Journal of Multivariate Analysis 101 (2010) 1959–1969

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Some equalities for estimations of variance components in a general linear model and its restricted and transformed models

Yongge Tian [a,*], Chunmei Liu [b]

[a] China Economics and Management Academy, Central University of Finance and Economics, Beijing 100081, China
[b] School of Information Management and Engineering, Shanghai University of Finance and Economics, Shanghai 200433, China

ARTICLE INFO

Article history:
Received 23 November 2008
Available online 1 June 2010

AMS 2000 subject classifications:
62J05
62J10
62H12
15A09

Keywords:
Linear regression model
Restricted model
Transformed model
Sub-sample model
Reduced model
Simple estimator
Minimum norm quadratic unbiased estimator
Equality for estimators
Matrix rank method

ABSTRACT

For the unknown positive parameter σ^2 in a general linear model $\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma}\}$, the two commonly used estimations are the simple estimator (SE) and the minimum norm quadratic unbiased estimator (MINQUE). In this paper, we derive necessary and sufficient conditions for the equivalence of the SEs and MINQUEs of the variance component σ^2 in the original model $\mathscr{M}$, the restricted model $\mathscr{M}_r = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta} \mid \mathbf{A}\boldsymbol{\beta} = \mathbf{b}, \sigma^2\boldsymbol{\Sigma}\}$, the transformed model $\mathscr{M}_t = \{\mathbf{A}\mathbf{y}, \mathbf{A}\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\}$, and the misspecified model $\mathscr{M}_m = \{\mathbf{y}, \mathbf{X}_0\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma}_0\}$.

1. Introduction

Consider a general linear model

$$\mathscr{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma}\}, \tag{1.1}$$

where $\mathbf{X} \in \mathbb{R}^{n\times p}$ is a known matrix of arbitrary rank, $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is an observable random vector with $E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}$ and $Cov(\mathbf{y}) = \sigma^2\boldsymbol{\Sigma}$, $\boldsymbol{\beta} \in \mathbb{R}^{p\times 1}$ is a vector of unknown parameters, $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is a known or unknown nonnegative definite matrix of arbitrary rank, and σ^2, called the variance component, is a positive unknown parameter.

In the investigation of (1.1), the following two cases usually occur:

(a) An extraneous information is available on the unknown parameter vector $\boldsymbol{\beta}$ in the form of a consistent linear matrix equation

$$\mathbf{A}\boldsymbol{\beta} = \mathbf{b}, \tag{1.2}$$

* Corresponding author.
E-mail addresses: yongge.tian@gmail.com, yongge@mail.shufe.edu.cn (Y. Tian), cmliu@mail.shufe.edu.cn (C. Liu).

0047-259X/$ – see front matter
doi:10.1016/j.jmva.2010.04.011

where $\mathbf{A}$ is an $m \times p$ known matrix with $\text{rank}(\mathbf{A}) = m$ and $\mathbf{b}$ is an $m \times 1$ known vector. This restriction often occurs, for example, in hypothesis testing on the parameter vector in (1.1). The model (1.1) subject to (1.2) is called a restricted linear model. In such a case, the model in (1.1) together with (1.2) can be written in the following compact form

$$\mathscr{M}_r = \{\mathbf{y},\ \mathbf{X}\boldsymbol{\beta} \mid \mathbf{A}\boldsymbol{\beta} = \mathbf{b},\ \sigma^2\boldsymbol{\Sigma}\}. \tag{1.3}$$

(b) A linear transformation of (1.1) is

$$\mathscr{M}_t = \{\mathbf{A}\mathbf{y}, \mathbf{A}\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'\}, \tag{1.4}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$ is a known matrix of arbitrary rank. This happens for regression models with grouped observations, aggregated data or missing data, natural restrictions to parameters, sub-sample models, reduced models, etc. Some special cases of (1.4) are given below:

(i) Partition the model matrix $\mathbf{X}$ in (1.1) as $\mathbf{X} = \begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{bmatrix}$, where $\mathbf{X}_1 \in \mathbb{R}^{n_1\times p}$ and $\mathbf{X}_2 \in \mathbb{R}^{n_2\times p}$, and let the transformation matrix in (1.4) be $\mathbf{A} = [\mathbf{I}_{n_1}, \mathbf{0}]$ and $\mathbf{A} = [\mathbf{0}, \mathbf{I}_{n_2}]$, respectively. Then, (1.4) reduces to the following two sub-sample models:

$$\mathscr{M}_{s1} = \{\mathbf{y}_1, \mathbf{X}_1\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma}_{11}\} \quad \text{and} \quad \mathscr{M}_{s2} = \{\mathbf{y}_2, \mathbf{X}_2\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma}_{22}\}, \tag{1.5}$$

where $\mathbf{y}_1 \in \mathbb{R}^{n_1\times 1}$, $\mathbf{y}_2 \in \mathbb{R}^{n_2\times 1}$, and $\boldsymbol{\Sigma}_{11} \in \mathbb{R}^{n_1\times n_1}$ and $\boldsymbol{\Sigma}_{22} \in \mathbb{R}^{n_2\times n_2}$ are the upper-left and lower-right corners of the covariance matrix $\boldsymbol{\Sigma}$, respectively.

(ii) If $\mathbf{A} = \mathbf{X}'$, then (1.4) becomes

$$\mathscr{M}_t = \{\mathbf{X}'\mathbf{y}, \mathbf{X}'\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{X}'\boldsymbol{\Sigma}\mathbf{X}\}. \tag{1.6}$$

(iii) If $\mathbf{A} = \mathbf{E}_{\boldsymbol{\Sigma}}$, then (1.4) reduces to

$$\mathscr{M}_t = \{\mathbf{E}_{\boldsymbol{\Sigma}}\mathbf{y}, \mathbf{E}_{\boldsymbol{\Sigma}}\mathbf{X}\boldsymbol{\beta}, \mathbf{0}\}, \tag{1.7}$$

that is to say, $\mathbf{E}_{\boldsymbol{\Sigma}}\mathbf{y} = \mathbf{E}_{\boldsymbol{\Sigma}}\mathbf{X}\boldsymbol{\beta}$ holds with probability 1. The equality is called the natural restriction to the parameters in (1.1) in the literature.

(iv) Partition the mean vector $\mathbf{X}\boldsymbol{\beta}$ in (1.1) as $\mathbf{X}\boldsymbol{\beta} = \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_2\boldsymbol{\beta}_2$, where $\mathbf{X}_1 \in \mathbb{R}^{n\times p_1}$ and $\mathbf{X}_2 \in \mathbb{R}^{n\times p_2}$ with $p_1 + p_2 = p$, and let the transformation matrix $\mathbf{A}$ in (1.4) be $\mathbf{A} = \mathbf{E}_{\mathbf{X}_2}$ and $\mathbf{A} = \mathbf{E}_{\mathbf{X}_1}$ respectively. Then (1.4) becomes

$$\mathscr{M}_{r1} = \{\mathbf{E}_{\mathbf{X}_2}\mathbf{y}, \mathbf{E}_{\mathbf{X}_2}\mathbf{X}_1\boldsymbol{\beta}_1, \sigma^2\mathbf{E}_{\mathbf{X}_2}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_2}\}, \qquad \mathscr{M}_{r2} = \{\mathbf{E}_{\mathbf{X}_1}\mathbf{y}, \mathbf{E}_{\mathbf{X}_1}\mathbf{X}_2\boldsymbol{\beta}_2, \sigma^2\mathbf{E}_{\mathbf{X}_1}\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_1}\}, \tag{1.8}$$

respectively, both of which are called correctly reduced versions of (1.1), see Nurhonen and Puntanen [3], and Puntanen [4].

Because the original model (1.1), the restricted model (1.3) and the transformed model (1.4) are different in structure, estimations derived from (1.1), (1.3) and (1.4) are not necessarily equal. In particular, if only (1.4) is given, the unknown parameters in (1.1) can only be estimated through (1.4). In this case, it is necessary to compare algebraic and statistical properties of estimations derived from (1.1), (1.3) and (1.4).

As is well known, two commonly used estimators for the mean vector $\mathbf{X}\boldsymbol{\beta}$ in (1.1) are the ordinary least-squares estimator (OLSE) and the best linear unbiased estimator (BLUE); while the two commonly used estimators for the unknown variance component σ^2 in (1.1) are the simple estimator (SE) and the minimum norm quadratic unbiased estimator (MINQUE), both of which are derived from the OLSE and BLUE of $\mathbf{X}\boldsymbol{\beta}$ in (1.1), respectively. The two estimators are not necessarily the same in general. Thus, it would be of interest to study the relations between the SEs and MINQUEs, in particular, to give necessary and sufficient conditions for the SEs and MINQUEs to equal. This topic was also considered in the literature. For example, Groß [1] gave necessary and sufficient conditions for the SE and MINQUE of σ^2 in (1.1) to equal; Wang et al. [11] compared the SE and MINQUE of σ^2 in (1.1) through the MSEs of the estimators; Zhang [12] gave some necessary and sufficient conditions for the MINQUEs of σ^2 in (1.1) and (1.4) to equal. The purpose of this paper is to derive identifying conditions for the SEs and MINQUEs of the variance component σ^2 in (1.1), (1.3) and (1.4) to equal.

