\gamma(t)\big), v_t\big(\gamma(t)\big)\big\rangle \, \mathrm{d}\eta(x, \gamma) \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \int_{\Gamma_T} \big\langle \nabla_x \zeta\big(\gamma(t)\big), v_t\big(\gamma(t)\big)\big\rangle \, \mathrm{d}\eta_x^s(\gamma) c_s(x) \, \mathrm{d}\mu_s(x) \, \mathrm{d}s \\
&\quad + \int_{\mathbb{R}^d} \int_{\Gamma_T} \zeta\big(\gamma(t)\big) \, \mathrm{d}\eta_x^t(\gamma) c_t(x) \, \mathrm{d}\mu_t(x) \\
&= \int_{\mathbb{R}^d} \big\langle \nabla_x \zeta(x), v_t(x)\big\rangle \, \mathrm{d}\mu_t(x) + \int_{\mathbb{R}^d} \zeta(x) \bigg(\int_{\Gamma_T} \mathrm{d}\eta_x^t(\gamma) \bigg) c_t(x) \, \mathrm{d}\mu_t(x),
\end{aligned}$$

then using again that $\mu_t \ll \mu_t^\eta$ and that $\eta_x^t(\Gamma_T) = 1$ for μ_t^η-a.e. x, we get (3). $\square$

Remark 4.2. Observe that in the case when $D_x \cdot v_t(x) \ll \mathscr{L}^d$ with a density $\operatorname{div} v_t(x) \in L^\infty((0, T) \times \mathbb{R}^d)$, taking $c_t(x) := -\operatorname{div} v_t(x)$, Eq. (3.1) becomes,

$$\partial_t \mu_t(x) + v_t(x) \cdot \nabla_x \mu_t(x) = 0, \tag{4.7}$$

and then Theorem 4.1 provides a probabilistic representation for positive measure-valued solutions of (4.7) of the form $\mu_t = \omega_t \mathscr{L}^d$, with $\omega \in L^1([0, T]; L^1(\mathbb{R}^d))$. So, at this point, it would be interesting to understand to what extent a probabilistic representation for this kind solutions of (4.7) is still valid if we relax our hypothesis on the vector field v, i.e. if $\operatorname{div}_x v_t(x) \notin L^\infty((0,T)\times\mathbb{R}^d)$ (and notice that it is necessary to have $D_x \cdot v_t(x) \ll \mathscr{L}^d$ to define the distribution $v_t(x)\cdot\nabla_x\mu_t(x)$), but this is still an open problem.

In the remaining part of this section we give a generalization of Theorem 4.1 (that we will apply in the proof of Theorem 5.9) to measure-valued solutions of the transport equation that are only *locally finite* on $\mathbb{R}^d$, but under the additional assumption of boundedness on v.

Given a compact set $K\subset\mathbb{R}^d$, we denote by $\Gamma_T^K\subset\Gamma_T$ the set:

$$\Gamma_T^K := \big\{\gamma\in\Gamma_T\colon\ \gamma(0)\in K\big\}.$$

Theorem 4.3. *Let $\mu_t\in L^\infty([0,T];\mathcal{M}^+(\mathbb{R}^d))$ be a locally narrowly continuous solution of the continuity equation*:

$$\partial_t\mu_t + D_x\cdot(v_t\mu_t) = c_t\mu_t \quad \text{in } (0,T)\times\mathbb{R}^d, \tag{4.8}$$

for a suitable Borel vector field $v_t(x)=v(t,x)\in L^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ and a Borel bounded scalar function $c_t(x)=c(t,x)$. Then there exists a positive measure η in $\mathbb{R}^d\times\Gamma_T$ that is finite on $K\times\Gamma_T^K$ for any compact $K\subset\mathbb{R}^d$ and such that

(i) *η is concentrated on the set of $\gamma\in AC^p(0,T;\mathbb{R}^d)$ that are solutions of the ODE $\dot\gamma(t)=v_t(\gamma(t))$ for $\mathscr{L}^1$-a.e. $t\in(0,T)$ with $\gamma(0)=x$;*

(ii) *$\mu_s\ll\mu_s^\eta$ for every s in $[0,T]$, and the measure μ_s^η is given by the disintegration theorem, i.e.*

$$\eta=\int_{\mathbb{R}^d}\eta_x^s\,\mathrm{d}\mu_s^\eta(x)\quad \text{with } \mu_s^\eta=(e_s)_\#\eta; \tag{4.9}$$

