Journal of Computational and Applied Mathematics 235 (2011) 2523–2550

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A generic framework for stochastic Loss-Given-Default

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ARTICLE INFO

Article history:
Received 10 February 2010
Received in revised form 31 October 2010

Keywords:
Loss-Given-Default
Lévy process
Factor model
Basel II

ABSTRACT

In this document a method is discussed to incorporate stochastic Loss-Given-Default (LGD) in factor models, i.e. structural models for credit risk. The general idea exhibited in this text is to introduce a common dependence of the LGD and the probability of default (PD) on a latent variable, representing the systemic risk. Though our theory can be applied to any arbitrary firm-value model and any underlying distribution for the LGD, provided its support is a compact subset of [0, 1], special attention is given to the extension of the well-known cases of the Gaussian copula framework and the shifted Gamma one-factor model (a particular case of the generic one-factor Lévy model), and the LGD is modeled by a Beta distribution, in accordance with rating agency models and the Credit Metrics model.

In order to introduce stochastic LGD, a monotonically decreasing relation is derived between the loss rate L, i.e. the loss as a percentage of the total exposure, and the standardized log-return R of the obligor's asset value, which is assumed to be a function of one or more systematic and idiosyncratic risk factors. The property that the relation is decreasing guarantees that the LGD is negatively correlated to R and hence positively correlated to the default rate. From this relation, expressions are then derived for the cumulative distribution function (CDF) and the expected value of the loss rate and the LGD, conditionally on a realization of the systematic risk factor(s). It is important to remark that all our results are derived under the large homogeneous portfolio (LHP) assumption and that they are fully consistent with the IRB approach outlined by the Basel II Capital Accord.

We will demonstrate the impact of incorporating stochastic LGD and using models based on skew and fat-tailed distributions in determining adequate capital requirements. Furthermore, we also skim the potential application of the proposed framework in a credit risk environment. It will turn out that both building blocks, i.e. stochastic LGD and fat-tailed distributions, separately, increase the projected loss and thus the required capital charge. Hence, the aggregation of a model based on a fat-tailed underlying distribution that accounts for stochastic LGD will lead to sound capital requirements.

1. Introduction

For long time, researchers and practitioners in the field of (portfolio) credit risk have spent more time and resources on modeling default risk and default dependence than they have on what is called Loss-Given-Default (LGD). According to the Basel II Capital Accord, LGD is the fraction of Exposure-At-Default (EAD) that will not be recovered following default. Traditional pricing and rating models generally combine a stochastic, or at least time-dependent, default rate with a time-invariant and constant LGD. For instance, even now, index and single name traders are still using a fixed 40% recovery

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doi:10.1016/j.cam.2010.11.006

assumption for their present value calculations, which causes problems in the CDO context, where the newly emerging [60%, 100%] super duper tranche would have[1] no value under traditional models.

Moreover, the financial crises of the past decade and their devastating impact on the global economy provided empirical evidence for the existence of significant positive correlation between the default rate and the LGD in a specific period (cf. [1]). This has led to the consensus that the current models are inadequate, calling for new methods taking into account non-constant LGD. Under Basel II, for instance, banks and other financial institutions are now recommended to calculate the *Downturn LGD*, which reflects economic downturn conditions where necessary to capture the relevant risks, i.e. the realized recovery rates should be lower than average during times of high default rates, to avoid underestimating the expected loss (cf. Basel Committee on Banking [2,3]). The main reason for this requirement is that the Vasicek model (cf. [4]) used in the Basel Capital Accord does not have systematic correlation between probability of default (PD) and loss given default (LGD), which would underestimate downturn risk.

In the literature, several ways have yet been proposed to obtain a non-constant LGD that is correlated with the default risk, extending each of the three popular approaches for modeling default risk, i.e. reduced form models, firm-value models and the copula approach. The former two approaches model the LGD by assuming it is driven by a latent variable that is correlated with the latent variable driving default (cf. [5–7]), whereas the latter aim at directly modeling the spot recovery, i.e. the recovery upon default (cf. [8,9]). An extensive overview of the literature can be found in [10].

In this document a method is discussed to incorporate stochastic LGD in structural models for default risk, as introduced in the seminal papers in [11,12], by introducing a common dependence of the LGD and the PD on a latent variable, representing the systemic risk. Though our theory can be applied to any arbitrary firm-value model and any underlying distribution for the LGD, provided its support is a compact subset of [0, 1], special attention is given to the extension of the well-known cases of the Gaussian copula framework and the shifted Gamma one-factor model, i.e. a particular case of the generic one-factor Lévy model (cf. [13]). Furthermore in this text, w.r.t. the above implementations, we will model the LGD by a Beta distribution, in line with rating agency models and the Credit Metrics model.

In order to introduce stochastic LGD, a monotonically decreasing relation is derived between the loss rate L, i.e. the loss as a percentage of the total exposure, and the standardized log-return R of the obligor's asset value, which is assumed to be a function of one or more systematic and idiosyncratic risk factors. As a starting point, we use the work of Tasche [6] and Joocheol et al. [7]. The property that the relation is decreasing, guarantees that the LGD is negatively correlated to R and hence positively correlated to the default rate. From this relation, expressions are next derived for the cumulative distribution function (CDF) and the expected value of the loss rate and the LGD, conditionally on a realization of the systematic risk factor(s). It is important to remark that all our results are derived under the large homogeneous portfolio (LHP) assumption and that they are fully consistent with the IRB approach outlined by the Basel II Capital Accord.

We will demonstrate the impact of incorporating stochastic LGD and using models based on skew and fat-tailed distributions in determining adequate capital requirements. Furthermore, we also skim the potential application of the proposed framework in a credit risk environment. It will turn out that both building blocks, i.e. stochastic LGD and fat-tailed distributions, separately, increase the projected loss and thus the required capital charge. Hence, the aggregation of a model based on a fat-tailed underlying distribution that accounts for stochastic LGD will lead to sound capital requirements.

The text is organized as follows: Section 2 describes the general framework. We determine a relation between the loss rate L and the standardized log-return R of the debtor's asset value and, based on this relation, derive expressions for the CDF and the expected value of the loss rate L and the LGD, conditionally on a level of the systematic risk. Sections 3 and 4 discuss the extensions of the Normal one-factor model and the generic one-factor Lévy model, especially the (shifted) Gamma one-factor model. Section 5 forms the main contribution of this text and provides a generalization, allowing the loss rate L to depend on two risk drivers instead of one, breaking the comonotonicity between defaults and losses that is introduced by the relation described in Section 2. In Section 6 we demonstrate the impact of accounting for downturn (stochastic) LGD on the capital charge required under the Basel II Capital Accord and compare our model with the work of Amraoui and Hitier [14] and Andersen and Sidenius [10], who introduce stochastic LGD in structural models for CDO valuation, with the aim of flattening the base-correlation curve. Section 7 concludes the paper. Readers interested only in a specific formula, may skip the text and go straight to the Appendix at the end, where all the results are summarized.

2. General framework

2.1. Introduction

In this section we develop the general framework. Starting from a structural model for credit risk and under the large homogeneous portfolio (LHP) assumption, we will derive a model-independent relation between the cumulative loss rate L_t, i.e. the loss as a percentage of the total exposure, and the standardized log-return R of the obligor's asset value. Moreover, we will show that the results derived in this section are fully consistent with the Basel II framework.

Consider a portfolio with notional value N, consisting of M names with respective notionals $N_i, i = 1, \ldots, M$. Furthermore, assume that C is a macro-economic factor, common to all credits and that I_i is an idiosyncratic factor, specific

[1] Assuming a fixed 40% recovery, i.e. a fixed 60% LGD, the expected loss (and hence the value) of the [60%, 100%] super duper tranche will be equal to zero.

to the ith name. Finally, let $\rho \in (0,1)$ denote the correlation between the log-returns of any two names i and $j \neq i$, i.e. we assume equicorrelation between the log-returns. Following[2] Merton's model, obligor i defaults in a time period $[0,t]$ if the standardized log-return of the asset value $R_i \overset{\text{not.}}{=} R_i(C, I_i; \rho)$ hits a certain lower bound H_t^d, that is $1_{i,t}^d = 1 \Leftrightarrow R_i \leq H_t^d$, with $1_{i,t}^d$ the default indicator, equaling 1 if the ith name has defaulted in the interval $[0,t]$ and 0 otherwise. Furthermore, the corresponding default probability (PD) is given by $p_{i,t}^d = \Pr\left[1_{i,t}^d = 1\right] = \Pr\left[R_i \leq H_t^d\right]$. Mark that R_i is traditionally modeled by a Normal one-factor model, i.e.

$$R_i = \sqrt{\rho}C + \sqrt{1-\rho}I_i,$$

where the random variables C and I_i are assumed to be i.i.d. and follow a standard Normal distribution. The latter Gaussian one-factor model will be covered in Section 3. However, as mentioned before, in Section 4 of this text we will also examine the possibilities of incorporating stochastic LGD in the (shifted) Gamma one-factor model (cf. [13]). Though it should be mentioned that the different models discussed in this text are consistent w.r.t. ρ, in the sense that they all satisfy $\text{Corr}\left[R_i, R_j\right] = \rho$, for $i \neq j$, $\text{Corr}[R_i, C] = \sqrt{\rho}$ and $\text{Corr}[R_i, I_i] = \sqrt{1-\rho}$.

Any default by an arbitrary credit induces a loss to the investor. The fraction of the Exposure-At-Default (EAD) that will not be recovered in the event of a default is referred to as the Loss-Given-Default (LGD). To clarify this, let $PL_{i,t}$ denote the cumulative loss due to credit i until time t. Furthermore, assume that the PD[3] at time t is stochastic and given by $Q_{i,t}^d$, where $\mathrm{E}\left[Q_{i,t}^d\right] = p_{i,t}^d$. Finally, note that the loss is only positive in the case of a default. Then, the cumulative portfolio loss rate, i.e. the total loss as a percentage of the total exposure, until time t, is given by

$$L_t = \frac{PL_t}{N} = \frac{1}{N}\sum_{i=1}^{M} PL_{i,t} = \frac{1}{N}\sum_{i=1}^{M}\left[L_{i,t}\left|1_{i,t}^d = 1\right.\right] Q_{i,t}^d N_i,$$

with $L_{i,t}$ the cumulative loss rate at time t of a fixed obligor i. Denote the LGD of obligor i by $\text{LGD}_{i,t} = \left[L_{i,t}\left|1_{i,t}^d = 1\right.\right]$. In the sequel we assume that the LGD is time-independent and omit the subscript $_t$.

Now, under the LHP assumption, it holds that $Q_{i,t}^d \overset{d}{=} Q_t^d$ and $\text{LGD}_i Q_{i,t}^d \overset{d}{=} \text{LGD} Q_t^d$, for all $i = 1, 2, \ldots, M$, such that

$$PL_t \overset{d}{=} \text{LGD} Q_t^d N, \tag{1}$$

where $\overset{d}{=}$ stands for equality in distribution. Hence, in the LHP limit, due to the strong Law of large numbers, L_t is equal to the expected loss of one obligor, i.e.

$$L_t = \mathrm{E}\left[\text{LGD} Q_t^d\right], \tag{2}$$

which is traditionally set equal to = $\mathrm{E}[\text{LGD}]\, p_t^d$, justified by the assumption that the LGD and the PD are independent, where, the LGD is generally assumed to be constant. However, as declared before, there is sufficient empirical evidence for the existence of significant positive correlation between the latter variables, in a specific time period (cf. [1]).

The latter obviously complicates the computation of the expectation in the right-hand side of (2). The general idea exhibited in this text is to introduce a common dependence of the LGD and the PD on the latent variable C, representing the systematic risk, while assuming independence of the conditional LGD (CLGD) and the conditional PD (CPD) (cf. [6,7]). More specifically, we will determine the conditional expected loss rate, conditional on a common factor C and set

$$L_t = \kappa(C) = \mathrm{E}\left[\text{LGD} Q_t^d \middle| C\right] = \mathrm{E}[\text{LGD}\,|C]\, p_t^{d,C}, \tag{3}$$

where $p_t^{d,C}$ is the conditional expected default probability and κ a monotonically decreasing function of C. This implies (cf. [17]) that

$$\Pr[L_t \leq l] = \Pr[\kappa(C) \leq l] \overset{*}{=} 1 - \Pr\left[C \leq \kappa^{[-1]}(l)\right], \tag{4}$$

from which the Value-at-risk (VaR) can then be computed as

$$\text{VaR}_\alpha(L_t) = \kappa\left[F_C^{[-1]}(1-\alpha)\right]. \tag{5}$$

Furthermore, the downturn LGD (DLGD) is defined as (cf. [18])

$$\text{DLGD}_\alpha = \frac{\text{VaR}_\alpha(L_t)}{p_t^{d,F_C^{[-1]}(\alpha)}} = \mathrm{E}\left[\text{LGD}\left|C = F_C^{[-1]}(\alpha)\right.\right], \tag{6}$$

2 In this work we assume Merton's one-period model for default, i.e. default can only occur at the maturity T of the debt instrument. This has some obvious drawbacks, as compared to the first-passage models (cf. [15]) allowing default to occur at any time $t \in [0, T]$. However, this issue can easily be solved by introducing a time-dependent default barrier (cf. [16]).

3 Note that when speaking of the default time at time t, we mean default in the time period $[0, t]$.

with $F_X^{[-1]}$ the generalized inverse of the CDF of a random variable X. The second equality of (6) follows directly from substituting (5) into (3). Note, however, that equality $\overset{*}{=}$ and therefore Eqs. (3) and (5) are only valid if C is one-dimensional. Hence, if C is a vector of 2 or more common factors (cf. Section 5.4) there will generally be no other solution than to derive the VaR and DLGD through simulation, based on $\kappa(C)$ and thus on the expected value of the CLGD. Therefore, in this text, we will primarily be concerned with the derivation of the latter quantity.

2.2. Model framework

We are now ready to derive the model-independent relation between the loss rate and the standardized log-return R of the obligor's asset value. Assume that

$$L_t = \begin{cases} \Lambda > 0; & 1_t^d = 1; \\ 0; & 1_t^d = 0, \end{cases} \tag{7}$$

is the cumulative loss rate of an arbitrary name in the (homogeneous) portfolio. Notice that the random variable Λ is in fact the (time-independent) LGD and is distributed according to a law D, with bounded support $[\lambda_l, \lambda_u]$, where $0 \le \lambda_l \le \lambda_u \le 1$. A popular choice for D is the Beta(a, b) distribution, because the support of the latter is $[0, 1]$. Moreover, we have that $\Pr[L_t > 0] = \Pr\left[1_t^d = 1\right] = p_t^d$. Hence, it holds that $\Pr[L_t > 0] = \Pr\left[R \le H_t^d\right]$, where the risk factor $R \overset{\text{not.}}{=} R(C, I; \rho)$ is a function of a systematic risk factor C and an idiosyncratic risk factor I, satisfying $\text{Corr}[R, C] = \sqrt{\rho}$, $\text{Corr}[R, I] = \sqrt{1-\rho}$, and where H^d is the (possibly time dependent) default barrier, which triggers default when being hit by R.

From the above it follows that (cf. [6])

$$\begin{aligned} p_{l,t} &= \Pr[L_t \le l] \\ &= F_\Lambda(l) \cdot p_t^d + \left(1 - p_t^d\right), \end{aligned} \tag{8}$$

for all $l \in [0, 1]$ and for all $t \ge 0$, with $F_\Lambda(l) = \Pr\left[L_t \le l \,\middle|\, 1_t^d = 1\right]$ the CDF associated to the law D. But this implies that

$$l = \begin{cases} F_\Lambda^{[-1]}\left(\dfrac{p_{l,t} - \left(1 - p_t^d\right)}{p_t^d}\right); & 1 - p_t^d < p_{l,t} \le 1; \\ 0; & 0 \le p_{l,t} \le 1 - p_d, \end{cases}$$

with $F_\Lambda^{[-1]}$ the inverse of the CDF of Λ. Using the fact that $\forall l \in (0, 1],\ \exists!\, r \in \left[\inf(R), H_t^d\right) : p_{l,t} = 1 - F_R(r)$, leads to

$$L_t = h_1(R) = \begin{cases} F_\Lambda^{[-1]}\left(1 - \dfrac{F_R(R)}{p_t^d}\right); & \inf(R) \le R < H_t^d; \\ 0; & H_t^d \le R \le \sup(R), \end{cases} \tag{9}$$

hence, for $r \in \left[\inf(R), H_t^d\right)$ we have

$$r = h_1^{[-1]}(l) = F_R^{[-1]}\left(\left[1 - F_\Lambda(l)\right] p_t^d\right), \tag{10}$$

with $l \in (0, 1]$. By convention, we set $h_1^{[-1]}(0) = F_R^{[-1]}\left(p_t^d\right) = H_t^d$. Notice that $\frac{F_R(R)}{p_t^d} = \Pr\left[R \le R \,\middle|\, R \le H_t^d\right] \overset{\text{not.}}{=} F_{R|R \le H_t^d}(R)$. From this it is easy to verify that

$$1 - \frac{F_R(R)}{p_t^d} = 1 - F_{R|R \le H_t^d}(R) = F_\Lambda(\Lambda) \sim \text{Un}[0, 1], \tag{11}$$

such that $F_\Lambda^{[-1]}\left(1 - \frac{F_R(R)}{p_t^d}\right) = \Lambda$, for $\inf(R) \le R < H_t^d$, as required. Furthermore, using (9) and (11), Eq. (10) can be rewritten as follows

$$r = h_1^{[-1]}(l) = F_{R|R \le H_t^d}^{[-1]}\left(1 - F_\Lambda(l)\right), \quad l \in (0, 1]. \tag{12}$$

Recall, from the previous paragraph, that we assumed the LGD Λ to be time-independent, i.e. the parameters of the law D are independent of the time of default. Moreover, we assume that the latter law is also independent of the distribution of the risk factor R.[4] However, the latter distribution will influence the conditional distribution of the LGD, conditional on a realization of the systematic risk factor C or the idiosyncratic risk factor I. In this paper we only examine the situation of

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4 Unlike the distribution of the loss rate L_t, which depends both on time and the underlying factor model, through p_t^d, i.e. the unconditional probability of default before time t (cf. (8)).

common dependence of the LGD and the PD on the systematic factor C, thereby following Joocheol et al. [7], who state that *under the asymptotic single risk factor model the loss rate for a well diversified portfolio depends only on the (single) systematic risk factor and not on the idiosyncratic risk factors*. Though the reader is cordially invited to translate the theory to conditioning on I.