Throughout this paper, $\mathbb{R}^{m\times n}$ stands for the collection of all $m \times n$ real matrices. The symbols $\mathbf{A}'$, $r(\mathbf{A})$ and $\mathscr{R}(\mathbf{A})$ stand for the transpose, rank and range (column space) of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, respectively. The Moore–Penrose inverse of $\mathbf{A}$, denoted by $\mathbf{A}^+$, is defined to be the unique solution $\mathbf{X}$ satisfying the four matrix equations $\mathbf{AGA} = \mathbf{A}$, $\mathbf{GAG} = \mathbf{G}$, $(\mathbf{AG})' = \mathbf{AG}$ and $(\mathbf{GA})' = \mathbf{GA}$. Further, let $\mathbf{P}_{\mathbf{A}}$, $\mathbf{E}_{\mathbf{A}}$ and $\mathbf{F}_{\mathbf{A}}$ stand for the three orthogonal projectors $\mathbf{P}_{\mathbf{A}} = \mathbf{AA}^+$, $\mathbf{E}_{\mathbf{A}} = \mathbf{I}_m - \mathbf{AA}^+$ and $\mathbf{F}_{\mathbf{A}} = \mathbf{I}_n - \mathbf{A}^+\mathbf{A}$.

According to Rao [5], the linear model $\mathscr{M}$ in (1.1) is said to be consistent if $\mathbf{y} \in \mathscr{R}[\mathbf{X}, \boldsymbol{\Sigma}]$ holds with probability 1. In what follows, we assume that (1.1) is consistent. Based on this fact, we use the following result to characterize relations between two quadratic forms.

Lemma 1.1. *Assume that the matrix* $\mathbf{M}$ *is symmetric. Then*

$$\mathbf{y}'\mathbf{M}\mathbf{y} = 0 \quad \textit{for all } \mathbf{y} \in \mathscr{R}[\mathbf{X}, \boldsymbol{\Sigma}] \Leftrightarrow [\mathbf{X}, \boldsymbol{\Sigma}]'\mathbf{M}[\mathbf{X}, \boldsymbol{\Sigma}] = \mathbf{0}.$$

In particular,

(a) *Under* $\mathbf{MX} = \mathbf{0}$, *the equality* $\mathbf{y}'\mathbf{M}\mathbf{y} = 0$ *holds for all* $\mathbf{y} \in \mathscr{R}[\mathbf{X}, \boldsymbol{\Sigma}]$ *if and only if* $\boldsymbol{\Sigma}'\mathbf{M}\boldsymbol{\Sigma} = \mathbf{0}$.
(b) *Under the condition* $r[\mathbf{X}, \boldsymbol{\Sigma}] = n$, $\mathbf{y}'\mathbf{M}\mathbf{y} = 0$ *holds for all* $\mathbf{y}$ *if and only if* $\mathbf{M} = \mathbf{0}$.

Lemma 1.2. *Assume that the matrix* $\mathbf{M}$ *is symmetric. Then*

$$\begin{bmatrix}\mathbf{y}\\\mathbf{b}\end{bmatrix}'\mathbf{M}\begin{bmatrix}\mathbf{y}\\\mathbf{b}\end{bmatrix}=0\quad \text{for all } \begin{bmatrix}\mathbf{y}\\\mathbf{b}\end{bmatrix}\in\mathscr{R}\begin{bmatrix}\mathbf{X} & \mathbf{\Sigma}\\\mathbf{A} & 0\end{bmatrix}\Leftrightarrow\begin{bmatrix}\mathbf{X} & \mathbf{\Sigma}\\\mathbf{A} & 0\end{bmatrix}'\mathbf{M}\begin{bmatrix}\mathbf{X} & \mathbf{\Sigma}\\\mathbf{A} & 0\end{bmatrix}=\mathbf{0}.$$

In particular,

(a) *Under the condition* $\mathbf{M}\begin{bmatrix}\mathbf{X}\\\mathbf{A}\end{bmatrix}=\mathbf{0}$, *the equality* $\begin{bmatrix}\mathbf{y}\\\mathbf{b}\end{bmatrix}'\mathbf{M}\begin{bmatrix}\mathbf{y}\\\mathbf{b}\end{bmatrix}=0$ *holds for all* $\mathbf{y}\in\mathscr{R}[\mathbf{X},\mathbf{\Sigma}]$ *if and only if* $\begin{bmatrix}\mathbf{\Sigma}\\0\end{bmatrix}'\mathbf{M}\begin{bmatrix}\mathbf{\Sigma}\\0\end{bmatrix}=\mathbf{0}$.

(b) *Under the condition* $r\begin{bmatrix}\mathbf{X} & \mathbf{\Sigma}\\\mathbf{A} & 0\end{bmatrix}=n+m$, *the equality* $\begin{bmatrix}\mathbf{y}\\\mathbf{b}\end{bmatrix}'\mathbf{M}\begin{bmatrix}\mathbf{y}\\\mathbf{b}\end{bmatrix}=0$ *holds for all* $\mathbf{y}\in\mathscr{R}[\mathbf{X},\mathbf{\Sigma}]$ *if and only if* $\mathbf{M}=\mathbf{0}$.

One of the most fundamental quantities in linear algebra is the rank of a matrix, which is a well understood and easy to compute number. It has been realized in the past decades that the rank of matrix is a useful tool for simplifying complicated matrix expressions or equalities. In order to simplify various matrix expressions involving the Moore–Penrose inverses of matrices, we shall use the following rank formulas by Marsaglia and Styan [2, Theorem 19].

Lemma 1.3. *Let* $\mathbf{A}\in\mathbb{R}^{m\times n}$, $\mathbf{B}\in\mathbb{R}^{m\times k}$, $\mathbf{C}\in\mathbb{R}^{l\times n}$ *and* $\mathbf{D}\in\mathbb{R}^{l\times k}$. *Then*

$$r[\mathbf{A},\mathbf{B}]=r(\mathbf{A})+r(\mathbf{E_A}\mathbf{B})=r(\mathbf{B})+r(\mathbf{E_B}\mathbf{A}),\tag{1.9}$$

$$r\begin{bmatrix}\mathbf{A}\\\mathbf{C}\end{bmatrix}=r(\mathbf{A})+r(\mathbf{C}\mathbf{F_A})=r(\mathbf{C})+r(\mathbf{A}\mathbf{F_C}),\tag{1.10}$$

$$r\begin{bmatrix}\mathbf{A} & \mathbf{B}\\\mathbf{C} & \mathbf{0}\end{bmatrix}=r(\mathbf{B})+r(\mathbf{C})+r(\mathbf{E_B}\mathbf{A}\mathbf{F_C}).\tag{1.11}$$

If $\mathscr{R}(\mathbf{B})\subseteq\mathscr{R}(\mathbf{A})$ *and* $\mathscr{R}(\mathbf{C}')\subseteq\mathscr{R}(\mathbf{A}')$, *then*

$$r\begin{bmatrix}\mathbf{A} & \mathbf{B}\\\mathbf{C} & \mathbf{D}\end{bmatrix}=r(\mathbf{A})+r(\mathbf{D}-\mathbf{C}\mathbf{A}^{+}\mathbf{B}).\tag{1.12}$$

In general, the rank of the Schur complement $\mathbf{D}-\mathbf{C}\mathbf{A}^{+}\mathbf{B}$ can be calculated by the formula

$$r(\mathbf{D}-\mathbf{C}\mathbf{A}^{+}\mathbf{B})=r\begin{bmatrix}\mathbf{A}'\mathbf{A}\mathbf{A}' & \mathbf{A}'\mathbf{B}\\\mathbf{C}\mathbf{A}' & \mathbf{D}\end{bmatrix}-r(\mathbf{A}),\tag{1.13}$$

see Tian [7]. The following result was given by Tian [8].

Lemma 1.4. *Let* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *be two outer inverses of a matrix* $\mathbf{A}$, *i.e.,* $\mathbf{G}_1\mathbf{A}\mathbf{G}_1=\mathbf{G}_1$ *and* $\mathbf{G}_2\mathbf{A}\mathbf{G}_2=\mathbf{G}_2$. *Then*

$$r(\mathbf{G}_1-\mathbf{G}_2)=r\begin{bmatrix}\mathbf{G}_1\\\mathbf{G}_2\end{bmatrix}+r[\mathbf{G}_1,\mathbf{G}_2]-r(\mathbf{G}_1)-r(\mathbf{G}_2).\tag{1.14}$$

In order to simplify the matrix operations occurred in the estimations, we use the following equalities

$$(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}=\mathbf{E_X}(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}\mathbf{E_X},\tag{1.15}$$

$$\mathscr{R}[(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}]=\mathscr{R}(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})=\mathscr{R}(\mathbf{E_X}\mathbf{\Sigma}),\tag{1.16}$$

$$r(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})=r(\mathbf{E_X}\mathbf{\Sigma})=r[\mathbf{X},\mathbf{\Sigma}]-r(\mathbf{X}).\tag{1.17}$$

We also use the following simple results on ranges and ranks of matrices

$$r(\mathbf{M}-\mathbf{M}^2)=r(\mathbf{M})+r(\mathbf{I}_n-\mathbf{M})-n,\tag{1.18}$$

$$\mathscr{R}(\mathbf{A}_1)=\mathscr{R}(\mathbf{A}_2)\quad\text{and}\quad\mathscr{R}(\mathbf{B}_1)=\mathscr{R}(\mathbf{B}_2)\Rightarrow r[\mathbf{A}_1,\mathbf{B}_1]=r[\mathbf{A}_2,\mathbf{B}_2].\tag{1.19}$$

In order to derive closed-form formulas for ranks of partitioned matrices, we use the following three types of elementary block matrix operation (EBMO): (i) interchange two block rows (columns) in a partitioned matrix; (ii) multiply a block row by a nonsingular matrix from the left-hand side (multiply a block column by a nonsingular matrix from the right-hand side) in a partitioned matrix; (iii) multiply a block row by a matrix from the left-hand side and add it to another block row (multiply a block column by a matrix from the right-hand side and add it to another block column) in a partitioned matrix. It is obvious that the EBMOs do not change the rank of a matrix.