(iii) *for every $t\in[0,T]$, μ_t satisfies the following equality,*

$$\langle\mu_t,\varphi\rangle=\langle\mu_t^\eta,\varphi\rangle+\int_0^t\bigg(\int_{\mathbb{R}^d}\int_{\Gamma_T}\varphi\big(\gamma(t)\big)\,\mathrm{d}\eta_x^s(\gamma)c_s(x)\,\mathrm{d}\mu_s(x)\bigg)\,\mathrm{d}s, \tag{4.10}$$

for any test function $\varphi\in C_c^\infty(\mathbb{R}^d)$.

Conversely, any η satisfying (i), (ii) *and*

$$\int_0^T\int_{K\times\Gamma_T^K}\big(\big|v_t\big(\gamma(t)\big)\big|+\big|c_t\big(\gamma(t)\big)\big|\big)\,\mathrm{d}\eta(x,\gamma)\,\mathrm{d}t<+\infty, \tag{4.11}$$

for any compact set $K\subset\mathbb{R}^d$, induces via (4.10) *a solution of* (4.8)*, with $\mu_0=\gamma(0)_\#\eta$.*

We do not write down the proof, which develops along the same lines as in Theorem 4.1 with the obvious modifications, but observe that a suitable extension of Lemma 3.10 is necessary. We have to choose in the appropriate way the kernel ρ which produces the mollifiers ρ_ε, because in this case μ_t is *only locally finite*. First of all observe that we need a kernel $\rho>0$ so that $\mu_t*\rho_\varepsilon\neq 0$, because in the proof we define:

$$v_t^\varepsilon := \frac{(v_t\mu_t)*\rho_\varepsilon}{\mu_t*\rho_\varepsilon},\qquad c_t^\varepsilon := \frac{(c_t\mu_t)*\rho_\varepsilon}{\mu_t*\rho_\varepsilon},$$

furthermore we need a regularizing kernel ($\rho\in C^2(\mathbb{R}^d)$ is enough) because in the proof we apply Proposition 3.6 to $\mu_t^\varepsilon:=\mu_t*\rho_\varepsilon$. Let us spend a few lines in the description of such a ρ. Since we are supposing $v\in L^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ and $\mu\in L^\infty([0,T];\mathcal{M}^+(\mathbb{R}^d))$, we can apply Lemma 2.11 of [4] so to deduce for any $x\in\mathbb{R}^d$ and any $R\in\mathbb{R}$,

$$\mu_t\big(B_R(x)\big)\leqslant\mu_0\big(B_{R+Ct}(x)\big)\quad \text{for a fixed } t\in[0,T],\ C=\|v\|_{L^\infty((0,T)\times\mathbb{R}^d)},$$

and then the local upper bound:

$$\sup_{t\in[0,T]} \mu_t\big(B_R(x)\big) \leqslant \mu_0\big(B_{R+CT}(x)\big) := m_R(x).$$

Fix now a function $\phi \in C^2(\mathbb{R}^d)$ such that $\phi > 0$ in B_1 and $\phi \equiv 0$ in $\mathbb{R}^d \setminus B_1$, and take a countable covering of $\mathbb{R}^d$ with balls of radius 1 fixed and centres $\{x_N\}_{N\in\mathbb{N}}$. Set $\rho_N(x) := \phi(x - x_N)$ and define $\rho := \sum_{N\in\mathbb{N}} c_N \rho_N$, where c_N are chosen so that

$$\sum_{N=1}^{\infty} c_N \int_{\mathbb{R}^d} \rho_N(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1,$$

and

$$\|\rho\|_{C^2(\mathbb{R}^d)} = \sum_{N=1}^{\infty} m_1(x_N) c_N \|\rho_n\|_{C^2} = \|\phi\|_{C^2} \sum_{N=1}^{\infty} m_1(x_N) c_N < \infty.$$

Let us also observe that in this framework in which we handle locally-finite measures we have to replace the notions of tightness and narrow convergence in $\mathbb{R}^d \times \Gamma_T$ respectively with tightness in $K \times \Gamma_T^K$ for any compact set $K \subset \mathbb{R}^d$, and local narrow convergence, where local means that in the definition of convergence (see Section 2) we consider test-functions whose support is contained in $K \times \Gamma_T$, where $K \subset \mathbb{R}^d$ is compact.