Assuming a one-factor structural model for the obligor's asset value, the conditional distribution of the LGD, given $C = c$, is given by

$$\begin{aligned}\Pr\left[L_t \le l \,|\, R \le H_t^d, C = c\right] &= \Pr\left[h_1(R) \le l \,\middle|\, R \le H_t^d, C = c\right]\\ &\overset{*}{=} \Pr\left[R \ge h_1^{[-1]}(l) \,\middle|\, R \le H_t^d, C = c\right]\\ &= 1 - \frac{\Pr\left[R \le h_1^{[-1]}(l) \,\middle|\, C = c\right]}{\Pr\left[R \le H_t^d \,\middle|\, C = c\right]},\end{aligned} \tag{13}$$

where equality $\overset{*}{=}$ is explained by the fact that the function h_1 is monotonically decreasing in R (cf. (9)). Now, if we denote the conditional CDF of the LGD given $C = c$ by $F_{\Lambda|C=c}(\cdot)$ and the corresponding CPD by $p_t^{d,c}$, then, combining (8) and (13) yields that the conditional distribution of the loss rate L, given $C = c$, satisfies

$$\begin{aligned}\Pr\left[L_t \le l \,|C = c\right] &= F_{\Lambda|C=c}(l) \cdot p_t^{d,c} + \left(1 - p_t^{d,c}\right)\\ &= 1 - \Pr\left[R \le h_1^{[-1]}(l) \,\middle|\, C = c\right].\end{aligned} \tag{14}$$

Moreover, using relations (9) and (14), we can determine the expected value of the CLGD as

$$\begin{aligned}\mathrm{E}\left[L_t \,\middle|R \le H_t^d, C = c\right] &= \frac{\mathrm{E}\left[L_t \,|C = c\right]}{\Pr\left[R \le H_t^d \,\middle|\, C = c\right]}\\ &= \frac{\int_{l=\lambda_l}^{\lambda_u} \Pr\left[R \le h_1^{[-1]}(l) \,\middle|\, C = c\right] \mathrm{d}l}{\Pr\left[R \le H_t^d \,\middle|\, C = c\right]}.\end{aligned} \tag{15}$$

Note that the numerator in the right-hand side of the above expression equals $\kappa(c) = \mathrm{VaR}_{1-F_C(c)}$ (cf. (3) and (5)), thus (15) corresponds to $\mathrm{DLGD}_{1-F_C(c)}$. Hence, being able to compute the above quantities allows one to incorporate stochastic recovery rates in VaR and DLGD calculations (cf. (5) and (6)), but it also has many other practical applications, e.g. as pricing models for credit defaults swaps (CDSs) or collateralized debt obligations (CDOs) and scenario generators for analyzing and rating asset-backed securities (ABSs).

Finally, note that (15) is in line with the standard procedure, outlined in the Basel II capital framework, of setting the expected loss rate equal to $\mathrm{E}\left[L_t\right] = \mathrm{E}\left[L_t \,\middle|1_t^d = 1\right] \Pr\left[1_t^d = 1\right]$. Conditioning on $C = c$ and dividing both sides of the latter equality by $\Pr\left[1_t^d = 1 \,\middle|\, C = c\right]$ leads to (15).

In the next two sections, following rating agency practice, we will assume that Λ follows a Beta distribution, with parameters $a, b > 0$, hence $\lambda_l = 0$ and $\lambda_u = 1$.[5] Though it should be kept in mind that any distribution with bounded support is suited for Λ.[6] Readers interested in the formulas for a general distribution are referred to the Appendix at the end of this text. Hence,

$$L_t = h_1(R) = \begin{cases}\mathbb{B}_{a,b}^{[-1]}\left(1 - \dfrac{F_R(R)}{p_t^d}\right); & \inf(R) \le R < H_t^d;\\ 0; & H_t^d \le R \le \sup(R),\end{cases} \tag{16}$$

and

$$r = F_R^{[-1]}\left(\left[1 - \mathbb{B}_{a,b}(l)\right] p_t^d\right), \quad l \in (0, 1], \tag{17}$$

with $\mathbb{B}_{a,b}$ the CDF of the Beta distribution with parameters a and b. The latter can be calibrated based on historical data regarding the expected value and the variance of the loss rate L or the LGD Λ. However, assuming that 1 − E[LGD] is equal to the expected recovery conditional on default, to be consistent with single-name and index pricing, the LGD model must be calibrated such that 1 − E[LGD] is the same as the mid recovery, generally taken to be 40% w.r.t. CDSs (cf. [14]).

5 Let σ_X^2 and μ_X denote the variance and the expected value of the random variable X, then it is easy to verify that $\sigma_{L_t}^2 = p_t^d\left(\mu_{\mathrm{LGD}}^2 + \sigma_{\mathrm{LGD}}^2\right) - \left(\mu_{\mathrm{LGD}} p_t^d\right)^2$. By the properties of the Beta distribution, it then follows that $0 \le \mu_{\mathrm{LGD}}^2 p_t^d\left(1 - p_t^d\right) < \sigma_L^2 < \mu_{\mathrm{LGD}} p_t^d\left(1 - \mu_{\mathrm{LGD}} p_t^d\right) \le 0.25$.

6 Note that one can always construct a random variable Y with support $[\lambda_l, \lambda_u]$ from a Beta(a, b) distributed random variable X, using the transformation $Y = \lambda_l + (\lambda_u - \lambda_l)X$.

3. The Gaussian one-factor model

Recall that the Normal one-factor model models the asset value V of a borrower, where V is described by a geometric Brownian motion,

$$V_T = V_0 \exp\left[\mu(T) + \sigma(T) W_T\right] \stackrel{d}{=} V_0 \exp\left[\mu(T) + \sigma(T)\sqrt{T} Z\right], \tag{18}$$

with W a Wiener process and where the random variable $Z \sim \mathrm{N}(0, 1)$ satisfies

$$Z \stackrel{\text{not.}}{=} R(X, \xi; \rho) = \sqrt{\rho} X + \sqrt{1-\rho}\,\xi,$$

with $X, \xi \stackrel{\text{i.i.d.}}{\sim} \mathrm{N}(0, 1)$ and $\rho \in (0, 1)$. The random variable X denotes the systematic risk factor (C) which is common to each obligor and the random variable ξ represents the idiosyncratic risk factor (I) associated with each individual obligor. Finally, as indicated before, ρ determines the borrower's exposure to the systematic and idiosyncratic risk factors.

A borrower is said to default at time $t \geq 0$, if his financial situation deteriorates so dramatically that V_T hits a predetermined lower bound B_t^d, which (as can be seen from (18)) is equivalent to saying that Z hits some barrier $H_t^d \in \mathbb{R}$. From (16) it then follows that

$$L_t = h_1(Z) = \begin{cases} \mathbb{B}_{a,b}^{[-1]}\left(1 - \dfrac{\Phi(Z)}{\Phi(H_t^d)}\right); & -\infty \leq Z < H_t^d; \\ 0; & H_t^d \leq Z \leq +\infty, \end{cases} \tag{19}$$

hence $h_1^{[-1]}(l) = \Phi^{[-1]}\left[\left(1 - \mathbb{B}_{a,b}(l)\right)\Phi\left(H_t^d\right)\right]$, for $l \in (0, 1]$, where Φ is the CDF of the standard Normal distribution. Using Eqs. (13)–(15) this leads to

$$\Pr\left[L_t \leq l \,\middle|\, Z \leq H_t^d, X = x\right] = 1 - \frac{\Phi\left[\frac{h_1^{[-1]}(l) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right]}{\Phi\left[\frac{H_t^d - \sqrt{\rho}x}{\sqrt{1-\rho}}\right]}, \tag{20}$$

$$\Pr\left[L_t \leq l \,|\, X = x\right] = 1 - \Phi\left[\frac{h_1^{[-1]}(l) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right], \tag{21}$$

and

$$\mathrm{E}\left[L_t \,\middle|\, Z \leq H_t^d, X = x\right] = \frac{\int_{l=0}^{1} \Phi\left[\frac{h_1^{[-1]}(l) - \sqrt{\rho}x}{\sqrt{1-\rho}}\right] \mathrm{d}l}{\Phi\left(\frac{H_t^d - \sqrt{\rho}x}{\sqrt{1-\rho}}\right)}, \tag{22}$$

with h_1 given by (19).

4. Generic one-factor Lévy model

4.1. The generic one-factor Lévy model

The generic one-factor Lévy model (cf. [13]) is comparable to and in fact a generalization of the Normal one-factor model. However, instead of describing the name's asset value by a geometric Brownian motion, we will now model the latter with an exponential Lévy model, i.e.

$$V_T = V_0 \exp\left[A_T\right], \tag{23}$$

where the standardized log-return is modeled by a Lévy process $A = \{A_t : t \geq 0\}$, satisfying

$$A_t \stackrel{\text{not.}}{=} R(Y, \chi; \rho) = Y_{\rho \frac{t}{T}} + \chi_{(1-\rho)\frac{t}{T}}, \tag{24}$$

with $\rho \in (0, 1)$ and Y and χ i.i.d. Lévy processes, based on the same mother infinitely divisible distribution L, satisfying $\mathrm{E}[Y_t] = 0$ and $\mathrm{Var}[Y_t] = t$, for all $t \geq 0$. Then the Lévy process A will also be based on the law L, but generally will not be identically distributed to Y and χ. Moreover, $\mathrm{E}\left[A_T\right] = 0$ and $\mathrm{Var}\left[A_T\right] = 1$, in line with the Normal one-factor model. Each of the Lévy models discussed in this text will be silently assumed to be parameterized, in order to satisfy the latter properties.

In the above equation, Y_ρ is a systematic risk factor, common to all borrowers, $\chi_{1-\rho}$ is an idiosyncratic risk factor and ρ determines the exposure to the latter risk factors. It can be shown that $\mathrm{Corr}\left[A_T, Y\rho\right] = \sqrt{\rho}$ and $\mathrm{Corr}\left[A_T, \chi_{1-\rho}\right] = \sqrt{1-\rho}$.

A borrower defaults if A_T hits a predetermined barrier H_t^d. Hence, from (16) it follows that

$$L_t = h_1(A_T) = \begin{cases} \mathbb{B}_{a,b}^{[-1]}\left(1 - \dfrac{F_{A_T}(A_T)}{F_{A_T}(H_t^d)}\right); & \inf(A_T) \le A_T < H_t^d; \\ 0; & H_t^d \le A_T \le \sup(A_T), \end{cases} \tag{25}$$

thus $h_1^{[-1]}(l) = F_{A_T}^{[-1]}\left[\left(1 - \mathbb{B}_{a,b}(l)\right) F_{A_T}\left(H_t^d\right)\right]$, for $l \in (0, 1]$. From this we obtain, using Eqs. (13)–(15),

$$\Pr\left[L_t \le l \middle| A_T \le H_t^d, Y_\rho = y_\rho\right] = 1 - \frac{\Pr\left[A_T \le h_1^{[-1]}(l) \middle| Y_\rho = y_\rho\right]}{\Pr\left[A_T \le H_t^d \middle| Y_\rho = y_\rho\right]}, \tag{26}$$

$$\Pr\left[L_t \le l \middle| Y_\rho = y_\rho\right] = 1 - \Pr\left[A_T \le h_1^{[-1]}(l) \middle| Y_\rho = y_\rho\right], \tag{27}$$

and

$$\mathrm{E}\left[L_t \middle| A_T \le H_t^d, Y_\rho = y_\rho\right] = \frac{\int_{l=0}^1 \Pr\left[A_T \le h_1^{[-1]}(l) \middle| Y_\rho = y_\rho\right] \mathrm{d}l}{\Pr\left[A_T \le H_t^d \middle| Y_\rho = y_\rho\right]}, \tag{28}$$

with h given by (25) and F_{A_T} the CDF of the random variable A_T.

4.2. The Gamma one-factor model

From now on, we will assume that Y and χ are i.i.d. shifted Gamma processes, i.e. $Y = \{Y_t = t\mu_g - G_t; t \ge 0\}$, where G is a Gamma process, with shape parameter $\alpha_g > 0$ and scale parameter $\beta_g > 0$. Setting $\beta_g = \sqrt{\alpha_g}$ and $\mu_g = \frac{\alpha_g}{\beta_g}$ ensures that $\mathrm{E}[A_T] = 0$ and $\mathrm{Var}[A_T] = 1$ (cf. [13]). Note that the processes Y and χ are based on the Gamma $\left(\alpha_g, \beta_g\right)$ distribution, whereas the process A is based on the Gamma $\left(\frac{\alpha_g}{T}, \beta_g\right)$ distribution. Indeed, from the fact that a Gamma distribution is infinitely divisible it follows that

$$\begin{aligned} Y_t &\overset{d}{=} t\mu_g - X_{Y_t} \\ \chi_t &\overset{d}{=} t\mu_g - X_{\chi_t}, \end{aligned} \tag{29}$$

for $t \ge 0$, which implies, using (24), that

$$A_t = Y_{\rho\frac{t}{T}} + \chi_{(1-\rho)\frac{t}{T}} \overset{d}{=} \mu_g \frac{t}{T} - \left[X_{Y_{\rho\frac{t}{T}}} + X_{\chi_{(1-\rho)\frac{t}{T}}}\right] \overset{d}{=} \mu_g \frac{t}{T} - X_{A_t}, \tag{30}$$

with $X_{Y_t}, X_{\chi_t} \sim$ Gamma $\left(t\alpha_g, \beta_g\right)$ and $X_{A_t} \sim$ Gamma $\left(\alpha_g \frac{t}{T}, \beta_g\right)$. Hence, $A_T \in \left[-\infty, \mu_g\right]$, since $X_{Y_\rho}, X_{\chi_{1-\rho}}, X_{A_T} > 0$. Notice that the fact that Y and χ are i.i.d. Lévy processes implies that the random variables X_{Y_t} and X_{χ_t} are independent.

From the previous paragraph, together with (25), it then follows that in the case of the Gamma one-factor model

$$L_t = h_1(A_T) = \begin{cases} \mathbb{B}_{a,b}^{[-1]}\left(1 - \dfrac{1 - \Gamma_{\alpha_g,\beta_g}\left(\mu_g - A_T\right)}{1 - \Gamma_{\alpha_g,\beta_g}\left(\mu_g - H_t^d\right)}\right); & -\infty \le A_T < H_t^d; \\ 0; & H_t^d \le A_T \le \mu_g, \end{cases} \tag{31}$$

such that $h_1^{[-1]}(l) = \mu_g - \Gamma_{\alpha_g,\beta_g}^{[-1]}\left[1 - \left\{1 - \Gamma_{\alpha_g,\beta_g}\left(\mu_g - H_t^d\right)\right\}\left\{1 - \mathbb{B}_{a,b}(l)\right\}\right]$, for $l \in (0, 1]$, with $\Gamma_{m,n}$ the CDF of a Gamma distribution with shape parameter $m > 0$ and scale parameter $n > 0$. Using Eqs. (13)–(15) this leads to

$$\Pr\left[L_t \le l \middle| A_T \le H_t^d, Y_\rho = y_\rho\right] = 1 - \frac{1 - \Gamma_{(1-\rho)\alpha_g,\beta_g}\left[(1-\rho)\mu_g + y_\rho - h_1^{[-1]}(l)\right]}{1 - \Gamma_{(1-\rho)\alpha_g,\beta_g}\left[(1-\rho)\mu_g + y_\rho - H_t^d\right]}, \tag{32}$$

$$\Pr\left[L_t \le l \middle| Y_\rho = y_\rho\right] = \Gamma_{(1-\rho)\alpha_g,\beta_g}\left[(1-\rho)\mu_g + y_\rho - h_1^{[-1]}(l)\right], \tag{33}$$

and

$$\mathrm{E}\left[L \middle| A_T \le H_t^d, Y_\rho = y_\rho\right] = \frac{1 - \int_{l=0}^1 \Gamma_{(1-\rho)\alpha_g,\beta_g}\left[(1-\rho)\mu_g + y_\rho - h_1^{[-1]}(l)\right] \mathrm{d}l}{1 - \Gamma_{(1-\rho)\alpha_g,\beta_g}\left[(1-\rho)\,\mu_g + y_\rho - H_t^d\right]}, \tag{34}$$

with h_1 given by (31). Note that we can truncate the latter integral at $\epsilon = h_1\left((1-\rho)\,\mu_g + y_\rho\right)$, as the integrand is zero for all $l \ge h_1\left((1-\rho)\,\mu_g + y_\rho\right)$.