2. Equality for the SE and MINQUE in the original model

Lemma 2.1. *Let $\mathscr{M}$ be as given in (1.1). Then,*

(a) *The SE of σ^2 in (1.1), denoted by $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$, is given by*

$$\mathrm{SE}_{\mathscr{M}}(\sigma^2) = \frac{1}{f}\mathbf{y}'\mathbf{E_X}\mathbf{y}, \quad f = r[\mathbf{X}, \mathbf{\Sigma}] - r(\mathbf{X}) > 0. \tag{2.1}$$

(b) *The MINQUE of σ^2 in (1.1), denoted by $\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$, is given by*

$$\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = \frac{1}{f}\mathbf{y}'(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}\mathbf{y}, \quad f = r[\mathbf{X}, \mathbf{\Sigma}] - r(\mathbf{X}) > 0. \tag{2.2}$$

The SE of σ^2 in (2.1) is not necessarily unbiased for σ^2 in (1.1). If the random vector $\mathbf{y}$ in (1.1) is normally distributed, then $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$ has $\chi^2(f)$-distribution if and only if $\mathbf{\Sigma}\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X}\mathbf{\Sigma} = \mathbf{\Sigma}\mathbf{E_X}\mathbf{\Sigma}$, see Rao and Mitra [6]. More discussion on the distributions of $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$ and $\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$ can be found in Groß [1]. Also note from (2.1) and (2.2) that the SE of the variance component σ^2 involves no $\mathbf{\Sigma}$, but does the MINQUE. Hence, the SE is only an available estimator for σ^2 when $\mathbf{\Sigma}$ is unknown or partially known. The following result gives identifying conditions for the two estimators to equal.

Theorem 2.2. *Let $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$ and $\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$ be as given in (2.1) and (2.2). Then,*

(a) *The following statements are equivalent:*
  (i) $\mathrm{SE}_{\mathscr{M}}(\sigma^2) = \mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$ *holds with probability* 1.
  (ii) $(\mathbf{E_X}\mathbf{\Sigma})^2 = \mathbf{E_X}\mathbf{\Sigma}$, *i.e.,* $\mathbf{E_X}\mathbf{\Sigma}$ *is idempotent.*
  (iii) $r\begin{bmatrix} \mathbf{I}_n - \mathbf{\Sigma} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{bmatrix} = n + 2r(\mathbf{X}) - r[\mathbf{X}, \mathbf{\Sigma}]$.

(b) *Under the condition $r[\mathbf{X}, \mathbf{\Sigma}] = n$, the following statements are equivalent:*
  (i) $\mathrm{SE}_{\mathscr{M}}(\sigma^2) = \mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$ *holds with probability* 1.
  (ii) $\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X} = \mathbf{E_X}$.
  (iii) $r\begin{bmatrix} \mathbf{I}_n - \mathbf{\Sigma} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{bmatrix} = 2r(\mathbf{X})$.

Proof. Note that both $\mathbf{E_X}$ and $(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}$ are symmetric and $\mathbf{E_X}\mathbf{X} = \mathbf{0}$ and $(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}\mathbf{X} = \mathbf{0}$. Hence, it can be derived from (2.1), (2.2) and Lemma 1.1 that $\mathrm{SE}_{\mathscr{M}}(\sigma^2) = \mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$ holds with probability 1 if and only if $\mathbf{\Sigma}\mathbf{E_X}\mathbf{\Sigma} = \mathbf{\Sigma}(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}\mathbf{\Sigma}$. Applying (1.12), (1.15) to the difference of both sides of the equality and simplifying by EBMOs, we obtain

$$\begin{aligned}
r[\mathbf{\Sigma}\mathbf{E_X}\mathbf{\Sigma} - \mathbf{\Sigma}(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}\mathbf{\Sigma}] &= r\begin{bmatrix} \mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X} & \mathbf{E_X}\mathbf{\Sigma} \\ \mathbf{\Sigma}\mathbf{E_X} & \mathbf{\Sigma}\mathbf{E_X}\mathbf{\Sigma} \end{bmatrix} - r(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X}) \\
&= r\begin{bmatrix} \mathbf{0} & \mathbf{E_X}\mathbf{\Sigma} - \mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X}\mathbf{\Sigma} \\ \mathbf{\Sigma}\mathbf{E_X} & \mathbf{0} \end{bmatrix} - r(\mathbf{E_X}\mathbf{\Sigma}) \\
&= r[\mathbf{E_X}\mathbf{\Sigma} - (\mathbf{E_X}\mathbf{\Sigma})^2] \\
&= r(\mathbf{E_X}\mathbf{\Sigma}) + r(\mathbf{I}_n - \mathbf{E_X}\mathbf{\Sigma}) - n \quad (\text{by } (1.18)) \\
&= r[\mathbf{X}, \mathbf{\Sigma}] - r(\mathbf{X}) + r(\mathbf{I}_n - \mathbf{\Sigma} + \mathbf{X}\mathbf{X}^{+}\mathbf{\Sigma}) - n \\
&= r[\mathbf{X}, \mathbf{\Sigma}] - 2r(\mathbf{X}) + r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{\Sigma} \\ \mathbf{X} & \mathbf{\Sigma} - \mathbf{I}_n \end{bmatrix} - n \quad (\text{by } (1.12)) \\
&= r[\mathbf{X}, \mathbf{\Sigma}] - 2r(\mathbf{X}) + r\begin{bmatrix} \mathbf{0} & \mathbf{X}' \\ \mathbf{X} & \mathbf{\Sigma} - \mathbf{I}_n \end{bmatrix} - n.
\end{aligned}$$

Setting the right-hand sides to zero gives (a). Result (b) follows from (a). □

The equivalences of (i) and (ii) in Theorem 2.2(a) and (b) were given by Groß [1]. Theorem 2.2 shows that under the conditions in (ii) and (iii), we can use the SE instead of the MINQUE, while the SE has the same optimal statistical properties as the MINQUE. If $\mathbf{\Sigma}$ is unknown or partial known, then the two equalities in (ii) of Theorem 2.2(a) and (b) are in fact two matrix equations for $\mathbf{\Sigma}$ to satisfy, while the two equalities in (iii) of Theorem 2.2(a) and (b) are two matrix rank equations for $\mathbf{\Sigma}$ to satisfy.

3. Equalities for the SEs and the MINQUEs in the original model and its restricted models

A popular transformation in the literature on the restricted model in (1.3) is combining (1.1) and (1.2) into the following implicitly restricted model

$$\mathscr{M}_c = \{\mathbf{y}_c, \mathbf{X}_c\boldsymbol{\beta}, \sigma^2\mathbf{\Sigma}_c\}, \tag{3.1}$$

where $\mathbf{y}_c = \begin{bmatrix} \mathbf{y} \\ \mathbf{b} \end{bmatrix}$, $\mathbf{X}_c = \begin{bmatrix} \mathbf{X} \\ \mathbf{A} \end{bmatrix}$ and $\mathbf{\Sigma}_c = \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

Lemma 3.1. *Let $\mathscr{M}_c$ be as given in (3.1). Then,*

(a) *The SE of σ^2 in (3.1) is given by*

$$\mathrm{SE}_{\mathscr{M}_c}(\sigma^2) = \frac{1}{f_c}\mathbf{y}_c'\mathbf{E}_{\mathbf{X}_c}\mathbf{y}_c, \quad f_c = r[\mathbf{X}_c, \mathbf{\Sigma}_c] - r(\mathbf{X}_c) > 0. \tag{3.2}$$

(b) *The MINQUE of σ^2 in (3.1) is given by*

$$\mathrm{MINQUE}_{\mathscr{M}_c}(\sigma^2) = \frac{1}{f_c}\mathbf{y}_c'(\mathbf{E}_{\mathbf{X}_c}\mathbf{\Sigma}_c\mathbf{E}_{\mathbf{X}_c})^{+}\mathbf{y}_c, \quad f_c = r[\mathbf{X}_c, \mathbf{\Sigma}_c] - r(\mathbf{X}_c) > 0. \tag{3.3}$$

Also note that $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$ and $\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$ in (2.1) and (2.2) can be represented as

$$\mathrm{SE}_{\mathscr{M}}(\sigma^2) = \frac{1}{f}\mathbf{y}'\mathbf{E}_{\mathbf{X}}\mathbf{y} = \frac{1}{f}\mathbf{y}_c'\begin{bmatrix}\mathbf{E}_{\mathbf{X}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{y}_c, \tag{3.4}$$

$$\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = \frac{1}{f}\mathbf{y}'(\mathbf{E}_{\mathbf{X}}\mathbf{\Sigma}\mathbf{E}_{\mathbf{X}})^{+}\mathbf{y} = \frac{1}{f}\mathbf{y}_c'\begin{bmatrix}(\mathbf{E}_{\mathbf{X}}\mathbf{\Sigma}\mathbf{E}_{\mathbf{X}})^{+} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{y}_c. \tag{3.5}$$

Applying Lemma 1.2 to the estimators in (3.2)–(3.5) gives the following results.