5. Applications

In this section we intend to provide some applications for the theory of *probabilistic solutions* (see definition below) developed in the previous part of the paper, and in particular we want to see how we can apply it to deduce uniqueness results for solutions of the transport equation from uniqueness of the characteristics of v.

Let us give first a precise definition of what we intend for "probabilistic solution".

Definition 5.1 *(Probabilistic solutions).* Let v be a Borel vector field and c a Borel bounded scalar function defined in $(0,T) \times \mathbb{R}^d$, and let $\mu = \{\mu_t\}_{t\in[0,T]}$ be a family of measures with finite total variation that is a distributional solution of:

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = c_t \mu_t \quad \text{in } (0,T) \times \mathbb{R}^d. \tag{5.1}$$

Then, we say that μ is a *probabilistic solution* of (5.1) if there exists a measure $\eta \in \mathcal{M}(\Gamma_T \times \mathbb{R}^d)$ such that

(i) η is representable as $\eta = \int_{\mathbb{R}^d} \eta_x \,\mathrm{d}\mu_0(x)$, with $\mu_0 = (e_0)_\# \eta$ and $\eta_x \geqslant 0$ for μ_0-a.e. x;
(ii) μ_t can be represented in terms of η in the sense of Theorem 4.1.

Observe that, in the previous definition, in condition (i) the measures η_x are obtained from the measure η through disintegration w.r.t. μ_0, and the fundamental requirement is that η_x must be positive for μ_0-a.e. x, while a priori no hypothesis on the sign of μ_0 is made. In this sense with Definition 5.1 we are giving an extension to "possibly signed" measures μ of the notion of probabilistic solutions of (5.1), implicitly given for positive μ in Theorem 4.1.

For clarify, let us rewrite the direct implication of Theorem 4.1 in terms of Definition 5.1:

Theorem 5.2. *Let $\mu_t : [0,T] \to \mathcal{M}_b^+(\mathbb{R}^d)$ be a narrowly continuous solution of the continuity equation,*

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = c_t \mu_t \quad \textit{in } (0,T) \times \mathbb{R}^d, \tag{5.2}$$

for a suitable Borel vector field $v_t(x) = v(t,x)$ and a Borel bounded scalar function $c_t(x) = c(t,x)$ satisfying:

$$\int_0^T \int_{\mathbb{R}^d} \big|v_t(x)\big|^p \,\mathrm{d}\mu_t(x)\,\mathrm{d}t < +\infty \quad \textit{for some } p \geqslant 1.$$

Then μ is a probabilistic solution of (5.2).

Now as a first step we want to show the relationship between renormalized solutions and probabilistic solutions. The theory of renormalized solutions has been first developed by Di Perna and Lions for vector fields with Sobolev regularity, and only recently improved by Ambrosio for BV-vector fields, so we refer the reader to [19] and [3] and the bibliography contained therein for a deep study of renormalized solutions, and we only recall the definition.

The definition below is not verbatim the definition of renormalized solutions that is given in [19] and [3], but the reader can easily verify that it is just the same one noticing that, under the hypotheses of Definition 5.3, the distribution $v \cdot \nabla_x \mu$ (that is present in the definition given by Di Perna–Lions) is well defined as $D_x \cdot (v\mu) - \mu \operatorname{div}_x v$.

Definition 5.3 *(Renormalized solutions).* Let $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field such that $D_x \cdot v_t = \operatorname{div} v_t \mathcal{L}^d$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$, with $\operatorname{div} v \in L^1_{\mathrm{loc}}((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$. Let $\mu \in L^\infty((0,T); L^\infty(\mathbb{R}^d))$ such that $\int_0^T |v_t||\mu_t| \, \mathrm{d}t < +\infty$ and assume that $\partial_t \mu + D_x \cdot (v\mu) \in L^1_{\mathrm{loc}}((0,T) \times \mathbb{R}^d)$. Then, we say that μ is a *renormalized solution* of (5.1) if

$$\partial_t \beta(\mu_t) + D_x \cdot \big(\beta(\mu_t) v_t\big) = -L\beta(\mu_t) D_x \cdot v_t + \beta'(\mu_t) c_t \mu_t \quad \forall \beta \in C^1(\mathbb{R}), \tag{5.3}$$

with $L\beta(s) = s\beta'(s) - \beta(s)$.