Note that in the special case where $\alpha_g = 1$, the random variable $\mu_g - A_T \overset{d}{=} X_{A_T}$ follows a standard exponential distribution (assuming $\beta_g = \sqrt{\alpha_g}$ and $\mu_g = \frac{\alpha_g}{\beta_g}$) and hence has the so-called memoryless property. This fact, together with the observation that the CPD is equal to 1 when $Y_\rho \le H_t^d - (1-\rho)\,\mu_g$, implies that, for any $\kappa > 0$,

$$\begin{aligned} \mathrm{E}\left[L \,\middle|\, A_T \le H_t^d, Y_\rho = H_t^d - (1-\rho)\,\mu_g - \kappa\right] &= \mathrm{E}\left[L \,\middle|\, Y_\rho = H_t^d - (1-\rho)\,\mu_g - \kappa\right] \\ &= \mathrm{E}\left[\mathbb{B}_{a,b}^{[-1]}\left(\Pr\left[X_{A_T} \le X_{\chi_{1-\rho}} + \kappa\right]\right)\right]. \end{aligned} \tag{35}$$

Hence, below the Armageddon point $Y_\rho = H_t^d - (1-\rho)\,\mu_g$, where the CPD is equal to 1, the conditional expected LGD is equal to the conditional expected loss and independent of the level of the default barrier, which determines the unconditional default probability. This observation no longer holds when $\alpha_g \neq 1$, as the Gamma distribution is generally not memoryless. Furthermore, notice that $\mathbb{B}_{a,b}^{[-1]}\left(\Pr\left[X_{A_T} \le X_{\chi_{1-\rho}} + \epsilon\right]\right)$ is not necessarily Beta distributed, since X_{A_T} and $X_{\chi_{1-\rho}}$ are generally not identically distributed.

5. Generalization

5.1. Note on comonotonicity between defaults and losses

An obvious drawback of the relation $L = h_1(R)$ (cf. (9)) is that default and LGD are comonotonic, i.e. both are driven by one and the same random variable R, i.e. the standardized log-return of the obligor's asset value. This is to some extent acceptable, in the sense that it is likely that a global or regional economic downturn will cause firm-values to spiral downwards, leading to a significant increase in default rates and loss rates, but on the other hand it is possible that at some point in time there are other (e.g. sector-related) factors which counteract or even suppress the negative impact of the global downturn, causing the loss rate to decrease or even be zero, despite the firm's high default rate (invoked by the macro-economic environment).

In order to break the comonotonicity between defaults and losses, one may consider the loss rate L to be a function of both the standardized log-return $R(C, I; \rho)$ and an additional variable J, where the former is driven by macro-economic and idiosyncratic factors and the latter represents certain *additional* events, that may counteract the negative impact of a default on the loss rate. We use the notation $R(C, I; \rho)$ to stress the fact that the R is a function of a systematic risk C and an idiosyncratic risk I. Moreover, we assume that C, I and J are independent and identically distributed (i.i.d.) random variables. For obvious reasons, we will refer to this model as *the three-factor model*. However, an immediate shortfall of this method is that the additional effects are independent of the overall economy.

An alternative that solves this problem, following Frye and Hillebrand [5,19], is to consider two risk factors $R_1(C, I; \rho_1)$ and $R_2(C, J; \rho_2)$, where as above, the common risk factor C and the idiosyncratic risk factors I and J are i.i.d. random variables and ρ_1 measures the exposure of the former risk factor to C and I, whereas ρ_2 measures the exposure of the latter factor to C and J. The factor $R_1(C, I; \rho_1)$ corresponds to the standardized log-return of the credit's asset value, whereas the loss rate is fully determined by $R_2(C, J; \rho_2)$, with no obvious intuitive interpretation. In the sequel, we will abbreviate the latter factors by R_1 and R_2. Note that R_1 and R_2 are both driven by the systematic risk factor C and hence are (positively) correlated. In this text, the above framework will be referred to as *the* Hillebrand [19]-*type model*.

We will go even one step further and make the loss rate L dependent on both R_1 and R_2. It will turn out that the procedure discussed in Section 1, as well as the two methods described in the previous paragraphs are all special cases of our more general framework.

5.2. A non-comonotonic extension

In order to provide a consistent generalization of the above framework, unlike Hillebrand [19], we let the loss rate L be a function of both random variables $R_1\,(C, I; \rho_1)$ and $R_2\,(C, J; \rho_2)$, with $\mathrm{Corr}\,[R_i, C] = \sqrt{\rho_i}$, $i = 1, 2$, $\mathrm{Corr}\,[R_1, I] = \sqrt{1-\rho_1}$ and $\mathrm{Corr}\,[R_2, J] = \sqrt{1-\rho_2}$. More specifically, we propose to model the loss rate as

$$L_t = h_2(R_3) = \begin{cases} F_\Lambda^{[-1]}\left(1 - \dfrac{F_{R_3,R_1}\left(R_3, H_t^d\right)}{p_t^d}\right); & \inf(R_1) \le R_1 < H_t^d; \\ 0; & H_t^d \le R_1 \le \sup(R_1), \end{cases} \tag{36}$$

and

$$r_3 = h_2^{[-1]}(l) = F_{R_3 \left| R_1 \le H_t^d \right.}^{[-1]}\left[1 - F_\Lambda(l)\right], \tag{37}$$

for $l \in (0, 1]$, where $R_3 \overset{\text{not.}}{=} R_3\,(R_1, R_2; \rho_3)$ satisfies

$$\begin{aligned} &\text{(a) } R_3\,(R_1, R_2; 1) = R_1; \\ &\text{(b) } R_3\,(R_1, R_2; 0) = R_2; \\ &\text{(c) } \mathrm{Corr}\,(R_3, R_i) \in \left[\mathrm{Corr}\,(R_1, R_2)\,, 1\right], \end{aligned} \tag{38}$$

with Corr (R_3, R_1) increasing in ρ_3 and Corr (R_3, R_2) decreasing in ρ_3, for a given pair (ρ_1, ρ_2). Furthermore, as can be seen from the first two requirements, the function $R_3(R_1, R_2; \rho_3)$ must be such that setting $\rho_3 = 1$ puts us in the framework of Section 1, whereas $\rho_3 = 0$ corresponds to the Hillebrand [19]-type model. Finally, setting $\rho_3 \in (0, 1)$ and $\rho_2 = 0$ gives the three-factor model.

Note, from the third requirement, that the dependence between R_3 on the one hand and R_1 and R_2 on the other is bounded from below by Corr $[R_1, R_2]$. Hence Corr (R_3, R_1) will generally not be equal to $\sqrt{\rho_3}$ and Corr (R_3, R_2) will generally not be equal to $\sqrt{1-\rho_3}$. Indeed, for $\rho_3 \in (0, 1)$, the dependence of the loss rate's driver R_3 on R_1 and R_2 (and thus on the systematic risk factor C and the idiosyncratic factors I and J) will be a non-trivial function of the triplet (ρ_1, ρ_2, ρ_3). Furthermore, for $\rho_3 = 1$, the exposure of R_3 to C and I is exclusively measured by ρ_1 and for $\rho_3 = 0$ the exposure to C and J is fully determined by ρ_2. Also, note that, if $\rho_1 = \rho_2 = 0$, the default rates and the loss rates are, generally, no longer influenced by the systematic risk C and hence are independent between obligors. However, they are still dependent within each debtor, due to the common dependence on the idiosyncratic factor I.

Furthermore, as was the case for the function h_1, here again it can be shown that

$$1 - \frac{F_{R_3,R_1}\left(R_3, H_t^d\right)}{p_t^d} = 1 - F_{R_3 \left| R_1 \le H_t^d \right.}(R_3) = F_\Lambda(\Lambda) \sim \mathrm{Un}[0, 1],$$

for $\inf(R_1) \le R_1 < H_t^d$. Finally, note that, even if the joint CDF of (R_3, R_1) is known, there will generally not exist a closed form solution to $F^{[-1]}_{R_3 \left| R_1 \le H_t^d \right.}(\cdot)$. Hence the inverse $h_2^{[-1]}(l)$ must be determined numerically.

Using Eqs. (36) and (37), the reader may verify that

$$\begin{aligned} \Pr\left[L_t \le l \middle| R_1 \le H_t^d, C = c\right] &= \Pr\left[R_3 \ge h_2^{[-1]}(l) \middle| R_1 \le H_t^d, C = c\right] \\ &= 1 - \frac{\Pr\left[R_3 \le h_2^{[-1]}(l), R_1 \le H_t^d \middle| C = c\right]}{\Pr\left[R_1 \le H_t^d \middle| C = c\right]} \end{aligned} \tag{39}$$

$$\Pr\left[L_t \le l \middle| C = c\right] = 1 - \Pr\left[R_3 \le h_2^{[-1]}(l), R_1 \le H_t^d \middle| C = c\right], \tag{40}$$

and

$$\mathrm{E}\left[L_t \middle| R_1 \le H_t^d, C = c\right] = \frac{\int_{l=\lambda_l}^{\lambda_u} \Pr\left[R_3 \le h_2^{[-1]}(l), R_1 \le H_t^d \middle| C = c\right] \mathrm{d}l}{\Pr\left[R_1 \le H_t^d \middle| C = c\right]}. \tag{41}$$

Notice that the default probability, conditional on $C = c$, is independent of R_2. This is due to the conditional independence of R_1 and R_2.

We conclude that in order to be able to apply the proposed generalization in a specific factor model the main task is to determine an appropriate function $R_3(R_1, R_2; \rho_3)$, satisfying the requirements in (38) and preferably such that the joint distributions of (R_3, R_1) and $[(R_3, R_1) | C = c]$ are known. Eqs. (42) and (53), as discussed in the next two sections, are our proposals for the latter function, in the Gaussian one-factor model and the shifted Gamma one-factor model, respectively.

Furthermore, following prior practice, below we will again assume that the LGD Λ follows a Beta distribution.

5.3. Normal one-factor model

In the case of the Normal one-factor model, we suggest to use

$$Z_3 \overset{\text{not.}}{=} Z_3(Z_1, Z_2; \rho_3) = \sqrt{\rho_3} Z_1 + \sqrt{1-\rho_3} Z_2, \tag{42}$$

where

$$Z_i \overset{\text{not.}}{=} Z_i(X, \xi_i; \rho_i) = \sqrt{\rho_i} X + \sqrt{1-\rho_i} \xi_i, \tag{43}$$

for $i = 1, 2$, with $X, \xi_1, \xi_2 \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, 1)$ and $\rho_1, \rho_2, \rho_3 \in [0, 1]$. The factor Z_1 corresponds to the standardized log-return of the debtor's asset value, i.e. an obligor defaults if $Z_1 \le H_t^d$, whereas Z_2 describes the influence of the additional effects on the loss rate. Note that Z_1 and Z_2 are dependent, through the dependence on X. Finally, note that Z_3 satisfies (38).

Notice that the above equations can also be expressed in terms of Wiener processes. Indeed, let W, $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(2)}$ be independent Wiener processes, i.e. $W = \{W_t; t \ge 0\}$ and $W_t \sim \mathrm{N}(0, t)$, then it follows, from the well-known scaling property $\sqrt{c} W_t = W_{ct}$, that

$$Z_3 \overset{d}{=} W^{(1)}_{\rho_3 T} + W^{(2)}_{(1-\rho_3)T}, \tag{44}$$

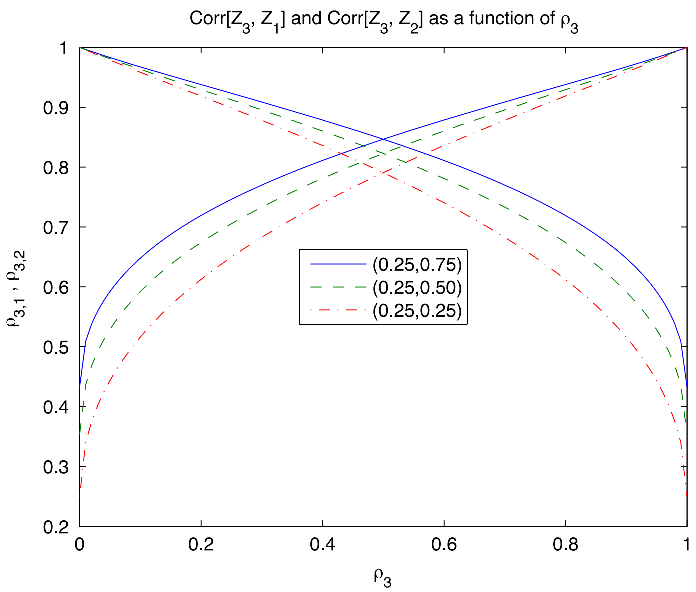

Fig. 1. Pearson correlation between Z_3 and $\{Z_1, Z_2\}$.

with

$$W_t^{(i)} = W_{\rho_i \frac{t}{T}} + \widetilde{W}^{(i)}_{(1-\rho_i)\frac{t}{T}}, \tag{45}$$

for $i = 1, 2$.

From (43) we conclude that the random vector $(Z_1, Z_2)^T$ is bivariate normally distributed, with mean $\boldsymbol{\mu} = (0, 0)^T$ and linear correlation coefficient $\rho = \sqrt{\rho_1 \rho_2} \geq 0$. From this it immediately follows that

$$Z_3 \sim \mathrm{N}\left(0,\, 1 + 2\sqrt{\rho_1 \rho_2 \rho_3 (1 - \rho_3)}\right).$$

Furthermore, it can be shown that

$$\begin{aligned} \mathrm{Corr}\,(Z_3, Z_1) &= \frac{\sqrt{\rho_3} + \sqrt{\rho_1 \rho_2 (1-\rho_3)}}{\sqrt{1 + 2\sqrt{\rho_1 \rho_2 \rho_3 (1-\rho_3)}}} \geq 0; \\ \mathrm{Corr}\,(Z_3, Z_2) &= \frac{\sqrt{1-\rho_3} + \sqrt{\rho_1 \rho_2 \rho_3}}{\sqrt{1 + 2\sqrt{\rho_1 \rho_2 \rho_3 (1-\rho_3)}}} \geq 0. \end{aligned} \tag{46}$$

In line with the third requirement of (38), the above two expressions are bounded from below by $\mathrm{Corr}\,(Z_1, Z_2) = \sqrt{\rho_1 \rho_2}$. Moreover, they are symmetric in ρ_1 and ρ_2. Fig. 1 depicts the above linear correlation coefficients as a function of ρ_3 for several pairs (ρ_1, ρ_2). It is apparent that $\mathrm{Corr}\,(Z_3, Z_1) = \mathrm{Corr}\,(Z_3, Z_2)$ for $\rho_3 = 0.50$, independently of the pair (ρ_1, ρ_2). Hence, ρ_3 is the dominant driver of the latter correlations.