Theorem 3.2. *Let $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$ and $\mathrm{SE}_{\mathscr{M}_c}(\sigma^2)$ be as given in (2.1) and (3.1). Then,*

(a) *$f\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_c\mathrm{SE}_{\mathscr{M}_c}(\sigma^2)$ holds with probability 1 if and only if $r\begin{bmatrix}\mathbf{A}'\mathbf{A} & \mathbf{X}'\mathbf{X} & \mathbf{0}\\ \mathbf{X}'\mathbf{X} & -\mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{\Sigma}\\ \mathbf{0} & \mathbf{\Sigma}\mathbf{X} & \mathbf{0}\end{bmatrix} = r(\mathbf{X}_c) + r(\mathbf{X})$.*

(b) *Under the condition $r\begin{bmatrix}\mathbf{X} & \mathbf{\Sigma}\\ \mathbf{A} & 0\end{bmatrix} = n + m$, $f\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_c\mathrm{SE}_{\mathscr{M}_c}(\sigma^2)$ holds with probability 1 if and only if $\mathscr{R}(\mathbf{X}') \cap \mathscr{R}(\mathbf{A}') = \{\mathbf{0}\}$.*

Proof. Note that both $\begin{bmatrix}\mathbf{E}_{\mathbf{X}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{X}_c = \mathbf{0}$ and $\mathbf{E}_{\mathbf{X}_c}\mathbf{X}_c = \mathbf{0}$. Hence, it can be seen from (3.2), (3.4) and Lemma 1.2(a) that $f\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_c\mathrm{SE}_{\mathscr{M}_c}(\sigma^2)$ holds with probability 1 if and only if

$$\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}'\begin{bmatrix}\mathbf{E}_{\mathbf{X}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}'\mathbf{E}_{\mathbf{X}_c}\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}.$$

Applying (1.13) to the difference of both sides of this equality and simplifying by EBMOs, we obtain

$$\begin{aligned} & r\left(\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}'\begin{bmatrix}\mathbf{E}_{\mathbf{X}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix} - \begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}'\mathbf{E}_{\mathbf{X}_c}\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}\right)\\ &= r\left([\mathbf{\Sigma}, \mathbf{0}]\mathbf{X}_c\mathbf{X}_c^{+}\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix} - \mathbf{\Sigma}\mathbf{X}\mathbf{X}^{+}\mathbf{\Sigma}\right)\\ &= r\begin{bmatrix}\mathbf{X}_c'\mathbf{X}_c & \mathbf{0} & \mathbf{X}'\mathbf{\Sigma}\\ \mathbf{0} & -\mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{\Sigma}\\ \mathbf{\Sigma}\mathbf{X} & \mathbf{\Sigma}\mathbf{X} & \mathbf{0}\end{bmatrix} - r(\mathbf{X}_c) - r(\mathbf{X})\\ &= r\begin{bmatrix}\mathbf{A}'\mathbf{A} & \mathbf{X}'\mathbf{X} & \mathbf{0}\\ \mathbf{X}'\mathbf{X} & -\mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{\Sigma}\\ \mathbf{0} & \mathbf{\Sigma}\mathbf{X} & \mathbf{0}\end{bmatrix} - r(\mathbf{X}_c) - r(\mathbf{X}). \end{aligned} \tag{3.6}$$

Setting the right-hand side of (3.6) to zero leads to (a). Result (b) is derived from Lemma 1.2(b). □

Theorem 3.3. *Let $\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$ and $\mathrm{MINQUE}_{\mathscr{M}_c}(\sigma^2)$ be as given in (3.5) and (3.3). Then,*

(a) *$f\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = f_c\mathrm{MINQUE}_{\mathscr{M}_c}(\sigma^2)$ holds with probability 1 if and only if $r\begin{bmatrix}\mathbf{X} & \mathbf{\Sigma}\\ \mathbf{A} & \mathbf{0}\end{bmatrix} = r\begin{bmatrix}\mathbf{X}\\ \mathbf{A}\end{bmatrix} + r[\mathbf{X}, \mathbf{\Sigma}] - r(\mathbf{X})$.*

(b) *Under the condition $r\begin{bmatrix}\mathbf{X} & \mathbf{\Sigma}\\ \mathbf{A} & 0\end{bmatrix} = n + m$, $f\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = f_c\mathrm{MINQUE}_{\mathscr{M}_c}(\sigma^2)$ holds with probability 1 if and only if $\mathscr{R}(\mathbf{X}') \cap \mathscr{R}(\mathbf{A}') = \{\mathbf{0}\}$.*

Proof. From Lemma 1.2(a), $f\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = f_c\mathrm{MINQUE}_{\mathscr{M}_c}(\sigma^2)$ holds with probability 1 if and only if

$$\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}'\begin{bmatrix}(\mathbf{E}_{\mathbf{X}}\mathbf{\Sigma}\mathbf{E}_{\mathbf{X}})^{+} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix} = \begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}'(\mathbf{E}_{\mathbf{X}_c}\mathbf{\Sigma}_c\mathbf{E}_{\mathbf{X}_c})^{+}\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}.$$

It is easy to verify that both sides of this equality are outer inverses of $\mathbf{\Sigma}^+$. In this case, applying (1.14) and simplifying by (1.10), (1.15)–(1.17) and (1.19) gives

$$
\begin{aligned}
& r\left(\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}'\begin{bmatrix}(\mathbf{E_X\Sigma E_X})^+ & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}-\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}'(\mathbf{E}_{\mathbf{X}_c}\mathbf{\Sigma}_c\mathbf{E}_{\mathbf{X}_c})^+\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}\right)\\
&= 2r\left[\mathbf{\Sigma}(\mathbf{E_X\Sigma E_X})^+\mathbf{\Sigma},\ [\mathbf{\Sigma},\mathbf{0}](\mathbf{E}_{\mathbf{X}_c}\mathbf{\Sigma}_c\mathbf{E}_{\mathbf{X}_c})^+\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}\right]-r[\mathbf{\Sigma}(\mathbf{E_X\Sigma E_X})^+\mathbf{\Sigma}]-r\left([\mathbf{\Sigma},\mathbf{0}](\mathbf{E}_{\mathbf{X}_c}\mathbf{\Sigma}_c\mathbf{E}_{\mathbf{X}_c})^+\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{0}\end{bmatrix}\right)\\
&= 2r\left[\mathbf{\Sigma E_X},\ [\mathbf{\Sigma},\mathbf{0}]\mathbf{E}_{\mathbf{X}_c}\right]-r(\mathbf{\Sigma E_X})-r\left([\mathbf{\Sigma},\mathbf{0}]\mathbf{E}_{\mathbf{X}_c}\right)\\
&= 2r\begin{bmatrix}\mathbf{\Sigma} & \mathbf{\Sigma} & \mathbf{0}\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}' & \mathbf{A}'\end{bmatrix}-r\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{X}'\end{bmatrix}-r\begin{bmatrix}\mathbf{\Sigma} & \mathbf{0}\\ \mathbf{X}' & \mathbf{A}'\end{bmatrix}-r(\mathbf{X})-r(\mathbf{X}_c)\\
&= 2r\begin{bmatrix}\mathbf{0} & \mathbf{\Sigma} & \mathbf{0}\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}' & \mathbf{A}'\end{bmatrix}-r\begin{bmatrix}\mathbf{\Sigma}\\ \mathbf{X}'\end{bmatrix}-r\begin{bmatrix}\mathbf{\Sigma} & \mathbf{0}\\ \mathbf{X}' & \mathbf{A}'\end{bmatrix}-r(\mathbf{X})-r(\mathbf{X}_c)\\
&= r\begin{bmatrix}\mathbf{X} & \mathbf{\Sigma}\\ \mathbf{A} & \mathbf{0}\end{bmatrix}-r\begin{bmatrix}\mathbf{X}\\ \mathbf{A}\end{bmatrix}-r[\mathbf{X},\mathbf{\Sigma}]+r(\mathbf{X}).
\end{aligned}
\tag{3.7}
$$

Setting the right-hand side of (3.7) to zero leads to (a). Result (b) is derived from Lemma 1.2(b). □

4. Equalities for the SEs and the MINQUEs in the original model and its transformed models

Lemma 4.1. *Let $\mathscr{M}_t$ be as given in (1.4). Then,*
(a) *The SE of σ^2 in (1.4) is given by*

$$\mathrm{SE}_{\mathscr{M}_t}(\sigma^2)=\frac{1}{f_t}\mathbf{y}'\mathbf{A}'\mathbf{E_{AX}Ay},\quad f_t=r[\mathbf{AX},\mathbf{A\Sigma}]-r(\mathbf{AX})>0. \tag{4.1}$$

(b) *The MINQUE of σ^2 in (1.4) is given by*

$$\mathrm{MINQUE}_{\mathscr{M}_t}(\sigma^2)=\frac{1}{f_t}\mathbf{y}'\mathbf{A}'(\mathbf{E_{AX}A\Sigma A'E_{AX}})^+\mathbf{Ay},\quad f_t=r[\mathbf{AX},\mathbf{A\Sigma}]-r(\mathbf{AX})>0. \tag{4.2}$$

Theorem 4.2. *Let $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$ and $\mathrm{SE}_{\mathscr{M}_t}(\sigma^2)$ be as given in (2.1) and (4.1). Then the following statements are equivalent:*
(a) *$f\,\mathrm{SE}_{\mathscr{M}}(\sigma^2)=f_t\mathrm{SE}_{\mathscr{M}_t}(\sigma^2)$ holds with probability 1.*
(b) *$\mathbf{\Sigma E_X\Sigma}=\mathbf{\Sigma A'E_{AX}A\Sigma}$.*
(c) *$r(\mathbf{P'NP})=r(\mathbf{X})+r(\mathbf{AX})$, where $\mathbf{P}=\mathrm{diag}(\mathbf{X},\mathbf{AX},\mathbf{\Sigma})$ and $\mathbf{N}=\begin{bmatrix}\mathbf{I}_n & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & -\mathbf{I}_n & \mathbf{I}_n\\ \mathbf{I}_n & \mathbf{I}_n & \mathbf{I}_n-\mathbf{A}'\mathbf{A}\end{bmatrix}$.*