It has to be noticed that the definition of renormalized solution makes sense also when μ is not a distributional solution, i.e. when μ is not locally summable, and in this case the notion of probabilistic solution has no meaning. In the case when both notions make sense the next proposition points out a simple relationship between the two kind of solutions: we consider the equation,

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \tag{5.4}$$

and we are able to show that any distributional renormalized solution of (5.4) is also a probabilistic one. The contrary is false in general (as we will discuss in Remark 5.5), and an interesting open problem is to understand under which conditions in the class of distributional solutions of (5.4) we have the coincidence between the sets of renormalized and probabilistic solutions.

Proposition 5.4. *Let v be a vector field that satisfies the hypotheses of Definition* 5.3 *and of Theorem* 4.1. *Then any distributional renormalized solution ω of* (5.4) *is also a probabilistic one* (*to be precise, $\mu = \omega \mathcal{L}^d$ is a probabilistic solution, according to Definition* 5.1).

Proof. Let v be as in the statement, and take ω a renormalized distributional solution of (5.4). Then we can write $\omega = \omega^+ - \omega^-$, where ω^+ and ω^- are respectively the positive and the negative part of ω. Let us define $\beta_\varepsilon(s) := \sqrt{\varepsilon^2 + (s^+)^2} - \varepsilon$, and observe that for any $\varepsilon > 0$,

$$\beta_\varepsilon(s) \in C^1(\mathbb{R}), \quad \beta'_\varepsilon(s) \in [0,1] \quad \text{and} \quad L\beta_\varepsilon(s) \in [0,\varepsilon].$$

Furthermore pointwise $\beta_\varepsilon(s) \to s^+$ as $\varepsilon \to 0$, so if we pass to the limit as $\varepsilon \to 0$ in (5.3) with β_ε instead of β, we can deduce that ω^+ is a renormalized (and then $\mu^+ = \omega^+ \mathcal{L}^d$ a distributional) solution of (5.4), and the same holds for ω^- (taking $\beta_\varepsilon(s) := \sqrt{\varepsilon^2 + (s^-)^2} - \varepsilon$ and applying the same argument used for ω^+). From their definition $\mu^+ = \omega^+ \mathcal{L}^d$ and $\mu^- = \omega^- \mathcal{L}^d$ are positive measure-valued solutions, so we can apply Theorem 4.1 to each one of them to find η^+ and η^- such that for any test function φ,

$$\langle \mu_t^+, \varphi \rangle = \langle (\mu^+)_t^\eta, \varphi \rangle \quad \text{and} \quad \langle \mu_t^-, \varphi \rangle = \langle (\mu^-)_t^\eta, \varphi \rangle. \tag{5.5}$$

Using the disintegration theorem we can write,

$$\eta^+ = \int_{\mathbb{R}^d} \eta_x^+ \omega_0^+(x) \, \mathrm{d}x \quad \text{and} \quad \eta^- = \int_{\mathbb{R}^d} \eta_x^- \omega_0^-(x) \, \mathrm{d}x,$$

so, if we define,

$$\eta := \int_{\mathbb{R}^d} \big(\eta_x^+ \chi_{\{\omega_0^+ \geqslant 0\}}(x) + \eta_x^- \chi_{\{\omega_0^- > 0\}}(x)\big) \, \mathrm{d}\mathcal{L}^d(x), \tag{5.6}$$

we can immediately realize that $\eta \geqslant 0$ and furthermore,

$$
\begin{aligned}
\int_{\mathbb{R}^d} \mu_t \varphi \, \mathrm{d}\mathcal{L}^d(x) &= \int_{\mathbb{R}^d} \mu_t^+ \varphi \, \mathrm{d}\mathcal{L}^d(x) - \int_{\mathbb{R}^d} \mu_t^- \varphi \, \mathrm{d}\mathcal{L}^d(x) \\
&= \int_{\mathbb{R}^d} \int_{\Gamma_T} \varphi\big(\gamma(t)\big) \, \mathrm{d}\big(\eta_x^+ \omega_0(x) - \eta_x^- \omega_0(x)\big) \, \mathrm{d}\mathcal{L}^d(x) \\
&= \big\langle \mu_t^\eta, \varphi \big\rangle.
\end{aligned}
$$

The previous chain of equalities simply tells us that μ is a probabilistic solution that can be represented by the measure η given by (5.6). □

Observe that for the non-homogeneous equation the argument used in the proof of the previous proposition does not lead to the same conclusion.