Now, $X, \xi_1, \xi_2 \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, 1)$ implies that[7]

$$(Z_3, Z_1)' \sim \mathrm{N}^2\left(\boldsymbol{\mu}_{3,1}, \Sigma_{3,1}\right), \tag{47}$$

and

$$(Z_3, Z_1 \,|X = x)' \sim \mathrm{N}^2\left(\boldsymbol{\mu}^x_{3,1}, \Sigma^x_{3,1}\right), \tag{48}$$

with

$$\begin{aligned} \boldsymbol{\mu}_{3,1} &= (0, 0)'; \\ \Sigma_{3,1} &= \begin{pmatrix} \sigma_3^2 & \rho_{3,1}\sigma_3\sigma_1 \\ \rho_{3,1}\sigma_3\sigma_1 & \sigma_1^2 \end{pmatrix}; \\ \sigma_1 &= 1; \end{aligned}$$

[7] Since both $(Z_3, Z_1)^T$ and $(Z_3, Z_1 \,|X = x)^T$ can be decomposed as $\mathbb{A}\mathbf{Z} + \boldsymbol{\mu}$, with $\mathbf{Z} = (X, \xi_1, \xi_2)^T$ a random vector whose components are independent standard Normal random variables, $\mathbb{A} \in \mathbb{R}^{2\times 3}$ and $\boldsymbol{\mu} \in \mathbb{R}^{2\times 1}$.

$$\sigma_3 = \sqrt{1 + 2\sqrt{\rho_1\rho_2\rho_3(1-\rho_3)}};$$
$$\rho_{3,1} = \frac{\sqrt{\rho_3} + \sqrt{\rho_1\rho_2(1-\rho_3)}}{\sqrt{1 + 2\sqrt{\rho_1\rho_2\rho_3(1-\rho_3)}}};$$
$$\boldsymbol{\mu}^x_{3,1} = \left(\left[\sqrt{\rho_1\rho_3} + \sqrt{\rho_2\,(1-\rho_3)}\right]x,\ \sqrt{\rho_1}x\right)';$$
$$\Sigma^x_{3,1} = \begin{pmatrix} \left(\sigma^x_3\right)^2 & \rho^x_{3,1}\sigma^x_3\sigma^x_1 \\ \rho^x_{3,1}\sigma^x_3\sigma^x_1 & \left(\sigma^x_1\right)^2 \end{pmatrix};$$
$$\sigma^x_1 = \sqrt{1-\rho_1};$$
$$\sigma^x_3 = \sqrt{\rho_3\,(1-\rho_1) + (1-\rho_3)\,(1-\rho_2)};$$
$$\rho^x_{3,1} = \frac{1}{\sqrt{1 + \frac{(1-\rho_2)(1-\rho_3)}{\rho_3(1-\rho_1)}}}.$$

By $N^2(\boldsymbol{\mu}, \Sigma)$ we denote the bivariate Normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix Σ. Write $\Phi^2_{\boldsymbol{\mu},\Sigma}$ for the CDF of the latter, then, from (36), we get

$$L_t = h_2(Z_3) = \begin{cases} \mathbb{B}^{[-1]}_{a,b}\left(1 - \dfrac{\Phi^2_{\boldsymbol{\mu}_{3,1},\Sigma_{3,1}}\left(Z_3, H^d_t\right)}{\Phi\left(H^d_t\right)}\right); & -\infty \le Z_1 < H^d_t; \\ 0; & H^d_t \le Z_1 \le +\infty. \end{cases} \tag{49}$$

Note that the inverse $z_3 = h_2^{[-1]}(l)$, for $l \in (0, 1]$, can be computed numerically efficiently, thanks to efficient algorithms to compute the bivariate Normal CDF (cf. [20]).

Then, using (39), (40) and (66), it is a trivial task to verify that

$$\Pr\left[L_t \le l \,\middle|\, Z_1 \le H^d_t, X = x\right] = 1 - \frac{\Phi^2_{\boldsymbol{\mu}^x_{3,1},\Sigma^x_{3,1}}\left(h_2^{[-1]}(l), H^d_t\right)}{\Phi\left(\frac{H^d_t - \sqrt{\rho_1}x}{\sqrt{1-\rho_1}}\right)}, \tag{50}$$

$$\Pr\left[L_t \le l \,|\, X = x\right] = 1 - \Phi^2_{\boldsymbol{\mu}^x_{3,1},\Sigma^x_{3,1}}\left(h_2^{[-1]}(l), H^d_t\right), \tag{51}$$

and

$$\mathrm{E}\left[L_t \,\middle|\, Z_1 \le H^d_t, X = x\right] = \frac{\int_{l=0}^{1} \Phi^2_{\boldsymbol{\mu}^x_{3,1},\Sigma^x_{3,1}}\left(h_2^{[-1]}(l), H^d_t\right) \mathrm{d}l}{\Phi\left(\frac{H^d_t - \sqrt{\rho_1}x}{\sqrt{1-\rho_1}}\right)}. \tag{52}$$

5.4. Gamma one-factor model

In case of the Gamma one-factor model, to be consistent with the previous section, we suggest to use

$$A^{(3)}_T \overset{\text{not.}}{=} A^{(3)}_T\left(A^{(1)}, A^{(2)}; \rho_3\right) = A^{(1)}_{\rho_3 T} + A^{(2)}_{(1-\rho_3)T}, \tag{53}$$

where

$$A^{(i)}_t \overset{\text{not.}}{=} A^{(i)}_t\left(Y, \chi^{(i)}; \rho_i\right) = Y_{\rho_i \frac{t}{T}} + \chi^{(i)}_{(1-\rho_i)\frac{t}{T}}, \tag{54}$$

for $i = 1, 2$ and $t \in [0, T]$, with $\rho_1, \rho_2, \rho_3 \in [0, 1]$ and $Y, \chi^{(1)}, \chi^{(2)}$ i.i.d. shifted Gamma processes, with shape and scale parameters chosen such $\mathrm{E}\left[A^{(i)}_t\right] = 0$ and $\mathrm{Var}\left[A^{(i)}_t\right] = t$. The latter implies that $\mathrm{E}\left[A^{(3)}_T\right] = 0$ and $\mathrm{Var}\left[A^{(3)}_T\right] = 1 + 2\min\left\{\rho_1\rho_3, \rho_2\,(1-\rho_3)\right\}$.

Finally, notice that (53) can not generally be expressed in the form of (42), due to the fact that the Gamma distribution, and hence the (shifted) Gamma process, do not satisfy the required scaling property. More specifically, recall from (30) that $A^{(i)}_t \overset{d}{=} \mu_g \frac{t}{T} - X_{A_t}$, with $A_t \sim \mathrm{Gamma}\left(\alpha_g \frac{t}{T}, \beta_g\right)$. From this it is apparent that there will generally exist no function g such that $g(c)A^{(i)}_t \overset{d}{=} A^{(i)}_{ct}$.

The process $A^{(1)}$ describes the standardized log-return of the credit, i.e. an obligor defaults if $A^{(1)}_T \le H^d_t$, whereas the process $A^{(2)}$ describes the additional effects influencing the loss rate. The latter two risk factors are dependent, since they

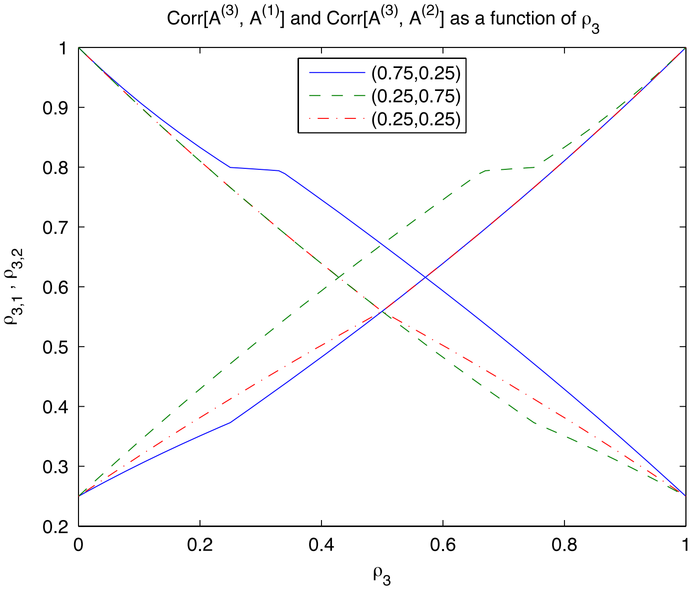

Fig. 2. Pearson correlation between $A_T^{(3)}$ and $\left\{A_T^{(1)}, A_T^{(2)}\right\}$.

are both driven by the common process Y, as compared to a common factor X in the Normal one-factor model. Note that we actually only need to know the state of the latter process at times ρ_1 (for the default driver) and $\rho_1\rho_3$, $\rho_2(1-\rho_3)$ (for the loss driver). Hence, our model can also be regarded as one with 3 mutually correlated systematic risk factors Y_{ρ_1}, $Y_{\rho_1\rho_3}$ and $Y_{\rho_2(1-\rho_3)}$, where the loss rate's driver $A_T^{(3)}$ is implicitly related to the former one, by (56), and explicitly related to the latter two, by (53). This is both an advantage as well as an unpleasant side-effect and certainly increases the analytical complexity w.r.t. determining the VaR and DLGD, as mentioned in Section 2.1.

Finally, notice that $A_T^{(3)}$ satisfies (38), where

$$\begin{aligned}\mathrm{Corr}\left(A_T^{(3)}, A_T^{(1)}\right) &= \frac{\rho_3 + \min\left\{\rho_1, \rho_2\left(1-\rho_3\right)\right\}}{\sqrt{1+2\min\left\{\rho_1\rho_3, \rho_2\left(1-\rho_3\right)\right\}}} \geq 0;\\ \mathrm{Corr}\left(A_T^{(3)}, A_T^{(2)}\right) &= \frac{\left(1-\rho_3\right) + \min\left\{\rho_1\rho_3, \rho_2\right\}}{\sqrt{1+2\min\left\{\rho_1\rho_3, \rho_2\left(1-\rho_3\right)\right\}}} \geq 0.\end{aligned} \tag{55}$$

In line with the third requirement of (38), the above two expressions are bounded from below by $\mathrm{Corr}\,(Z_1, Z_2) = \min\{\rho_1, \rho_2\}$. However, contrary to the Normal one-factor model, the above expressions are not symmetric in ρ_1 and ρ_2. Fig. 2 depicts the above linear correlation coefficients, as a function of ρ_3 for several pairs (ρ_1, ρ_2). Obviously, $\mathrm{Corr}\,(Z_3, Z_1) = \mathrm{Corr}\,(Z_3, Z_2)$, for $\rho_3 = 0.50$, only if $\rho_1 = \rho_2$. Note also that $\mathrm{Corr}\,(Z_3, Z_1)$, for the pair (ρ_1, ρ_2), behaves as $\mathrm{Corr}\,(Z_3, Z_2)$, for the pair (ρ_2, ρ_1). Furthermore, as compared to the Normal one-factor model, in the case of the shifted Gamma model the former correlation coefficient increases much more slowly and the latter correlation coefficient decreases much more quickly as a function of ρ_3. Hence in the latter model, ρ_3 dominates the dependence less between $A_T^{(3)}$ on the one hand and $A_T^{(1)}$ and $A_T^{(2)}$ on the other.

Given (53) and (54), and assuming the LGD Λ follows a Beta distribution, the relation between the loss rate and the risk factor $A^{(3)}$ is given by

$$L_t = h_2\left(\mathrm{A}_\mathrm{T}^{(3)}\right) = \begin{cases}\mathbb{B}_{a,b}^{[-1]}\left(1 - \dfrac{F_{A_T^{(3)},A_T^{(1)}}\left(A_T^{(3)}, H_t^d\right)}{1-\Gamma_{\alpha_g,\beta_g}\left(\mu_g - H_t^d\right)}\right); & -\infty \leq A_T^{(1)} < H_t^d;\\ 0; & H_t^d \leq A_T^{(1)} \leq \mu_g,\end{cases} \tag{56}$$

where the joint CDF of the couple $\left(A_T^{(3)}, A_T^{(1)}\right)$ is generally unknown. Indeed, let $G_s \sim \mathrm{Gamma}\left(\alpha_g s, \beta_g\right)$, for $s > 0$, then it follows from (53) and (54) that

$$A_T^{(3)} \stackrel{d}{=} \mu_g - \left[G_{\rho_3} + G_{1-\rho_3}\right],$$

which is not necessarily equal in distribution to $\mu_g - G_1$, since

$$\mathrm{Corr}\left[G_{\rho_3}, G_{1-\rho_3}\right] = \mathrm{Corr}\left[A_{\rho_3 T}^{(1)}, A_{(1-\rho_3)T}^{(2)}\right] = \frac{\min\left\{\rho_1\rho_3, \rho_2\left(1-\rho_3\right)\right\}}{\sqrt{\rho_3\left(1-\rho_3\right)}} \geq 0.$$

Hence, unlike the Normal one-factor model, the distribution of the random variate $A_T^{(3)}$ that drives the loss rate, and thus also of the pair $\left(A_T^{(3)}, A_T^{(1)}\right)$, is generally unknown.[8] Therefore, in order to be able to determine (the inverse of) $h_2\left(A_T^{(3)}\right)$ we have to numerically estimate the argument

$$\frac{F_{A_T^{(3)},A_T^{(1)}}\left(A_T^{(3)}, H_t^d\right)}{1-\Gamma_{\alpha_g,\beta_g}\left(\mu_g - H_t^d\right)} = F_{A_T^{(3)}\left|A_T^{(1)}\le H_t^d\right.}\left(A_T^{(3)}\right). \tag{57}$$

Once we have constructed the above conditional CDF, it is straightforward to determine the inverse $a_3 = h_2^{[-1]}(l)$, for $l \in (0, 1]$.

Finally, let $y(\omega)$ be a particular realization of the process Y, then, Eqs. (39), (40) and (66) lead to

$$\Pr\left[L_t \le l \left| A_T^{(1)} \le H_t^d, Y = y(\omega)\right.\right] = 1 - \frac{\Pr\left[A_T^{(3)} \le h_2^{[-1]}(l), A_T^{(1)} \le H_t^d \,|Y = y(\omega)\right]}{\Pr\left[A_T^{(1)} \le H_t^d \,|Y = y(\omega)\right]}$$
$$= \frac{\Gamma_{\varrho_1\alpha_g,\beta_g}\left[g\left(u_1, v_1, w_1\right)\right] - \Gamma_{\varrho_1\alpha_g,\beta_g,\varrho_2\alpha_g,\beta_g,\rho}\left[g\left(u_1, v_1, w_1\right), g\left(u_2, v_2, w_2\right)\right]}{1-\Gamma_{\varrho_2\alpha_g,\beta_g}\left[g\left(u_2, v_2, w_2\right)\right]}, \tag{58}$$

$$\begin{aligned}\Pr\left[L_t \le l\,|Y = y(\omega)\right] = {} & \Gamma_{\varrho_1\alpha_g,\beta_g}\left[g\left(u_1, v_1, w_1\right)\right] + \Gamma_{\varrho_2\alpha_g,\beta_g}\left[g\left(u_2, v_2, w_2\right)\right] \\ & - \Gamma_{\varrho_1\alpha_g,\beta_g,\varrho_2\alpha_g,\beta_g,\rho}\left[g\left(u_1, v_1, w_1\right), g\left(u_2, v_2, w_2\right)\right],\end{aligned} \tag{59}$$

and

$$\begin{aligned}&\mathrm{E}\left[L_t \left| A_T^{(1)} \le H_t^d, Y = y(\omega)\right.\right] \\ &= 1 - \frac{\int_{l=0}^{1}\left(\Gamma_{\varrho_1\alpha_g,\beta_g}\left[g\left(u_1, v_1, w_1\right)\right] - \Gamma_{\varrho_1\alpha_g,\beta_g,\varrho_2\alpha_g,\beta_g,\rho}\left[g\left(u_1, v_1, w_1\right), g\left(u_2, v_2, w_2\right)\right]\right)\mathrm{d}l}{1-\Gamma_{\varrho_2\alpha_g,\beta_g}\left[g\left(u_2, v_2, w_2\right)\right]},\end{aligned} \tag{60}$$

with

$$\begin{aligned}&\varrho_1 = (1-\rho_1)\,\rho_3 + (1-\rho_2)\,(1-\rho_3) \in (0,1)\,,\\ &\varrho_2 = (1-\rho_1) \in (0,1)\,,\\ &\rho = \rho_3\sqrt{\frac{\varrho_2}{\varrho_1}} \in (0,1)\,,\\ &u_i = \varrho_i\mu_g, \quad i = 1,2,\\ &v_1 = h_2^{(-1)}(l),\\ &v_2 = H_t^d,\\ &w_1 = y_{\rho_1\rho_3} + y_{\rho_2(1-\rho_3)},\\ &w_2 = y_{\rho_1},\\ &g\left(u, v, w\right) = u - v + w,\end{aligned}$$

and $\Gamma_{m,n}$ the CDF of a Gamma distribution with shape parameter $m > 0$ and scale parameter $n > 0$ and $\Gamma_{m_1,n_1,m_2,n_1,\rho}$ the joint CDF of a pair (G_1, G_2), with $\mathrm{Corr}\left[G_1, G_2\right] = \rho \in (0, 1)$, where G_i follows a Gamma distribution with shape parameter $m_i > 0$ and scale parameter $n_i > 0$.

In order to compute the latter bivariate CDF we use five parameter bivariate Gamma CDF in [22], i.e. let $G_i \sim \mathrm{Gamma}\left(m_i, n_i\right),\ i = 1, 2$, with $\mathrm{Corr}\left[G_1, G_2\right] = \rho \ge 0$, then, provided that $m_1 < m_2$,

$$\begin{aligned}\Gamma_{m_1,n_1,m_2,n_2,\rho}\left[g_1, g_2\right] &= \Pr\left[G_1 \le g_1, G_2 \le g_2\right],\\ &= \psi\sum_{j=0}^{+\infty}\sum_{k=0}^{+\infty} c_{j,k} H\left(m_1 + k, \frac{n_1 g_1 1_{g_1>0}}{1-\eta}\right)\\ &\quad\times H\left(m_2 + j + k, \frac{n_2 g_2 1_{g_2>0}}{1-\eta}\right); \quad 0 \le \rho < \sqrt{\frac{m_1}{m_2}},\end{aligned} \tag{61}$$

[8] Though it is possible, in some cases, to determine the CDF of a sum of correlated Gamma random variables. Alouini et al. [21], among others, consider the case of equal shape parameters.