Proof. Note that both $\mathbf{E_X}$ and $\mathbf{E_X}(\mathbf{E_X\Sigma E_X})^+\mathbf{E_X}$ are symmetric and $\mathbf{E_XX}=\mathbf{0}$ and $\mathbf{E_{AX}AX}=\mathbf{0}$. Hence, it can be seen from (2.1), (4.1) and Lemma 1.1 that $f\,\mathrm{SE}_{\mathscr{M}}(\sigma^2)=f_t\mathrm{SE}_{\mathscr{M}_t}(\sigma^2)$ holds with probability 1 if and only if $\mathbf{\Sigma E_X\Sigma}=\mathbf{\Sigma A'E_{AX}A\Sigma}$, as required for (b). Applying (1.12), $\mathbf{X}^+=(\mathbf{X'X})^+\mathbf{X}'$ and $(\mathbf{AX})^+=[(\mathbf{AX})'(\mathbf{AX})]^+(\mathbf{AX})'$ to the difference of both sides of the equality and simplifying by EBMOs, we obtain

$$
\begin{aligned}
& r(\mathbf{\Sigma E_X\Sigma}-\mathbf{\Sigma A'E_{AX}A\Sigma})\\
&= r[\mathbf{\Sigma}^2-\mathbf{\Sigma A'A\Sigma}-\mathbf{\Sigma XX^+\Sigma}+\mathbf{\Sigma A'AX}(\mathbf{AX})^+\mathbf{A\Sigma}]\\
&= r\{\mathbf{\Sigma}^2-\mathbf{\Sigma A'A\Sigma}-\mathbf{\Sigma X}(\mathbf{X'X})^+\mathbf{X'\Sigma}+\mathbf{\Sigma A'AX}[(\mathbf{AX})'(\mathbf{AX})]^+(\mathbf{AX})'\mathbf{A\Sigma}\}\\
&= r\begin{bmatrix}\mathbf{X'X} & \mathbf{0} & \mathbf{X'\Sigma}\\ \mathbf{0} & -(\mathbf{AX})'\mathbf{AX} & (\mathbf{AX})'\mathbf{A\Sigma}\\ \mathbf{\Sigma X} & \mathbf{\Sigma A'AX} & \mathbf{\Sigma}^2-\mathbf{\Sigma A'A\Sigma}\end{bmatrix}-r(\mathbf{X})-r(\mathbf{AX})\\
&= r(\mathbf{P'NP})-r(\mathbf{X})-r(\mathbf{AX}).
\end{aligned}
$$

Setting the right-hand side to zero leads to the equivalence of (b) and (c). □

The results in Theorem 4.2 seem complicated under the general assumptions in (1.1) and (1.4). However, if the transformation matrix $\mathbf{A}$ is given with some special forms, or the covariance matrix $\mathbf{\Sigma}$ is positive definite, then the rank equality in Theorem 4.2(c) can simplify further.

Corollary 4.3. *Let $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$ and $\mathrm{SE}_{\mathscr{M}_t}(\sigma^2)$ be as given in (2.1) and (4.1) and assume that $r[\mathbf{X},\mathbf{\Sigma}]=n$. Then the following statements are equivalent:*
(a) *$f\,\mathrm{SE}_{\mathscr{M}}(\sigma^2)=f_t\mathrm{SE}_{\mathscr{M}_t}(\sigma^2)$.*

(b) $\mathbf{E_X} = \mathbf{A}'\mathbf{E_{AX}}\mathbf{A}$.
(c) $r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}' \\ \mathbf{X} & \mathbf{I}_n - \mathbf{A}'\mathbf{A} \end{bmatrix} = r(\mathbf{X}) + r(\mathbf{AX})$.

Proof. It can be seen from Lemma 1.1(b) that under the condition $r[\mathbf{X}, \mathbf{\Sigma}] = n$, the equality $f\,\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_t\mathrm{SE}_{\mathscr{M}_t}(\sigma^2)$ holds if and only if the equality in (b) holds. Applying (1.12) to $\mathbf{E_X} - \mathbf{A}'\mathbf{E_{AX}}\mathbf{A}$ and simplifying by EBMOs, we obtain

$$\begin{aligned} r(\mathbf{E_X} - \mathbf{A}'\mathbf{E_{AX}}\mathbf{A}) &= r\{\mathbf{I}_n - \mathbf{A}'\mathbf{A} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{+}\mathbf{X}' + \mathbf{A}'\mathbf{AX}[(\mathbf{AX})'(\mathbf{AX})]^{+}(\mathbf{AX})'\mathbf{A}\} \\ &= r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{0} & \mathbf{X}' \\ \mathbf{0} & -(\mathbf{AX})'\mathbf{AX} & (\mathbf{AX})'\mathbf{A} \\ \mathbf{X} & \mathbf{A}'\mathbf{AX} & \mathbf{I}_n - \mathbf{A}'\mathbf{A} \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{AX}) \\ &= r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{X} & \mathbf{X}' \\ \mathbf{0} & \mathbf{0} & (\mathbf{AX})'\mathbf{A} \\ \mathbf{X} & \mathbf{X} & \mathbf{I}_n - \mathbf{A}'\mathbf{A} \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{AX}) \\ &= r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{X} & \mathbf{X}' \\ \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{X} & \mathbf{X}' \\ \mathbf{X} & \mathbf{X} & \mathbf{I}_n - \mathbf{A}'\mathbf{A} \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{AX}) \\ &= r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}' \\ \mathbf{X} & \mathbf{I}_n - \mathbf{A}'\mathbf{A} \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{AX}). \end{aligned}$$

Hence, (b) and (c) are equivalent. □

Corollary 4.3 shows that the covariance matrix $\mathbf{\Sigma}$ in (1.1) is not included in the verification of the equality in (a). If the transformed matrix $\mathbf{A}$ satisfies $\mathbf{A}'\mathbf{A} = \mathbf{I}_n$, then we have the following result.

Corollary 4.4. *Let* $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$ *and* $\mathrm{SE}_{\mathscr{M}_t}(\sigma^2)$ *be as given in* (2.1) *and* (4.1) *and assume* $\mathbf{A}'\mathbf{A} = \mathbf{I}_n$. *Then,*

(a) $f\,\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_t\mathrm{SE}_{\mathscr{M}_t}(\sigma^2)$ *holds with probability* 1 *if and only if* $r(\mathbf{\Sigma X}) = r(\mathbf{X})$, *i.e.,* $\mathscr{R}(\mathbf{X}'\mathbf{\Sigma}) = \mathscr{R}(\mathbf{X}')$.
(b) $f\,\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_t\mathrm{SE}_{\mathscr{M}_t}(\sigma^2)$ *if* $r[\mathbf{X}, \mathbf{\Sigma}] = n$.

Theorem 4.5. *Let* $\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$ *and* $\mathrm{MINQUE}_{\mathscr{M}_t}(\sigma^2)$ *be as given in* (2.2) *and* (4.2). *Then,*

(a) $f\,\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = f_t\mathrm{MINQUE}_{\mathscr{M}_t}(\sigma^2)$ *holds with probability* 1 *if and only if* $f = f_t$.
(b) $\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = \mathrm{MINQUE}_{\mathscr{M}_t}(\sigma^2)$ *holds with probability* 1 *if* $r(\mathbf{A}) = n$.

Proof. Note that both $(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}$ and $\mathbf{A}'(\mathbf{E_{AX}}\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{E_{AX}})^{+}\mathbf{A}$ are symmetric, and

$$(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}\mathbf{X} = \mathbf{0} \quad \text{and} \quad \mathbf{A}'(\mathbf{E_{AX}}\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{E_{AX}})^{+}\mathbf{AX} = \mathbf{0}.$$

Hence, it can be seen from (2.2), (4.2) and Lemma 1.1(b) that $f\,\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = f_t\mathrm{MINQUE}_{\mathscr{M}_t}(\sigma^2)$ holds with probability 1 if and only if

$$\mathbf{\Sigma}(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}\mathbf{\Sigma} = \mathbf{\Sigma}\mathbf{A}'(\mathbf{E_{AX}}\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{E_{AX}})^{+}\mathbf{A}\mathbf{\Sigma}. \tag{4.3}$$