Remark 5.5. Let us see now, with a simple argument, why we cannot expect that the contrary implication in Proposition 5.4 is true, i.e. why is false that every probabilistic solution of (5.4) is also a renormalized one. Indeed, suppose for simplicity $D_x \cdot v = 0$, and take a solution μ bounded from below. Then $\mu + k$ is still a solution for every constant k, furthermore for k sufficiently large $\mu + k$ is positive and then it has a probabilistic representation. If we suppose, by contradiction, that any probabilistic solution is also a renormalized one, we can immediately deduce that $\mu + k$ and then μ is a renormalized solution of our equation, while it is not generally true that any solution bounded from below, even for a vector field with null divergence, has the renormalization property (see for instance the counterexample built by Depauw in [18]).

In the following two theorems we will illustrate how to deduce uniqueness results for some classes of solutions ω of the initial value problem for the homogeneous transport equation from the uniqueness of the characteristics of v. When supposing the uniqueness of the characteristic curves associated to v we will always mean that there exists a vector field $\tilde{v}$ $\mathcal{L}^d$-equivalent to v, i.e. satisfying $\mathcal{L}^d(\{\tilde{v}_t(x) \neq v_t(x)\}) = 0$, for $\mathcal{L}^1$-a.e. $t \in (0, T)$ and such that the characteristic curves associated to $\tilde{v}$ are unique. Since we will consider distributional solutions, taking $\tilde{v}$ instead of v in (3.15) does not affect the value of the solutions, but allows us to deduce uniqueness results for ω using the theory developed in the previous section.

Theorem 5.6. *Let $v : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field satisfying the following conditions*:

(i) *$D_x \cdot v_t = \operatorname{div}_x v_t \mathcal{L}^d$ for $\mathcal{L}^1$-a.e. $t \in (0, T)$ with $\operatorname{div}_x v \in L^1_{\mathrm{loc}}((0, T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$,*
(ii) *there exists a vector field $\tilde{v}$ for which $\mathcal{L}^d(\{\tilde{v}_t(x) \neq v_t(x)\}) = 0$ for $\mathcal{L}^1$-a.e. $t \in (0, T)$, and such that for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$, for all s there exists* at most one *characteristic curve $X(t, s, x)$ in $[0, T]$ associated to $\tilde{v}$.*

Then, for any initial datum $\omega_0 \in L^\infty(\mathbb{R}^d)$, there exists at most one *$\omega \in L^\infty([0, T]; (L^1 \cap L^\infty)(\mathbb{R}^d))$ distributional renormalized solution of,*

$$
\partial_t \omega_t + D_x \cdot (v_t \omega_t) = 0 \quad \text{in } (0, T) \times \mathbb{R}^d, \quad \omega(0, \cdot) = \omega_0, \tag{5.7}
$$

and satisfying

$$
\int_0^T \int_{\mathbb{R}^d} \big|v_t(x)\big|^p \big|\omega_t(x)\big| \, \mathrm{d}x \, \mathrm{d}t < +\infty \quad \text{for some } p \geqslant 1.
$$

Proof. Let v and ω be as in the statement of the theorem. We want to prove that ω, if exists, is unique.

By the proof of Proposition 5.4 we know that also ω^+ and ω^- are renormalized solutions of (5.7). If $\eta^+ \in \mathcal{M}_b^+(\mathbb{R}^d \times \Gamma_T)$ is the measure associated to $\mu^+ = \omega^+ \mathcal{L}^d$ by Theorem 4.1, by hypothesis (ii), we know that for $s \in (0, T)$ holds $(\eta^+)_x^s = \delta_{X(\cdot, s, x)}$, where $X(\cdot, s, x)$ is the unique solution of:

$$
\dot{X}(t, s, x) = v_t\big(X(t, s, x)\big) \quad \text{in } (0, T), \quad X(s, s, x) = x.
$$

So (4.3) for μ^+ reduces to

$$\mu_t^+ = (e_t)_\# \eta^+,$$

and this gives uniqueness for μ^+ once we fix the initial value $\mu_0^+ = \omega_0^+ \mathcal{L}^d$.