Table 1
Basel II capital charges.

PD	Charge_α	
	E [LGD] = 10	E [LGD] = 60
1	2.10	12.61
5	3.22	19.32
10	4.05	24.28
25	4.94	29.61

with

$$\eta = \rho\sqrt{\frac{m_2}{m_1}} \in [0, 1),$$

$$\psi = \frac{(1-\eta)^{m_2}}{\Gamma(m_1)\Gamma(m_2 - m_1)},$$

$$c_{j,k} = \frac{\eta^{j+k}\Gamma(m_2 - m_1 + j)}{j!k!\Gamma(m_2 + j + k)},$$

where $\Gamma(\cdot)$ is the Gamma function and $H(\cdot, \cdot)$ is the lower incomplete Gamma function, defined as

$$H(a, z) = \int_{s=0}^{z} s^{a-1}\mathrm{e}^{-s}\mathrm{d}s.$$

Note that the requirement $m_1 < m_2$ is necessary in order to avoid the poles $0, -1, -2, \ldots$ of the Gamma function in computing ψ and $c_{j,k}$. The limit case where $m_1 = m_2 = m$, which occurs if $\rho_1 = \rho_2$ or $\rho_3 = 1$, can be dealt with using the four parameter bivariate Gamma CDF in [22], i.e.,

$$\Gamma_{m,n_1,m,n_2,\rho}[g_1, g_2] = \widetilde{\psi}\sum_{k=0}^{+\infty}\widetilde{c}_k H\left(m + k, \frac{n_1 g_1 1_{g_1>0}}{1-\rho}\right) H\left(m + k, \frac{n_2 g_2 1_{g_2>0}}{1-\rho}\right); \quad 0 \le \rho < 1, \tag{62}$$

with

$$\widetilde{\psi} = \frac{(1-\rho)^m}{\Gamma(m)},$$

$$\widetilde{c}_k = \frac{\rho^k}{k!\Gamma(m+k)}.$$

Finally, it is left as an exercise to the reader to verify that the association parameter $\eta = \rho_3$, if $\varrho_1 > \varrho_2$ (or equivalently $\rho_1 > \rho_2$) and that $\eta = \rho_3\frac{\varrho_2}{\varrho_1} = \frac{1}{1+\frac{(1-\rho_2)(1-\rho_3)}{\rho_3(1-\rho_1)}}$, if $\varrho_1 \le \varrho_2$. Hence, by construction, we always have $\eta \in (0, 1)$. Note that the boundary cases $\rho_3 = 1$ or $\rho_2 = 1$ imply $\eta = 1$, whereas $\rho_3 = 0$ or $\rho_1 = 1$ imply $\eta = 0$.

6. Numerical results

In this section we implement stochastic recovery in the Normal one-factor model and the (shifted) Gamma one-factor model. In a first, Basel II-oriented, experiment, we will examine the sensitivity of the required capital charge, i.e. $\text{VaR}_\alpha(L_t) - \text{E}[L]$, and the DLGD, at level $\alpha = 99.99\%$, to changes in the expected value of the LGD (assumed to be Beta(a, b) distributed), the PD and the correlation coefficients $\rho_i, i = 1, 2, 3$. In a second, CDO base-correlation curve-oriented, experiment we will compare the relation between the CLGD and the systematic factor induced by our framework to the corresponding LGD(C)-curves according to Amraoui and Hitier [14] and Andersen and Sidenius [10].

6.1. Basel II: sensitivity analysis

In this section we will look at the sensitivity of the required capital charge, i.e. $\text{VaR}_\alpha(L_t) - \text{E}[L]$, and the DLGD, at level $\alpha = 99.99\%$, to changes in the expected value of the LGD, the PD and the correlation coefficients $\rho_i,\ i = 1, 2, 3$; both under the Normal one-factor model and the Gamma one-factor model. Furthermore, we will compare the model values for the required capital to the requirements under the Basel II Capital Accord. The latter are summarized in Table 1 (all figures in %), for varying PDs and varying (expected) LGDs. Note that the original formulas use $\alpha = 99.9\%$ instead of 99.99% (cf. Basel Committee on Banking Supervision [23]).

Unless specifically stated otherwise, the following parameters will be used throughout this section:

- $T = 1$ year;
- $\text{PD} \overset{\text{not.}}{=} p_T^d = 25\%$;

Table 2
Capital charge, VaR and DLGD at level $\alpha = 99.99\%$ for the Normal one-factor model ($\rho 1 = \rho_2$ fixed).

PD	CPD	ρ_3	E [LGD] = 10			E [LGD] = 60		
			Charge_α	VaR_α	DLGD_α	Charge_α	VaR_α	DLGD_α
		0	5.15	5.25	22.62	15.06	15.66	70.16
1	22.01	50	5.40	5.50	24.46	14.87	15.78	71.34
		100	3.34	3.34	15.63	13.62	14.22	64.62
		0	7.99	8.49	22.84	23.19	26.19	70.40
5	37.20	50	8.61	9.10	24.96	23.01	26.53	71.67
		100	5.49	5.99	16.09	21.17	24.17	64.98
		0	10.59	11.59	23.79	29.56	35.56	70.46
10	50.47	50	11.65	12.60	25.01	29.53	36.14	71.71
		100	7.51	8.50	16.84	27.09	33.09	65.56
		0	15.19	17.69	23.84	37.76	52.76	70.96
25	74.36	50	17.60	19.93	26.80	38.53	54.00	72.63
		100	11.55	14.03	18.87	34.88	49.88	67.08

- $\text{E}[\text{LGD}] = 60\%$;
- $\text{Var}[\text{LGD}] = 1\%$;
- $\rho_3 = 50\%$;
- $\alpha_g = 1$,

where the latter is the shape parameter of the Gamma one-factor model. The above parametrization implies that $a = 0.8$, $b = 7.2$, $\beta_g = \sqrt{\alpha_g} = 1$ and $\mu_g = \frac{\alpha_g}{\beta_g} = 1$. Furthermore, the exposures to the common risk factor(s) are assumed to be equal and set according to the rule for sovereign exposures, outlined in the Basel II Capital Accord (2003), i.e. $\rho_1 = \rho_2 = \rho_{se}(\text{PD})$, where

$$\rho_{se} \overset{\text{not.}}{=} \rho_{se}(\text{PD}) = \frac{1 - \text{e}^{-50\text{PD}}}{1 - \text{e}^{-50}} 0.12 + \left(1 - \frac{1 - \text{e}^{-50\text{PD}}}{1 - \text{e}^{-50}}\right) 0.24, \tag{63}$$

which yields $\rho_{se}(0.25) = 12\%$, under the above parameter setting.

Finally, note that, as a consequence of setting the confidence level at 99.99%, the CPD of the Gamma one-factor model will be equal to 100%, for every $\text{PD} \in [0, 1]$, provided $\rho_{se}(\text{PD}) > 4.56\text{e}{-4}$. To see this, recall that $\text{VaR}_\alpha(L_t) = \kappa \left[F_C^{[-1]}(1-\alpha)\right]$, with C the common factor. In case of the Gamma one-factor model, using the above parameter setting, the latter is represented by $Y_{\rho_{se}}$ and it can be shown that

$$F_{Y_{\rho_{se}}}^{[-1]}(1-\alpha) = \rho_{se}\mu_g - \Gamma^{[-1]}_{\alpha_g \rho_{se}, \beta_g}(\alpha). \tag{64}$$

Moreover we already mentioned that the CPD of the Gamma one-factor model will be equal to 100%, independent of the idiosyncratic factor, if the realization $y_{\rho_{se}}$ of the common factor satisfies $y_{\rho_{se}} \le \rho_{se}\mu_g - H_T^d$, which, by (64), is equivalent to $\Gamma^{[-1]}_{\alpha_g \rho_{se}, \beta_g}(\alpha) \ge H_T^d = \mu_g - \Gamma^{[-1]}_{\alpha_g, \beta_g}[1 - \text{PD}]$. The latter, on its turn, is equivalent to $\text{PD} \le 1 - \Gamma_{\alpha_g, \beta_g}\left[\mu_g - \Gamma^{[-1]}_{\alpha_g \rho_{se}, \beta_g}(\alpha)\right]$, from which the above assertion follows, since $\mu_g - \Gamma^{[-1]}_{\alpha_g \rho_{se}, \beta_g}(\alpha) = 0$, for $\alpha = 99.99\%$, if $\rho_{se}(\text{PD}) > 4.56\text{e}{-4}$.

Tables 2 and 3 (all figures in %) show the required capital charges, the VaR of the loss rate and the DLGD, with $\alpha = 99.99\%$, for varying PDs and varying expected LGDs at different levels of ρ_3, with $\rho_1 = \rho_2 = \rho_{se(\text{PD})}$, under the Normal one-factor model and the Gamma one-factor model, respectively. Tables 4 and 5 (all figures in %) give the results for $\rho_3 = 1$ and $\rho_1 = \rho_2 = k\rho_{se(\text{PD})}$, for $k = 1, 2, 4$, providing an insight into the impact of the dependence on the systematic factor(s).

Firstly, all tables clearly show that the capital charges under a model that takes DLGD into account are (significantly) higher than their Basel II equivalents (cf. Table 1), irrespective of the applied correlations. Note, in this regard, that the DLGD is always higher than the expected LGD, where the latter difference increases with the PD. Hence, the potential for realized LGDs to be higher than average during times of high default rates may be a material source of unaccounted credit losses for some exposures or portfolios.

Moreover, the relative difference between the expected LGD and the DLGD is much higher at lower values of the former. This can be seen by comparing the figures, obtained by the Normal (Gamma) one-factor model, for E[LGD] = 10%, to those for E[LGD] = 60%. In the former case, the DLGD is approximately between 1.5 (2.4) and 2.5 (5.3) times E[LGD], whereas in the latter case, the DLGD is approximately between 1.08 (1.18) and 1.17 (1.4) times E[LGD]. The latter comparison also reveals that the (heavy-tailed) Gamma one-factor model produces substantially higher DLGDs, VaRs and capital charges than the Normal one-factor model.

Secondly, as for the impact of ρ_3, and hence a selection between the comonotonic [6] model ($\rho_3 = 1$), intermediate models ($\rho_3 \in (0, 1)$) and the Hillebrand [19]-type model ($\rho_3 = 0$), from Tables 2 and 3 it is apparent that the Tasche model systematically produces lower capital charges and lower DLGDs, whereas the figures produced by intermediate models and

Table 3
Capital charge, VaR and DLGD at level $\alpha = 99.99\%$ for the Gamma one-factor model ($\rho 1 = \rho_2$ fixed).

PD	CPD	ρ_3	E [LGD] = 10			E [LGD] = 60		
			Charge$_\alpha$	VaR$_\alpha$	DLGD$_\alpha$	Charge$_\alpha$	VaR$_\alpha$	DLGD$_\alpha$
1	100	0	38.60	38.62	38.62	78.63	78.77	78.77
		100	23.88	23.95	23.95	70.40	70.79	70.79
5	100	0	48.50	48.59	48.59	79.63	80.27	80.27
		100	34.68	34.93	34.93	77.17	78.65	78.65
10	100	0	51.50	51.68	51.68	80.19	81.45	81.45
		100	38.98	39.44	39.44	76.33	79.02	79.02
25	100	0	52.11	52.63	52.63	80.33	83.61	83.61
		100	45.48	46.60	46.60	73.88	80.26	80.26

Table 4
Capital charge and DLGD at level $\alpha = 99.99\%$ for the Normal one-factor model ($\rho 3 = 1$ fixed).

PD	CPD	$k \times \rho_{se}$	E [LGD] = 10		E [LGD] = 60	
			Charge$_\alpha$	DLGD$_\alpha$	Charge$_\alpha$	DLGD$_\alpha$
1	22.01	1	3.34	15.63	13.62	34.62
	49.13	2	9.22	18.97	35.35	67.07
	97.52	4	17.96	23.15	72.13	74.58
5	37.20	1	5.49	16.09	21.17	64.98
	61.45	2	11.77	19.97	38.62	67.73
	93.24	4	19.06	24.25	65.70	73.69
10	50.47	1	7.51	16.84	27.09	65.56
	73.49	2	14.74	21.41	44.49	68.71
	96.51	4	22.36	26.40	66.52	75.14
25	74.36	1	11.55	18.87	34.88	67.08
	90.59	2	20.31	25.16	49.47	71.17
	99.58	4	28.44	31.76	63.05	78.38

Table 5
Capital charge and DLGD at level $\alpha = 99.99\%$ for the Gamma one-factor model ($\rho 3 = 1$ fixed).

PD	CPD	$k \times \rho_{se}$	E [LGD] = 10		E [LGD] = 60	
			Charge$_\alpha$	DLGD$_\alpha$	Charge$_\alpha$	DLGD$_\alpha$
1	100	1	23.88	23.95	70.40	70.79
		2	30.25	30.34	77.90	78.43
		4	37.29	37.40	77.89	78.48
5	100	1	34.68	34.93	77.17	78.65
		2	39.36	39.75	76.71	79.02
		4	50.20	50.68	76.89	79.91
10	100	1	38.98	39.44	76.33	79.02
		2	43.99	44.72	75.54	79.78
		4	50.54	51.48	76.88	82.21
25	100	1	45.48	46.60	73.88	80.26
		2	50.22	52.00	72.23	82.60
		4	52.58	54.92	75.18	89.20

the Hillebrand [19]-type model are comparable, with a slight tendency for intermediate models to produce marginally higher values. On the other hand, the relative increase of the capital charge, the VaR and the DLGD as a function of PD appears to be proportional to ρ_3 and hence is maximal at $\rho_3 = 100\%$. This is, of course, due to the fact that ρ_3 dominates the dependence between defaults and losses. Finally, note that the sensitivities w.r.t. ρ_3 and the PD, discussed in this paragraph, hold for both the Normal one-factor model and the Gamma one-factor model and that, especially in the former case, the relative increase of the capital charge and the DLGD, as a function of these parameters, is again more significant at lower values of E[LGD]. From this we conclude that inducing a higher correlation between defaults and losses, will result in a lower capital charge, for a given PD, but, simultaneously, will cause capital charges and DLGDs to increase faster as a function of PD.

Next, Tables 4 and 5, depicting the dependence of the capital charge, VaR and DLGD w.r.t. ρ_{se}, i.e. the exposure to the systematic risk(s), show no surprises. In line with one's expectations, increasing the latter exposure, and thereby the dependence between obligors, results in higher capital charges and DLGDs. Note, again, that the increase as a function of PD of the above quantities is more significant in the Normal one-factor model. This is of course due to the fact that the absolute figures are substantially lower in the latter case, but can also be attributed to the different dependency structure underlying both models (cf. Figs. 1 and 2). Notice, finally, that the case $\rho_3 = 50\%$ (or in general $\rho_3 \in (0, 1)$) is not covered in the case of the Gamma one-factor model (cf. Table 3), in order to avoid having to deal with 2 systematic risk factors. Though it is theoretically possible to express the expected loss and hence the VaR by a function of two systematic factors (cf. [17]).

Finally, as already mentioned repeatedly before, the Gamma one-factor model, due to the apparent skewness and fat-tailedness of its underlying distribution, produces substantially higher capital charges and DLGDs as compared to the Normal one-factor model. Hence, besides accounting for stochastic LGD, when assessing a portfolio's credit risk, using stochastic processes based on an underlying distribution that takes skewness and fat tail-behavior into account, adds an additional level of safety to determining an adequate capital requirement.

6.2. CDO base correlation

In this section we will briefly compare our stochastic LGD approach to two models used in practice, i.e. the Amraoui–Hitier [14] model (BNP Paribas) and the Andersen–Sidenius [10] model (Bank of America). Both models are initially developed within a CDO valuation-context, with the goal of flattening the so called base-correlation curve. The latter is found by calibrating, e.g., the Normal one-factor model, to the price of a first loss tranche (FLT), i.e. to the sum of all standardized tranches up to an attachment point (e.g. the 0%–6% FLT follows from the sum of 0%–3% and 3%–6% tranches). The curve of correlations obtained by calibrating to first loss tranches is called the base-correlation curve and turns out to be much smoother and more stable than that obtained by calibrating to plain tranches.