It is easy to verify that both $\mathbf{\Sigma}(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}\mathbf{\Sigma}$ and $\mathbf{\Sigma}\mathbf{A}'(\mathbf{E_{AX}}\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{E_{AX}})^{+}\mathbf{A}\mathbf{\Sigma}$ are outer inverses of $\mathbf{\Sigma}^{+}$. In this case, applying (1.14) to the difference of both sides of (4.3) and simplifying by EBMOs gives

$$\begin{aligned} &r[\mathbf{\Sigma}(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}\mathbf{\Sigma} - \mathbf{\Sigma}\mathbf{A}'(\mathbf{E_{AX}}\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{E_{AX}})^{+}\mathbf{A}\mathbf{\Sigma}] \\ &\quad = 2r[\mathbf{\Sigma}(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X})^{+}\mathbf{\Sigma}, \mathbf{\Sigma}\mathbf{A}'(\mathbf{E_{AX}}\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{E_{AX}})^{+}\mathbf{A}\mathbf{\Sigma}] - r(\mathbf{E_X}\mathbf{\Sigma}\mathbf{E_X}) - r(\mathbf{E_{AX}}\mathbf{A}\mathbf{\Sigma}\mathbf{A}'\mathbf{E_{AX}}) \\ &\quad = 2r[\mathbf{\Sigma}\mathbf{E_X}, \mathbf{\Sigma}\mathbf{A}'\mathbf{E_{AX}}] - r(\mathbf{E_X}\mathbf{\Sigma}) - r(\mathbf{E_{AX}}\mathbf{A}\mathbf{\Sigma}) \quad (\text{by (1.17) and (1.19)}) \\ &\quad = 2r\begin{bmatrix} \mathbf{X} & \mathbf{0} & \mathbf{\Sigma} \\ \mathbf{0} & \mathbf{AX} & \mathbf{A}\mathbf{\Sigma} \end{bmatrix} - r[\mathbf{X}, \mathbf{\Sigma}] - r[\mathbf{AX}, \mathbf{A}\mathbf{\Sigma}] - r(\mathbf{X}) - r(\mathbf{AX}) \quad (\text{by (1.9) and (1.10)}) \\ &\quad = 2r\begin{bmatrix} \mathbf{X} & \mathbf{0} & \mathbf{\Sigma} \\ \mathbf{0} & \mathbf{AX} & \mathbf{0} \end{bmatrix} - r[\mathbf{X}, \mathbf{\Sigma}] - r[\mathbf{AX}, \mathbf{A}\mathbf{\Sigma}] - r(\mathbf{X}) - r(\mathbf{AX}) \\ &\quad = r[\mathbf{X}, \mathbf{\Sigma}] + r(\mathbf{AX}) - r[\mathbf{AX}, \mathbf{A}\mathbf{\Sigma}] - r(\mathbf{X}) \\ &\quad = f - f_t. \end{aligned}$$

Setting the right-hand side to zero results in (a). Result (b) is a direct consequence of (a). □

The condition $f_t = f$ is easy to verify, under which we can use $\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$ instead of $\mathrm{MINQUE}_{\mathscr{M}_t}(\sigma^2)$. In what follows, we give some applications of the previous theorems and corollaries to the sub-sample models and reduced models in (1.5) and (1.8).

Let

$$f_{s1} = r[\mathbf{X}_1, \mathbf{\Sigma}_{11}] - r(\mathbf{X}_1) > 0, \quad \text{and} \quad f_{s2} = r[\mathbf{X}_2, \mathbf{\Sigma}_{22}] - r(\mathbf{X}_2) > 0.$$

Then it can be derived from (4.1) that the SEs of σ^2 in the two sub-sample models in (1.5) are given by

$$\mathrm{SE}_{\mathscr{M}_{s1}}(\sigma^2) = \frac{1}{f_{s1}}\mathbf{y}_1'\mathbf{E}_{\mathbf{X}_1}\mathbf{y}_1, \qquad \mathrm{SE}_{\mathscr{M}_{s2}}(\sigma^2) = \frac{1}{f_{s2}}\mathbf{y}_2'\mathbf{E}_{\mathbf{X}_2}\mathbf{y}_2, \tag{4.4}$$

and from (4.2) that the MINQUEs of σ^2 in the two sub-sample models in (1.5) are given by

$$\mathrm{MINQUE}_{\mathscr{M}_{s1}}(\sigma^2) = \frac{1}{f_{s1}}\mathbf{y}_1'(\mathbf{E}_{\mathbf{X}_1}\mathbf{\Sigma}_{11}\mathbf{E}_{\mathbf{X}_1})^{+}\mathbf{y}_1, \qquad \mathrm{MINQUE}_{\mathscr{M}_{s2}}(\sigma^2) = \frac{1}{f_{s2}}\mathbf{y}_2'(\mathbf{E}_{\mathbf{X}_2}\mathbf{\Sigma}_{22}\mathbf{E}_{\mathbf{X}_2})^{+}\mathbf{y}_2. \tag{4.5}$$

Applying Theorems 4.2 and 4.5 to (4.4) and (4.5) gives the following results.

Theorem 4.6. *Let* $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$, $\mathrm{SE}_{\mathscr{M}_{s1}}(\sigma^2)$ *and* $\mathrm{SE}_{\mathscr{M}_{s2}}(\sigma^2)$ *be as given in* (2.1) *and* (4.4), *and let* $\mathbf{\Sigma}_1 = [\mathbf{\Sigma}_{11}, \mathbf{\Sigma}_{12}]$ *and* $\mathbf{\Sigma}_2 = [\mathbf{\Sigma}_{21}, \mathbf{\Sigma}_{22}]$. *Then,*

(a) $f\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_{s1}\mathrm{SE}_{\mathscr{M}_{s1}}(\sigma^2)$ *holds with probability* 1 *if and only if*

$$r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{0} & \mathbf{X}'\mathbf{\Sigma} \\ \mathbf{0} & -\mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{\Sigma}_1 \\ \mathbf{\Sigma}'\mathbf{X} & \mathbf{\Sigma}_1'\mathbf{X}_1 & \mathbf{\Sigma}_2'\mathbf{\Sigma}_2 \end{bmatrix} = r(\mathbf{X}) + r(\mathbf{X}_1).$$

(b) $f\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_{s2}\mathrm{SE}_{\mathscr{M}_{s2}}(\sigma^2)$ *holds with probability* 1 *if and only if*

$$r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{0} & \mathbf{X}'\mathbf{\Sigma} \\ \mathbf{0} & -\mathbf{X}_2'\mathbf{X}_2 & \mathbf{X}_2'\mathbf{\Sigma}_2 \\ \mathbf{\Sigma}'\mathbf{X} & \mathbf{\Sigma}_2'\mathbf{X}_2 & \mathbf{\Sigma}_1'\mathbf{\Sigma}_1 \end{bmatrix} = r(\mathbf{X}) + r(\mathbf{X}_2).$$

(c) *Under the condition that* $\mathbf{\Sigma}$ *is positive definite,*

$$f\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_{s1}\mathrm{SE}_{\mathscr{M}_{s1}}(\sigma^2) \Leftrightarrow \mathscr{R}(\mathbf{X}_1') \cap \mathscr{R}(\mathbf{X}_2') = \{\mathbf{0}\} \quad \text{and} \quad r(\mathbf{X}_2) = n_2,$$
$$f\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_{s2}\mathrm{SE}_{\mathscr{M}_{s2}}(\sigma^2) \Leftrightarrow \mathscr{R}(\mathbf{X}_1') \cap \mathscr{R}(\mathbf{X}_2') = \{\mathbf{0}\} \quad \text{and} \quad r(\mathbf{X}_1) = n_1.$$

Theorem 4.7. *Let* $\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$, $\mathrm{MINQUE}_{\mathscr{M}_{s1}}(\sigma^2)$ *and* $\mathrm{MINQUE}_{\mathscr{M}_{s2}}(\sigma^2)$ *be as given in* (2.2) *and* (4.5). *Then*

(a) $f\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = f_{s1}\mathrm{MINQUE}_{\mathscr{M}_{s1}}(\sigma^2)$ *holds with probability* 1 *if and only if* $f = f_{s1}$.
(b) $f\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = f_{s2}\mathrm{MINQUE}_{\mathscr{M}_{s2}}(\sigma^2)$ *holds with probability* 1 *if and only if* $f = f_{s2}$.
(c) *Under the condition that* $\mathbf{\Sigma}$ *is positive definite,*

$$f\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = f_{s1}\mathrm{MINQUE}_{\mathscr{M}_{s1}}(\sigma^2) \Leftrightarrow \mathscr{R}(\mathbf{X}_1') \cap \mathscr{R}(\mathbf{X}_2') = \{\mathbf{0}\} \quad \text{and} \quad r(\mathbf{X}_2) = n_2,$$
$$f\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = f_{s2}\mathrm{MINQUE}_{\mathscr{M}_{s2}}(\sigma^2) \Leftrightarrow \mathscr{R}(\mathbf{X}_1') \cap \mathscr{R}(\mathbf{X}_2') = \{\mathbf{0}\} \quad \text{and} \quad r(\mathbf{X}_1) = n_1.$$

Let $f = r[\mathbf{X}, \mathbf{\Sigma}] - r(\mathbf{X}) > 0$. Then it can be derived from (4.1) and (4.2) that the SEs of σ^2 in the two reduced models in (1.8) are given by

$$\mathrm{SE}_{\mathscr{M}_{r1}}(\sigma^2) = \frac{1}{f}\mathbf{y}'\mathbf{E}_{(\mathbf{E}_{\mathbf{X}_2}\mathbf{X}_1)}\mathbf{y}, \qquad \mathrm{SE}_{\mathscr{M}_{r2}}(\sigma^2) = \frac{1}{f}\mathbf{y}'\mathbf{E}_{(\mathbf{E}_{\mathbf{X}_1}\mathbf{X}_2)}\mathbf{y}, \tag{4.6}$$

while the MINQUEs of σ^2 in the two reduced models in (1.8) are given by

$$\mathrm{MINQUE}_{\mathscr{M}_{r1}}(\sigma^2) = \frac{1}{f}\mathbf{y}'\mathbf{E}_{\mathbf{X}_2}(\mathbf{E}_{(\mathbf{E}_{\mathbf{X}_2}\mathbf{X}_1)}\mathbf{E}_{\mathbf{X}_2}\mathbf{\Sigma}\mathbf{E}_{\mathbf{X}_2}\mathbf{E}_{(\mathbf{E}_{\mathbf{X}_2}\mathbf{X}_1)})^{+}\mathbf{E}_{\mathbf{X}_2}\mathbf{y}, \tag{4.7}$$

$$\mathrm{MINQUE}_{\mathscr{M}_{r2}}(\sigma^2) = \frac{1}{f}\mathbf{y}'\mathbf{E}_{\mathbf{X}_1}(\mathbf{E}_{(\mathbf{E}_{\mathbf{X}_1}\mathbf{X}_2)}\mathbf{E}_{\mathbf{X}_1}\mathbf{\Sigma}\mathbf{E}_{\mathbf{X}_1}\mathbf{E}_{(\mathbf{E}_{\mathbf{X}_1}\mathbf{X}_2)})^{+}\mathbf{E}_{\mathbf{X}_1}\mathbf{y}. \tag{4.8}$$