The same argument applied to $\mu^- = \omega^- \mathcal{L}^d$ gives uniqueness for μ^-. So at the end we have also uniqueness for $\mu = \mu^+ - \mu^-$, and consequently for ω. $\square$

Remark 5.7 *(An open problem).* A well known condition on v that guarantees existence and uniqueness of the characteristics of v is the Osgood condition (see for instance [21] and [15]), and *an interesting open problem is to understand whether the Osgood condition implies the renormalization property*, because this may lead to existence results under the assumptions of Theorem 5.6.

In the next theorem we exhibit an existence and uniqueness result for the solutions of the Cauchy problem for the homogeneous transport equation in divergence form in the class of *possibly signed* measure-valued solutions absolutely continuous with respect to the Lebesgue measure. So to simplify the proof of this result we first prove an auxiliary proposition in which we localize the problem, but this localization argument does not play a relevant role from the conceptual point of view, indeed immediately after in Theorem 5.9 we explain how to extend the result to the whole $\mathbb{R}^d$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a C^1 boundary. We look for distributional measure-valued solutions $\mu = \omega \mathcal{L}^d$ of,

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = 0 \quad \text{in } (0,T) \times \Omega, \qquad \mu(0,\cdot) = \mu_0, \tag{5.8}$$

where $\mu_0 = \omega_0 \mathcal{L}^d$ with $\omega_0 \in L^\infty(\Omega)$ under the additional condition that $v_t \omega_t$ has no flow on the boundary $\partial\Omega$.

Observe that we can think both the vector field v_t and the solutions ω_t as defined in the whole space but concentrated on $\bar{\Omega}$. So, we may apply the global results of Theorem 4.1 in the proof of the next theorem, and we are requiring the null-flow condition essentially because we want the distributional divergence of the vector field v_t, for a fixed time t, to be a measure absolutely continuous with respect to the Lebesgue measure and without any contribution arising from boundary components. In fact, the Gauss–Green formula for a Lipschitz vector field B on a bounded domain Ω with C^1-boundary says that $\int_\Omega \Phi \, \mathrm{d}(D_x \cdot B) = \int_\Omega \langle B, \nabla \Phi \rangle \, \mathrm{d}\mathcal{L}^d + \int_{\partial\Omega} \Phi (B \cdot \nu) \, \mathrm{d}\mathcal{H}^{d-1}$ for any test function Φ, where ν is the outer normal on $\partial\Omega$ and $\mathcal{H}^{d-1}$ is the Hausdorff measure on $\partial\Omega$.

Proposition 5.8. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a C^1 boundary. Let $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel bounded vector field satisfying the following conditions*:

(i) $v(t,x) = 0$ *when* $x \notin \Omega$;
(ii) $D_x \cdot v_t = \operatorname{div}_x v_t \mathcal{L}^d$ *for* $\mathcal{L}^1$*-a.e.* $t \in (0,T)$ *with* $\operatorname{div}_x v \in L^\infty((0,T) \times \mathbb{R}^d)$;
(iii) *there exists a vector field $\tilde{v}$ for which $\mathcal{L}^d(\{\tilde{v}_t(x) \neq v_t(x)\}) = 0$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$, and such that for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$, for all s there exists* at most one *characteristic curve $X(t,s,x)$ in* $[0,T]$ *associated to $\tilde{v}$.*

Then for any $\omega_0 \in L^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \omega_0 \subseteq \Omega$, *there exists* at most one $\omega \in L^\infty([0,T]; L^\infty(\Omega))$ *such that $\mu = \omega \mathcal{L}^d$ is a distributional solution of* (5.8) *and $v_t \omega_t$ has no flow on the boundary $\partial\Omega$.*

Proof. Let $\mu = \omega \mathcal{L}^d$ with $\omega \in L^\infty([0,T]; L^\infty(\Omega))$ be a distributional solution of the initial value problem (5.8) such that $v_t \omega_t$ has no flow on the boundary $\partial\Omega$, then we have to prove that under the assumptions above ω is unique.