We will start with a short discussion of the latter approaches and indicate the significant differences and shortcomings w.r.t. our approach. Already by construction, our framework turns out to be both more general and more flexible. Next, we will show that our model can easily reproduce the results generated by either of the former two models. In this respect, in a CDO valuation-context it suffices to concentrate on the CLGD. Indeed, CDO pricing models are typically based on a Monte-Carlo simulation, where in each iteration a level for the systematic factor(s) C is drawn, that is then used as an input in the generation of the assets' standardized return $R_1\left(C, I; \rho_1\right)$ and, assuming stochastic LGD, the generation of the LGD conditional on C. Hence, we only need to verify if our model can reproduce random LGD(C)-curves generated by any of the above discussed models. This then, heuristically, provides evidence to state that our framework can also be used to derive stable (flat) base-correlation curves, with the same rate of success as the Amraoui–Hitier model and the Andersen–Sidenius model.

6.2.1. A brief review

Recall that according to our framework (cf. Section 5.2)

$$\text{LGD} = \Lambda = h_2(R_3) = F_{\Lambda}^{[-1]}\left(1 - \frac{F_{R_3,R_1}\left(R_3, H_t^d\right)}{p_t^d}\right), \tag{65}$$

and

$$\mathrm{E}\left[LGD \middle| C = c\right] = \frac{\int_{l=\lambda_l}^{\lambda_u} \Pr\left[R_3 \le h_2^{[-1]}(l), R_1 \le H_t^d \middle| C = c\right] \mathrm{d}l}{\Pr\left[R_1 \le H_t^d \middle| C = c\right]}, \tag{66}$$

provided that $\inf(R_1) \le R_1 < H_t^d$, where the random variable Λ is distributed according to a law D with bounded support $[\lambda_l, \lambda_u] \subseteq [0, 1]$. Hence, our model can deal with any possible scenario of lower and/or upper bounds on the LGD. Moreover, it is possible to cover all of these scenarios under the assumption that Λ follows a Beta (a, b) distribution, having support $[0, 1]$, by simply transforming the latter, such that

$$\text{LGD} = \lambda_l + (\lambda_u - \lambda_l)\,\Lambda,$$

where the parameters $a, b > 0$ are set in order to match predetermined values, e.g. based on historical observations, of the expected value and variance of the LGD. Note that, theoretically,

$$\begin{aligned} a &= \frac{\mu_\Lambda}{\sigma_\Lambda^2}\left[\mu_\Lambda\left(1 - \mu_\Lambda\right)\right]; \\ b &= \frac{1 - \mu_\Lambda}{\sigma_\Lambda^2}\left[\mu_\Lambda\left(1 - \mu_\Lambda\right)\right], \end{aligned} \tag{67}$$

with

$$\begin{aligned} \mu_\Lambda &= \frac{\mu_{\text{LGD}} - \lambda_l}{\lambda_u - \lambda_l}; \\ \sigma_\Lambda^2 &= \left(\frac{\sigma_{\text{LGD}}}{\lambda_u - \lambda_l}\right)^2, \end{aligned}$$

with μ_X the expected value of X and σ_X^2 the variance of X, for any random variable X.

6.2.2. Amraoui and Hitier [14]

In this work, the LGD is linked directly to the common factor of the underlying Normal one-factor default model through a deterministic function based on a recovery mark down (MD) argument and an appropriate transformation of the (un)conditional default probability.

Assume, as before, that default is governed by the random variable $R_1(C, I; \rho_1)$. Furthermore, let $\overline{\text{Rec}}$ denote the stripping recovery, e.g. 40%, and $g_{\rho_1}(p; c)$ the CPD, given $C = c$, corresponding to the unconditional default probability p. Then, the LGD, conditional on $C = c$, is given by

$$\text{LGD}(c) = (1 - \text{Rec}(c)) = \left(1 - \widetilde{\text{Rec}}\right) \frac{g_{\rho_1}\left(\tilde{p}_t^d; c\right)}{g_{\rho_1}\left(p_t^d; c\right)}, \tag{68}$$

with $\tilde{p}_t^d = \frac{1-\overline{\text{Rec}}}{1-\widetilde{\text{Rec}}} p_t^d$, such that the expected recovery conditional on default is the same as the mid recovery $\overline{\text{Rec}}$, and $0 \le \widetilde{\text{Rec}} \le \overline{\text{Rec}}$ a recovery mark down.

Note that, contrary to our method, the above model does not require the specification of the recovery distribution. This clearly reduces the analytical complexity, but comes at the cost of reduced model flexibility. Furthermore, mark that $\text{LGD}(c) \in [0, 1 - \widetilde{\text{Rec}}]$ and is hence bounded from above by $1 - \widetilde{\text{Rec}}$, but there is no obvious way to bound the LGD from below, unlike our model.

6.2.3. Andersen and Sidenius [10]

In line with the framework discussed in this text, the latter authors achieve stochastic LGD through a (multi-)factor model, sharing one (or more) latent variable(s) with the structural model for default, which is then mapped onto [0, 1] by some function $\mathcal{F}$. In the case of the standard Gaussian copula model, $\mathcal{F}$ is set to be the standard Normal CDF, leading to

$$\begin{aligned} Z_1 &= \sqrt{\theta_{as}} X + \sqrt{1-\theta_{as}} \xi_1; \\ Z_2 &= \mu_{as} + \vartheta_{as} X + \nu_{as} \xi_2; \\ \text{LGD} &= l_{\max} \left(1 - \Phi\left[Z_2\right]\right), \end{aligned} \tag{69}$$

with $l_{\max} \in (0, 1]$ an upper bound on the LGD, $\theta_{as} \in (0, 1)$, $\mu_{as} \in \mathbb{R}$, $\vartheta_{as} \in \mathbb{R}$, $\nu_{as} > 0$ and $X, \xi_1, \xi_2 \overset{\text{i.i.d.}}{\sim} \text{N}(0, 1)$. Recall that default occurs when Z_1 hits the default barrier H_t^d. Furthermore, notice that $\Phi[Z_2]$ describes the recovery rate (Rec). Note, also, that (69) is related to the Hillebrand [19]-type model, which, apart from the final mapping onto the LGD, can be recovered by setting $\mu_{as} = 0$, $\nu_{as} = \sqrt{1-\vartheta_{as}^2}$, assuming $\vartheta_{as} \in [0, 1]$. Notice, however, that under the latter two models, contrary to our framework (65) and the Amraoui–Hitier model (68), there is no explicit link between the LGD and the unconditional default probability p_t^d.

Furthermore, assuming that $L_t = 1 - \text{Rec}$ and that $\Pr\left[L_t = 0 \,\middle|\, 1_t^d = 0\right] = 1$, it follows from Andersen and Sidenius [10] that

$$\text{E}\left[\text{LGD} \,|X = x\right] = \int_{l=0}^{l_{\max}} \Phi\left[\frac{\Phi^{[-1]}\left[1-l\right] - \mu_{as} - \vartheta_{as} x}{\nu_{as}}\right] \text{d}l. \tag{70}$$

It should be mentioned that the latter authors actually derive an expression for the expected value of the Recovery-Given-Default (RGD). In this respect, the above requirements w.r.t. L_t are necessary to ensure that $\text{E}[\text{LGD}\,|X] = 1 - \text{E}[RGD\,|X]$. Note that the latter expression closely resembles the corresponding formula in the Hillebrand [19]-type setting, under the Normal one-factor model, of our general framework (cf. Appendix A.2.3). The only difference is in the use of $\Phi^{[-1]}[1-l]$ instead of $h_2^{[-1]}(l)$.

Some concluding remarks. Note, from (69), that it is unclear what the distribution of the LGD is. Furthermore, though the factor $l_{\max}$ allows one to bound the LGD from above, also in this model there is no obvious consistent solution to bound the LGD from below. Moreover, note from (69) that Z_2 will generally not be standard Normally distributed and, as such, $\Phi[Z_2]$ will not be uniform on the entire interval [0, 1]. Hence, even when $l_{\max} = 1$, the LGD will be restricted to a region $\mathcal{L} \subseteq [0, 1]$. Some of the latter issues may be solved, following the methodology exhibited in this text, by an additional mapping of the image of the function $\mathcal{F}$ through the quantile function of a distribution with bounded support $[\lambda_l, \lambda_u] \subseteq [0, 1]$. Finally, as for the use of the standard Normal distribution to map the recovery onto the interval [0, 1], the authors are aware that *the use of a cumulative Gaussian distribution may be unconventional*, but note that *the more traditional choice of a specification in terms of the beta-distribution leads to a less tractable model*. However, it is our opinion that the latter statement is rather arbitrary.

6.2.4. Numerical results

In this section, we examine the impact of the systematic factor X (Y_{ρ_1}) on the conditional expected value of the LGD, in case of the Normal one-factor model (Gamma one-factor model). In the former case, the LGD(C)-curves will be compared to the ones obtained with the models discussed in Sections 6.2.2 and 6.2.3.

The following parameters are used:

- PD = 10%;
- E[LGD] = 60%;
- Var[LGD] = 4%;
- $\lambda_l = 0\%$;
- $\lambda_u = 90\%$;
- $\rho_1 = 20\%$;
- $\rho_2 = 20\%$;
- $\rho_3 \in \{0\%, 50\%, 100\%\}$;
- $\alpha_g = 1$.

Moreover, regarding our framework, we will assume that $\Lambda \sim \text{Beta}(a, b)$, with $a \approx 2.333$ and $b \approx 1.167$ (cf. (67)). Concerning the Amraoui–Hitier model, the above parametrization implies that $\widetilde{\text{Rec}} = 10\%$ and $\overline{\text{Rec}} = 40\%$. Furthermore, the parameters of the Andersen–Sidenius model are set in order to resemble the Hillebrand [19]-type model, i.e.

- $\mu_{as} = 0$;
- $\vartheta_{as} = \sqrt{\rho_2}$;
- $\nu_{as} = \sqrt{1 - \vartheta_{as}^2}$.

The resulting LGD(C)-curves, for the Normal one-factor model, are shown in Fig. 4. Fig. 3 gives the corresponding CPDs $g(p; c)$ (cf. (68)). For illustrative purposes, in case of the Amraoui–Hitier model, we plotted both the case without a mark down (MD) and the case with a MD. In the former case, though the LGD is dependent on X, it is still bounded from above by one minus the mid recovery, i.e. $1 - \overline{\text{Rec}} = 60\%$, and even a $\rho_1 = 100\%$ scenario would not be able to give spreads for the [60% 100%] super duper tranche. The MD approach, as well as the Andersen–Sidenius model and the framework presented in this text clearly overcome this problem, generating CLGDs in the entire spectrum between λ_l and λ_u that are consistent with the movement of the systematic risk X, i.e. losses are negatively correlated to X (and hence to R_1).

Furthermore, notice that both the Amraoui–Hitier MD approach and the comonotonic [6] model, i.e. $\rho_3 = 1$, generate (unrealistically) high CLGDs at levels of X corresponding to an almost zero CPD. This is a direct consequence of the rather low exposure $\rho_1 = 20\%$ to the systematic risk. This was already acknowledged by the former authors, who state that *When default correlation tends to* 0%, *the curve tends to be flat and the recovery is fixed. When the correlation tends to* 100%, *the curves are steep functions, and the recovery density has a Dirac at* $\widetilde{R}$, *with weights such that the recovery conditional on default is* $\overline{R}$. It is apparent that the Andersen–Sidenius model, the Hillebrand [19]-type model and intermediate $\rho_3 \in (0, 1)$ are less prone to this effect, and produce more realistic, or at least more traditional, sigmoid LGD(C)-curves.

Finally, mark that the latter type of models appear to generate LGD(C)-curves that (generally) lie between the corresponding curves produced by the Amraoui–Hitier model and the Andersen–Sidenius model. Hence, the latter can, to some extent, be regarded as bounds on the former. It is also clear that the former's LGD(C)-curves closely resemble the ones by Andersen–Sidenius. Hence, it makes sense to implement our framework within CDO valuation models to obtain flat(ter) base correlation curves.

W.r.t. the shifted Gamma one-factor model, Figs. 6 and 7, respectively, show the CPD and the CLGD as a function of the common factor Y_{ρ_1} of the default driver $A_T^{(1)}$. Note that, given the above parameter setting, in the case $\rho_3 = 50\%$ we actually have two correlated systematic factors $Y_{\frac{\rho_1}{2}}$ and Y_{ρ_1}. In order to clearly isolate the impact of the latter, the former is replaced by its conditional expected value, conditional on the latter, i.e. by

$$\mathrm{E}\left[Y_{\frac{\rho_1}{2}} \,\middle|\, Y_{\rho_1} = y_{\rho_1}\right] = \frac{\int_{y=y_{\rho_1}-\frac{\rho_1}{2}\mu_g}^{\frac{\rho_1}{2}\mu_g} y\gamma_{\frac{\rho_1}{2}\alpha_g,\beta_g}\left(\frac{\rho_1}{2}\mu_g - y_{\rho_1} + y\right)\gamma_{\frac{\rho_1}{2}\alpha_g,\beta_g}\left(\frac{\rho_1}{2}\mu_g - y\right)\mathrm{d}y}{\gamma_{\rho_1\alpha_g,\beta_g}\left(\rho_1\mu_g - y_{\rho_1}\right)}, \tag{71}$$

for an arbitrary realization y_{ρ_1} of Y_{ρ_1}, where $\gamma_{m,n}$ is the probability density function of the Gamma distribution with shape parameter m and scale parameter n. The latter is graphed at Fig. 5. Note the very strong positive correlation, which can be shown to be $\frac{1}{\sqrt{2}}$, between both risk factors.

Now, returning to Figs. 6 and 7, several remarks can be made. First of all, both the CPD and the CLGD increase much faster as a function of the systematic risk than their equivalents under the Normal one-factor model. As compared to the latter framework, in the former case, the curves are convex/concave, instead of showing the typical sigmoid shape. This explains why the DLGDs and the capital charges generated by the latter model are much higher than their Normal one-factor equivalents (cf. Section 6.1).

Furthermore, notice, in this respect, that the CPD almost immediately hits 100% and that the models with $\rho_3 > 0$ show some unexpected behavior to the right of this Armageddon point. Indeed, in the area where the CPD is less than 100%, under the latter models, the CLGD is positively correlated to Y_{ρ_1}. Moreover, comparing the curves corresponding to $\rho_3 = 50\%$ and $\rho_3 = 100\%$ shows that this effect is manifested more significantly for increasing values of the latter parameter, reaching its peak at the comonotonic (Tasche) model. However, though unexpected and potentially unwelcome, one should not worry over this behavior, as it is completely ruled out after multiplying the CLGD with the CPD, to obtain the conditional loss

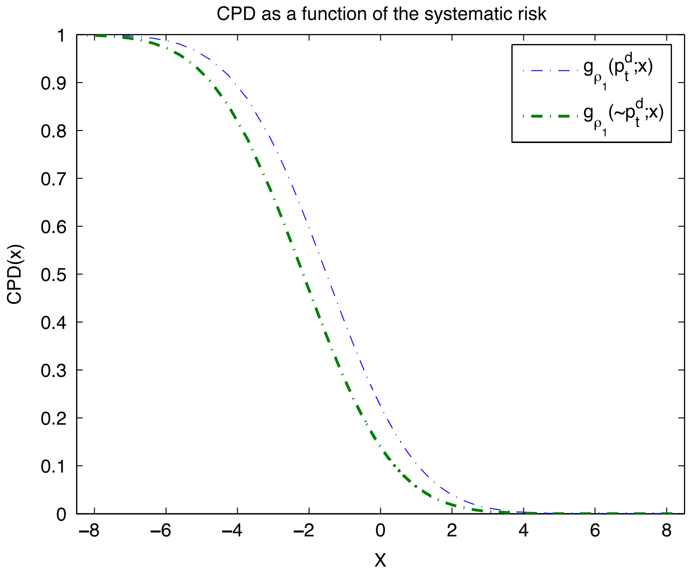

Fig. 3. CPD as a function of the systematic risk X (Normal one-factor model).

rate (CL), which, in the end, is the true variable of interest. The latter is shown, as a function of Y_{ρ_1}, in Fig. 8. Note that the previously discussed effect has vanished completely.

Finally, to get some intuition about the combined effect of the pair $\left(Y_{\rho_1}, Y_{\frac{\rho_1}{2}}\right)$ of risk factors, in Figs. 9 and 10, we plotted the CLGD and the CL as a function of the latter pair. From the former figure, it is apparent that Y_{ρ_1} is dominating the CLGD, which is obviously negatively correlated to the latter, whereas $Y_{\frac{\rho_1}{2}}$ has almost no impact, unless at very small values, where it appears to be positively correlated to the CLGD. The conclusion changes when looking at the CL, which appears to be more sensitive to $Y_{\frac{\rho_1}{2}}$. Furthermore, the CL is clearly negatively correlated to both $Y_{\frac{\rho_1}{2}}$ and Y_{ρ_1}, reaching a maximum when both factors approach their respective infimum.