Corollary 4.8. *Let* $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$, $\mathrm{SE}_{\mathscr{M}_{r1}}(\sigma^2)$ *and* $\mathrm{SE}_{\mathscr{M}_{r2}}(\sigma^2)$ *be as given in* (2.1) *and* (4.6). *Then,*

(a) $\mathrm{SE}_{\mathscr{M}}(\sigma^2) = \mathrm{SE}_{\mathscr{M}_{r1}}(\sigma^2)$ *holds with probability* 1 *if and only if*

$$r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{0} & \mathbf{X}'\mathbf{\Sigma} \\ \mathbf{0} & -\mathbf{X}_1'\mathbf{E}_{\mathbf{X}_2}\mathbf{X}_1 & \mathbf{X}_1'\mathbf{E}_{\mathbf{X}_2}\mathbf{\Sigma} \\ \mathbf{\Sigma}\mathbf{X} & \mathbf{\Sigma}\mathbf{E}_{\mathbf{X}_2}\mathbf{X}_1 & \mathbf{\Sigma}\mathbf{P}_{\mathbf{X}_2}\mathbf{\Sigma} \end{bmatrix} = 2r(\mathbf{X}) - r(\mathbf{X}_2).$$

(b) $\mathrm{SE}_{\mathscr{M}}(\sigma^2) = \mathrm{SE}_{\mathscr{M}_{r2}}(\sigma^2)$ *holds with probability* 1 *if and only if*

$$r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{0} & \mathbf{X}'\boldsymbol{\Sigma} \\ \mathbf{0} & -\mathbf{X}_2'\mathbf{E}_{\mathbf{X}_1}\mathbf{X}_2 & \mathbf{X}_2'\mathbf{E}_{\mathbf{X}_1}\boldsymbol{\Sigma} \\ \boldsymbol{\Sigma}\mathbf{X} & \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_1}\mathbf{X}_2 & \boldsymbol{\Sigma}\mathbf{P}_{\mathbf{X}_1}\boldsymbol{\Sigma} \end{bmatrix} = 2r(\mathbf{X}) - r(\mathbf{X}_1).$$

(c) $\mathrm{SE}_{\mathscr{M}}(\sigma^2) = \mathrm{SE}_{\mathscr{M}_{r1}}(\sigma^2) = \mathrm{SE}_{\mathscr{M}_{r2}}(\sigma^2)$ *if* $\boldsymbol{\Sigma}$ *is positive definite.*

Corollary 4.9 (*Zhang [12]*). *Let* $\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2)$, $\mathrm{MINQUE}_{\mathscr{M}_{r1}}(\sigma^2)$ *and* $\mathrm{MINQUE}_{\mathscr{M}_{r2}}(\sigma^2)$ *be as given in* (2.2), (4.7) *and* (4.8). *Then the equalities*

$$\mathrm{MINQUE}_{\mathscr{M}}(\sigma^2) = \mathrm{MINQUE}_{\mathscr{M}_{r1}}(\sigma^2) = \mathrm{MINQUE}_{\mathscr{M}_{r2}}(\sigma^2)$$

hold with probability 1.

5. Equalities for the SEs and the MINQUEs of σ^2 in the original model and its misspecified models

In the theory of regression analysis, assumptions on error terms play an important role. Suppose that the model matrix $\mathbf{X}$ in (1.1) is misspecified as $\mathbf{X}_0$, and the covariance matrix $\boldsymbol{\Sigma}$ in (1.1) is misspecified as $\boldsymbol{\Sigma}_0$. Then the corresponding misspecified model of (1.1) is given by

$$\mathscr{M}_m = \{\mathbf{y}, \mathbf{X}_0\boldsymbol{\beta}, \sigma^2\boldsymbol{\Sigma}_0\}. \tag{5.1}$$

In this case, the SE of σ^2 in (5.1) is

$$\mathrm{SE}_{\mathscr{M}_m}(\sigma^2) = \frac{1}{f_m}\mathbf{y}'\mathbf{E}_{\mathbf{X}_0}\mathbf{y}, \quad f_m = r[\mathbf{X}_0, \boldsymbol{\Sigma}_0] - r(\mathbf{X}_0) > 0, \tag{5.2}$$

and the MINQUE of σ^2 in (5.1) is

$$\mathrm{MINQUE}_{\mathscr{M}_m}(\sigma^2) = \frac{1}{f_m}\mathbf{y}'\mathbf{E}_{\mathbf{X}_0}(\mathbf{E}_{\mathbf{X}_0}\boldsymbol{\Sigma}_0\mathbf{E}_{\mathbf{X}_0})^{+}\mathbf{E}_{\mathbf{X}_0}\mathbf{y}, \quad f_m = r[\mathbf{X}_0, \boldsymbol{\Sigma}_0] - r(\mathbf{X}_0) > 0. \tag{5.3}$$

Because (5.1) is a misspecified model of (1.1), the estimator in (5.3) is not necessarily a minimum norm quadratic unbiased estimator of σ^2. In this section, we give necessary and sufficient conditions for the SEs and MINQUEs in (2.1), (2.2), (5.2) and (5.3) to be equal.

Theorem 5.1. *Let* $\mathrm{SE}_{\mathscr{M}}(\sigma^2)$ *and* $\mathrm{SE}_{\mathscr{M}_m}(\sigma^2)$ *be as given in* (2.1) *and* (5.2). *Then,*

(a) *The following statements are equivalent:*
  (i) $f\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_m\mathrm{SE}_{\mathscr{M}_m}(\sigma^2)$ *holds with probability* 1.
  (ii) $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}(\mathbf{X}_0)$ *and* $\mathscr{R}(\mathbf{X}_0'\mathbf{X}) \subseteq \mathscr{R}(\mathbf{X}_0'\boldsymbol{\Sigma})$.
(b) *Under the condition* $r[\mathbf{X}, \boldsymbol{\Sigma}] = n$, *the following statements are equivalent:*
  (i) $f\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_m\mathrm{SE}_{\mathscr{M}_m}(\sigma^2)$ *holds with probability* 1.
  (ii) $\mathscr{R}(\mathbf{X}) = \mathscr{R}(\mathbf{X}_0)$.

Proof. Note that both $\mathbf{E}_{\mathbf{X}}$ and $\mathbf{E}_{\mathbf{X}_0}$ are symmetric. Hence, it can be seen from (2.1), (5.2) and Lemma 1.1 that $f\mathrm{SE}_{\mathscr{M}}(\sigma^2) = f_m\mathrm{SE}_{\mathscr{M}_m}(\sigma^2)$ holds with probability 1 if and only if

$$[\mathbf{X}, \boldsymbol{\Sigma}]'\mathbf{E}_{\mathbf{X}}[\mathbf{X}, \boldsymbol{\Sigma}] = [\mathbf{X}, \boldsymbol{\Sigma}]'\mathbf{E}_{\mathbf{X}_0}[\mathbf{X}, \boldsymbol{\Sigma}]. \tag{5.4}$$

Comparing both sides leads to

$$\mathbf{X}'\mathbf{E}_{\mathbf{X}_0}\mathbf{X} = \mathbf{0}, \qquad \mathbf{X}'\mathbf{E}_{\mathbf{X}_0}\boldsymbol{\Sigma} = \mathbf{0}, \qquad \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_0}\boldsymbol{\Sigma}. \tag{5.5}$$