We can think that all the measures involved in our arguing are defined on the whole space but concentrated on $\bar{\Omega}$, so we can apply the global results of the previous sections, and in particular Theorem 4.1.

We can write $\omega \mathcal{L}^d = \nu_1 - \nu_2$, where

$$\nu_1 := \frac{1}{2}\left(\omega \mathcal{L}^d + \frac{\omega_\infty}{\delta}\tau\right) \quad \text{and} \quad \nu_2 := \frac{1}{2}\left(\frac{\omega_\infty}{\delta}\tau - \omega \mathcal{L}^d\right),$$

$$\omega_\infty := \|\omega\|_{L^\infty}, \qquad \delta := \mathrm{e}^{-\int_0^T \|[\operatorname{div} v_s]^+\|_\infty \, \mathrm{d}s}$$

and $\tau \ll \mathcal{L}^d$ is a solution of the transport equation such that its density $\bar{\tau} \in L^\infty([0,T]; L^1(\Omega))$ and $\bar{\tau} \geqslant \delta > 0$. The existence of such a solution τ in the case of a Lipschitz vector field v is given by formula (3.14), where we take $\rho(x) \equiv 1$ (see Remark 3.8), and observe that from the same formula (3.14) we get the following estimate on τ:

$$\mathrm{e}^{-\int_0^t \|[\operatorname{div} v_s]^+\|_\infty \,\mathrm{d}s} \leqslant \bar{\tau}_t \leqslant \mathrm{e}^{\int_0^t \|[\operatorname{div} v_s]^-\|_\infty \,\mathrm{d}s}. \tag{5.9}$$

In our case we can find a suitable τ as a weak limit in the following way: we take a sequence $(z_t^n)_{n\in\mathbb{N}}$ of locally Lipschitz (in the space variable) vector fields such that z_t^n strongly converges to $\tilde{v}_t$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$ as $n \to +\infty$. Then we consider an increasing sequence of open sets $\Omega_n \subset \Omega$ such that $\mathcal{L}^d(\Omega \setminus \Omega_n) \to 0$ when $n \to +\infty$, and a sequence of Lipschitz functions Ψ_n such that $\Psi_n \equiv 1$ on $\bar{\Omega}_n$ and $\Psi_n \equiv 0$ on $\partial\Omega \cup (\mathbb{R}^n \setminus \Omega)$. It comes out that for any $n \in \mathbb{N}$ $\Psi_n z_n$ is a Lipschitz vector field equal to zero on $\partial\Omega$, and furthermore the sequence $\Psi_n z_n$ still strongly converges to $\tilde{v}_t$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$ as $n \to +\infty$.

We denote by τ^n the function given by (3.14) where $\rho(x) \equiv 1$ and with X^n (the unique characteristic curve associated to $v^n = \Psi_n z_n$) instead of X. Finally we choose τ to be the weak limit in the sense of distributions of the sequence (τ^n) as $n \to +\infty$.

Observe that ν_1 and ν_2 are positive measure-valued solutions of the equation, so the uniqueness of the characteristics implies (through the probabilistic representation, as we detailed discussed in the proof of Theorem 5.6) separately uniqueness results for both ν_1 and ν_2 with the corresponding initial conditions, and then uniqueness for ω with initial condition ω_0. □

The results of Proposition 5.8 can be extended to $\mathbb{R}^d$ under the same assumptions on v, that is we can prove the following theorem:

Theorem 5.9. *Let $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel bounded vector field satisfying the following conditions*:

(i) *$D_x \cdot v_t = \operatorname{div}_x v_t \mathcal{L}^d$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$ with $\operatorname{div}_x v \in L^\infty((0,T) \times \mathbb{R}^d)$,*
(ii) *there exists a vector field $\tilde{v}$ for which $\mathcal{L}^d(\{\tilde{v}_t(x) \neq v_t(x)\}) = 0$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$, and such that for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$, for all s there exists* at most one *characteristic curve $X(t,s,x)$ in $[0,T]$ associated to $\tilde{v}$.*

Then for any $\omega_0 \in L^\infty(\mathbb{R}^d)$ there exists at most one *$\omega \in L^\infty([0,T]; L^\infty(\mathbb{R}^d))$ such that $\mu = \omega\mathcal{L}^d$ is a distributional solution of*:

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = 0 \quad \text{in } (0,T) \times \mathbb{R}^d, \quad \mu(0,\cdot) = \mu_0. \tag{5.10}$$

The proof is just the same as the previous proposition, but we have to take care of the fact that a density $w \in L^\infty((0,T); L^\infty(\mathbb{R}^d))$ gives in $\mathbb{R}^d$ for a fixed time t a σ-finite measures $w_t\mathcal{L}^d$, then in this case in the proof we will apply Theorem 4.3 instead of Theorem 4.1.