7. Conclusion

In this text we have introduced a flexible framework to extend the traditional structural models for assessing (portfolio) credit risk and pricing financial derivatives in order to take into account stochastic LGD, regardless of the latter's distribution, provided its support is a compact subset of $[0, 1]$. Our method essentially relies on a common dependence of the LGD and the PD on a latent variable, representing the systemic risk. The latter dependence is governed by three correlation parameters ρ_1, ρ_2 and ρ_3 which allow one to capture a whole spectrum of stochastic LGD models.

We argued that most of the existing models for stochastic LGD can be cast in the framework presented in this text and, moreover, that the latter framework produces results, w.r.t. capital charges and DLGDs, that are consistent both with the Basel II Capital Accord and with recently developed models. Furthermore, it is apparent that, construction-wise, our framework induces greater flexibility and that regarding numerical efficiency, it certainly does not underperform currently existing methods.

Not only does our framework produce results consistent with existing methods, it also sheds light on the impact of using stochastic processes based on fat-tailed underlying distributions. The dramatic increase of the capital charges and the DLGD, caused by the latter models, accentuates the importance of granting sufficient weight to extreme downward movements of the economy, whether globally, regionally or within a certain industry. Traditional models, based on the Normal distribution, are clearly incapable of capturing these shocks and as such, even though accounting for stochastic LGD, often severely underestimate the required economic capital. A paradigm which has been proven right by the current global financial crisis. Therefore, we claim that extending the contemporary models to be able to deal with downturn LGD is insufficient. Only models based on the combined force of a fat-tailed underlying distribution and stochastic LGD can yield adequate capital requirements.

Appendix

In this Appendix we summarize the expressions for

- the relation h_2 between L_t and R_3;
- the conditional distribution function of the loss given default, conditional on $C = c$;
- the conditional expected value of the loss given default, conditional on $C = c$,

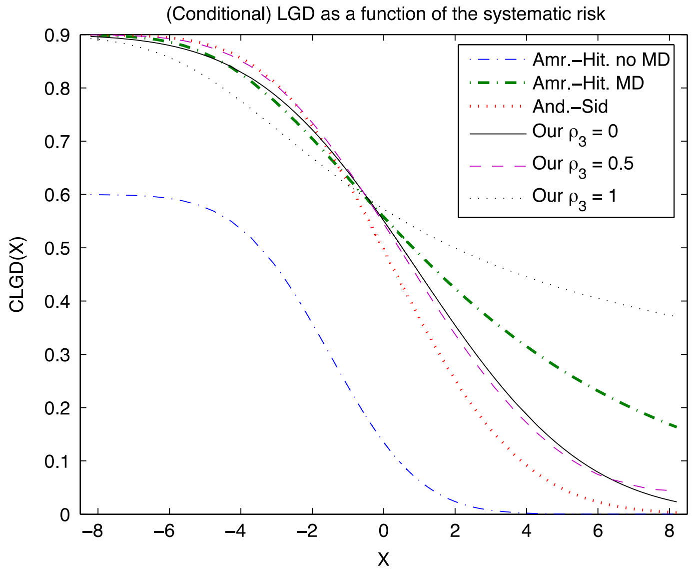

Fig. 4. CLGD as a function of the systematic risk X (Normal one-factor model).

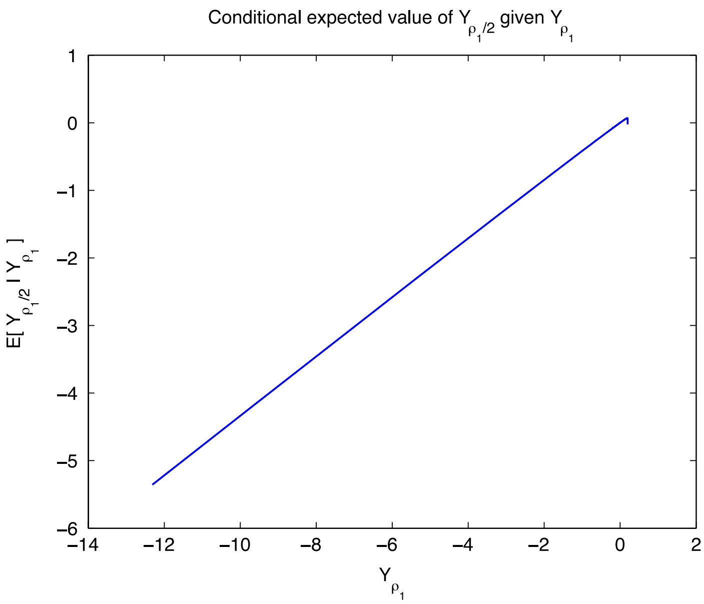

Fig. 5. Conditional expected value of $Y_{\frac{\rho_1}{2}}$ given Y_{ρ_1} (Gamma one-factor model).

in

(1) the general case, i.e. $\rho_1 \in [0, 1)$ and $\rho_2, \rho_3 \in [0, 1]$;

(2) the comonotonic case, i.e. $\rho_1 \in [0, 1)$ and $\rho_3 = 1$;

(3) the model of Hillebrand [19], i.e. $\rho_2 \in [0, 1)$ and $\rho_3 = 0$;

(4) the *three-factor* model, i.e. $\rho_1 \in [0, 1)$, $\rho_3 \in [0, 1]$ and $\rho_2 = 0$.

Appendix A.1 provides the model independent formulas, assuming the LGD follows a distribution F_Λ with support $[\lambda_l, \lambda_u] \subseteq [0, 1]$. Appendices A.2 and A.3, in turn, implement the formulas for the Normal one-factor model and the Gamma one-factor model, assuming the LGD follows a distribution F_Λ with support $[\lambda_l, \lambda_u] \subseteq [0, 1]$.

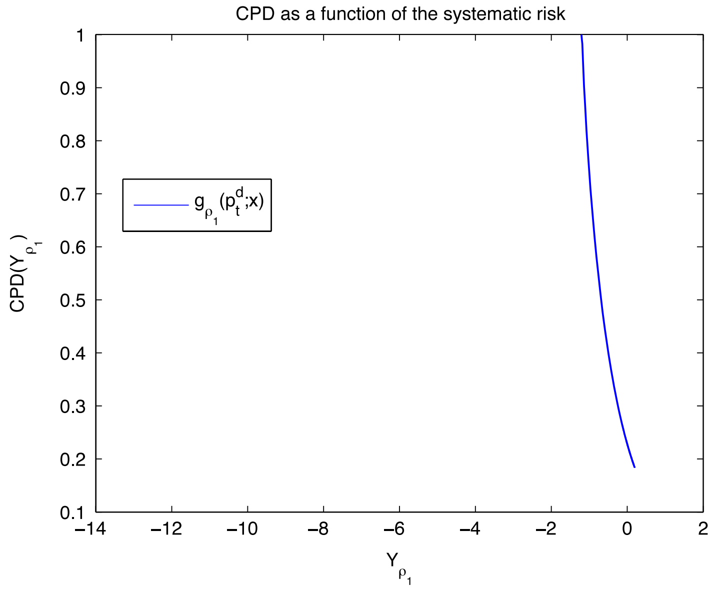

Fig. 6. CPD as a function of the systematic risk Y_{ρ_1} (Gamma one-factor model).

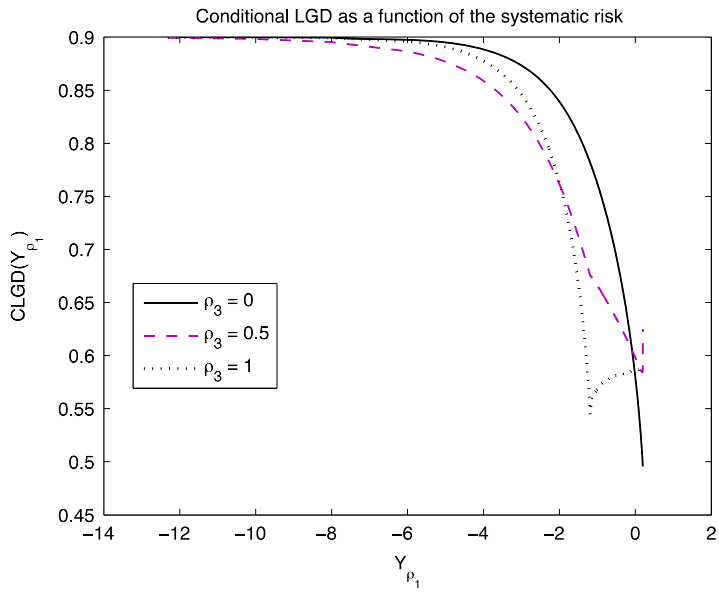

Fig. 7. CLGD as a function of the systematic risk Y_{ρ_1} (Gamma one-factor model).

A.1. Model independent formulas

A.1.1. The general case

Note that

$$R_3 \overset{\text{not.}}{=} R_3\left(R_1, R_2; \rho_3\right),$$

with

$$R_1 \overset{\text{not.}}{=} R_1\left(C, I; \rho_1\right);$$
$$R_2 \overset{\text{not.}}{=} R_2\left(C, J; \rho_2\right).$$

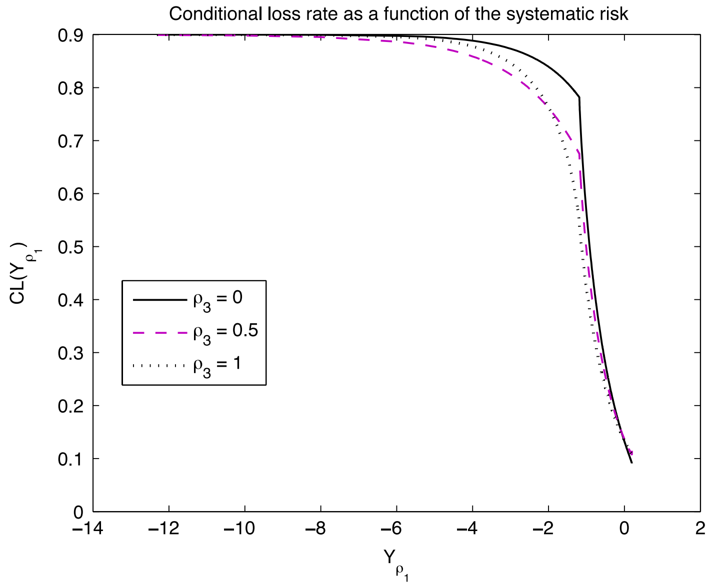

Fig. 8. Conditional loss rate as a function of the systematic risk Y_{ρ_1} (Gamma one-factor model).

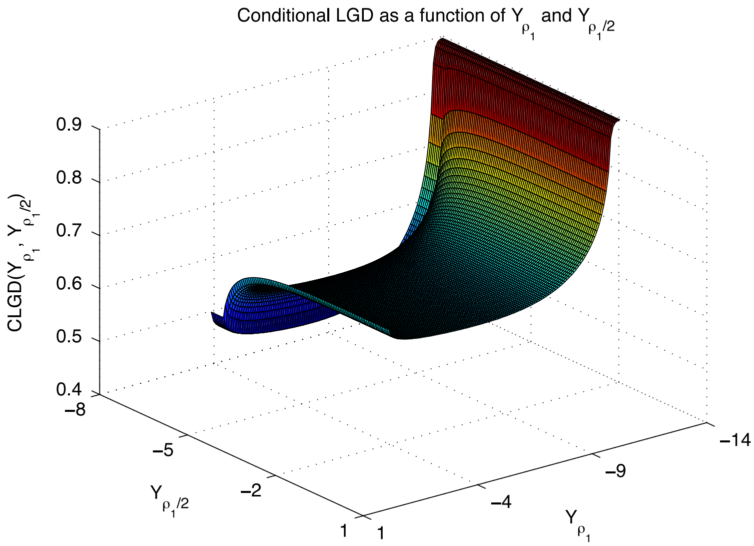

Fig. 9. CLGD as a function of the systematic risks Y_{ρ_1} and $Y_{\frac{\rho_1}{2}}$ (Gamma one-factor model).

We have,

$$L_t = h_2(R_3) = \begin{cases} F_{\Lambda}^{[-1]}\left(1 - \dfrac{F_{R_3,R_1}\left(R_3, H_t^d\right)}{p_t^d}\right); & \inf(R_1) \le R_1 < H_t^d; \\ 0; & H_t^d \le R_1 \le \sup(R_1), \end{cases}$$

$$\Pr\left[L_t \le l \middle| R_1 \le H_t^d, C = c\right] = 1 - \frac{\Pr\left[R_3 \le h_2^{[-1]}(l), R_1 \le H_t^d \middle| C = c\right]}{\Pr\left[R_1 \le H_t^d \middle| C = c\right]},$$

$$\Pr\left[L_t \le l \middle| C = c\right] = 1 - \Pr\left[R_3 \le h_2^{[-1]}(l), R_1 \le H_t^d \middle| C = c\right],$$

$$\mathrm{E}\left[L_t \middle| R_1 \le H_t^d, C = c\right] = \frac{\int_{l=\lambda_l}^{\lambda_u} \Pr\left[R_3 \le h_2^{[-1]}(l), R_1 \le H_t^d \middle| C = c\right] \mathrm{d}l}{\Pr\left[R_1 \le H_t^d \middle| C = c\right]}.$$

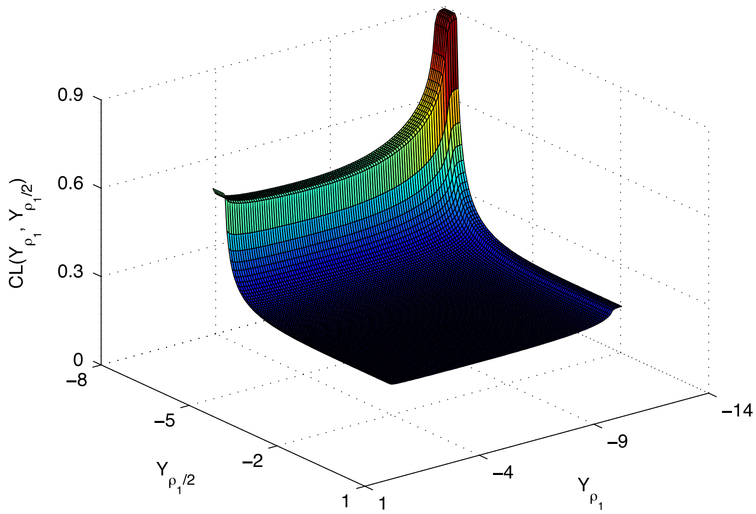

Fig. 10. Conditional loss rate as a function of the systematic risks Y_{ρ_1} and $Y_{\frac{\rho_1}{2}}$ (Gamma one-factor model).

A.1.2. The comonotonic case [6]

Note that $R_3 \equiv R_1$.

We have,

$$L_t = h_2(R_3) = \begin{cases} F_{\Lambda}^{[-1]}\left(1 - \dfrac{F_{R_3}(R_3)}{p_t^d}\right); & \inf(R_1) \le R_1 < H_t^d; \\ 0; & H_t^d \le R_1 \le \sup(R_1), \end{cases}$$

$$\Pr\left[L_t \le l \middle| R_1 \le H_t^d, C = c\right] = 1 - \frac{\Pr\left[R_3 \le h_2^{[-1]}(l) \middle| C = c\right]}{\Pr\left[R_1 \le H_t^d \middle| C = c\right]},$$

$$\Pr\left[L_t \le l \middle| C = c\right] = 1 - \Pr\left[R_3 \le h_2^{[-1]}(l) \middle| C = c\right],$$

$$\mathrm{E}\left[L_t \middle| R_1 \le H_t^d, C = c\right] = \frac{\int_{l=\lambda_l}^{\lambda_u} \Pr\left[R_3 \le h_2^{[-1]}(l) \middle| C = c\right] \mathrm{d}l}{\Pr\left[R_1 \le H_t^d \middle| C = c\right]}.$$

A.1.3. Hillebrand [19]-type model

Note that $R_3 \equiv R_2$.

We have,

$$L_t = h_2(R_3) = \begin{cases} F_{\Lambda}^{[-1]}\left(1 - \dfrac{F_{R_3,R_1}\left(R_3, H_t^d\right)}{p_t^d}\right); & \inf(R_1) \le R_1 < H_t^d; \\ 0; & H_t^d \le R_1 \le \sup(R_1), \end{cases}$$

$$\Pr\left[L_t \le l \middle| R_1 \le H_t^d, C = c\right] = 1 - \Pr\left[R_3 \le h_2^{[-1]}(l) \middle| C = c\right],$$

$$\Pr\left[L_t \le l \middle| C = c\right] = \Pr\left[L_t \le l \middle| R_1 \le H_t^d, C = c\right] p_t^{d,c} + \left(1 - p_t^{d,c}\right),$$

$$\mathrm{E}\left[L_t \middle| R_1 \le H_t^d, C = c\right] = \int_{l=\lambda_l}^{\lambda_u} \Pr\left[R_3 \le h_2^{[-1]}(l) \middle| C = c\right] \mathrm{d}l.$$

Note, from the second equation, that in the Hillebrand [19]-type model the conditional distribution of the LGD, given a level of the common factor, is independent of the random variable R_1 driving default.