The first equality in (5.5) is obviously equivalent to $\mathbf{E}_{\mathbf{X}_0}\mathbf{X} = \mathbf{0}$, i.e., $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}(\mathbf{X}_0)$. In this case, the second equality in (5.5) holds as well. Applying (1.13) and simplifying by EBMOs and $\mathscr{R}(\mathbf{X}) \subseteq \mathscr{R}(\mathbf{X}_0)$, we obtain

$$\begin{aligned} r(\boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{E}_{\mathbf{X}_0}\boldsymbol{\Sigma}) &= r(\boldsymbol{\Sigma}\mathbf{X}\mathbf{X}^{+}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}\mathbf{X}_0\mathbf{X}_0^{+}\boldsymbol{\Sigma}) \\ &= r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{0} & \mathbf{X}'\boldsymbol{\Sigma} \\ \mathbf{0} & -\mathbf{X}_0'\mathbf{X}_0 & \mathbf{X}_0'\boldsymbol{\Sigma} \\ \boldsymbol{\Sigma}\mathbf{X} & \boldsymbol{\Sigma}\mathbf{X}_0 & \mathbf{0} \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_0) \\ &= r\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{0} & \mathbf{X}'\boldsymbol{\Sigma} \\ \mathbf{X}_0'\mathbf{X} & -\mathbf{X}_0'\mathbf{X}_0 & \mathbf{X}_0'\boldsymbol{\Sigma} \\ \mathbf{0} & \boldsymbol{\Sigma}\mathbf{X}_0 & \mathbf{0} \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_0) \end{aligned}$$

$$
\begin{aligned}
&= r\begin{bmatrix} \mathbf{0} & \mathbf{X}'\mathbf{X}_0 & \mathbf{0} \\ \mathbf{X}_0'\mathbf{X} & -\mathbf{X}_0'\mathbf{X}_0 & \mathbf{X}_0'\boldsymbol{\Sigma} \\ \mathbf{0} & \boldsymbol{\Sigma}\mathbf{X}_0 & \mathbf{0} \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_0) \\
&= r\begin{bmatrix} \mathbf{X}_0'\mathbf{X}_0 & \mathbf{X}_0'\mathbf{X} & \mathbf{X}_0'\boldsymbol{\Sigma} \\ \mathbf{X}'\mathbf{X}_0 & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\Sigma}\mathbf{X}_0 & \mathbf{0} & \mathbf{0} \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_0) \\
&= r[\mathbf{X}_0'\mathbf{X}_0\ \mathbf{X}_0'\mathbf{X}, \mathbf{X}_0'\boldsymbol{\Sigma}] + r[\mathbf{X}_0'\mathbf{X}, \mathbf{X}_0'\boldsymbol{\Sigma}] - r(\mathbf{X}) - r(\mathbf{X}_0) \\
&= r[\mathbf{X}_0'\mathbf{X}, \mathbf{X}_0'\boldsymbol{\Sigma}] - r(\mathbf{X}) \\
&= r[\mathbf{X}_0'\mathbf{X}, \mathbf{X}_0'\boldsymbol{\Sigma}] - r(\mathbf{X}_0'\mathbf{X}).
\end{aligned}
$$

Setting the rank equality to zero leads to the equivalence of (i) and (ii) in (a). Under the condition $r[\mathbf{X}, \boldsymbol{\Sigma}] = n$, (5.3) is equivalent to $\mathbf{E_X} = \mathbf{E}_{\mathbf{X}_0}$, i.e., $\mathscr{R}(\mathbf{X}) = \mathscr{R}(\mathbf{X}_0)$, establishing the equivalence of (i) and (ii) in (b). □

Theorem 5.2. *Let* $\text{MINQUE}_{\mathscr{M}}(\sigma^2)$ *and* $\text{MINQUE}_{\mathscr{M}_m}(\sigma^2)$ *be as given in* (2.2) *and* (5.3) *with* $\mathbf{X} = \mathbf{X}_0$. *Then,*

(a) *The following statements are equivalent:*
 (i) $f\,\text{MINQUE}_{\mathscr{M}}(\sigma^2) = f_m\text{MINQUE}_{\mathscr{M}_m}(\sigma^2)$ *holds with probability* 1.
 (ii) $(\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X})^+$ *is a g-inverse of* $\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X}$.
 (iii) $r[\boldsymbol{\Sigma}_0\mathbf{E_X}(\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0)\mathbf{E_X}\boldsymbol{\Sigma}_0] = r(\mathbf{E_X}\boldsymbol{\Sigma}_0) - r(\mathbf{E_X}\boldsymbol{\Sigma})$.

(b) *Under the condition* $r[\mathbf{X}, \boldsymbol{\Sigma}] = n$, *the following statements are equivalent:*
 (i) $f\,\text{MINQUE}_{\mathscr{M}}(\sigma^2) = f_m\text{MINQUE}_{\mathscr{M}_m}(\sigma^2)$ *holds with probability* 1.
 (ii) $\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X} = \mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X}$.
 (iii) $r\begin{bmatrix} \boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma} & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{bmatrix} = 2r(\mathbf{X})$.

Proof. Note that both $\mathbf{E_X}(\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X})^+\mathbf{E_X}$ and $\mathbf{E_X}(\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X})^+\mathbf{E_X}$ are symmetric. Therefore, it can be seen from (2.2), (5.3) and Lemma 1.1 that $f\,\text{MINQUE}_{\mathscr{M}}(\sigma^2) = f_m\text{MINQUE}_{\mathscr{M}_m}(\sigma^2)$ holds with probability 1 if and only if

$$\boldsymbol{\Sigma}\mathbf{E_X}(\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X})^+\mathbf{E_X}\boldsymbol{\Sigma} = \boldsymbol{\Sigma}\mathbf{E_X}(\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X})^+\mathbf{E_X}\boldsymbol{\Sigma}. \tag{5.6}$$

Pre- and post-multiplying $\mathbf{E_X}$ on both sides of (5.6) gives

$$\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X} = \mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X}(\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X})^+\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X}. \tag{5.7}$$

Pre- and post-multiplying $\boldsymbol{\Sigma}\mathbf{E_X}(\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X})^+$ and $(\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X})^+\mathbf{E_X}\boldsymbol{\Sigma}$ on both sides of (5.7) gives (5.6). Thus, (5.6) and (5.7) are equivalent. Applying (1.13) to (5.7) and simplifying by EBMOs, we obtain

$$
\begin{aligned}
&r[\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X} - \mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X}(\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X})^+\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X}] \\
&\quad = r\begin{bmatrix} (\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X})^3 & \mathbf{E_X}\boldsymbol{\Sigma}_0(\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X}) \\ (\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X})\boldsymbol{\Sigma}_0\mathbf{E_X} & \mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X} \end{bmatrix} - r(\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X}) \\
&\quad = r\begin{bmatrix} \mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X} - \mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X} & \mathbf{0} \\ \mathbf{0} & \mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X} \end{bmatrix} - r(\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X}) \\
&\quad = r[\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X}(\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0)\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X}] + r(\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X}) - r(\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X}) \\
&\quad = r[\boldsymbol{\Sigma}_0\mathbf{E_X}(\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0)\mathbf{E_X}\boldsymbol{\Sigma}_0] + r(\mathbf{E_X}\boldsymbol{\Sigma}) - r(\mathbf{E_X}\boldsymbol{\Sigma}_0).
\end{aligned}
$$

By setting the right-hand sides equal to zero, we obtain the equivalence in (a). Under the condition $r[\mathbf{X}, \boldsymbol{\Sigma}] = n$, the equality $f\,\text{MINQUE}_{\mathscr{M}}(\sigma^2) = f_m\text{MINQUE}_{\mathscr{M}_m}(\sigma^2)$ holds with probability 1 if and only if

$$\mathbf{E_X}(\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X})^+\mathbf{E_X} = \mathbf{E_X}(\mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X})^+\mathbf{E_X},$$

which is equivalent to $\mathbf{E_X}\boldsymbol{\Sigma}\mathbf{E_X} = \mathbf{E_X}\boldsymbol{\Sigma}_0\mathbf{E_X}$ by (1.15), establishing the equivalence of (i) and (ii) in (b). Applying (1.11) to the equality leads to the equivalence of (ii) and (iii) in (b). □

6. Concluding remarks

We gave necessary and sufficient conditions for some equalities of the SEs and MINQUEs of the variance component σ^2 in (1.1), (1.3) and (1.4) to hold through simplifying matrix expressions associated with the equalities by ranks of matrices. The results obtained demonstrate a variety of new properties of the SEs and MINQUEs of the variance components.

In regression analysis, various possible equalities and decompositions of quadratic forms of random variables can be proposed, and necessary and sufficient conditions for the equalities and decompositions to hold need to be established. In these events, the matrix rank method can efficiently be used to characterize various equalities and decompositions of quadratic forms. Some recent work on the matrix rank method in showing the well-known Cochran's theorem and its extensions was given by Tian and Styan [9,10].

Acknowledgments

The authors thank two anonymous referees for their helpful comments and suggestions on this paper.

References

[1] J. Groß, A note on equality of MINQUE and simple estimator in the general Gauss–Markov model, Statist. Probab. Lett. 35 (1997) 335–339.
[2] G. Marsaglia, G.P.H. Styan, Equalities and inequalities for ranks of matrices, Linear Multilinear Algebra 2 (1974) 269–292.
[3] M. Nurhonen, S. Puntanen, A property of partitioned generalized regression, Comm. Statist. Theory Methods 21 (1992) 1579–1583.
[4] S. Puntanen, Some matrix results related to a partitioned singular linear model, Comm. Statist. Theory Methods 25 (1996) 269–279.
[5] C.R. Rao, Unified theory of linear estimation, Sankhyā, Ser. A 33 (1971) 371–394.
[6] C.R. Rao, S.K. Mitra, Generalized Inverse of Matrices and its Applications, Wiley, New York, 1971.
[7] Y. Tian, More on maximal and minimal ranks of Schur complements with applications, Appl. Math. Comput. 152 (2004) 675–692.
[8] Y. Tian, Rank equalities related to outer inverses of matrices and applications, Linear Multilinear Algebra 49 (2002) 269–288.
[9] Y. Tian, G.P.H. Styan, Cochran's statistical theorem for outer inverses of matrices and matrix quadratic forms, Linear Multilinear Algebra 53 (2005) 387–392.
[10] Y. Tian, G.P.H. Styan, Cochran's statistical theorem revisited, J. Statist. Plann. Inference 136 (2006) 2659–2667.
[11] S. Wang, M. Wu, W. Ma, Comparison of MINQUE and simple estimate of the error variance in the general linear models, Acta Math. Appl. Sin. Ser. B 19 (2003) 13–18.
[12] B. Zhang, The BLUE and MINQUE in Gauss–Markoff model with linear transformation of the observable variables, Acta Math. Sci. Ser. B 27 (2007) 203–210.