For completeness let us write a uniqueness statement like Theorem 5.6, but for non-negative measure-valued solutions instead for renormalized solutions, whose proof easily follows as a simple application of Theorem 4.1.

Theorem 5.10. *Let c be a Borel bounded scalar function defined in $[0,T] \times \mathbb{R}^d$. Let $v : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field satisfying the following conditions*:

(i) *$D_x \cdot v_t = \operatorname{div}_x v_t \mathcal{L}^d$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$ with $\operatorname{div}_x v \in L^1_{\mathrm{loc}}((0,T); L^1_{\mathrm{loc}}(\mathbb{R}^d))$,*
(ii) *there exists a vector field $\tilde{v}$ for which $\mathcal{L}^d(\{\tilde{v}_t(x) \neq v_t(x)\}) = 0$ for $\mathcal{L}^1$-a.e. $t \in (0,T)$, and such that for all s $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ there exists* at most one *characteristic curve $X(t,s,x)$ in $[0,T]$ associated to $\tilde{v}$.*

Then, for any initial datum $\mu_0 \in \mathcal{M}_b^+$, there exists at most one *$\mu = \{\mu_t\}_{t\in(0,T)} \subset \mathcal{M}_b^+$ narrowly continuous solution of*

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = c_t \mu_t \quad \text{in } (0,T) \times \mathbb{R}^d$$

and satisfying

$$\int_0^T \int_{\mathbb{R}^d} \left|v_t(x)\right|^p \,\mathrm{d}\mu_t(x)\,\mathrm{d}t < +\infty \quad \text{for some } p \geqslant 1.$$

Remark 5.11. At this point it may be useful a comparison between our uniqueness results given in Theorem 5.9 and uniqueness results proved by Bahouri and Chemin in [9]. As a byproduct of a uniqueness result within Besov spaces, they are able to prove that if $v \in L^1_{\mathrm{loc}}(\mathbb{R}^+; LL(\mathbb{R}^d))$ is a divergence-free vector field, $g \in L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{M}(\mathbb{R}^d))$ and $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$, then there exists a unique solution $\mu \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{M}(\mathbb{R}^d))$ of

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = g \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \quad \mu(0, \cdot) = \mu_0. \tag{5.11}$$

Let us recall that the Log-Lipschitz regularity is a particular case of the Osgood condition. To be precise, the space of Log-Lipschitz functions $LL(\mathbb{R}^d)$ is the space of bounded functions u on $\mathbb{R}^d$ such that

$$\|u\|_{LL(\mathbb{R}^d)} := \|u\|_{L^\infty(\mathbb{R}^d)} + \sup_{0<|x-y|\leqslant 1} \frac{|u(x) - u(y)|}{|x-y|(1 - \log|x-y|)} < +\infty.$$

If we suppose that μ_1 and μ_2 are two solutions of (5.11), then $\mu_1 - \mu_2$ solves,

$$\partial_t \mu_t + D_x \cdot (v_t \mu_t) = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \quad \mu(0, \cdot) = 0,$$

and for solutions of the initial value problem above both uniqueness results of Theorem 5.8 and of Bahouri and Chemin are available. The result of Bahouri and Chemin is stronger in the sense that is within the class $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{M}(\mathbb{R}^d))$ without further restrictions, but the hypotheses on v are also stronger. Indeed, in Theorem 5.8, first we are supposing less than the divergence-free assumption on v (because we require only that $D_x \cdot v_t = \operatorname{div} v_t \mathcal{L}^d$ for $\mathcal{L}^1$-a.e. $t \in (0, T)$, with $\operatorname{div}_x v \in L^\infty((0, T) \times \mathbb{R}^d)$), and second (in hypotheses (ii)) we are also supposing less than the Log-Lipschitz regularity.

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