A.1.4. The three-factor model

Note that $R_3 \overset{\text{not.}}{=} R_3\,(R_1, J;\, \rho_3)$.

Note that R_3 and R_1 are still dependent, even conditional on $C = c$, since they also share the random variable I, i.e. within a fixed obligor, defaults and losses are dependent through the shared idiosyncratic factor. Hence, setting $\rho_2 = 0$ only changes the dependency structure between R_1 and R_3 (without necessarily making them independent) and therefore the formulas from Appendix A.1.1 can't generally be simplified.

A.2. Normal one-factor model

A.2.1. The general case

Note that

$$Z_3 \overset{\text{not.}}{=} Z_3\left(Z_1, Z_2; \rho_3\right) = \sqrt{\rho_3} Z_1 + \sqrt{1-\rho_3} Z_2,$$

with, for $i = 1, 2$,

$$Z_i \overset{\text{not.}}{=} Z_i\left(X, \xi_i; \rho_i\right) = \sqrt{\rho_i} X + \sqrt{1-\rho_i}\xi_i,$$

where X, ξ_1 and ξ_2 are i.i.d. standard Normally distributed. We have,

$$L_t = h_2(Z_3) = \begin{cases} F_\Lambda^{[-1]}\left(1 - \dfrac{\Phi^2_{\mu_{3,1},\Sigma_{3,1}}\left(Z_3, H_t^d\right)}{\Phi\left(H_t^d\right)}\right); & -\infty \le Z_1 < H_t^d; \\ 0; & H_t^d \le Z_1 \le +\infty, \end{cases}$$

$$\Pr\left[L_t \le l \,\middle|\, Z_1 \le H_t^d, X = x\right] = 1 - \frac{\Phi^2_{\mu^x_{3,1},\Sigma^x_{3,1}}\left(h_2^{[-1]}(l), H_t^d\right)}{\Phi\left(\frac{H_t^d - \sqrt{\rho_1}x}{\sqrt{1-\rho_1}}\right)},$$

$$\Pr\left[L_t \le l \,|\, X = x\right] = 1 - \Phi^2_{\mu^x_{3,1},\Sigma^x_{3,1}}\left(h_2^{[-1]}(l), H_t^d\right),$$

$$\mathrm{E}\left[L_t \,\middle|\, Z_1 \le H_t^d, X = x\right] = \frac{\int_{l=\lambda_l}^{\lambda_u} \Phi^2_{\mu^x_{3,1},\Sigma^x_{3,1}}\left(h_2^{[-1]}(l), H_t^d\right) \mathrm{d}l}{\Phi\left(\frac{H_t^d - \sqrt{\rho_1}x}{\sqrt{1-\rho_1}}\right)},$$

with,

$$\boldsymbol{\mu}_{3,1} = (0, 0)';$$

$$\Sigma_{3,1} = \begin{pmatrix} \sigma_3^2 & \rho_{3,1}\sigma_3\sigma_1 \\ \rho_{3,1}\sigma_3\sigma_1 & \sigma_1^2 \end{pmatrix};$$

$$\sigma_1 = 1;$$

$$\sigma_3 = \sqrt{1 + 2\sqrt{\rho_1\rho_2\rho_3(1-\rho_3)}};$$

$$\rho_{3,1} = \frac{\sqrt{\rho_3} + \sqrt{\rho_1\rho_2(1-\rho_3)}}{\sqrt{1 + 2\sqrt{\rho_1\rho_2\rho_3(1-\rho_3)}}};$$

$$\boldsymbol{\mu}^x_{3,1} = \left(\left[\sqrt{\rho_1\rho_3} + \sqrt{\rho_2\left(1-\rho_3\right)}\right]x, \sqrt{\rho_1}x\right)';$$

$$\Sigma^x_{3,1} = \begin{pmatrix} \left(\sigma_3^x\right)^2 & \rho^x_{3,1}\sigma^x_3\sigma^x_1 \\ \rho^x_{3,1}\sigma^x_3\sigma^x_1 & \left(\sigma^x_1\right)^2 \end{pmatrix};$$

$$\sigma^x_1 = \sqrt{1-\rho_1};$$

$$\sigma^x_3 = \sqrt{\rho_3\left(1-\rho_1\right) + \left(1-\rho_3\right)\left(1-\rho_2\right)};$$

$$\rho^x_{3,1} = \frac{1}{\sqrt{1 + \frac{(1-\rho_2)(1-\rho_3)}{\rho_3(1-\rho_1)}}}.$$

A.2.2. The comonotonic case [6]

Note that $Z_3 \equiv Z_1$.

We have,

$$L_t = h_2(Z_3) = \begin{cases} F_\Lambda^{[-1]}\left(1 - \dfrac{\Phi(Z_3)}{\Phi(H_t^d)}\right); & -\infty \le Z_1 < H_t^d; \\ 0; & H_t^d \le Z_1 \le +\infty, \end{cases}$$

$$\Pr\left[L_t \le l \middle| Z_1 \le H_t^d, X = x\right] = 1 - \frac{\Phi\left[\frac{h_2^{[-1]}(l)-\sqrt{\rho_1}x}{\sqrt{1-\rho_1}}\right]}{\Phi\left[\frac{H_t^d-\sqrt{\rho_1}x}{\sqrt{1-\rho_1}}\right]},$$

$$\Pr\left[L_t \le l | X = x\right] = 1 - \Phi\left[\frac{h_2^{[-1]}(l) - \sqrt{\rho_1}x}{\sqrt{1-\rho_1}}\right],$$

$$\mathrm{E}\left[L_t \middle| Z_1 \le H_t^d, X = x\right] = \frac{\int_{l=\lambda_l}^{\lambda_u} \Phi\left[\frac{h_2^{[-1]}(l)-\sqrt{\rho_1}x}{\sqrt{1-\rho_1}}\right] \mathrm{d}l}{\Phi\left(\frac{H_t^d-\sqrt{\rho_1}x}{\sqrt{1-\rho_1}}\right)}.$$

A.2.3. Hillebrand [19]-type model

Note that $Z_3 \equiv Z_2$. Hence, $\rho_{3,1} = \rho_{3,2} = \sqrt{\rho_1\rho_2}$, $\rho_{3,1}^x = 0$ and $\boldsymbol{\mu}_{3,1}^x = \left(\sqrt{\rho_2}x, \sqrt{\rho_1}x\right)'$.
We have,

$$L_t = h_2(Z_3) = \begin{cases} F_\Lambda^{[-1]}\left(1 - \frac{\Phi^2_{\boldsymbol{\mu}_{3,1},\Sigma_{3,1}}\left(Z_3, H_t^d\right)}{\Phi\left(H_t^d\right)}\right); & -\infty \le Z_1 < H_t^d; \\ 0; & H_t^d \le Z_1 \le +\infty, \end{cases}$$

$$\Pr\left[L_t \le l | Z_1 \le H_t^d, X = x\right] = 1 - \Phi\left(\frac{h_2^{[-1]}(l) - \sqrt{\rho_2}x}{\sqrt{1-\rho_2}}\right),$$

$$\Pr\left[L_t \le l | X = x\right] = \Pr\left[L_t \le l \middle| Z_1 \le H_t^d, X = x\right] p_t^{d,c} + \left(1 - p_t^{d,c}\right),$$

$$\mathrm{E}\left[L_t \middle| Z_1 \le H_t^d, X = x\right] = \int_{l=\lambda_l}^{\lambda_u} \Phi\left(\frac{h_2^{[-1]}(l) - \sqrt{\rho_2}x}{\sqrt{1-\rho_2}}\right) \mathrm{d}l.$$

A.2.4. The three-factor model

Note that $Z_3 = \sqrt{\rho_3}Z_1 + \sqrt{1-\rho_3}J$. The formulas from Appendix A.2.1 apply, with $\rho_{3,1} = \sqrt{\rho_3}$, $\rho_{3,2} = \sqrt{1-\rho_3}$, $\rho_{3,1}^x = \frac{\sqrt{\rho_3}}{\sqrt{\rho_3 + \frac{1-\rho_3}{1-\rho_1}}}$ and $\boldsymbol{\mu}_{3,1}^x = \left(\sqrt{\rho_1\rho_3}x, \sqrt{\rho_1}x\right)'$.

A.3. Gamma one-factor model

A.3.1. The general case

Note that

$$A_T^{(3)} \overset{\text{not.}}{=} A_T^{(3)}\left(A_T^{(1)}, A_T^{(2)}; \rho_3\right) = A^{(1)}_{\rho_3 T} + A^{(2)}_{(1-\rho_3)T},$$

with, for $i = 1, 2$,

$$A_t^{(i)} \overset{\text{not.}}{=} A_t^{(i)}\left(Y, \chi^{(i)}; \rho_i\right) = Y_{\rho_i \frac{t}{T}} + \chi^{(i)}_{(1-\rho_i)\frac{t}{T}},$$

where Y, $\chi^{(1)}$ and $\chi^{(2)}$ are i.i.d. shifted Gamma processes, with parameters α_g, $\beta_g = \sqrt{\alpha_g}$ and $\mu_g = \frac{\alpha_g}{\beta_g}$. We have,

$$L_t = h_2\left(\mathrm{A}_\mathrm{T}^{(3)}\right) = \begin{cases} F_\Lambda^{[-1]}\left(1 - \frac{F_{A_T^{(3)}, A_T^{(1)}}\left(A_T^{(3)}, H_t^d\right)}{1 - \Gamma_{\alpha_g,\beta_g}\left(\mu_g - H_t^d\right)}\right); & -\infty \le A_T^{(1)} < H_t^d; \\ 0; & H_t^d \le A_T^{(1)} \le \mu_g. \end{cases}$$

$$\Pr\left[L_t \le l \middle| A_T^{(1)} \le H_t^d, Y = y(\omega)\right]$$
$$= \frac{\Gamma_{\varrho_1\alpha_g,\beta_g}\left[g\left(u_1, v_1, w_1\right)\right] - \Gamma_{\varrho_1\alpha_g,\beta_g,\varrho_2\alpha_g,\beta_g,\rho}\left[g\left(u_1, v_1, w_1\right), g\left(u_2, v_2, w_2\right)\right]}{1 - \Gamma_{\varrho_2\alpha_g,\beta_g}\left[g\left(u_2, v_2, w_2\right)\right]},$$

$$\Pr\left[L_t \le l | Y = y(\omega)\right]$$
$$= \Gamma_{\varrho_1\alpha_g,\beta_g}\left[g\left(u_1, v_1, w_1\right)\right] + \Gamma_{\varrho_2\alpha_g,\beta_g}\left[g\left(u_2, v_2, w_2\right)\right] - \Gamma_{\varrho_1\alpha_g,\beta_g,\varrho_2\alpha_g,\beta_g,\rho}\left[g\left(u_1, v_1, w_1\right), g\left(u_2, v_2, w_2\right)\right],$$

$$\mathrm{E}\left[L_t \left| A_T^{(1)} \leq H_t^d, Y = y(\omega) \right.\right] = \lambda_u - \lambda_l - \frac{\int_{l=\lambda_l}^{\lambda_u} \left(\Gamma_{\varrho_1\alpha_g,\beta_g}\left[g\left(u_1, v_1, w_1\right)\right] - \Gamma_{\varrho_1\alpha_g,\beta_g,\varrho_2\alpha_g,\beta_g,\rho}\left[g\left(u_1, v_1, w_1\right), g\left(u_2, v_2, w_2\right)\right]\right) \mathrm{d}l}{1 - \Gamma_{\varrho_2\alpha_g,\beta_g}\left[g\left(u_2, v_2, w_2\right)\right]},$$

with

$$\begin{aligned}
&\varrho_1 = (1-\rho_1)\,\rho_3 + (1-\rho_2)\,(1-\rho_3) \in (0,1)\,,\\
&\varrho_2 = (1-\rho_1) \in (0,1)\,,\\
&\rho = \rho_3\sqrt{\frac{\varrho_2}{\varrho_1}} \in (0,1)\,,\\
&u_i = \varrho_i \mu_g, \quad i = 1, 2,\\
&v_1 = h_2^{(-1)}(l),\\
&v_2 = H_t^d,\\
&w_1 = y_{\rho_1\rho_3} + y_{\rho_2(1-\rho_3)},\\
&w_2 = y_{\rho_1},\\
&g\left(u, v, w\right) = u - v + w,
\end{aligned}$$

$\Gamma_{m,n}$ the CDF of Gamma distribution with shape parameter $m > 0$ and scale parameter $n >$ and $\Gamma_{m_1,n_1,m_2,n_1,\rho}$ the joint CDF of a pair (G_1, G_2), with $\mathrm{Corr}\,[G_1, G_2] = \rho$, where G_i follows a Gamma distribution with shape parameter $m_i > 0$ and scale parameter $n_i > 0$.

A.3.2. The comonotonic case [6]

Note that $A_T^{(3)} \equiv A_T^{(1)}$.
We have,

$$L_t = h_2\left(A_T^{(3)}\right) = \begin{cases} F_\Lambda^{[-1]}\left(1 - \dfrac{1 - \Gamma_{\alpha_g,\beta_g}\left(\mu_g - A_T^{(3)}\right)}{1 - \Gamma_{\alpha_g,\beta_g}\left(\mu_g - H_t^d\right)}\right); & -\infty \leq A_T^{(1)} < H_t^d;\\ 0; & H_t^d \leq A_T^{(1)} \leq \mu_g, \end{cases}$$

$$\Pr\left[L_t \leq l \left| A_T^{(1)} \leq H_t^d, Y_{\rho_1} = y_{\rho_1} \right.\right] = 1 - \frac{1 - \Gamma_{(1-\rho_1)\alpha_g,\beta_g}\left[(1-\rho_1)\mu_g + y_{\rho_1} - h_2^{[-1]}(l)\right]}{1 - \Gamma_{(1-\rho_1)\alpha_g,\beta_g}\left[(1-\rho_1)\mu_g + y_{\rho_1} - H_t^d\right]},$$

$$\Pr\left[L_t \leq l \left| Y_{\rho_1} = y_{\rho_1} \right.\right] = \Gamma_{(1-\rho_1)\alpha_g,\beta_g}\left[(1-\rho_1)\mu_g + y_{\rho_1} - h_2^{[-1]}(l)\right],$$

$$\mathrm{E}\left[L_t \left| A_T^{(1)} \leq H_t^d, Y_{\rho_1} = y_{\rho_1} \right.\right] = \frac{\lambda_u - \lambda_l - \int_{l=\lambda_l}^{\lambda_u} \Gamma_{(1-\rho_1)\alpha_g,\beta_g}\left[(1-\rho_1)\mu_g + y_{\rho_1} - h_2^{[-1]}(l)\right] \mathrm{d}l}{1 - \Gamma_{(1-\rho_1)\alpha_g,\beta_g}\left[(1-\rho_1)\,\mu_g + y_{\rho_1} - H_t^d\right]}.$$

A.3.3. Hillebrand [19]-type model

Note that $A_T^{(3)} \equiv A_T^{(2)}$.
We have,

$$L_t = h_2\left(\mathrm{A}_\mathrm{T}^{(3)}\right) = \begin{cases} F_\Lambda^{[-1]}\left(1 - \dfrac{F_{A_T^{(3)},A_T^{(1)}}\left(A_T^{(3)}, H_t^d\right)}{1 - \Gamma_{\alpha_g,\beta_g}\left(\mu_g - H_t^d\right)}\right); & -\infty \leq A_T^{(1)} < H_t^d;\\ 0; & H_t^d \leq A_T^{(1)} \leq \mu_g, \end{cases}$$

$$\Pr\left[L_t \leq l \left| A_T^{(1)} \leq H_t^d, Y_{\rho_2} = y_{\rho_2} \right.\right] = \Gamma_{(1-\rho_2)\alpha_g,\beta_g}\left[(1-\rho_2)\mu_g + y_{\rho_2} - h_2^{[-1]}(l)\right],$$

$$\Pr\left[L_t \leq l \left| Y_{\rho_2} = y_{\rho_2} \right.\right] = \Pr\left[L_t \leq l \left| A_T^{(1)} \leq H_t^d, Y_{\rho_2} = y_{\rho_2} \right.\right] p_t^{d,c} + \left(1 - p_t^{d,c}\right),$$

$$\mathrm{E}\left[L_t \left| A_T^{(1)} \leq H_t^d, Y_{\rho_2} = y_{\rho_2} \right.\right] = \lambda_u - \lambda_l - \int_{l=\lambda_l}^{\lambda_u} \Gamma_{(1-\rho_2)\alpha_g,\beta_g}\left[(1-\rho_2)\mu_g + y_{\rho_2} - h_2^{[-1]}(l)\right] \mathrm{d}l.$$

A.3.4. The three-factor model

Note that $A_T^{(3)} = A_{\rho_3 T}^{(1)} + \chi_{1-\rho_3}^{(2)}$. The formulas from Appendix A.3.1 apply, with $\rho_2 = 0$, where, by the properties of the Gamma process, $y_0 = 0$.

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